Correction in Banerjee, Kaniel, Kremer (2009) - Price Drift as an Outcome of Differences in Higher-Order Beliefs

For Example 3 in the paper, there is an error in the calculation on page 3718 when we write[1]

$$\mathbb{E}_{i,1}\left[P_2\right] = \frac{1}{4}\left(S_i + U\right) - \frac{\gamma}{2}\frac{1}{b}\left(P_1 - \frac{a}{2}\left(S_i + U_i\right)\right) = \frac{1}{4}\left(S_i + U\right) - \frac{\gamma}{2}Z_1. \tag{1}$$

Specifically, since investor i cannot directly observe Z_1, the second equality does not hold, and so we cannot plug it in when calculating the expectation $\mathbb{E}\left[P_2 - \mathbb{E}_1\left[P_2\right]|P_1 - P_0\right]$. As a result, in this example, when aggregate supply follows a random walk, prices do not exhibit drift.

The non-existence of drift in this example is a consequence of the assumption that aggregate supply shocks follow a random walk i.e., $Z_{t+1} = Z_t + z_{t+1}$. In particular, we show below, that when we allow for general persistence in supply shocks i.e., $Z_{t+1} = \alpha Z_t + z_{t+1}$, then prices exhibit drift in Example 3 when $\alpha < 1$ and Σ_z is sufficiently small.

We chose to model noise as a random walk not because we believe it is the natural model for noise but because we believed it would clearly highlight our effect. Specifically, when $\alpha < 1$, prices in the rational expectations benchmark mechanically exhibit reversals. We were hoping to convey the intuition for our results more transparently by shutting this mechanical effect down. However, in the special knife-edge case of $\alpha = 1$ there is in fact no drift or reversal.

To gain some intuition, it is useful to reconsider the expressions for prices in our economy:

$$P_2 = \bar{\mathbb{E}}\left[V\right] - \gamma\Sigma_V Z_2, \text{ and} \tag{2}$$

$$\begin{aligned} P_1 &= \left(\Sigma_V^{-1} + \Sigma_{P_2}^{-1}\right)^{-1}\left\{\Sigma_V^{-1}\bar{\mathbb{E}}\left[V\right] + \Sigma_{P_2}^{-1}\bar{\mathbb{E}}_1\left[P_2\right] - \gamma Z_1\right\} &(3)\\ &= \left(\Sigma_V^{-1} + \Sigma_{P_2}^{-1}\right)^{-1}\left\{\Sigma_V^{-1}P_2 + \Sigma_{P_2}^{-1}\bar{\mathbb{E}}_1\left[P_2\right] + \gamma\left(Z_2 - Z_1\right)\right\} &(4)\end{aligned}$$

This implies that

$$P_2 - P_1 = \left(\Sigma_V^{-1} + \Sigma_{P_2}^{-1}\right)^{-1}\left\{\Sigma_{P_2}^{-1}\left(P_2 - \bar{\mathbb{E}}_1\left[P_2\right]\right) - \gamma\left(Z_2 - Z_1\right)\right\}. \tag{5}$$

[1] We thank Albert "Pete" Kyle and Yajun Wang for bringing this to our attention.

When $Z_{t+1} = \alpha Z_t + z_{t+1}$, this implies that:

$$\mathbb{E}\left[P_2 - P_1 | P_1 - P_0\right] \propto \mathbb{E}\left[\Sigma_{P_2}^{-1}\left(P_2 - \bar{\mathbb{E}}_1\left[P_2\right]\right) + \gamma\left(Z_1 - Z_2\right) \middle| P_1 - P_0\right] \tag{6}$$

$$= \mathbb{E}\left[\begin{array}{c} \Sigma_{P_2}^{-1}\left(\bar{\mathbb{E}}\left[V\right] - \gamma\Sigma_V Z_2 - \bar{\mathbb{E}}_1\left[\bar{\mathbb{E}}\left[V\right] - \gamma\Sigma_V Z_2\right]\right) \\ +\gamma\left(1-\alpha\right)Z_1 \end{array} \middle| P_1 - P_0\right] \tag{7}$$

$$= \mathbb{E}\left[\begin{array}{c} \Sigma_{P_2}^{-1}\Sigma_V\left\{\Sigma_{\varepsilon}^{-1}\left(\bar{S} - \bar{\mathbb{E}}_1\left[\bar{S}\right]\right) - \gamma\alpha\left(Z_1 - \bar{\mathbb{E}}_1\left[Z_1\right]\right)\right\} \\ +\gamma\left(1-\alpha\right)Z_1 \end{array} \middle| P_1 - P_0\right] \tag{8}$$

Since P_1 provides information about $\bar{S}$ and Z_1, differences of opinion have offsetting effects on the $\left(\bar{S} - \bar{\mathbb{E}}_1\left[\bar{S}\right]\right)$ and $\left(Z_1 - \bar{\mathbb{E}}_1\left[Z_1\right]\right)$ errors. In the example, $\alpha = 1$ and these effects exactly offset each other, which leads to no price drift. However, as the next section illustrates, when $\alpha < 1$, the two effects do not offset perfectly, and so one can generate drift when $\Sigma_Z \to 0$. Specifically, the above expression highlights that when α and Σ_z are sufficiently small, then prices exhibit drift iff

$$\mathbb{E}\left[\left(\bar{S} - \bar{\mathbb{E}}_1\left[\bar{S}\right]\right) \middle| P_1 - P_0\right] = \kappa\left(P_1 - P_0\right) \tag{9}$$

for some positive κ. As such, we believe that the underlying economic intuition for our results are sound.[2]

Example 3 with Generalized Persistence

In the example, $V \sim N\left(0, 1/\tau_V\right)$, $\varepsilon_i \sim N\left(0, 1/\tau_\varepsilon\right)$. Suppose the aggregate supply of the asset be given by Z_1 and $Z_2 = \alpha Z_1 + z_2$, where $Z_1, z_2 \sim N\left(0, 1/\tau_z\right)$ and independent of each other and V and ε_i. In this example, $\tau_V = \tau_\varepsilon = 1$. Let $\bar{S} \equiv \int_i S_i di$. The objective distribution of $\bar{S} = V$, but according to investor i's interpretation, we have

$$\bar{S}_i \equiv_i \frac{1}{2}\left(S_i + U_i\right). \tag{10}$$

Date 2 price. At time 2, the conditional expectations are

$$\mathbb{E}_{i,1}\left[V\right] = \frac{\tau_\varepsilon}{\tau_V + \tau_\varepsilon} S_i = \frac{1}{2} S_i, \quad \Sigma_V \equiv \text{var}_{i,1}\left[V\right] = \frac{1}{\tau_V + \tau_\varepsilon} = \frac{1}{2}. \tag{11}$$

[2] For instance, with i.i.d. noise, $\alpha = 0$. In this case, the above condition is sufficient for drift when $\Sigma_z \to 0$.

Market clearing at date 1 implies that

$$
\begin{aligned}
P_2 &= \int_i \mathbb{E}_{i,1}\left[V\right] di - \gamma \Sigma_V Z_2 && (12)\\
&= \frac{\tau_\varepsilon}{\tau_V + \tau_\varepsilon}\bar{S} - \gamma \Sigma_V Z_2 && (13)\\
&= \frac{1}{2}\bar{S} - \frac{\gamma}{2}Z_2. && (14)
\end{aligned}
$$

The objective distribution of the price is

$$
P_2 = \frac{1}{2}V - \frac{\gamma}{2}Z_2, \tag{15}
$$

while investor i's subjective beliefs about the date 2 price is

$$
P_2 =_i \frac{1}{2}\left(\frac{1}{2}\left(S_i + U_i\right)\right) - \frac{\gamma}{2}Z_2. \tag{16}
$$

Date 1 price. Conjecture that the date 1 price is of the form

$$
P_1 = a\bar{S} + bZ_1. \tag{17}
$$

According to investor i, this has a distribution

$$
P_1 =_i a\left(\frac{1}{2}\left(S_i + U_i\right)\right) + bZ_1. \tag{18}
$$

Investor i observes U_i,S_i and P_1. This implies:

$$
\begin{aligned}
\mathbb{E}_{i,1}\left[\bar{S}|U_i, S_i, P_1\right] &= \frac{1}{2}\left(S_i + U_i\right) && (19)\\
\mathbb{E}_{i,1}\left[Z_1|U_i, S_i, P_1\right] &= \frac{1}{b}\left(P_1 - a\mathbb{E}_{i,1}\left[\bar{S}|U_i, S_i, P_1\right]\right) && (20)\\
&= \frac{1}{b}\left(P_1 - \frac{a}{2}\left(S_i + U_i\right)\right) && (21)
\end{aligned}
$$

The above expressions imply

$$
\begin{aligned}
\mathbb{E}_{i,1}\left[P_2|U_i, S_i, P_1\right] &= \frac{1}{2}\mathbb{E}_{i,1}\left[\bar{S}|U_i, S_i, P_1\right] - \frac{\gamma}{2}\alpha\mathbb{E}_{i,1}\left[Z_1|U_i, S_i, P_1\right] && (22)\\
&= \frac{1}{4}\left(S_i + U_i\right) - \frac{\alpha\gamma}{2}\left(\frac{1}{b}\left(P_1 - \frac{a}{2}\left(S_i + U_i\right)\right)\right) && (23)
\end{aligned}
$$

and

$$\mathrm{var}_{i,1}\left[P_2|U_i,S_i,P_1\right]=\Sigma_{P_2}=\frac{1}{4}\frac{\gamma^2}{\tau_z}. \tag{24}$$

This implies

$$\bar{\mathbb{E}}_1\left[P_2\right]=\frac{1}{4}\bar{S}-\frac{\alpha\gamma}{2}\left(\frac{1}{b}\left(P_1-\frac{a}{2}\left(\bar{S}\right)\right)\right) \tag{25}$$

and so

$$P_1=\frac{\Sigma_V^{-1}\bar{\mathbb{E}}_1\left[V\right]+\Sigma_{P_2}^{-1}\bar{\mathbb{E}}_1\left[P_2\right]-\gamma Z_1}{\Sigma_V^{-1}+\Sigma_{P_2}^{-1}} \tag{26}$$

$$=\bar{S}\left(\frac{1}{2}-\frac{a\alpha\gamma\tau_z+b\tau_z}{2b\gamma^2+4b\tau_z}\right)-\frac{\left(\gamma^3+2\alpha\gamma\tau_z\right)}{2\gamma^2+4\tau_z}Z_1 \tag{27}$$

This implies the price coefficients a and b can be solved for using the system of two equations:

$$a=\frac{1}{2}-\frac{a\alpha\gamma\tau_z+b\tau_z}{2b\gamma^2+4b\tau_z},\quad b=-\frac{\left(\gamma^3+2\alpha\gamma\tau_z\right)}{2\gamma^2+4\tau_z} \tag{28}$$

which implies the solution is:

$$a=\frac{\left(\gamma^2+\tau_z\right)\left(\gamma^2+2\alpha\tau_z\right)}{2\left(\gamma^2+2\tau_z\right)\left(\gamma^2+\alpha\tau_z\right)},\quad b=-\frac{\gamma^3+2\alpha\gamma\tau_z}{2\left(\gamma^2+2\tau_z\right)} \tag{29}$$

- Note that for the special case of random walk (as in the paper) $\alpha=1$, and the above reduce to

$$a=\frac{1}{2},\quad b=-\frac{\gamma}{2} \tag{30}$$

- For the other extreme of i.i.d. aggregate supply shocks (i.e., $\alpha=0$), the above reduce to

$$a=\frac{\gamma^2+\tau_z}{2\left(\gamma^2+2\tau_z\right)},\quad b=-\frac{\gamma^3}{2\left(\gamma^2+2\tau_z\right)}. \tag{31}$$

To summarize,

$$P_2=\frac{1}{2}\bar{S}-\frac{\gamma}{2}Z_2 \tag{32}$$

$$P_1=a\bar{S}+bZ_1, \tag{33}$$

where

$$a=\frac{\left(\gamma^2+\tau_z\right)\left(\gamma^2+2\alpha\tau_z\right)}{2\left(\gamma^2+2\tau_z\right)\left(\gamma^2+\alpha\tau_z\right)},\quad b=-\frac{\gamma^3+2\alpha\gamma\tau_z}{2\left(\gamma^2+2\tau_z\right)}. \tag{34}$$

Now,

$$
\begin{aligned}
\mathbb{E}\left[P_2 - P_1 | P_1 - P_0\right] &= \mathbb{E}\left[\left(\frac{1}{2} - a\right)\bar{S} - \frac{\gamma}{2}z_2 - \left(\alpha\frac{\gamma}{2} + b\right)Z_1 | P_1 - P_0\right] && (35)\\
&\propto \operatorname{cov}\left(\left(\frac{1}{2} - a\right)\bar{S} - \frac{\gamma}{2}z_2 - \left(\alpha\frac{\gamma}{2} + b\right)Z_1, a\bar{S} + bZ_1\right) && (36)\\
&= \left(\frac{1}{2} - a\right)a\operatorname{var}\left(\bar{S}\right) - \left(\alpha\frac{\gamma}{2} + b\right)b\operatorname{var}\left(Z_1\right) && (37)\\
&= \frac{(\alpha-1)\gamma^2\left(\gamma^2 + 2\alpha\tau_z\right)\left(\gamma^6 + \left(\alpha^2 - 1\right)\gamma^2\tau_z^2 + 2\alpha\gamma^4\tau_z - \tau_z^3\right)}{4\tau_z\left(\gamma^2 + 2\tau_z\right)^2\left(\gamma^2 + \alpha\tau_z\right)^2} \equiv \rho && (38)
\end{aligned}
$$

Note that there is price drift iff $\rho > 0$, since

$$
\mathbb{E}\left[P_2 - P_1 | P_1 - P_0\right] = \frac{\rho}{\operatorname{var}\left[P_1 - P_0\right]}\left(P_1 - P_0\right). \qquad (39)
$$

- Note that when $\alpha = 1$, the above implies $\rho = 0$. This is the result in your note.
- When $\alpha = 0$, we have
$$
\rho = \frac{\tau_z^3 + \gamma^2\tau_z^2 - \gamma^6}{4\tau_z\left(\gamma^2 + 2\tau_z\right)^2} \qquad (40)
$$
which implies that $\rho > 0$ when τ_z is sufficiently large.
- More generally, we can show that $\rho > 0$ iff τ_z is sufficiently large. Mathematica implies:
$$
\tau_z > \frac{\left(1 + \sqrt{5}\right)\gamma^2}{2} \qquad (41)
$$
is sufficient. A weaker sufficient condition is $\tau_z > 2\gamma^2$.

The Figure highlights that when $\alpha < 1$ and τ_z is sufficiently large, prices exhibit drift.

Figure 1: A plot of ρ vs. α

Other parameters are $\tau_V = \tau_\varepsilon = \gamma = 1$.