# Strategic trading and unobservable information acquisition

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**Abstract**

We allow a strategic trader to choose when to acquire information about an asset's payoff, instead of endowing her with it. When trading is sufficiently frequent, the market can break down — we show there cannot exist equilibria if information costs are lumpy and acquisition cannot be publicly observed. In any conjectured equilibrium, the trader has two profitable deviations: (i) she can pre-emptively acquire information, or (ii) she can delay acquisition to defer the acquisition cost without incurring a loss in trading gains. We explore the robustness of our results and suggest approaches to reconcile strategic trading with endogenous information acquisition.

JEL: D82, D84, G12, G14

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# 1 Introduction

The canonical strategic trading framework, introduced by Kyle (1985), is foundational for understanding how markets incorporate private information. The vast literature that builds on this framework provides many important insights into how informed investors trade strategically on their private information in a variety of market settings and the consequences for asset prices.[1] The framework has also been used to guide a large body of empirical analysis and policy recommendations about liquidity, price informativeness, market design, and disclosure regulation.

However, a key limitation of this setting is that the strategic trader is endowed with private information before trading begins, instead of acquiring it endogenously at a time of her choosing. This assumption is restrictive because the value of acquiring private information can change over time and with economic conditions. For instance, a falling real estate market may lead a bank to acquire loan-level data on its mortgage-backed securities in order to revalue its positions. A consolidation wave in a particular industry may lead a merger arbitrage hedge fund to investigate remaining firms as potential targets. As such, it is natural that investors optimize not only how they trade on their private information, but also when they acquire such information.

To study this behavior, we extend the continuous-time Kyle (1985) framework to allow for unobservable, costly information acquisition. A single risky asset is traded by a risk-neutral, strategic trader and a mass of noise traders. The asset payoff is publicly revealed at a random time, and can depend on publicly observable "news."[2,3] A risk-neutral market maker competitively sets the asset's price, conditional on the public news and aggregate order flow. In contrast to the previous literature, we do not assume the strategic trader is endowed with private information. Instead, she can choose to pay a fixed cost (á la Grossman and Stiglitz (1980)) to acquire payoff relevant information at any point in time. Crucially, the market maker cannot observe or immediately detect when the trader acquires information — instead, this must be inferred from the observed order flow.[4]

[1]The literature is truly vast — as of July 2019, the paper has over 10,400 citations on Google Scholar.

[2]The assumption of a random horizon is largely for tractability and is not qualitatively important for our primary results. What is key is that a random horizon induces the trader to discount future profits. We expect our results to carry over to settings with fixed horizon that feature discounting for other reasons (e.g., if the trader has a subjective discount factor or the risk-free rate is nonzero). Section 4.5 discusses how our results are affected when there is no discounting.

[3]The (potential) presence of payoff-relevant public news does not drive any of our results. Rather, entertaining the possibility of public signals allows us to parsimoniously nest a broad class of settings that have been considered in the literature.

[4]As we shall see, the unobservability of acquisition is key for our results. In a companion paper (Banerjee and Breon-Drish, 2018), we consider a setting where entry (and the associated information acquisition) is publicly observable, and explore the implications of allowing the strategic trader to choose when to enter a

We show that allowing for unobservable, dynamic information acquisition can lead to equilibrium breakdown. First, we find that there cannot exist any equilibria in which information acquisition follows a pure strategy. Importantly, this implies that the standard, strategic trading equilibrium (as in Kyle (1985) and other related papers) do not arise as an endogenous outcome of costly information acquisition. Second, we also rule out the existence of equilibria in which the strategic trader mixes between acquiring information at different points in time. These include equilibria that involve "discrete" mixing in which the trader acquires at a countable collection of times and/or "continuous" mixing in which the trader acquires information with a given intensity over some interval of time.

We then explore the robustness of our results to a number of alternative settings. First, we show that risk-neutrality of the strategic trader and competitiveness of the market maker are not critical for our results — the non-existence of equilibrium arises even when these assumptions are relaxed. Second, we argue that non-existence arises even if the trader starts with an initial, endowed information advantage and can optimally choose when to acquire additional information. Next, we establish that non-existence is not an artifact of the continuous-time setting, but arises in discrete time when trading is sufficiently frequent. Fourth, we also show that without discounting, non-existence can obtain unless the cost of information is sufficiently low. Finally, we show that the assumption of "lumpy" information costs is important: when the cost of information acquisition does not have a fixed component, we find that an equilibrium with endogenous, unobservable costly information acquisition can be sustained.[5]

Our non-existence results uncover two key economic forces that apply more generally to settings with dynamic information acquisition. First, if a trader can acquire information earlier than the market anticipates without being detected, she can exploit her informational advantage by trading against a pricing rule that is insufficiently responsive. We refer to this as the **pre-emption deviation**. While particularly transparent in our analysis, this force is likely to rule out pure-strategy acquisition equilibria with delay in more general settings (e.g., with multiple strategic or non-strategic traders).

Second, because she optimally smooths her trades over time, a strategic trader's gains from being informed over any *short* period are small when trading opportunities are frequent. In continuous time this leads to "trade timing indifference" — over any finite interval of time, an informed strategic trader is indifferent between trading along her equilibrium strategy or

new trading opportunity.

[5]Specifically, we consider the special case of our framework studied by Caldentey and Stacchetti (2010), and show that an equilibrium exists if the strategic trader can pay a flow cost to dynamically optimize the precision of a flow of signals, but there is no fixed cost involved when going from zero precision (i.e., no information) to strictly positive precision.

refraining from trade over that interval and then trading optimally going forward.[6] However, this implies that instead of acquiring information at the time prescribed by (a conjectured) equilibrium, she can instead wait over an interval, and then acquire information. This **delay deviation** is strictly profitable since the trader does not incur a loss in trading gains, but benefits (in present value) by delaying the cost of information acquisition.

When information acquisition is unobservable, the pre-emption deviation applies quite generally and restricts the set of possible equilibria to those in which information is acquired immediately (with positive probability). We show that the delay deviation also applies more generally (e.g., when the market maker is not competitive, when the strategic trader is risk-averse, or when she is initially endowed with some information), since any equilibrium must satisfy trade timing indifference. Moreover, while trade timing indifference does not hold exactly in a discrete time setting, we show that a similar delay deviation obtains when the period between trades is sufficiently small — in this case, although the loss in trading gains from delay is non-zero, it is of a smaller order of magnitude than the gain from delaying the acquisition cost.

In the canonical, continuous time Kyle (1985) model without discounting, we show that the market breaks down unless information costs are sufficiently low. Recall that the Kyle trading equilibrium implicitly assumes that information is acquired immediately with probability one, since the market maker *knows* the strategic trader is informed when the market opens. However, if information is sufficiently costly and information choices are endogenous, the trader prefers to deviate by never acquiring information. But never acquiring information cannot be an equilibrium either: in this case, the price does not respond to order flow and the trader should deviate by acquiring information and trading to make arbitrarily large profits.

Finally, we show that when the cost of information acquisition does not have a fixed component (i.e., when the trader dynamically chooses a signal precision subject to a flow cost), an equilibrium with endogenous information acquisition can be sustained. Specifically, we consider a generalization of Caldentey and Stacchetti (2010) where the strategic trader can optimally choose the precision of a flow of private information, but does not incur a fixed cost of acquisition. As a result, the delay deviation is no longer strictly profitable. In this case, we establish the existence of an equilibrium with endogenous, unobservable information acquisition. This result implies that one approach to reconciling standard strategic trading

[6] This property of the standard Kyle (1985) framework was first noted by Back (1992). However, it also applies to settings in which the strategic trader is risk-averse, or the market maker is risk averse and/or not perfectly competitive. Economically, timing indifference arises because, in equilibrium, there cannot be any predictability in the level or slope of the price function if the trader refrains from trading. If there were, then she could deviate from any proposed trading strategy by waiting and then exploit such predictability.

models with endogenous information acquisition is to restrict attention to such "smooth" information acquisition technologies which do not involve lumpy costs. Whether such a restriction is appropriate is an empirical question. For instance, it may be reasonable to approximate increased attention or scrutiny by existing market participants using such smooth information acquisition. On the other hand, entry into new markets or asset classes are more likely to involve fixed, or lumpy, costs of information acquisition.

Our paper is at the intersection of two closely related literatures. The first follows Kyle (1985) and focuses on strategic trading by investors who are *exogenously* endowed with information about the asset payoff (e.g., see Back (1992), Back and Baruch (2004), Caldentey and Stacchetti (2010), and Collin-Dufresne and Fos (2016)). The second follows Grossman and Stiglitz (1980) and studies endogenous information acquisition in financial markets. However, in contrast to our setting, these papers restrict investors to acquire information (or commit to a sequence of signals) before the start of trade.[7] A notable exception is Banerjee and Breon-Drish (2018), who allow for dynamic information acquisition and entry that is detectable by the market maker. Our paper is also related to a recent literature that studies markets in which some participants face uncertainty about the existence or informedness of others (e.g., Chakraborty and Yılmaz (2004), Alti, Kaniel, and Yoeli (2012), Li (2013), Banerjee and Green (2015), Back et al. (2017), Wang and Yang (2016)).

Since our framework nests a number of influential models of strategic trading in the literature, our non-existence results highlight an important limitation on applying such models to economic settings in which costly, unobservable information acquisition is an important feature. Our analysis implies that: (i) a reduced-form approach that treats the information acquisition decision as unmodeled and simply endows the trader with information may be inconsistent, since the resulting, standard financial market equilibrium cannot arise as an endogenous outcome with information acquisition, and (ii) an approach that explicitly models information acquisition may not be capable of sustaining an overall equilibrium in which the trading equilibria resemble those studied in the literature. Our analysis also proposes settings when endogenous information acquisition can be reconciled with strategic trading including when there are only flow costs of information acquisition, or when information acquisition is detectable.

The rest of the paper is organized as follows. Section 2 presents the general framework for our analysis. Section 3 presents the main results of the paper, starting with a discussion

[7]For instance, Back and Pedersen (1998) and Holden and Subrahmanyam (2002) allow investors to precommit to receiving signals at particular dates, while Kendall (2018), Dugast and Foucault (2018), and Huang and Yueshen (2018) incorporate a time-cost of information. Veldkamp (2006) considers a sequence of one-period information acquisition decisions. In all these cases, the information choices occur before trading and so acquisition is effectively a static decision.

of the key economic forces and then establishing sufficient conditions for trade timing indifference to obtain. Section 4 explores the robustness of our non-existence results to alternate assumptions: (i) alternate preferences (or competitive environment) for the market maker, (ii) risk-aversion of the strategic trader, (iii) the case where the strategic trader is initially endowed with some information, (iv) non-continuous trading opportunities, (v) no discounting, and (vi) smooth information acquisition technologies without fixed costs. Section 5 concludes by discussing some implications for future work. Unless otherwise mentioned, all proofs are in the Appendix.

# 2 Model

Our framework is based on the continuous-time Kyle (1985) model with random horizon in Back and Baruch (2004) and Caldentey and Stacchetti (2010). Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which is defined an $(n+1)$-dimensional standard Brownian motion $\bar{W} = (W_1, \ldots, W_n, W_Z)$ with filtration $\mathcal{F}_t^W$, independent random variables $S$ and $T$ and independent $m$-dimensional random vector $\nu_0$. Let $\mathcal{F}_t$ denote the augmentation of the filtration $\sigma(\nu_0, \{\bar{W}_s\}_{\{0 \le s \le t\}})$. Suppose that the random variable $T$ is exponentially distributed with rate $r$, and that $S \in \mathcal{S} \equiv \text{Support}(S)$ and $\nu_0$ have finite second moments. Finally, let $W = (W_1, \ldots, W_n)$ denote the first $n$ elements of $\bar{W}$.

There is an $m \geq 0$ dimensional vector of publicly-observable signals $\nu_t = (\nu_{1t}, \ldots, \nu_{mt})$ with initial value $\nu_0$ and which follows

$$d\nu = \mu(t, \nu)\, dt + \Sigma(t, \nu)\, dW_t,$$

where $\mu_t = (\mu_{1t}, \ldots, \mu_{mt})$ and $\Sigma_t = (\Sigma'_{1t}, \ldots, \Sigma'_{mt})$ denote the vector of drifts and matrix of diffusion coefficients. Suppose that $\mu$ and $\Sigma$ are such that there exists a unique strong solution to this set of stochastic differential equations (SDEs).[8]

There are two assets: a risky asset and a risk-free asset with interest rate normalized to zero. The risky asset pays off a terminal value at random time $T$. We assume that, given knowledge of the signal $S$ and the history of $\nu_t$, the conditional expected value $v_t$ of this payoff as of time $t$ is

$$v_t = f(t, \nu_t, S)$$

for some function $f$ that, for each $s \in \mathcal{S}$, is continuously differentiable in $t$ and twice

[8]See for instance Theorem 5.2.9 in Karatzas and Shreve (1998) who present Lipschitz and growth conditions on the coefficients that are sufficient to deliver this result in a Markovian setting.

continuously differentiable in $\nu$. We assume further that $f$ is such that $v_t$ is a martingale for an agent informed of $S$.[9] There is a single, risk-neutral strategic trader who can pay cost $c$ at any time $\tau$ to observe $S$.[10] Let $X_t$ denote the cumulative holdings of the trader, and suppose the initial position $X_0 = 0$. Following Back (1992), we require the trading strategy to be a semi-martingale adapted to her filtration (we discuss this in more detail below), which is a minimal condition for stochastic integration with respect to $X$ to be well-defined.

In addition to the strategic trader, there are noise traders who hold $Z_t$ shares of the asset at time $t$, where

$$dZ_t = \sigma_Z(t, \nu, Z)\, dW_{Zt}, \tag{1}$$

with $\sigma_Z(\cdot) > 0$ such that there exists a unique strong solution to this SDE. We explicitly allow for the possibility that the noise volatility depends on the news process, as well as the current cumulative noise trader holdings.

Finally, there is a competitive, risk neutral market maker who sets the price of the risky asset equal to the conditional expected payoff given the public information set. Let $\mathcal{F}_t^P$ denote the public information filtration, which is that induced from observing the aggregate order flow process $Y_t = X_t + Z_t$ and the public news $\nu_t$, i.e., $\mathcal{F}_t^P$ is the augmentation of the filtration $\sigma(\{\nu_t, Y_t\})$. Crucially, the mere fact that the trader has (or has not) acquired information is not directly observable; rather, as part of updating her beliefs about the asset value, the market maker must also use the public signals and the order flow to update about whether the trader has, in fact, acquired information. The price at time $t < T$ is given by

$$P_t = \mathbb{E}\left[v_t \middle| \mathcal{F}_t^P\right]. \tag{2}$$

We will place more structure on the class of pricing rules in Subsection 3.2 when we specialize to Markovian equilibria.Let $\mathcal{T}$ denote the set of $\mathcal{F}_t^P$ stopping times. We require that the trader's information acquisition time $\tau \in \mathcal{T}$. That is, we require acquisition to depend only on public information up to that point. Let $\mathcal{F}_t^I$ denote the augmentation of the filtration $\sigma(\mathcal{F}_t^P \cup \sigma(S))$. Thus, $\mathcal{F}_t^I$ represents the trader's information set, post-information acquisition.

[9] A simple set of sufficient conditions is: for each $s \in \mathcal{S}$, $f_t + f_\nu \cdot \mu + \frac{1}{2} tr(f_{\nu\nu}\Sigma\Sigma') = 0$ and $\mathbb{E}[\int_0^\infty f_\nu' \Sigma\Sigma' f_\nu du] < \infty$, which guarantee that $v$ is a (square integrable) martingale. Note also that because all market participants are risk-neutral, it is without loss of generality, economically, that $v_t$ is a martingale, that $v_t$ represents the conditional expected value of the asset rather than the value itself, and that we treat $v_T$ as the terminal value.

[10] The assumption that the trader observes a a single, though possibly multidimensional, "lump" of information is not crucial for our results. What is crucial is that obtaining information involves paying a fixed cost. As such, our results are robust to information technologies that provide the trader with a future flow of signals in return for an upfront cost of $c$. We discuss this in more detail in Section 4.

To be *admissible*, we require, as a minimal condition, that the trader's trading strategy be a semi-martingale adapted to her filtration. That is, her pre-acquisition trading strategy must be adapted to $\mathcal{F}_t^P$ and her post-acquisition strategy adapted to $\mathcal{F}_t^I$. We will place additional structure on the class of admissible trading strategies in Subsection 3.2 when we specialize to Markovian equilibria.

Note that to condense notation, we will often denote the time-$t$ public conditional expectation operator as $\mathbb{E}_t[\cdot]$ and the time-$t$ conditional expectation operator for a trader who has observed $S = s$ as $\mathbb{E}_t^s[\cdot]$. If super- and sub-scripts are omitted, the expectation should be understood as an unconditional one. We will also use the standard notation that a function or stochastic process evaluated at $t^-$ denotes the left limit, e.g., $X_{t^-} = \lim_{s \uparrow t} X_s$.

Our definition of equilibrium is standard, but modified to account for endogenous information acquisition.

**Definition 1.** *An equilibrium is a (i) probability distribution over stopping times in $\mathcal{T}$ and an admissible trading strategy $X_t$ for the trader and (ii) a price process $P_t$, such that, given the trader's strategy the price process satisfies (2) and, given the price process, the trading strategy and acquisition strategy maximize the expected profit*

$$\mathbb{E}\left[(v_T - P_T)\, X_T + \int_{[0,T]} X_{u^-} dP_u\right]. \tag{3}$$

The definition of equilibrium allows the trader to follow mixed acquisition strategies by randomizing over stopping times in $\mathcal{T}$. That is, at the beginning of the game, the trader randomly chooses a stopping time according to some probability distribution, and follows the realized strategy for the duration of the game.[11] Importantly, note that such a strategy can involve both "continuous" mixing in which the trader acquires information with a given intensity over an interval of times, as well as "discrete" mixing in which the trader acquires at a countable collection of times. If the probability distribution over $\mathcal{T}$ is degenerate, then the equilibrium is one of pure-strategy acquisition.

Finally, we let $J^s(t, \cdot), s \in \{U\} \cup Support(S)$ denote the value function for a trader of type $s$ and note that it can be expressed as

$$J^s(t, \cdot) = \mathbb{E}\left[(v_T - P_T)\, X_T + \int_{[t,T]} X_{u^-} dP_u\right]$$

[11] There are multiple, equivalent ways of defining randomization over stopping times. Aumann (1964) introduced the notion of randomizing by choosing a stopping time according to some probability distribution at the start of the game. Touzi and Vieille (2002) treat randomization by identifying the stopping strategy with an adapted, non-decreasing, right-continuous processes on $[0, 1]$ that represents the cdf of the time that stopping occurs. Shmaya and Solan (2014) show, under weak conditions, that these definitions are equivalent.

$$
\begin{aligned}
&= \mathbb{E}\left[\int_{[t,T]} (v_u - P_{u^-})\, dX_u + [X, v-P]_{[t,T]} + \int_{[t,T]} X_{u^-} dv_u\right] \\
&= E\left[\int_{[t,\infty)} e^{-r(u-t)} (v_u - P_{u^-})\, dX_u + \int_{[t,\infty)} e^{-r(u-t)} d[X, v-P]_u + \int_{[t,\infty)} e^{-r(u-t)} X_{u^-} dv_u\right]
\end{aligned}
$$

where the first equality follows from the integration by parts formula for semi-martingales and the second from the fact that $T$ is independently exponentially distributed.

It is important to note that our baseline setup nests a number of existing settings in the literature. For example, Back and Baruch (2004) consider the case in which $S \in \{0, 1\}$ has a binomial distribution, there are no publicly observable signals, and $v_t = f(t, \nu_t, S) = S$. In the special case of Caldentey and Stacchetti (2010) in which time is continuous and there is no ongoing flow of private information, $S \sim \mathcal{N}(0, \Sigma_0)$ and $v_t = f(t, \nu_t, S) = S$.[12] In contrast to these earlier models, in which the insider is endowed with private information about the asset value, our focus is on allowing her to acquire information at a time of her choosing. Banerjee and Breon-Drish (2018) also allow for dynamic information acquisition (and entry) and consider a special case in which $S \in \{l, h\}$, $\nu$ follows a geometric Brownian motion with zero drift, and $v_t = f(t, \nu_t, S) = \nu\xi_S$, where $\xi$ is a constant that depends on the realization of $S$. Importantly, however, they assume that the entry decision can be detected by the market maker.

# 3 Main Results

This section presents the main economic results of our paper. The first subsection highlights the key economic forces that give rise to non-existence of equilibria in our setting while placing minimal restrictions on the class of candidate equilibria under consideration. First, we show in a very general sense that there cannot exist pure strategy equilibria in which acquisition occurs after the beginning of the game. Intuitively, given a proposed equilibrium acquisition date, the strategic trader finds it optimal to deviate by preemptively acquiring information earlier and trading against an unresponsive pricing rule. Second, we show that in any setting with "trade timing indifference," (defined below) the strategic trader can profitably deviate by waiting — while her expected trading gains are unaffected due to the indifference condition, she benefits by delaying the cost of acquisition. This rules

[12] Note that because the strategic trader receives only a "lump" of private information, our model is not subject to the Caldentey and Stacchetti (2010) critique that the continuous-time equilibrium is not the limit of corresponding discrete time equilibria. However, all of the results below easily extend to time-varying signal $S_t$ when "acquiring information" entails paying $c$ to either perfectly observe $S_t$ at time $\tau$, or to observe it from from time $\tau$ forward. In fact, we show below that our nonexistence results extend to an analogous discrete-time model if the time between trading rounds is sufficiently small.

out the existence of both pure strategy equilibria with immediate acquisition and mixed strategy equilibria. The second subsection establishes conditions under which trade timing indifference arises naturally.

## 3.1 Key Economic Forces

We begin by considering equilibria in which information acquisition follows a pure strategy. Our first observation is immediate: never acquiring information cannot be an equilibrium.

**Proposition 1.** *There does not exist an equilibrium in which the trader follows a pure acquisition strategy that, with probability 1, never acquires information That is, in which* $\mathbb{P}(\tau = \infty) = 1$.

*Proof.* In an equilibrium in which the strategic trader never acquires information, the price is insensitive to order flow at all times. But this allows the trader to deviate from the conjectured equilibrium strategy by acquiring information, trading at an arbitrarily large rate with zero price impact, and generating unboundedly large profits. Since acquisition is unobservable by the market maker, she cannot respond to the deviation by adjusting the price impact. □

A similar argument rules out pure strategy equilibria in which information is acquired with some delay.

**Proposition 2.** *(Pre-emption Deviation) There does not exist an equilibrium in which the trader follows a pure acquisition strategy that acquires information after time* $t = 0$ *with positive probability. That is, in which* $\tau \geq 0$ *with* $\mathbb{P}(\tau > 0) > 0$.

*Proof.* Again, suppose that there does exist such an equilibrium. Then in equilibrium, the order flow is completely uninformative about $S$ prior to time $\tau$ and therefore the pricing rule is insensitive to order flow before $\tau$. But in the event $\{\tau > 0\}$ (which occurs with strictly positive probability), the strategic trader can once again profitably deviate by unobservable acquiring information prior to $\tau$ and trading at an arbitrarily large rate with zero price impact, thereby generating unbounded profits. □

Intuitively, the previous two results follow from the fact that when acquisition cannot be detected, the strategic trader cannot commit to acquiring information at a future date: she always finds it profitable to deviate by pre-empting herself and acquiring information earlier. The inability to commit to the equilibrium strategy is reminiscent of the durable-good monopolist's inability to commit to high prices in the future (see Coase (1972) and Gul et al. (1986)). However, in our setting, the lack of commitment leads to non-existence of pure-strategy equilibria with delay in information acquisition. Moreover, this incentive to preemptively acquire information is likely to apply more generally, e.g., in settings with multiple investors, in discrete time, and in settings with a fixed terminal date.

The above results imply that with unobservable acquisition, the only remaining candidates for a equilibrium are those in which (i) the trader follows a pure acquisition strategy that acquires immediately, $\mathbb{P}(\tau = 0) = 1$, or (ii) the trader follows a mixed acquisition strategy. Note that in a non-degenerate mixed strategy equilibrium, any stopping time $\tau$ in the set of stopping times on which the trader's acquisition strategy places positive probability must have strictly positive probability of acquisition in *any* neighborhood of zero (i.e., for *any* $\Delta > 0$, $\mathbb{P}(\tau \in (0,\Delta)) > 0)$). If any such stopping times did not satisfy this, then, conditional on that stopping time being realized from the original mixing randomization, the trader could profitably deviate by preempting the conjectured strategy as in Proposition 2. Because both of the remaining candidate equilibria have $\mathbb{P}(\tau \in [0,\Delta)) > 0)$, if we can rule out equilibria with such a property, then we have eliminated all candidate equilibria. Our next result establishes that if, in equilibrium, an informed trader's problem exhibits **"trade timing indifference**," then we can, in fact, rule out such equilibria.

Before doing so, we precisely define trade timing indifference:

**Definition 2.** An equilibrium features ***trade timing indifference*** if at any date $t$ and for each $\Delta > 0$, the change in expected profit for an informed trader over the interval $[t, t+\Delta)$, if she does not trade over this interval and follows her conjectured equilibrium strategy afterwards, is zero. i.e., if $dX_t^s = 0$, $t \in [t,(t+\Delta))$ implies[13]

$$\mathbb{E}_t^s\left[e^{-r\Delta} J^s\left((t+\Delta)^-, \cdot\right) - J^s\left(t^-, \cdot\right)\right] = 0. \tag{4}$$

As first noted by Back (1992), the above is a key feature of continuous-time Kyle models: over any finite interval of time, the trader is indifferent between playing her equilibrium trading strategy or refraining from trade over that interval and then trading optimally from that time forward. Economically, this result arises because an equilibrium pricing rule must be such that the trader does not perceive any predictability in the price level or price impact if she refrains from trading. Otherwise, she would have a profitable deviation from her conjectured equilibrium *trading* strategy. As we shall establish in Section 3.2, it arises

[13]There are two points in this definition that are worth clarifying. First, the $^-$ signs in (4) arise because we have not restricted the trader to smooth strategies. In principle, she could acquire at time $t$ and then immediately submit a discrete order. Because $J(t,\cdot)$ represents her forward-looking expected profit, we must evaluate at $t^-$ to capture any discrete trade at $t$ in the value function. Secondly, strictly speaking we require (4) to hold at any $\left\{\mathcal{F}_t^P\right\}$ stopping time $\tau$. Of course, since $\tau = t$ is a well defined stopping time for any given $t$, this implies (4).

naturally in any (Markovian) equilibrium of our model. Moreover, as we highlight in Section 4, it also arises in other related settings.

The next result establishes the non-existence of equilibria that feature trade timing indifference and endogenous information acquisition.

**Proposition 3.** *(Delay Deviation) Fix any* $\Delta > 0$*. There does not exist an equilibrium in which (i) trade timing indifference holds, and (ii) the strategic trader acquires information with positive probability in* $[0, \Delta)$*, i.e., in which* $\mathbb{P}(\tau \in [0, \Delta)) > 0$*).*

The argument relies on the following deviation, which, in fact, holds at all times of conjectured acquisition, even those outside of $[0, \Delta)$: instead of acquiring information at stopping time $\tau$, the strategic trader can instead *wait* over the interval $[\tau, \tau + \Delta)$, during which she does not acquire and does not trade, and then acquire information. Given that future periods are discounted (due to the stochastic horizon $T$), she benefits from delaying the cost of acquisition, but forgoes trading gains. Due to trade timing indifference, the expected loss in trading gains is zero. However, since discounted trading costs are of order $\Delta$, the deviation leaves the trader strictly better off.

*Proof.* Suppose there is an equilibrium and let $\tau \in \mathcal{T}$ be the trader's acquisition strategy. Let $\bar{J}(t, \cdot)$ denote the gross expected profit from acquiring information as of time $t$ given that one has not acquired information previously

$$\bar{J}(t, \cdot) = \mathbb{E}\left[J^S(t, \cdot) \,\middle|\, \mathcal{F}_t^P\right].$$

Notice that we must have $J^U(\tau^-, \cdot) \leq \bar{J}(\tau^-, \cdot) - c$. Consider the following deviation by the trader: do not acquire information at $\tau$, do not trade in $[\tau, \tau + \Delta)$, and then acquire at $t = \tau + \Delta$ and follow the conjectured equilibrium trading strategy from that point forward. The expected profit from this deviation is

$$\begin{aligned}
\bar{\Pi}_{d\tau} &\equiv e^{-r\Delta}\mathbb{E}_\tau[\bar{J}((\tau + \Delta)^-, \cdot) - c] - J^U(\tau^-, \cdot) \\
&\geq e^{-r\Delta}\mathbb{E}_\tau[\bar{J}((\tau + \Delta)^-, \cdot) - c] - (\bar{J}(\tau^-, \cdot) - c) &(5)\\
&= (1 - e^{-r\Delta})c + \mathbb{E}_\tau[e^{-r\Delta}\bar{J}((\tau + \Delta)^-, \cdot) - \bar{J}(\tau^-, \cdot)] &(6)\\
&= \left(1 - e^{-r\Delta}\right)c > 0, &(7)
\end{aligned}$$

where the final equality follows from the observation that trade timing indifference implies

$$\mathbb{E}_\tau\left[\mathbb{E}_\tau^s\left[e^{-r\Delta}J^s\left((\tau + \Delta)^-, \cdot\right) - J^s\left(\tau^-, \cdot\right)\right]\right] = 0. \qquad (8)$$

Hence, by deviating the trader benefits by paying the cost later, which is valuable due to discounting, but she forgoes trading profits in the interim. □

To summarize, there are two economic forces that generate non-existence of strategic trading equilibria with costly, unobservable information acquisition. First, there do not exist equilibria with delayed acquisition due to a "pre-emption deviation": the strategic trader is better off by deviating and acquiring information before she is expected to since this allows her to trade aggressively against a sub-optimally responsive pricing rule. Second, with trade timing indifference, there do not exist equilibria with immediate acquisition due to a "delay deviation": in this case, the strategic trader can profitably deviate by waiting before becoming informed.

In the next sub-section, we establish sufficient conditions under which any strategic trading equilibria must feature trade timing indifference.

## 3.2 Sufficient conditions for trade timing indifference

In this subsection, we show that any Markovian equilibrium of our model must feature trade timing indifference. This, together with Proposition 3 implies that there cannot be an equilibrium with costly information acquisition in our framework. We proceed in two steps. First, we will assume that there exists a solution to each informed trader type's Hamilton-Jacobi-Bellman (HJB) equation. It is important to note that we are not assuming that the HJB equation characterizes the value function, nor that the trader finds it optimal to trade smoothly, but merely that there exists a sufficiently smooth function that satisfies the HJB equation. Second, we will show that in any equilibrium, if it were to exist, an informed trader's value function is, in fact, characterized by the HJB equation and her optimal trading strategy is absolutely continuous i.e., $dX_t = \theta(\cdot)\, dt$, where $\theta(\cdot)$ denotes the trading rate. We then show that in order for the value function to be finite, this requires the equilibrium to feature trade timing indifference.

We begin by introducing some additional notation and structure for the market maker's pricing rule. Following the literature, we consider Markovian equilibria in which the asset price is a function of the exogenous public signals $\nu_t$, as well as an arbitrary (but finite) number $\ell$ of endogenous state variables $p_t$ that follow a Markovian diffusion and which keep track of the market maker's beliefs about $S$. In particular, we consider pricing rules of the form

$$P_t = g(t, \nu_t, p_t)$$

where $g$ is continuously differentiable in $t$ and twice continuously differentiable in $(\nu, p)$ and is strictly increasing in both $p$and $\nu$. The monotonicity condition is a normalization that ensures that increases in $p$ and $\nu$ represent "good news." There are $\ell > 0$ endogenous state variables $p_t$ with dynamics

$$dp = \alpha(t, p, \nu)\, dt + \Omega(t, p, \nu)\, dW + \mathbf{1} dY, \tag{9}$$

where $\alpha$ is an $\ell$-dimensional function, $\Omega$ is an $\ell \times m$matrix function, and $\mathbf{1}$ is an $\ell \times 1$ vector of ones such that there exists a unique strong solution to this SDE when $dY = dX + dZ$ and the trading strategy $X_t$ takes an admissible form (to be detailed below). We normalize $p_{0^-} = 0$. Without loss of generality and to simplify later notation, we also normalize the coefficients on $dY$ to be identically equal to one. Importantly, this does *not* imply that we restrict the price impact of a one unit trade to one dollar. Recalling that the function $g(\cdot)$ maps the state variables to the price, the overall dependence of the price on order flow is captured by $dP = \cdots + g_p \cdot \mathbf{1} dY$ where. Hence "Kyle's lambda" is $g_p \cdot \mathbf{1}$. We emphasize that the function $g$ and the coefficients $\alpha$ and $\Omega$ are equilibrium objects that, given an equilibrium trading strategy, are pinned down by the rationality of the pricing rule.

We also require a condition on the set of trading strategies in order to rule out doubling-type strategies (see, e.g., Back (1992)). In particular, a trading strategy is *admissible* if is a semi-martingale adapted to the strategic trader's filtration, and for all pricing rules satisfying the conditions specified above, we have

$$\mathbb{E}\left[\int_0^\infty \left(e^{-ru}(v_u - P_{u^-})\right)^2 d\left[Z, Z\right]_u\right] < \infty$$
$$\mathbb{E}\left[\int_0^\infty \left(e^{-ru} X_{u^-}\right)^2 d\left[v, v\right]_u\right] < \infty$$

where $[\cdot]$ denotes the quadratic (co)variation.

We now formalize our assumption on the HJB equation.

**Assumption 1.** *Suppose that in any conjectured equilibrium, for each $s \in$ support$(S)$, there exists a function $J^s(t, \nu, p)$ is continuously differentiable in $t$, twice continuously differentiable in $(\nu, p)$, that satisfies the HJB equation*[14]

$$0 = \sup_{\theta} \left\{ \begin{array}{c} -rJ^s + J^s_t + J^s_\nu \cdot \mu + J^s_p \cdot (\alpha + \mathbf{1}\theta) \\ +\frac{1}{2} tr\left(J^s_{\nu\nu} \Sigma\Sigma'\right) + \frac{1}{2} tr\left(J^s_{pp}(\Omega\Omega' + \mathbf{1}\sigma_z^2 \mathbf{1}')\right) \\ +tr\left(J_{\nu p} \Omega\Sigma'\right) + \theta\left(v_t - P_{t^-}\right) \end{array} \right\}, \tag{10}$$

[14]When applied to a vector or matrix, $'$, denotes the transpose.

*and when the order flow is generated by the conjectured equilibrium strategy, the integrability conditions*

$$\mathbb{E}\left[\int_0^\infty e^{-2rs} J_p'(s^-)\Omega\Omega' J_p(s^-)\, ds\right] < \infty$$
$$\mathbb{E}\left[\int_0^\infty e^{-2rs} J_\nu'(s^-)\Sigma\Sigma' J_\nu(s^-)\, ds\right] < \infty,$$

*and the transversality condition*

$$\lim_{t\to\infty} \mathbb{E}\left[e^{-rt} J(t, \nu_t, p_t)\right] = 0$$

*hold.*

Note that, if the HJB equation does in fact characterize the value function, the transversality condition implicitly provides information on what kinds of trading strategies are consistent with equilibrium (recalling that trades drive the $p_t$ variables), as it says that an optimal strategy exhausts all profitable trading opportunities if the trading game were to continue indefinitely (i.e., if $T$ tended to infinity). The integrability conditions, on the other hand, are essentially technical. Note also that Assumption 1 applies to the equilibria in existing models in the literature (e.g., Back and Baruch (2004), the continuous-time case of Caldentey and Stacchetti (2010), and the fixed-horizon, continuous-time model of Kyle (1985) with the appropriate modification).

We now establish that in any such equilibrium, if it were to exist, an informed trader's optimal trading strategy is absolutely continuous i.e., $dX_t = \theta(\cdot)\, dt$, where $\theta(\cdot)$ denotes the trading rate, and her value function is characterized by the HJB equation. Importantly, we also show that such an equilibrium must feature trade timing indifference. Note that the result below applies to both pure strategy equilibria in which information is immediately acquired and mixed strategy equilibria.

**Lemma 1.** *Suppose Assumption 1 holds. If there exists an overall equilibrium, then any optimal trading strategy for an informed trader is absolutely continuous and the value function for an informed trader is the solution to the HJB equation in Assumption 1, subject to the integrability and transversality conditions. Moreover, such an equilibrium must feature* ***trade timing indifference****.*

The optimality of the smooth trading strategy is established in the Appendix, and extends the arguments in Kyle (1985) and Back (1992) to our setting. Intuitively, if an informed trader does not trade smoothly she reveals her information too quickly. Moreover, the proof in the Appendix establishes that the $J^s$ are all solutions of the HJB equation (10). Because this equation is linear in $\theta$, and $\theta$ is unconstrained, it follows that the sum of the coefficients on $\theta$ must be identically zero and therefore the sum of the remaining terms must also equal zero i.e.,

$$\begin{array}{c} -rJ^s + J^s_t + J^s_\nu \cdot \mu + J^s_p \cdot \alpha \\ +\frac{1}{2}\mathrm{tr}\left(J^s_{\nu\nu}\Sigma\Sigma'\right) + \frac{1}{2}\mathrm{tr}\left(J^s_{pp}(\Omega\Omega' + \Lambda\sigma_z^2\Lambda')\right) + \mathrm{tr}\left(J_{\nu p}\Omega\Sigma'\right) \end{array} = 0 \tag{11}$$

But the above is simply the expected differential of the value function of an informed investor under the assumption that her trading rate at $t$ is zero i.e., $\theta^s_t = 0$. This establishes trade timing indifference.

The following result summarizes the implications of the earlier results.

**Theorem 1.** *There does not exist an equilibrium in which the trader follows a pure information acquisition strategy in which information is acquired after $t = 0$ with positive probability. Moreover, if Assumption 1 holds, there does not exist an equilibrium with costly information acquisition.*

The results in this section rule out the existence of equilibrium in the case of unobservable information acquisition, under standard regularity conditions. The deviation arguments apply generally to a large class of models that feature discounting (e.g., Back and Baruch (2004), Chau and Vayanos (2008), Caldentey and Stacchetti (2010)), which implies that the trading equilibria in these models do not naturally arise as a consequence of costly dynamic information acquisition. While these models provide useful intuition for how exogenous (or costless), private information gets incorporated into prices, our analysis recommends caution when considering settings with endogenous information acquisition. Moreover, as we discuss in the next section, our results are qualitatively similar in related settings. As such, reconciling existing models of strategic trading with unobservable, costly information acquisition is difficult.

# 4 Robustness

As the last section highlighted, pre-emption and delay are both robust, economically important forces that arise in dynamic settings. This section focuses on the robustness of the delay deviation to alternate settings, since establishing the existence of such a deviation is sufficient to rule out both pure and mixed strategy equilibria. Section 4.1 considers the case of a monopolistic market maker (e.g., as in Wang and Yang (2016)) and Section 4.2 considers the case of a risk-averse strategic trader. In both cases, we demonstrate that equilibria feature trade timing indifference and, consequently, our non-existence results apply directly.

Section 4.3 establishes that the delay deviation also applies to a setting in which the trader is initially endowed with some information. Section 4.4 establishes that our results do not depend on continuous trading, by showing that in the discrete time setting of Caldentey and Stacchetti (2010), there does not exist an equilibrium with pure strategy information acquisition when the time between trading rounds is sufficiently small. In Section 4.5, we relax the assumption that the payoff in future periods is discounted, by endogenizing information acquisition in the benchmark Kyle (1985) model. We show that an equilibrium with pure strategy acquisition cannot be sustained in this setting unless the cost of information acquisition is sufficiently low. Finally, Section 4.6 highlights the role of the information acquisition technology by considering the case of flow costs. In this case, we characterize conditions under which an unobservable acquisition equilibrium can be sustained.

## 4.1 Preferences of the market maker

A possible concern about our non-existence results is that they are a consequence of the assumption that the market maker is perfectly competitive and set the price as described in (2). Recent work in related models suggests that alternate assumptions about the market maker (e.g., that she is a profit maximizing monopolist) may help restore existence. For instance, Wang and Yang (2016) establish non-existence of equilibria when a risk-neutral, competitive market maker faces uncertainty about the existence of a strategic trader, but show that existence can be restored when the market maker is monopolistic.

However, the analogous approach does not recover existence of equilibria in our setting. Note that the arguments in Propositions 1, 2, and 3 do not rely on whether the price is the perfectly competitive one given by (2). Similarly, Assumption 1 and consequently Lemma 1 do not make use of the fact that the price is set competitively and therefore apply to settings in which the market maker maximizes an arbitrary objective function (e.g., she is risk-averse, imperfectly competitive, etc.) or even if she were to set prices based on an arbitrary, exogenous rule. As such, the pre-emption and delay deviations apply to a larger class of models in which the market maker is not necessarily risk-neutral nor competitive. The key feature that is required for our delay argument is trade timing indifference, which arises as long as price changes and price-impact changes are not predictable when the informed strategic trader does not trade.

## 4.2 Risk aversion of the strategic trader

In this subsection, we explore the effect of allowing the strategic trader to be risk-averse. Let $v_t = f(t, \nu_t, S)$ be the payoff of the asset if it liquidates at time $t$ – with risk aversion it is important that $v_t$ is interpreted as the value itself rather than merely the expeted value. Suppose that a trader with $S = s$ has utility $u(T, w_T^s)$ over her terminal wealth $w_T^s$, where

$$w_t^s = \int_0^t (v_u^s - P_u)\, dX_u^s, \tag{12}$$

and let her continuation value function be denoted by[15]

$$J^s(t, w_t, \nu_t, p_t) = \sup_{X^s} \mathbb{E}_t \left[ \int_t^\infty e^{-r(\ell - t)} u(\ell, w_\ell)\, d\ell \right], \tag{13}$$

conditional on the economy not having ended as of $t$. We argue that the delay deviation of 3 also applies since the equilibrium must feature trade timing indifference.

Arguments analogous to Lemma 1 imply that in equilibrium, any optimal trading strategy for an informed trader is absolutely continuous (i.e., $dX_t^s = \theta_t^s dt$), and the value function above satisfies the following HJB equation:

$$0 = \sup_\theta \left\{ \begin{array}{c} r\left(u(t, w_t) - J^s\right) + J_t^s + J_\nu^s \cdot \mu + J_p^s \cdot (\alpha + \mathbf{1}\theta) \\ +\frac{1}{2}\mathrm{tr}\left(J_{\nu\nu}^s \Sigma\Sigma'\right) + \frac{1}{2}\mathrm{tr}\left(J_{pp}^s(\Omega\Omega' + \mathbf{1}\sigma_z^2\mathbf{1}')\right) \\ +\mathrm{tr}\left(J_{\nu p}\Omega\Sigma'\right) + \theta\left(v_t - P_{t^-}\right) J_w \end{array} \right\}, \tag{14}$$

As before, the above problem is linear in $\theta$ and so a finite optimum requires:

$$J_p^s \cdot \mathbf{1} + (v_t - P_{t^-})\, J_w = 0, \text{ and} \tag{15}$$

$$\left\{ \begin{array}{c} r\left(u(t, w_t) - J^s\right) + J_t^s + J_\nu^s \cdot \mu + J_p^s \cdot \alpha \\ +\frac{1}{2}\mathrm{tr}\left(J_{\nu\nu}^s \Sigma\Sigma'\right) + \frac{1}{2}\mathrm{tr}\left(J_{pp}^s(\Omega\Omega' + \mathbf{1}\sigma_z^2\mathbf{1}')\right) \\ +\mathrm{tr}\left(J_{\nu p}\Omega\Sigma'\right) \end{array} \right\} = 0. \tag{16}$$

But the latter equation reflects the change in expected utility over an instant $dt$, accounting for the possibility that the economy ends with probability $r\,dt$ (in which case, the change is $u - J^s$ because the trader gets to consume her terminal wealth but loses future profitable trading opportunities). Because the change in expected utility, given no trade, is zero, this implies that the trader must be indifferent between her posited optimal strategy and not trading over an interval and then following her optimal strategy i.e., that the equilibrium must feature trade timing indifference. As before, this implies there cannot be an equilibrium

[15]Notice that there is a subtle difference with the value functions that we considered in the risk-neutral case above. In that case, we defined the value function as the expected trading profits from time $t$ onward, rather than the expected terminal wealth. When the trader is risk-neutral this distinction is immaterial since maximizing future trading profits and maximizing terminal wealth are equivalent. However, if the trader is not risk-neutral, we must explicitly keep track of her expected utility over the level of terminal wealth itself.

with unobservable information acquisition, even when the strategic trader is risk-averse.

### 4.3 Initial information endowment

Note that our nonexistence results largely survive if the trader is initially endowed with some private information and can choose to pay cost $c > 0$ to observe some additional piece(s) of information at a time of her choosing. In this case, it is easy to show that as long as Assumption 1 holds, then such a trader always has a profitable "delay deviation" from any conjectured information acquisition strategy. Under Assumption 1 this once again rules out the existence of any equilibria (pure or mixed strategy) with costly information acquisition. Even in the absence of Assumption 1, we suspect, but have not been able to prove, that a pre-emption deviation argument analogous to the one we considered above also rules out pure strategy equilibria with acquisition after $t = 0$.

### 4.4 Discrete time

Next, we consider unobservable information acquisition in the discrete-time model of Caldentey and Stacchetti (2010). Note that, as before, Propositions 1 and 2 apply to this setting: never acquiring information, or acquiring it with a delay, cannot be an equilibrium. The rest of the section establishes that information acquisition at date zero is not an equilibrium when the length between trading rounds is sufficiently small. Even though trade timing indifference does not arise in a discrete time model, the argument follows that of Proposition 3: instead of acquiring immediately, the strategic trader can wait for a period of length $\Delta$ and re-evaluate her decision. The expected gain from delaying acquisition is of order $\Delta$, but the expected loss from not trading in the first period is of order smaller than $\Delta$. As a result, when $\Delta$ is sufficiently small, the deviation is strictly profitable.

Recall the special case of Caldentey and Stacchetti (2010) in which there is a "lump" of initial private information and no ongoing flow of private information. Time is discrete and trade takes place at dates $t_n = n\Delta$ for $n \geq 0$ and $\Delta > 0$. The risky asset pays off $V \sim \mathcal{N}(0, \Sigma_0)$ immediately after trading round $T$, where $T$ is random. Specifically, $T = \eta\Delta$, where $\eta$ is geometrically distributed with failure probability $\rho = e^{-r\Delta}$.[16] The risk-neutral strategic trader observes $V$. Let $x_n$ denote her trade at date $t_n$. There are noise traders who submit iid trades $z_t \sim \mathcal{N}(0, \Sigma_z)$ with $\Sigma_z = \sigma_z^2\Delta$. Let $y_n = x_n + z_n$ denote the time $n$ order flow. Competitive risk-neutral market makers set the price $p_n$ in each trading round equal

[16] There are at least two different distributions that are often referred to as "the" geometric distribution. The one we use here is supported on the nonnegative integers $n \in \{0, 1, 2, \dots\}$ and has probability mass function $f_n = \rho^n(1-\rho)$ and cdf $F_n = 1 - \rho^{n+1}$.

to the conditional expected value. Following Caldentey and Stacchetti (2010), we focus on linear, Markovian equilibria in which the time $t_n$ price depends only on $p_{n-1}$ and $y_n$. Let $\bar{V}_n$ and $\Sigma_n$ denote the market maker's conditional expectation and variance, immediately *before* the time $t_n$ trading round. So, $\bar{V}_n = p_{n-1}$. Finally, set $p_{-1} = \mathbb{E}[V] = 0$.

Caldentey and Stacchetti (2010) show that if the trader is informed of the asset payoff there exists an equilibrium in which the asset price and trading strategy are given by

$$\begin{aligned} p_n(\bar{V}_n, y_n) &= \bar{V}_n + \lambda_n y_n \\ x_n(V, \bar{V}_n) &= \beta_n(V - \bar{V}_n), \end{aligned}$$

the trader's expected profit is

$$\Pi_n(p_{n-1}, V) = \alpha_n(V - p_{n-1})^2 + \gamma_n,$$

and the coefficients are characterized by the difference equations described in the proof of the following result. The ex-ante expected profit from acquiring information immediately before the $t = 0$ trading round is

$$\begin{aligned} \bar{\Pi}_0 \equiv \mathbb{E}[\Pi_0(p_{-1}, V)] - c &= \alpha_0 \mathbb{E}[(V - p_{-1})^2] + \gamma_0 - c \\ &= \alpha_0 \Sigma_0 + \gamma_0 - c. \end{aligned}$$

We would like to compare this to the expected profit if the trader deviates by remaining uninformed for the $n = 0$ trading round, not trading, and then acquiring immediately before round $n = 1$. Supposing that she does so, trades $x = 0$ units at time zero, and then follows the prescribed equilibrium trading strategy in the following rounds, the expected deviation profit is

$$\bar{\Pi}_{d0} = \mathbb{E}[x(V - p_0(\bar{V}_0, x + z_0))] + \mathbb{E}\left[\sum_{n=1}^{\infty} \rho^n (V - p_n) x_n - \rho c\right] \tag{17}$$

$$= \rho\left(\alpha_1(\Sigma_0 + \lambda_0^2 \Sigma_z) + \gamma_1 - c\right). \tag{18}$$

The following result establishes that such a deviation is profitable when the time between trading dates $\Delta$ is sufficiently small.

**Proposition 4.** *For all $\Delta > 0$ sufficiently small,*

$$\frac{\bar{\Pi}_{d0} - \bar{\Pi}_0}{\Delta} = \frac{-\frac{1}{2}\frac{(\beta_0^{\Delta})^3 \Sigma_0^2}{\sigma_z^2 \Delta} + (1 - e^{-r\Delta})c}{\Delta} > 0,$$

*and therefore there does not exist an equilibrium in which the trader follows a pure acquisition strategy.*

The above result suggests that the conclusions of Section 3.1 do not rely on the assumption of continuous trading. As such, when the time between trading dates is sufficiently small, the trading equilibrium in the discrete time setting of Caldentey and Stacchetti (2010) cannot arise endogenously as an outcome of unobservable, costly information acquisition. Given the compelling economic forces behind the result, we conjecture, but have not been able to prove, that similar arguments rule out mixed strategy acquisition when the time between trading rounds is sufficiently small. Note that, as in all of the above results, a positive discount rate (which is induced by the random horizon, though could be introduced explicitly) plays a crucial role in non-existence of equilibrium: the profitable deviation arises because by delaying acquisition, the present value of the information cost is (strictly) lower. However, as we show next, the delay deviation may also restrict the existence of equilibria even in settings without discounting.

### 4.5 No discounting

In this subsection, we study unobservable information acquisition in the continuous-time version of Kyle (1985). As in the case of a random horizon considered in the benchmark model, we first establish that the only candidate pure strategy equilibria involve acquisition at the beginning of the trading game. We then show that when the cost of information is sufficiently high, there does not exist a pure strategy equilibrium with acquisition at time zero, and therefore there does not exist any pure strategy equilibrium. The intuition for this result is a stronger version of the delay deviation in Proposition 3: when the cost of information is sufficiently high, it is profitable for the trader to deviate by *never* acquiring information.

Briefly, recall the setting. Trading takes place on the interval $[0, 1]$. That is, following Kyle (1985) and Back (1992) the terminal date $T$ is fixed and normalized to 1. As the end of trading the asset pays off $v$, where $v \sim \mathcal{N}(0, \Sigma_0)$. There is a risk-neutral strategic trader who maximizes her expected trading profit and at any time can choose to pay $c > 0$ to observe $v$. As above, we assume that she trades smoothly so that her cumulative holdings $X_t = \int_0^t \theta_u \, du$ for trading rate $\theta_u$. There are noise traders whose cumulative holdings $Z_t$ follow a Brownian motion with variance $\sigma_Z^2$. Prices are set by a risk neutral market maker who observes the cumulative order flow $Y_t = X_t + Z_t$ and sets $P_t = \mathbb{E}[v|\mathcal{F}_{Pt}]$. Because there is no news process, public information includes only the order flow so $\mathcal{F}_t^P$ is the augmentation of the filtration $\sigma(\{Y_t\})$. In the notation in the text, the conditional expected payoff is $f(t, S, \nu) = S$, where

$S = v$ , and the endogenous state variable is simply the cumulative order flow itself, $p_t = Y_t$.

An identical argument to that in Propositions 1 and 2 immediately implies that any pure-strategy equilibrium cannot involve acquisition after time zero. Otherwise the strategic trader has a profitable deviation from any conjectured equilibrium since she can preemptively acquire information, trade against an unresponsive pricing rule, and make unboundedly large profits.

Now, suppose that there is a pure strategy equilibrium in which the trader acquires information immediately at $t = 0$. In such an equilibrium, the pricing rule and the trader's post-acquisition value function are those from Kyle (1985) (or the special case of Back (1992) with normally-distributed payoff).

Hence,

$$P(t, y) = \lambda y,$$

where $\lambda = \sqrt{\frac{\Sigma_0}{\sigma_Z^2}}$ and

$$J^v(t, y) = \frac{1}{2}\sqrt{\frac{\sigma_Z^2}{\Sigma_0}}\left(v - \sqrt{\frac{\Sigma_0}{\sigma_Z^2}}y\right)^2 + \frac{1}{2}\sqrt{\Sigma_0\sigma_Z^2}(1 - t).$$

The ex-ante (gross) expected profit from being informed is therefore

$$\overline{J}(0, 0) = \mathbb{E}[J^v(0, 0)] = \sqrt{\Sigma_0\sigma_Z}$$

We would like to compare the above to the expected payoff if the trader deviates, remains uninformed for the duration of the trading game, and trades against the posited equilibrium price function. Following the argument of Proposition 1 in Back (1992), it is straightforward to construct the trader's value function under this deviation. It is

$$J^{d,U}(t, y) = J^0(t, y) = \frac{1}{2}\sqrt{\frac{\Sigma_0}{\sigma_Z^2}}y^2 + \frac{1}{2}\sqrt{\Sigma_0\sigma_Z^2}(1 - t).$$

At time zero, this becomes

$$J^{d,U}(0, 0) = \frac{1}{2}\sqrt{\Sigma_0\sigma_Z^2},$$

which is half of the informed trader's ex-ante gross profit. Notice that because the trader is risk-neutral, under the given price process her optimal trading profit depends only on her conditional expectation of the asset value (as well as the cumulative order flow and the calendar time), so her value function is identical to that for a trader who acquires information but for whom the realized signal does not change her prior expectation ($v = 0$). The fact that this trading profit is not zero is a consequence of the fact that in a dynamic model the trader expects profitable trading opportunities to arise in the future when the realized noise trade pushes the price away from zero.

With the above value functions, we arrive immediately at the following.

**Proposition 5.** *Suppose that Assumption 1 holds, and $c > \frac{1}{2}\sqrt{\Sigma_0 \sigma_Z^2}$. Then, there does not exist an equilibrium in which information acquisition follows a pure strategy.*

*Proof.* We have already ruled out pure strategy equilibria in which the trader acquires with some delay or never acquires. Suppose that there is an equilibrium in which the trader acquires information immediately. The analysis of Back (1992) implies that under the stated assumptions the asset price and the informed trader's value functions must be of the form above. Consider the trader's expected profit from unobservably deviating and never acquiring information. The net expected profit from this deviation is

$$J^{d,U}(0,0) - (\bar{J}(0,0) - c) = c - \frac{1}{2}\sqrt{\Sigma_0 \sigma_Z^2} > 0,$$

so the trader is better off by undertaking the deviation. □

Importantly, note that the above result implies that when $c > \frac{1}{2}\sqrt{\Sigma_0 \sigma_Z^2}$, the financial market equilibrium in Kyle (1985) cannot arise as a consequence of endogenous information acquisition.[17]

## 4.6 Alternate information acquisition technology

The deviation to delay acquisition also applies in settings where the trader chooses both the timing and precision of information as long as there is a "quasi-fixed" cost component of information technology (i.e., the cost of information, starting from *zero* precision, includes a non-zero "lump" component). However, when there is no such lump cost, the delay deviation is no longer strictly profitable. In this section, we illustrate this effect using an alternate information acquisition technology. Specifically, we assume that , instead of paying a lump cost to observe a piece (or pieces) of information, the trader can dynamically optimize the precision of a flow of signals.[18] We show that in this case, an equilibrium with endogenous information acquisition can be sustained.

[17] In fact, nonexistence holds even if we force the trader to make her acquisition decision at $t = 0$, as nothing in the proof above relied on deviating on intermediate dates.

[18] We thank Dmitry Orlov for suggesting that we explore this approach.

Consider the following setting that generalizes the continuous-time setting of Caldentey and Stacchetti (2010). The asset value $V_t$ is revealed at a random time $T$ distributed exponentially with rate $r > 0$. The asset value follows

$$dV = \sigma_V(t)dW_{Vt} \quad with \quad V_0 \sim \mathcal{N}(0, \Sigma_0)$$

where $W_{Vt}$ is a standard Brownian motion, and $V_0$ is distributed independently of all other random variables. There are noise traders that have cumulative holdings $Z_t = \int_0^t \sigma_Z(s)dW_{Zt}$. We allow, but do not require, the noise volatility $\sigma_Z(t)$ to vary with time. Let $X_t = \int_0^t \theta_s ds$ denote the strategic trader's cumulative holdings and $Y_t = X_t + Z_t$ the cumulative order flow. Let $\mathcal{F}_t^P = \sigma(\{Y_s\}_{0\le s\le t})$ denote the public information set.

The trader can observe a flow of signals of the form

$$dS_t = V_t dt + \sigma_S(\cdot)dW_{st}, \qquad S_0 = V_0 + \varepsilon,$$

where $W_{st}$ is a standard Brownian motion, independent of $W_{Vt}$ and $W_{Zt}$, and $\varepsilon \sim \mathcal{N}(0, \sigma_\varepsilon^2)$ is independently distributed. The precision process $\chi = 1/\sigma_S^2(\cdot)$ can be dynamically optimized by the trader, given her information set $\mathcal{F}_t^I = \sigma(\{S_s, Y_s\}_{0\le s\le t})$. The precision choice is subject to a flow cost incurred at rate $c(\chi)dt$. The cost function $c : [0, \infty) \to [0, \infty)$ is twice continuously differentiable with $c \geq 0$, $c' \geq 0, c'' > 0$; suppose further that $c'(0) = 0$ and $\lim_{\chi\to\infty} c'(\chi) \to \infty$ – this, along with the fact that $c'' > 0$, implies that $c'$ is a bijection of $[0, \infty)$ and has a well-defined, continuously differentiable inverse. The initial error $\varepsilon$ is independently normally distributed $\mathcal{N}(0, \sigma_{s0}^2)$, and for simplicity we take $\sigma_{s0}^2$ as exogenous since our focus is on optimizing the precision of the flow of signals, not of the initial, discrete signal. Let $v_t = \mathbb{E}[V|\mathcal{F}_t^I]$ denote the trader's conditional expected value.

We will search for a linear equilibrium in which the pricing rule, trading rate, and optimal precision take the form

$$\begin{aligned} dP &= \lambda(t)dY \\ \theta &= \beta(t)(v_t - P_t) \\ \chi &= \eta(t) \end{aligned}$$

for purely deterministic functions $\lambda, \beta, \eta$ and the strategic trader's value function takes the form

$$J(t, v_t, P) = \alpha(t)(v_t - P_t)^2 + \gamma(t),$$

for purely deterministic functions $\alpha$ and $\gamma$.[19] Finally, we require a standard admissibility condition on the trading strategy to rule out doubling-type strategies that accumulate unbounded losses followed by unbounded gains:

$$E\left[\int_0^\infty e^{-rs}|\beta(s)|\left(v_t-P_t\right)^2 ds\right]<\infty.$$

Using standard "conjecture and verify" methods, we demonstrate the following in the Appendix.

**Proposition 6.** *Denote $\Omega(t)\equiv var(V_t|\mathcal{F}_t^I)$, $\Psi(t)\equiv var(v_t|\mathcal{F}_t^P)$. Define $\Gamma(t)=\int_t^\infty \sigma_V^2(s)ds$, $\Phi(t)=\int_t^\infty e^{-2rs}\sigma_Z^2(s)ds$, and let $f(x)=[c']^{-1}(x)$ denote the inverse marginal cost of precision. Suppose $\Gamma(0)<\infty$ and $\Phi(0)<\infty$. . Define*

$$\lambda(0)=\sqrt{\frac{\Sigma_0+\Gamma_0}{\Phi_0}},$$

*and let $\Omega(t)$ be the unique, positive solution $\Omega(t)$ to the initial value problem*

$$\frac{d\Omega}{dt}=\sigma_V^2(t)-f\left(\tfrac{1}{2\lambda(0)}e^{rt}\,\Omega^2(t)\right)\Omega(t)^2,\qquad \Omega(0)=var(V_0|S_0).$$

*Supposing that $(\Sigma(0)+\Gamma(0))\frac{\Phi(t)}{\Phi(0)}>\Gamma(t)+\Omega(t)$ for all $t>0$ there is an equilibrium in which:*

$$\begin{aligned}
\lambda(t)&=\lambda(0)e^{-rt}\\
\Psi(t)&=(\Sigma(0)+\Gamma(0))\frac{\Phi(t)}{\Phi(0)}-\Gamma(t)-\Omega(t)\\
\beta(t)&=\frac{\lambda(t)\sigma_Z^2(t)}{\Psi(t)}\\
\eta(t)&=f\left(\tfrac{1}{2\lambda(0)}e^{rt}\,\Omega^2(t)\right)\\
\alpha(t)&=\frac{1}{2\lambda(0)}e^{rt}\\
\gamma(t)&=e^{rt}\left(\int_t^\infty\frac{1}{2}e^{-rs}\lambda(s)\sigma_Z^2(s)ds+\frac{1}{2\lambda(0)}\left(\int_t^\infty\sigma_V^2(s)ds+\Omega(t)\right)-\int_t^\infty e^{-rs}c(\eta(s))ds\right).
\end{aligned}$$

*In this equilibrium $\lim_{t\to\infty}\Omega(t)=0$ so that the trader learns the asset value perfectly in the*

[19] We emphasize that we are not restricting strategies to "best responses with deterministic coefficients". Rather, as we show in the proof, given the conjectured trading/acquisition strategy the market maker's optimal pricing rule has deterministic $\lambda(t)$, and given the conjectured pricing rule, the optimal trading/acquisition strategy has deterministic coefficients.

*limit.*

In this equilibrium $\lambda(t)$, as well as the public conditional uncertainty about the asset value $\Sigma(t) \equiv \mathrm{var}(V_t|\mathcal{F}_{Pt}) = \Psi(t) + \Omega(t)$, are, in fact, identical to that in the continuous-time model of Caldentey and Stacchetti (2010) in which there is no ongoing flow of information. It is also straightforward to show that the trader's ex-ante expected profit (gross of information acquisition costs) is identical.

These results may at first seem surprising; however, as Back and Pedersen (1998) show, in continuous-time Kyle models with a fixed amount of Gaussian private information to be observed by the trader, the timing of information arrival is irrelevant for the pricing rule and ex-ante expected profit. As we show in Lemma 4 in the Appendix, the trader optimally, endogenously learns the asset value perfectly in the limit, $\Omega(t) \to 0$, regardless of precision cost function (subject to the monotonicity and convexity requirements stated in the model setup). Hence, with respect to the the equilibrium pricing rule and expected profit, this setting is analogous to one in which she simply observes the entire trajectory of the asset value, $\{V_t\}_{0\leq t<\infty}$, at time-0. However, this equilibrium has distinct implications for the trader's optimal trading strategy and the dynamics of public uncertainty about the trader's valuation $\Psi(t)$, at any time $t < \infty$.

The above result suggests that one approach to reconciling strategic trading equilibria with endogenous information acquisition is to restrict the trader to "smooth" information acquisition technologies that do not include a fixed cost of information. Whether or not such restrictions are appropriate depends on the application of the model, and the nature of the information being acquired. In settings where the private informational advantage of a strategic trader arises from sustained effort, attention or scrutiny, the specification of information costs in this subsection may be a reasonable approximation. On the other hand, if becoming informed involves an upfront investment in new technology, human capital or expertise, then capturing fixed / lumpy costs of information acquisition is likely to be important. As such, one must carefully consider the application when introducing endogenous information acquisition to a model of strategic trading.

# 5 Concluding remarks

The Kyle-type strategic trading framework, in which a competitive, risk-neutral market maker sets prices in response to strategic trading by an informed trader, provides an important benchmark for understanding how markets incorporate private information. A key limitation of the standard setup is that the strategic trader is endowed with private information before trading begins, instead of acquiring it endogenously at a time of her choosing.

To explore the implications of endogenous information acquisition, we consider a strategic trading model in which the trader can choose *when* to acquire information about the asset payoff.

We show that the very existence of equilibrium depends crucially on whether the strategic trader can acquire information without being immediately detected. In particular, in a continuous-time, strategic trading setting with discounting, there cannot exist an equilibrium in which information acquisition follows a pure strategy. Moreover, we also rule out equilibria in which the strategic trader mixes between acquiring information at different times. Finally, we show that similar results arise when (i) the market maker is not competitive or is risk-averse, (ii) the strategic trader is risk-averse, (iii) the strategic trader is initially endowed with some information, (iv)trading is sufficiently frequent, but not continuous, or (v) there is no discounting, but information is sufficiently costly. These results suggest that equilibrium breakdown is feature of a broad class of models when information acquisition is endogenous and not detectable.

Our analysis highlights two economically important consequences of dynamic information acquisition by a strategic trader. First, if she can acquire information without being detected, a strategic trader has a strong incentive to deviate by preemptively acquiring information, because this allows her to trade against a (relatively) insensitive pricing rule. Second, because strategic traders smooth their trades over time, trading gains for a short time frame are (relatively) small when trading opportunities are sufficiently frequent. When information is costly and future payoffs are discounted, this creates an incentive to delay acquisition because the loss in trading gains is smaller than the benefit from delaying acquisition costs.

Understanding the key features of our setting that lead to equilibrium breakdown also provides insight into how to reconcile strategic trading and endogenous information acquisition. We conclude by summarizing these effects, discussing related literature and proposing avenues for future research.

**Information acquisition technologies.** As highlighted by Section 4.6, restricting the strategic trader to "smooth" information acquisition technologies without fixed / lumpy costs can potentially resolve our non-existence results by eliminating the possibility of the pre-emption deviation and eliminating the profitability of the delay deviation. While the model in Section 4.6 is stylized and meant as a "proof of concept," it would be interesting to explore (i) the robustness of the existence result, and (ii) the implications on information dynamics, in richer settings.

**Detectability of information acquisition**. Market breakdown arises because the strategic trader can deviate in her acquisition strategy without being detected. In settings where acquisition is detectable by other market participants, these deviations are no longer profitable and strategic trading equilibria can be sustained. For instance, Banerjee and Breon-Drish (2018) consider a setting **where** entry into new markets is observable, and show that allowing for flexibility in timing of information acquisition and entry leads to novel predictions for the likelihood of informed trading, entry dynamics and optimal precision choice.

**Commitment**. The deviations reflect an inability for the strategic trader to commit to an acquisition strategy. In contrast, when the trader can commit to an information strategy, standard strategic trading equilibria can be sustained (e.g., see Back and Pedersen (1998)). It would be useful to better understand how traders can commit to information acquisition in practice.

**Trading Frictions.** Our analysis suggests that accounting for trading frictions (e.g., restrictions on the trading rate, or costs of trading faster) might be another important aspect of understanding information acquisition in strategic trading environments. In particular, introducing trading frictions in a manner that eliminates trade timing indifference may restore the existence of equilibria.

# References


Alti, A., Kaniel, R., Yoeli, U., 2012. Why do institutional investors chase return trends? Journal of Financial Intermediation 21 (4), 694 – 721. 1

Aumann, R., 1964. Mixed and behavior strategies in infinite extensive games. In: Dresher, M., Shapley, L., Tucker, A. (Eds.), Advances in Game Theory. No. 52. Princeton, pp. 627–650. 11

Back, K., 1992. Insider trading in continuous time. Review of financial Studies 5 (3), 387–409. 6, 1, 2, 3.1, 3.2, 3.2, 4.5, 4.5, A.1

Back, K., Baruch, S., 2004. Information in securities markets: Kyle meets glosten and milgrom. Econometrica 72 (2), 433–465. 1, 2, 2, 3.2, 3.2

Back, K., Crotty, K., Li, T., 2017. Identifying information asymmetry in securities markets. The Review of Financial Studies 31 (6), 2277–2325. 1

Back, K., Pedersen, H., 1998. Long-lived information and intraday patterns. Journal of Financial Markets 1 (3), 385–402. 7, 4.6, 5

Banerjee, S., Breon-Drish, B. M., 2018. Dynamic information acquisition and entry into new markets. Working paper (http://ssrn.com/abstract=2846059). 4, 1, 2, 5

Banerjee, S., Green, B., 2015. Signal or noise? uncertainty and learning about whether other traders are informed. Journal of Financial Economics 117 (2), 398–423. 1

Caldentey, R., Stacchetti, E., January 2010. Insider trading with a random deadline. Econometrica 78 (1), 245–283. 5, 1, 2, 2, 12, 3.2, 3.2, 4, 4.4, 4.4, 4.6, 4.6, A.2, A.2, 20

Chakraborty, A., Yılmaz, B., 2004. Manipulation in market order models. Journal of financial Markets 7 (2), 187–206. 1

Chau, M., Vayanos, D., September 2008. Strong-form efficiency with monopolistic insiders. Review of Financial Studies 21 (5), 2275–2306. 3.2

Coase, R. H., 1972. Durability and monopoly. Journal of Law and Economics 15 (1), 143–149. 3.1

Collin-Dufresne, P., Fos, V., 2016. Insider trading, stochastic liquidity and equilibrium prices. Econometrica, Forthcoming. 1

Dugast, J., Foucault, T., 2018. Data abundance and asset price informativeness. Journal of Financial Economics. 7

Grossman, S. J., Stiglitz, J. E., 1980. On the impossibility of informationally efficient markets. American Economic Review 70 (3), 393–408. 1

Gul, F., Sonnenschein, H., Wilson, R., 1986. Foundations of dynamic monopoly and the coase conjecture. Journal of Economic Theory 39 (1), 155–190. 3.1

Holden, C. W., Subrahmanyam, A., January 2002. News events, information acquisition, and serial correlation. Journal of Business 75 (1), 1–32. 7

Huang, S., Yueshen, B. Z., 2018. Speed acquisition. 7

Karatzas, I., Shreve, S. E., 1998. Brownian Motion and Stochastic Calculus, 2nd Edition. Vol. 113 of Graduate Texts in Mathematics. Springer, New York. 8

Kendall, C., 2018. The time cost of information in financial markets. Journal of Economic Theory 176, 118–157. 7

Kyle, A. S., 1985. Continuous auctions and insider trading. Econometrica, 1315–1335. 1, 6, 2, 3.2, 3.2, 4, 4.5, 4.5

Li, T., 2013. Insider trading with uncertain informed trading. Available at SSRN 946324. 1

Liptser, R. S., Shiryaev, A. N., 2001. Statistics of Random Processes I: General Theory. Vol. 6 of Applications of Mathematics. Springer, New York. A.3.1

Protter, P. E., 2003. Stochastic Integration and Differential Equations, 2nd Edition. Vol. 21 of Applications of Mathematics. Springer, New York. A.1

Shmaya, E., Solan, E., 2014. Equivalence between random stopping times in continuous time, working paper. 11

Touzi, N., Vieille, N., 2002. Continuous-time dynkin games with mixed strategies. SIAM Journal on Control and Optimization 41 (4), 1073–1088. 11

Veldkamp, L. L., June 2006. Media frenzies in markets for financial information. American Economic Review 96 (3), 577–601. 7

Walter, W., 1998. Ordinary Differential Equations. No. 182 in Graduate Texts in Mathematics. Springer, New York. A.3.3

Wang, Y., Yang, M., 2016. Insider trading when there may not be an insider. Working paper (https://ssrn.com/abstract=2720736). 1, 4, 4.1

# A Proofs

## A.1 Proof of Lemma 1.

We will make use the following, equivalent, expressions for the trader's terminal wealth, the second of which follows from the integration by parts formula for semi-martingales

$$
\begin{aligned}
W_T &= (v_T - P_T)X_T + \int_0^T X_{s^-} dP_s \\
&= \int_0^\infty X_{s^-} dv_s + \int_0^T v_{s^-} dX_s + [X, v]_T - \int_0^T P_{s-} dX_s - [X, P]_T \\
&= \int_0^T (v_{s^-} - P_{s^-}) dX_s + \int_0^T d[X, v - P]_s + \int_0^T X_{s^-} dv_s
\end{aligned}
$$

where, for any $s$, $[\cdot]_s$ denotes the quadratic (co)variation over the interval $[0, s]$. Hence, under any information set that does not include $T$, we have from the independence of $T$

$$
\mathbb{E}[W_t|\cdot] = \mathbb{E}\left[\int_0^\infty e^{-rs}(v_{s^-} - P_{s^-}) dX_s + \int_0^\infty e^{-rs} d[X, v - P]_s + \int_0^\infty e^{-rs} X_{s^-} dv_s \middle| \cdot \right]
$$

The proof now proceeds in a way similar to that of Lemma 2 in Back (1992) and follows it closely. We show that the solution to the HJB equation provides an upper bound on the expected profit from any admissible trading strategy and that any absolutely continuous trading strategy that induces the value function to satisfy the transversality and integrability conditions achieves equality in upper the bound. For the rest of this proof, we suppress all arguments of the function $J$, other than time, where no confusion will result.

Without loss of generality, suppose the trader becomes informed at $t = 0$. The case in which she becomes informed at any other date is analogous. Consider an arbitrary semi-martingale strategy $X_s$. The generalized Ito's formula (Protter (2003), Thm. 33, Ch. 2) for semi-martingales implies that under the smoothness assumptions on $J$ (suppressing the $\nu$ and $p$ arguments for brevity), and recalling that jumps, if any, come from the $p$ process

$$
\begin{aligned}
e^{-rt}J(t) - J(0^-) = &\int_0^t e^{-rs}(-rJ(s^-) + J_s(s^-))\, ds + \int_0^t e^{-rs} J_\nu(s^-) \cdot d\nu + \int_0^t e^{-rs} J_p(s^-) \cdot dp \\
&+ \frac{1}{2}\int_0^t e^{-rs} \mathrm{tr}(J_{\nu\nu}(s^-) d[\nu^c, \nu^c]) + \frac{1}{2}\int_0^t e^{-rs} \mathrm{tr}(J_{pp}(s^-) d[p^c, p^c]) \\
&+ \int_0^t e^{-rs} \mathrm{tr}(J_{\nu p}(s^-) d[p^c, \nu^c]) + \sum_{0 \le s \le t} e^{-rs} \left(J(s) - J(s^-) - J_p(s^-) \cdot \Delta p_s\right),
\end{aligned}
$$

where the $c$ superscript denotes the continuous, local martingale portion of a given process.

The quadratic variations of the continuous portions can be written

$$
\begin{aligned}
[\nu^c, \nu^c]_t &= \int_0^t \Sigma\Sigma' ds \\
[p^c, p^c]_t &= \int_0^t d[\Omega W, \Omega W]_s ds + 2\int_0^t d[\Omega W, \mathbf{1}Y^c]_s ds + \int_0^t d[\mathbf{1}Y^c, \mathbf{1}Y^c]_s ds \\
&= \int_0^t \Omega\Omega' ds + 2\int_0^t \Omega d[W, X^c]_s \mathbf{1}' ds + \int_0^t \mathbf{1} d[Y^c, Y^c]_s \mathbf{1}' ds \\
[p, \nu]_t^c &= \int_0^t d[\Omega W, \Sigma W]_s ds + \int_0^t d[\mathbf{1}Y^c, \Sigma W]_s ds \\
&= \int_0^t \Omega\Sigma' ds + \int_0^t \mathbf{1} d[X^c, W]_s \Sigma'
\end{aligned}
$$

We have $[Y^c, Y^c]_t = [X^c, X^c]_t + 2[X^c, Z]_t + \int_0^t \sigma_Z^2 ds$.Therefore, the quadratic variations become

$$
\begin{aligned}
[\nu^c, \nu^c]_t &= \int_0^t \Sigma\Sigma' ds \\
[p^c, p^c]_t &= \int_0^t \Omega\Omega' ds + 2\int_0^t \Omega d[W, X^c]_s \mathbf{1}' ds + \int_0^t \mathbf{1} d[X^c, X^c]_s \mathbf{1}' ds \\
&\quad + 2\int_0^t \mathbf{1} d[X^c, Z]_s \mathbf{1}' ds + \int_0^t \mathbf{1}\sigma_Z^2 \mathbf{1}' ds \\
[p^c, \nu^c]_t &= \int_0^t \Omega\Sigma' ds + \int_0^t \mathbf{1} d[X^c, W]_s \Sigma' ds.
\end{aligned}
$$

Returning to the expression for $J(t) - J(0)$ above and also using the fact $\Delta p_s = \mathbf{1}\Delta X$ gives

$$
\begin{aligned}
e^{-rt}J(t) - J(0^-) &= \int_0^t e^{-rs} \Big( -rJ(s^-) + J_s(s^-) + J_\nu(s^-) \cdot \mu + J_p(s^-) \cdot \alpha + \frac{1}{2}\mathrm{tr}(J_{\nu\nu}\Sigma\Sigma') \\
&\qquad + \mathrm{tr}(J_{\nu p}\Omega\Sigma') + \frac{1}{2}\mathrm{tr}(J_{pp}(\Omega\Omega' + \mathbf{1}\sigma_Z^2\mathbf{1}')) \Big)\, ds \\
&+ \int_0^t e^{-rs} J_p(s^-) \cdot \mathbf{1} dX + \int_0^t e^{-rs} J_p(s^-) \left(\Omega dW + \mathbf{1} dZ\right) + \int_0^t e^{-rs} J_\nu(s^-) \Sigma dW \\
&+ \frac{1}{2}\int_0^t e^{-rs} \mathrm{tr}(J_{pp}(s^-)(\mathbf{1} d[X^c X^c]\mathbf{1}' + 2\Omega d[W, X^c]\mathbf{1}' + 2\mathbf{1} d[X^c, Z]\mathbf{1}')) \\
&+ \int_0^t e^{-rs} \mathrm{tr}(J_{\nu p}(s^-)\mathbf{1} d[X^c, W]\Sigma') + \sum_{0 \le s \le t} e^{-rs} \left(J(s) - J(s^-) - J_p(s^-) \cdot \mathbf{1}\Delta X_s\right).
\end{aligned}
$$

The HJB equation implies

$$J_p \cdot \mathbf{1} + v_t - P_{t^-} = 0 \tag{19}$$

$$\begin{aligned} &-rJ(s^-) + J_t(s^-) + J_\nu(s^-) \cdot \mu + J_p(s^-) \cdot \alpha + \frac{1}{2}\mathrm{tr}(J_{\nu\nu}\Sigma\Sigma') \\ &\qquad + \mathrm{tr}(J_{\nu p}\Omega\Sigma') + \frac{1}{2}\mathrm{tr}(J_{pp}(\Omega\Omega' + \mathbf{1}\sigma_Z^2\mathbf{1}')) = 0 \end{aligned} \tag{20}$$

Substituting first eq. (20) into the previous expression gives

$$\begin{aligned} e^{-rt}J(t) - J(0^-) = &\int_0^t e^{-rs}J_p(s^-) \cdot \mathbf{1}dX + \int_0^t e^{-rs}J_p(s^-)\left(\Omega dW + \mathbf{1}dZ\right) + \int_0^t e^{-rs}J_\nu(s^-)\Sigma dW \\ &+ \frac{1}{2}\int_0^t e^{-rs}\mathrm{tr}(J_{pp}(s^-)(\mathbf{1}d[X^cX^c]\mathbf{1}' + 2\Omega d[W,X^c]\mathbf{1}' + 2\mathbf{1}d[X^c,Z]\mathbf{1}')) \\ &+ \int_0^t e^{-rs}\mathrm{tr}(J_{\nu p}(s^-)\mathbf{1}d[X^c,W]\Sigma') + \sum_{0\le s\le t} e^{-rs}\left(J(s) - J(s^-) - J_p(s^-) \cdot \mathbf{1}\Delta X_s\right). \end{aligned}$$

Now substituting eq. (19)

$$\begin{aligned} e^{-rt}J(t) - J(0^-) = &-\int_0^t e^{-rs}(v_s - P_{s^-})(dX + dZ) + \int_0^t e^{-rs}J_p(s^-)\Omega dW + \int_0^t e^{-rs}J_\nu(s^-)\Sigma dW \\ &+ \frac{1}{2}\int_0^t e^{-rs}\mathrm{tr}(J_{pp}(s^-)(\mathbf{1}d[X^cX^c]\mathbf{1}' + 2\Omega d[W,X^c]\mathbf{1}' + 2\mathbf{1}d[X^c,Z]\mathbf{1}')) \\ &+ \int_0^t e^{-rs}\mathrm{tr}(J_{\nu p}(s^-)\mathbf{1}d[X^c,W]\Sigma') + \sum_{0<s\le t} e^{-rs}\left(J(s) - J(s^-) - (P_{s^-} - v_s)\Delta X_s\right). \end{aligned}$$

Rearrange

$$\begin{aligned} &\int_0^t e^{-rs}(v_s - P_{s^-})dX - J(0^-) \\ &= -e^{-rt}J(t) - \int_0^t e^{-rs}(v_s - P_{s^-})dZ + \int_0^t e^{-rs}J_p(s^-)\Omega dW + \int_0^t e^{-rs}J_\nu(s^-)\Sigma dW \\ &+ \frac{1}{2}\int_0^t e^{-rs}\mathrm{tr}(J_{pp}(s^-)(\mathbf{1}d[X^cX^c]\mathbf{1}' + 2\Omega d[W,X^c]\mathbf{1}' + 2\mathbf{1}d[X^c,Z]\mathbf{1}')) \\ &+ \int_0^t e^{-rs}\mathrm{tr}(J_{\nu p}(s^-)\mathbf{1}d[X^c,W]\Sigma') + \sum_{0\le s\le t} e^{-rs}\left(J(s) - J(s^-) - (P_{s^-} - v_s)\Delta X_s\right) \end{aligned}$$

Use the cyclic property of the trace and the fact that the trace of a scalar is itself to simplify the traces to matrix multiplication

$$
\begin{aligned}
&\int_0^t e^{-rs}(v_s - P_{s^-})dX - J(0^-) \\
&= -e^{-rt}J(t) - \int_0^t e^{-rs}(v_s - P_{s^-})dZ + \int_0^t e^{-rs}J_p(s^-)\Omega dW + \int_0^t J_\nu(s^-)\Sigma dW \\
&+ \int_0^t e^{-rs}\frac{1}{2}d[X^c, \mathbf{1}X^c]J_{pp}(s^-)\mathbf{1} + d[X^c, \Omega W]J_{pp}(s^-)\mathbf{1} + d[X^c, \mathbf{1}Z]J_{pp}(s^-)\mathbf{1})) \\
&+ \int_0^t e^{-rs}d[X^c, \Sigma W]J_{\nu p}(s^-)\mathbf{1} + \sum_{0\le s\le t} e^{-rs}\left(J(s) - J(s^-) - (P_{s^-} - v_s)\Delta X_s\right)
\end{aligned}
$$

Now, add $\int_{[0,t)} e^{-rs}d[X, v-P]_s + \int_{[0,t]} e^{-rs}X_{s^-}dv_s$ to both sides

$$
\begin{aligned}
&\int_0^t e^{-rs}(v_s - P_{s^-})dX + \int_0^t e^{-rs}d[X, v-P]_s + \int_{[0,t]} e^{-rs}X_{s^-}dv_s - J(0) \\
&= -e^{-rt}J(t) - \int_0^t e^{-rs}(v_s - P_{s^-})dZ + \int_0^t e^{-rs}J_p(s^-)\Omega dW + \int_0^t e^{-rs}J_\nu(s^-)\Sigma dW \\
&+ \int_0^t e^{-rs}(\frac{1}{2}d[X^c, \mathbf{1}X^c]J_{pp}(s^-)\mathbf{1} + d[X^c, \Omega W]J_{pp}(s^-)\mathbf{1} + d[X^c, \mathbf{1}Z]J_{pp}(s^-)\mathbf{1}) \\
&+ \int_0^t e^{-rs}d[X^c, \Sigma W]J_{\nu p}(s^-)\mathbf{1} + \sum_{0\le s\le t} e^{-rs}\left(J(s) - J(s^-) - (P_{s^-} - v_s)\Delta X_s\right) \\
&\quad + \int_0^t e^{-rs}d[X, v-P]_t + \int_0^t e^{-rs}X_{s^-}dv_s
\end{aligned} \tag{21}
$$

Note that the continuous, local martingale portion of $v_t - P_t$ (as differentials) is

$$
(f_\nu - g_\nu)\cdot\Sigma dW - g_p\cdot(\Omega dW + \mathbf{1}dY^c).
$$

Hence

$$
[X^c, (v-P)^c]_t = \int d[X, (f_\nu - g_\nu)\cdot\Sigma W - g_p\cdot\Omega W]^c - \int d[X, g_p\cdot\mathbf{1}X]^c - \int d[X, g_p\cdot\mathbf{1}Z]^c
$$

From eq. (19) we have

$$
\begin{aligned}
f_\nu - g_\nu &= -J'_{p\nu}\mathbf{1} = -J_{\nu p}\mathbf{1} \\
g_p &= J_{pp}\mathbf{1} > 0,
\end{aligned}
$$

where the inequality follows from the assumption that $g$ is increasing in the endogenous state variables.

Plugging in to the expression for the quadratic variation

$$
\begin{aligned}
[X^c, (v-P)^c]_t &= \int d[X, -\mathbf{1}' J_{p\nu} \Sigma W - \mathbf{1}' J_{pp} \Omega W]^c \\
&\quad - \int d[X, \mathbf{1}' J_{pp} \mathbf{1} X]^c - \int d[X, \mathbf{1}' J_{pp} \mathbf{1} Z]^c \\
&= \int -d[X, W]^c \Sigma' J_{\nu p} \mathbf{1} - \int d[X, W]^c \Omega' J_{pp} \mathbf{1} \\
&\quad - \int d[X, X]^c \mathbf{1}' J_{pp} \mathbf{1} - \int d[X, Z]^c \mathbf{1}' J_{pp} \mathbf{1}
\end{aligned} \tag{22}
$$

Plugging eq. (22) back into eq. (21) and again using $\Delta(v_s - p_s) = -\Delta p_s = -\mathbf{1}\Delta X$ gives

$$
\begin{aligned}
&\int_0^t e^{-rs}(v_s - P_{s^-})dX + \int_0^t e^{-rs} d[X, v-P]_s + \int_0^t e^{-rs} X_{s^-} dv_s - J(0) \\
&= -e^{-rt}J(t) - \int_0^t e^{-rs}(v_s - P_{s^-})dZ + \int_0^t e^{-rs} J_p(s^-)\Omega dW + \int_0^t e^{-rs} J_\nu(s^-)\Sigma dW + \int_0^t e^{-rs} X_{s^-} dv_s \\
&\quad - \frac{1}{2}\int_0^t e^{-rs} d[X^c, X^c] \mathbf{1}' J_{pp} \mathbf{1} + \sum_{0<s\le t} e^{-rs}\left(J(s) - J(s^-) - (P_s - v_s)\Delta X_s\right)
\end{aligned}
$$

Now, using (19) in the jump term and rearranging to group the stochastic integrals

$$
\begin{aligned}
&\int_0^t e^{-rs}(v_s - P_{s^-})dX + \int_0^t e^{-rs}[X, v-P]_s + \int_0^t e^{-rs} X_{s^-} dv_s - J(0^-) \\
&= -e^{-rt}J(t) - \frac{1}{2}\int_0^t e^{-rs} d[X^c, X^c] \mathbf{1}' J_{pp} \mathbf{1} + \sum_{0\le s\le t} e^{-rs}\left(J(s) - J(s^-) - J_p \cdot \mathbf{1}\Delta X_s\right) \\
&\quad - \int_0^t e^{-rs}(v_s - P_{s^-})dZ + \int_0^t e^{-rs} J_p(s^-)\Omega dW + \int_0^t e^{-rs} J_\nu(s^-)\Sigma dW + \int_0^t e^{-rs} X_{s^-} dv_s
\end{aligned} \tag{23}
$$

Let $x(t) = -\int_0^t e^{-rs}(v_s - P_{s^-})dZ + \int_0^t e^{-rs} J_p(s^-)\Omega dW + \int_0^t e^{-rs} J_\nu(s^-)\Sigma dW + \int_{[0,t]} e^{-rs} X_{s^-} dv_s$ denote the stochastic integrals. Recall that by assumption, $v_s$ is a martingale. Owing to the the admissibility condition on trading strategies and integrability conditions on $J$, $x(t)$ is a square integrable martingale. Hence, we can collect the stochastic integrals in the previous expression into a single martingale term, $x(t)$

$$
\begin{aligned}
&\int_0^t e^{-rs}(v_s - P_{s^-})dX_s + \int_{[0,t]} e^{-rs} d[X, v-P]_s + \int_0^t e^{-rs} X_{s^-} dv_s - J(0^-) \\
&= -e^{-rt}J(t) - \frac{1}{2}\int_0^t e^{-rs} d[X^c, X^c] \mathbf{1}' J_{pp} \cdot \mathbf{1}
\end{aligned}
$$

$$+ \sum_{0 \le s \le t} e^{-rs} \left( J(s) - J(s^-) - J_p \cdot \mathbf{1} \Delta X_s \right) + x(t). \tag{24}$$

Taking limits, we now have

$$\begin{aligned}
&\mathbb{E}\left[ \int_0^\infty e^{-rs}(v_s - P_{s^-}) dX_s + \int_{[0,\infty)} e^{-rs} d[X, v-P]_s + \int_{[0,\infty)} e^{-rs} X_{s^-} dv_s - J(0^-) \right] \\
&= \mathbb{E}\left[ \lim_{t\to\infty} \int_0^t e^{-rs}(v_s - P_{s^-}) dX_s + \int_0^t e^{-rs} d[X, v-P]_s + \int_{[0,t]} e^{-rs} X_{s^-} dv_s - J(0^-) \right] \\
&= \mathbb{E}\Bigg[ \lim_{t\to\infty} -e^{-rt} J(t) - \tfrac{1}{2} \textstyle\int_0^t e^{-rs} d[X^c, X^c] \mathbf{1}' J_{pp} \mathbf{1} \\
&\qquad\qquad + \sum_{0\le s\le t} e^{-rs} \left( J(s) - J(s^-) - J_p \cdot \mathbf{1} \Delta X_s \right) \Bigg] \\
&= \lim_{t\to\infty} \mathbb{E}\left[ -e^{-rt} J(t) - \tfrac{1}{2} \textstyle\int_0^t e^{-rs} d[X^c, X^c] \mathbf{1}' J_{pp} \mathbf{1} + \sum_{0\le s\le t} e^{-rs} \left( J(s) - J(s^-) - J_p \cdot \mathbf{1} \Delta X_s \right) + x(t) \right] \\
&= \lim_{t\to\infty} \mathbb{E}\left[ -\tfrac{1}{2} \textstyle\int_0^t e^{-rs} d[X^c, X^c] \mathbf{1}' J_{pp} \cdot \mathbf{1} + \sum_{0\le s\le t} e^{-rs} \left( J(s) - J(s^-) - J_p \cdot \mathbf{1} \Delta X_s \right) \right] \\
&\le 0
\end{aligned}$$

where the third line uses 24, the fourth line uses (i) the transversality condition on $J$ and $J \ge 0$, (ii) the fact that $x(t)$ is a square integrable martingale and (iii) the monotone convergence theorem, and the next-to-last line uses the transversality condition and the fact that $x(t)$ is a martingale. The final line uses the fact that $J_{pp} \cdot \mathbf{1} = g_p > 0$ by assumption on the pricing rule and that $[X^c, X^c]$ is a positive measure. Note that the inequality holds with equality if and only if the trading strategy is absolutely continuous, in which case $[X^c, X^c] \equiv 0$, $\Delta X \equiv 0$, and $J(s) - J(s^-) \equiv 0$.

This analysis establishes that the expected profit for any trading strategy is no greater than $J(0^-)$. Furthermore, any absolutely continuous strategy that induces the value function to satisfy the integrability and transversality conditions provides expected profit equal to $J(0^-)$ and and is therefore an optimal strategy. Any strategy that is not absolutely continuous and does not lead to a value function satisfying the transversality condition is strictly suboptimal.

□

## A.2 Proof of Proposition 4.

Caldentey and Stacchetti (2010) show that there exists an equilibrium in which the asset price and trading strategy are given by

$$
\begin{aligned}
p_n(\bar{V}_n, y_n) &= \bar{V}_n + \lambda_n y_n \\
x_n(V, \bar{V}_n) &= \beta_n(V - \bar{V}_n),
\end{aligned}
$$

the trader's expected profit is

$$
\Pi_n(p_{n-1}, V) = \alpha_n(V - p_{n-1})^2 + \gamma_n,
$$

and the constants are characterized by the difference equations

$$
\begin{aligned}
\Sigma_{n+1} &= \frac{\Sigma_n \Sigma_z}{\beta_n^2 \Sigma_n + \Sigma_z} \\
\beta_{n+1}\Sigma_{n+1} &= \rho \beta_n \Sigma_n \left( \frac{\Sigma_z^2}{\Sigma_z^2 - \beta_n^4 \Sigma_n^2} \right) \\
\lambda_n &= \frac{\beta_n \Sigma_n}{\beta_n^2 \Sigma_n + \Sigma_z} \\
\alpha_n &= \frac{1 - \lambda_n \beta_n}{2\lambda_n} \\
\rho \gamma_{n+1} &= \gamma_n - \frac{1 - 2\lambda_n \beta_n}{2\lambda_n(1 - \lambda_n \beta_n)} \lambda_n^2 \Sigma_z \\
\gamma_0 &= \sum_{k=0}^{\infty} \rho^k \left( \frac{1 - 2\lambda_k \beta_k}{2\lambda_k(1 - \lambda_k \beta_k)} \right) \lambda_k^2 \Sigma_z
\end{aligned}
$$

The ex-ante expected profit from acquiring information immediately before the $t = 0$ trading round is

$$
\begin{aligned}
\bar{\Pi}_0 \equiv \mathbb{E}[\Pi_0(p_{-1}, V)] - c &= \alpha_0 \mathbb{E}[(V - p_{-1})^2] + \gamma_0 - c \\
&= \alpha_0 \Sigma_0 + \gamma_0 - c.
\end{aligned}
$$

We would like to compare this to the expected profit if the trader deviates by remaining uninformed for the $n = 0$ trading round and then acquiring immediately before round $n = 1$. Supposing that she does so, trades $x$ units at time zero, and then follows the prescribed equilibrium trading strategy in the following rounds, the expected profit is

$$
\begin{aligned}
&\mathbb{E}[x(V - p_0(\bar{V}_0, x + z_0))] + \mathbb{E}\left[ \sum_{n=1}^{\infty} \rho^n (V - p_n) x_n - \rho c \right] \\
&= \mathbb{E}[x(V - \lambda_0(x + z_0))] + \rho \mathbb{E}\left[ \sum_{n=1}^{\infty} \rho^n (V - p_n) x_n - c \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbb{E}[x(V - \lambda_0(x + z_0))] + \rho\mathbb{E}[\Pi_1(p_0, V) - c] \\
&= \mathbb{E}[x(V - \lambda_0(x + z_0))] + \rho\mathbb{E}[\alpha_1(V - p_0)^2 + \gamma_1 - c] \\
&= \mathbb{E}[x(V - \lambda_0(x + z_0))] + \rho\mathbb{E}[\alpha_1(V - \lambda_0(x + z_0))^2 + \gamma_1 - c] \\
&= -x^2\lambda_0 + \rho\left(\alpha_1(\Sigma_0 + \lambda_0^2\Sigma_z + \lambda_0^2 x^2) + \gamma_1 - c\right),
\end{aligned}
$$

Take $x = 0$. This yields ex-ante deviation profits

$$
\bar{\Pi}_{d0} = \rho\left(\alpha_1(\Sigma_0 + \lambda_0^2\Sigma_z) + \gamma_1 - c\right).
$$

This deviation is profitable if and only if

$$
\begin{aligned}
&\bar{\Pi}_{d0} - \bar{\Pi}_0 > 0 \\
&\iff \rho\left(\alpha_1(\Sigma_0 + \lambda_0^2\Sigma_z) + \gamma_1 - c\right) - (\alpha_0\Sigma_0 + \gamma_0 - c) > 0 \\
&\iff (\rho\alpha_1 - \alpha_0)\Sigma_0 + \rho\gamma_1 - \gamma_0 + \rho\alpha_1\lambda_0^2\Sigma_z + (1 - \rho)c > 0.
\end{aligned}
$$

We have

$$
\begin{aligned}
\rho\alpha_1 - \alpha_0 &= \rho\frac{1 - \lambda_1\beta_1}{2\lambda_1} - \frac{1 - \lambda_0\beta_0}{2\lambda_0} \\
&= \rho\frac{1 - \frac{\beta_1^2\Sigma_1}{\beta_1^2\Sigma_1 + \Sigma_z}}{2\frac{\beta_1\Sigma_1}{\beta_1^2\Sigma_1 + \Sigma_z}} - \frac{1 - \frac{\beta_0^2\Sigma_0}{\beta_0^2\Sigma_0 + \Sigma_z}}{2\frac{\beta_0\Sigma_0}{\beta_0^2\Sigma_0 + \Sigma_z}} \\
&= \frac{\rho}{2}\frac{\Sigma_z}{\beta_1\Sigma_1} - \frac{1}{2}\frac{\Sigma_z}{\beta_0\Sigma_0} \\
&= \frac{1}{2}\Sigma_z\frac{1}{\beta_0\Sigma_0}\left(\frac{\rho\beta_0\Sigma_0}{\beta_1\Sigma_1} - 1\right) \\
&= \frac{1}{2}\Sigma_z\frac{1}{\beta_0\Sigma_0}\left(\frac{\Sigma_z^2 - \beta_0^4\Sigma_0^2}{\Sigma_z^2} - 1\right) \\
&= -\frac{1}{2}\frac{\beta_0^3\Sigma_0}{\Sigma_z},
\end{aligned}
$$

where the first equality substitutes in from the difference equation for $\alpha_n$, the second equality substitutes from the equation for $\lambda_n$, the third and fourth simplify and collect terms, the fifth equality uses the difference equation for $\beta_{n+1}\Sigma_{n+1}$, and the final equality simplifies and collects terms.

Similarly,

$$
\rho\gamma_1 - \gamma_0 = -\frac{1 - 2\lambda_0\beta_0}{2\lambda_0(1 - \lambda_0\beta_0)}\lambda_0^2\Sigma_z
$$

$$
\begin{aligned}
&= -\frac{1}{2}\frac{1-2\lambda_0\beta_0}{1-\lambda_0\beta_0}\lambda_0\Sigma_z \\
&= -\frac{1}{2}\frac{1-2\frac{\beta_0^2\Sigma_0}{\beta_0^2\Sigma_0+\Sigma_z}}{1-\frac{\beta_0^2\Sigma_0}{\beta_0^2\Sigma_0+\Sigma_z}}\lambda_0\Sigma_z \\
&= -\frac{1}{2}\frac{\Sigma_z-\beta_0^2\Sigma_0}{\Sigma_z}\lambda_0\Sigma_z \\
&= -\frac{1}{2}(\Sigma_z-\beta_0^2\Sigma_0)\lambda_0,
\end{aligned}
$$

where the first equality uses the difference equation for $\gamma_n$, the second equality cancels a $\lambda_0$, the third equality substitutes for $\lambda_0$, and the last two equalities simplify.

Furthermore, recalling from the calculations for $\rho\alpha_1-\alpha_0$ that $\rho\alpha_1 = \frac{\rho}{2}\frac{\Sigma_z}{\beta_1\Sigma_1}$ we have

$$
\begin{aligned}
\rho\alpha_1\lambda_0^2\Sigma_z &= \frac{\rho}{2}\frac{\Sigma_z}{\beta_1\Sigma_1}\lambda_0^2\Sigma_z \\
&= \frac{1}{2}\frac{\Sigma_z^2}{\beta_0\Sigma_0\left(\frac{\Sigma_z^2}{\Sigma_z^2-\beta_0^4\Sigma_0^2}\right)}\lambda_0^2 \\
&= \frac{1}{2}\frac{1}{\beta_0\Sigma_0}(\Sigma_z^2-\beta_0^4\Sigma_0^2)\lambda_0^2 \\
&= \frac{1}{2}\frac{1}{\beta_0\Sigma_0}(\Sigma_z^2-\beta_0^4\Sigma_0^2)\frac{\beta_0\Sigma_0}{\beta_0^2\Sigma_0+\Sigma_z}\lambda_0 \\
&= \frac{1}{2}(\Sigma_z^2-\beta_0^4\Sigma_0^2)\frac{1}{\beta_0^2\Sigma_0+\Sigma_z}\lambda_0,
\end{aligned}
$$

where the second equality substitutes from the difference equation for $\beta_n\Sigma_n$, the third equality simplifies, the fourth equality substitutes for $\lambda_0$, and the final equality simplifies.

Combining the most recent two displayed expressions, we have

$$
\begin{aligned}
\rho\gamma_1-\gamma_0+\rho\alpha_1\lambda_0^2\Sigma_z &= \frac{1}{2}(\Sigma_z^2-\beta_0^4\Sigma_0^2)\frac{1}{\beta_0^2\Sigma_0+\Sigma_z}\lambda_0 - \frac{1}{2}(\Sigma_z-\beta_0^2\Sigma_0)\lambda_0 \\
&= \frac{1}{2}\lambda_0\frac{1}{\beta_0^2\Sigma_0+\Sigma_z}(\Sigma_z^2-\beta_0^4\Sigma_0^2-(\Sigma_z-\beta_0^2\Sigma_0)(\beta_0^2\Sigma_0+\Sigma_z)) \\
&= \frac{1}{2}\lambda_0\frac{1}{\beta_0^2\Sigma_0+\Sigma_z}(\Sigma_z^2-\beta_0^4\Sigma_0^2-\beta_0^2\Sigma_z\Sigma_0-\Sigma_z^2+\beta_0^4\Sigma_0^2+\beta_0^2\Sigma_z\Sigma_0) \\
&= 0.
\end{aligned}
$$

It follows that

$$
\bar{\Pi}_{d0}-\bar{\Pi}_0 = (\rho\alpha_1-\alpha_0)\Sigma_0+\rho\gamma_1-\gamma_0+\rho\alpha_1\lambda_0^2\Sigma_z+(1-\rho)c
$$

$$= -\frac{1}{2}\frac{\beta_0^3\Sigma_0^2}{\Sigma_z} + (1-\rho)c. \tag{25}$$

We would like to study the behavior of the above expression as $\Delta \to 0$.

**Lemma 2.** *There exists a strictly increasing function $\psi$ such that*

$$\beta_0 = \frac{\sqrt{\Sigma_Z}}{\Sigma_0}\psi(\Sigma_0).$$

*Furthermore, $\psi(0) = 0$ and we have*

$$\psi(\Sigma_0) \leq [1-\rho]^{1/4}\sqrt{\Sigma_0}.$$

*Proof.* This proof leans heavily on the Appendix of Caldentey and Stacchetti (2010) but specialized to the case in which there is no flow of private information ($\Sigma_v \equiv 0$). As such, we point out only the essential differences in the analysis.

Define

$$A_n = \Sigma_n, \quad B_n = \frac{\beta_n\Sigma_n}{\sqrt{\Sigma_z}}$$

Then the difference equations for $\Sigma_n$ and $\beta_n\Sigma_n$ imply that $(A_{n+1}, B_{n+1}) = F(A_n, B_n)$, where

$$F_A(A_n, B_n) = \frac{A_n^2}{A_n + B_n^2}, \quad F_B(A_n, B_n) = \rho\left[\frac{A_n^2 B_n}{A_n^2 - B_n^4}\right].$$

Note that these are similar to those in Caldentey and Stacchetti (2010), with the exception that there is no $+1$ term in $F_A$ owing to the absence of a flow of private information $\Sigma_v$. Further, define

$$G_1(A) = 0, \quad G_2(A) = \sqrt{A}[1-\rho]^{1/4}, \quad G_3(A) = \sqrt{A},$$

where the function $G_1$ is defined so that $F_A(A, G_1(A)) = A$ and $G_2(A)$ is such that $F_B(A, G_2(A)) = G_2(A)$. (Note that there is a typo in this definition in Caldentey and Stacchetti (2010) which states $F_B(A, G_2(A)) = B$. However, this cannot hold in general since $B$ is on only one side of the equation.) Finally, $G_3(A)$ is defined so that a point $(A, B)$ is feasible (i.e., leads to a strictly positive value of $\Sigma_{n+1}\beta_{n+1}$ in its difference equation) if and only if $B < G_3(A)$.

These curves divide $\mathbb{R}_+^2$ into three mutually exclusive regions, the union of which comprises all of $\mathbb{R}_+^2$. First, define the infeasible region $R_5 = \{(A, B) : A \geq 0, B \geq G_3(A)\}$. Second, define region $R_1 = \{(A, B) : A \geq 0, G_2(A) < B < G_3(A)\}$. In this region, $F(A, B)$

is always to the left and higher than $(A, B)$ and the given expression for $F$ implies that starting the iteration $(A_{n+1}, B_{n+1}) = F(A_n, B_n)$ in $R_1$ will eventually lead the sequence to enter the infeasible region $R_5$. Finally, define $R_2 = \{(A, B) : A \geq 0, 0 = G_1(A) < B \leq G_2(A)\}$. Note that any sequence $(A_n, B_n)$ always remains feasible, as shown by Caldentey and Stacchetti (2010). Hence, any candidate $(A_0, B_0)$ must lie in $R_2$.

To put the above more clearly in the setting of Figure 1 in Caldentey and Stacchetti (2010), note that in our case, the function $G_1(A)$ is shifted identically downward to zero. This completely eliminates the regions $R_3$ and $R_4$ in their plot. Furthermore, the stationary point $(\hat{A}, \hat{B})$ in our case is defined by the point at which $G_1(A)$ and $G_2(A)$ intersect. This point is precisely $(0, 0)$. That is, with no flow of information to the insider, in the stationary limit the trader perfectly reveals her information and the market maker faces no residual uncertainty.

To complete the proof, we need to find a curve $\mathcal{C} \subset R_2$ such that $(0, 0) \in \mathcal{C}$ and $F(\mathcal{C}) \subset \mathcal{C}$. Note further that because for sequences in $R_2$, we have $(A_{n+1}, B_{n+1}) = (F_A(A_n, B_n), F_B(A_n, B_n)) < (A_n, B_n)$ we know that such a curve must be strictly increasing. Furthermore, because $F$ is continuous we know that such a curve exists. This curve can be defined by an increasing function $0 \leq \psi(A) \leq G_2(A)$ with $\psi(0) = 0$ so that $\mathcal{C} = \{(A, B) : A \geq 0, B = \psi(A)\}$.

Clearly if we take $B_0 = \psi(A_0)$ then the associated sequence always lies in $\mathcal{C}$ and we have $(A_n, B_n) \downarrow 0$, the stationary point. Hence, returning to the definitions of $A_0$ and $B_0$, this implies that we need to set

$$\begin{aligned} \frac{\beta_0 \Sigma_0}{\sqrt{\Sigma_Z}} &= \psi(\Sigma_0) \\ \Rightarrow \beta_0 &= \Psi(\Sigma_0) \equiv \frac{\sqrt{\Sigma_Z}}{\Sigma_0} \psi(\Sigma_0). \end{aligned}$$

The claimed inequality holds because it was shown above that $0 \leq \psi(A) \leq G_2(A)$. $\square$

We will now proceed with analyzing the behavior of the deviation profit in eq. (25) as $\Delta$ shrinks. To make clear the dependence of the various coefficients on $\Delta$, we write, e.g., $\beta_0^\Delta$ as applicable in the following. Recall that for a positive function of $\Delta > 0$, $h^\Delta$, we define $h^\Delta \sim O(\Delta^p)$ for $p > 0$ as $\Delta \to 0$ if and only if

$$\limsup_{\Delta \to 0} \frac{h^\Delta}{\Delta^p} < \infty.$$

The following result establishes a property of the limiting behavior of $\beta_0^\Delta$ as $\Delta \to 0$.

**Lemma 3.** *We have*

$$\beta_0^\Delta \sim O(\Delta^{3/4}), \quad \text{as } \Delta \to 0.$$

*Proof.* From Lemma 2 we know

$$\begin{aligned} 0 < \beta_0^\Delta &= \frac{\sqrt{\Sigma_Z}}{\Sigma_0}\psi(\Sigma_0) \\ &= \frac{\sigma_z\sqrt{\Delta}}{\Sigma_0}\psi(\Sigma_0) \\ &\leq \frac{\sigma_z\sqrt{\Delta}}{\Sigma_0}[1-\rho]^{1/4}\sqrt{\Sigma_0} \\ &= \frac{\sigma_z}{\sqrt{\Sigma_0}}\sqrt{\Delta}[1-e^{-r\Delta}]^{1/4}. \end{aligned}$$

We have $[1-e^{-r\Delta}]^{1/4} \sim O(\Delta^{1/4})$ from which it follows that

$$0 \leq \limsup_{\Delta\to 0} \frac{\beta_0^\Delta}{\Delta^{3/4}} \leq \frac{\sigma_z}{\sqrt{\Sigma_0}} \limsup_{\Delta\to 0} \frac{\sqrt{\Delta}[1-e^{-r\Delta}]^{1/4}}{\Delta^{3/4}} < \infty,$$

which establishes the result. □

Given these results, we now show that the main result: for all $\Delta > 0$ sufficiently small we have

$$\frac{\bar{\Pi}_{d0} - \bar{\Pi}_0}{\Delta} = \frac{-\frac{1}{2}\frac{(\beta_0^\Delta)^3\Sigma_0^2}{\sigma_z^2\Delta} + (1-e^{-r\Delta})c}{\Delta} > 0.$$

*Proof.* From Lemma 3 we know $\beta_0^\Delta \sim O(\Delta^{3/4})$. It follows that $(\beta_0^\Delta)^3 \sim O(\Delta^{9/4})$. Hence

$$\begin{aligned} \liminf_{\Delta\to 0}\left(-\frac{1}{2}\frac{(\beta_0^\Delta)^3\Sigma_0^2}{\sigma_z^2\Delta^2}\right) &= -\frac{1}{2}\frac{\Sigma_0^2}{\sigma_z^2}\limsup_{\Delta\to 0}\frac{(\beta_0^\Delta)^3}{\Delta^2} \\ &= -\frac{1}{2}\frac{\Sigma_0^2}{\sigma_z^2}\limsup_{\Delta\to 0}\frac{(\beta_0^\Delta)^3}{\Delta^{9/4}}\Delta^{1/4} \\ &= 0. \end{aligned}$$

Similarly since $\frac{1-e^{-r\Delta}}{\Delta} \to rc$, we have

$$\liminf_{\Delta\to 0}\frac{(1-e^{-r\Delta})c}{\Delta} = rc.$$

Combining the above two results yields

$$\liminf_{\Delta \to 0} \frac{\bar{\Pi}_{d0} - \bar{\Pi}_0}{\Delta} = rc > 0,$$

which establishes the result. $\square$

## A.3 Proof of Proposition 6.

The goal of this section is to establish the result in Proposition 6. Recall the conjectures in the text:

$$\begin{aligned} dP &= \lambda(t) dY \\ \theta &= \beta(t)(v_t - P_t) \\ \chi &= \eta(t) \end{aligned}$$

for purely deterministic functions $\lambda, \beta, \eta$, and a strategic trader value function of the form

$$J(t, v_t, P) = \alpha(t)(v_t - P_t)^2 + \gamma(t).$$

We need to show that (i) given the conjectured trading and information acquisition strategy, the optimal pricing rule take the conjectured form, and (ii) given the conjectured pricing rule, the optimal trading and information acquisition strategy take the conjectured form. We proceed by first considering the market maker's filtering problem, then the trader's optimization problem, and then jointly enforcing the optimality conditions derived for each agent.

### A.3.1 The market maker's problem

The market maker must filter the asset value from the order flow. Because the trader is better informed than the market maker, the law of iterated expectations implies that it is sufficient that she filter the trader's value estimate from order flow.

Using the conjectured precision $\eta(t)$, standard linear filtering arguments (e.g., Liptser and Shiryaev (2001)) deliver the dynamic's of the trader's conditional mean and variance as

$$\begin{aligned} dv_t &= \frac{\Omega(t)}{\sigma_s(\cdot)}(ds - v_t dt) = \sqrt{\eta(t)}\Omega(t) d\widehat{W}_{vt} \\ \frac{d\Omega}{dt} &= \sigma_V^2(t) - \eta(t)\Omega^2(t) \end{aligned}$$

where $\widehat{W}_{vt}$ is a Brownian motion under the trader's filtration.

The market maker's objective is to compute $\mathbb{E}[v_t|\mathcal{F}_{Pt}]$ from observing

$$dY = dX + dZ = \beta(t)(v_t - P_t)dt + \sigma_Z(t)dW_{Zt}$$

We can once again apply standard filtering theory to conclude that the market's maker's conditional mean and variance, $(\overline{v}, \Psi)$, are characterized by

$$d\overline{v}_t = \frac{\beta(t)\Psi(t)}{\sigma_Z^2(t)}\left(dY_t - \beta(t)(\overline{v}_t - P_t)dt\right)$$
$$\frac{d\Psi}{dt} = \eta(t)\Omega^2(t) - \frac{\beta^2(t)\Psi^2(t)}{\sigma_Z^2(t)}.$$

Enforcing efficient pricing $P_t = \overline{v}_t$ and then setting coefficients equal to the conjectured optimal pricing rule yields

$$\lambda(t) = \frac{\beta(t)\Psi(t)}{\sigma_Z^2(t)} \tag{26}$$

with

$$\frac{d\Psi}{dt} = \eta(t)\Omega^2(t) - \sigma_Z^2(t)\lambda^2(t). \tag{27}$$

### A.3.2 The trader's problem

We will use the HJB equation to determine the optimal strategy and value function coefficients.[20] Let $M_t = v_t - P_t$ and write the trader's conjectured value function as

$$J(t, M) = \alpha(t)M_t^2 + \gamma(t).$$

The trader's conditional mean and variance, under arbitrary precision $\chi(\cdot)$ are

$$dv_t = \frac{\Omega(t)}{\sigma_s(\cdot)}(ds - v_t dt) = \sqrt{\chi}\Omega(t)d\widehat{W}_{vt}$$
$$\frac{d\Omega}{dt} = \sigma_V^2(t) - \chi(\cdot)\Omega^2(t)$$

[20]It is straightforward to use standard verification arguments (see, e.g, Caldentey and Stacchetti (2010)) to show that the resulting strategies are in fact optimal and that the value function coefficients are as given in the Proposition. Details are available from the authors on request.

where $\widehat{W}_{vt}$ is a Brownian motion, independent of $W_{Zt}$, under the trader's filtration. Under her conjectured pricing rule, $M_t$ therefore follows

$$\begin{aligned} dM_t &= dv_t - dP_t \\ &= -\lambda(t)dY_t + \sqrt{\chi}\Omega(t)d\widehat{W}_{vt} \\ &- \lambda(t)\theta dt - \lambda(t)\sigma_Z(t)dW_{Zt} + \sqrt{\chi}\Omega(t)d\widehat{W}_{vt}. \end{aligned}$$

Suppressing the arguments of functions, the trader's HJB equation is

$$0 = \sup_{\theta\in\mathbb{R},\chi>0}\left\{-rJ + J_t - J_M\lambda\theta + \frac{1}{2}J_{MM}\left(\lambda^2\sigma_Z^2 + \chi\Omega^2\right) + \theta M - c(\chi)\right\}$$

The first-order condition with respect to $\theta$ and $\chi$ require

$$\begin{aligned} -J_M\lambda + M &= 0 \\ \frac{1}{2}J_{MM}\Omega^2 - c'(\chi) &= 0. \end{aligned}$$

Note that because of the convexity of $c(\cdot)$, the second-order condition that the Hessian with respect to $(\theta,\chi)$ is positive semi-definite is automatically satisfied. Using the conjectured value function and inverting the marginal cost $(c')^{-1} = f$ to solve explicitly for the precision (which is permissible because $c'' > 0$ and $c'$ is surjective onto $[0,\infty)$) yields

$$\begin{aligned} -2\alpha M\lambda + M &= 0 \\ \chi &= f\left(\alpha\Omega^2\right). \end{aligned} \tag{28}$$

Substituting the precision back into the HJB equation further yields

$$0 = -r\left(\alpha M^2 + \gamma\right) + \left(\alpha_t M^2 + \gamma_t\right) + \alpha\left(f\left(\alpha\Omega^2\right)\Omega^2 + \lambda^2\sigma_Z^2\right) - c\left(f\left(\alpha\Omega^2\right)\right). \tag{29}$$

Matching coefficients in eq. (28) and (29) yields the following system for $\alpha$, $\gamma$, and $\lambda$

$$\begin{aligned} 1 - 2\alpha\lambda &= 0 \\ \alpha_t - r\alpha &= 0 \end{aligned} \tag{30}$$

$$\gamma_t - r\gamma + \alpha\left(f\left(\alpha\Omega^2\right)\Omega^2 + \lambda^2\sigma_Z^2\right) - c\left(f\left(\alpha\Omega^2\right)\right) = 0.$$

Furthermore, matching the optimal precision with the initial conjecture yields

$$\chi = f\left(\alpha\Omega^2\right). \tag{31}$$

### A.3.3 Equilibrium conditions and derivation

Bringing together the results from the previous two subsections, an equilibrium is a collection of functions $(\lambda, \beta, \eta; \Omega, \Psi; \alpha, \gamma)$ that solve the system

$$\lambda(t) = \frac{\beta(t)\Psi(t)}{\sigma_Z^2(t)} \tag{32}$$

$$\frac{d\Psi}{dt} = \chi(t)\Omega^2(t) - \sigma_Z^2(t)\lambda^2(t) \tag{33}$$

$$\chi(t) = f\left(\alpha(t)\Omega^2(t)\right) \tag{34}$$

$$\frac{d\Omega}{dt} = \sigma_V^2(t) - \chi(t)\Omega(t)^2 \tag{35}$$

$$1 = 2\alpha(t)\lambda(t) \tag{36}$$

$$0 = \alpha_t(t) - r\alpha(t) \tag{37}$$

$$0 = \gamma_t(t) - r\gamma(t) + \alpha(t)\left(\chi(t)\Omega^2(t) + \lambda^2(t)\sigma_Z^2(t)\right) - c\left(\chi(t)\right) \tag{38}$$

It is immediate that

$$\lambda(t) = \lambda(0)e^{-rt}$$
$$\alpha(t) = \frac{1}{2\lambda(0)}e^{rt}.$$

so plugging into the $\chi$ equation gives

$$\chi(t) = f\left(\frac{1}{2\lambda(0)}e^{rt}\Omega^2(t)\right).$$

Similarly, the $\Omega$ differential equation becomes

$$\frac{d\Omega}{dt} = \sigma_V^2(t) - f\left(\frac{1}{2\lambda(0)}e^{rt}\Omega^2(t)\right)\Omega(t)^2. \tag{39}$$

which specifies $\Omega$ in terms of exogenous objects, up to the constant $\lambda(0)$.

To solve for $\Psi(t)$, let $\Sigma(t) = \mathrm{var}(V_t|\mathcal{F}_{Pt})$ be the public conditional variance of the asset value itself. The law of total variance implies $\Sigma(t) = \Psi(t) + \Omega(t)$. Adding the differential equations for $\Psi$ and $\Omega$ gives a differential equation in $\Sigma$

$$\begin{aligned}\frac{\partial \Sigma}{\partial t} &= \sigma_V^2(t) - \sigma_Z^2(t)\lambda^2(t) \\ &= \sigma_V^2(t) - \lambda^2(0)e^{-2rt}\sigma_Z^2(t)\end{aligned}$$

from which direct integration yields

$$\begin{aligned}\Sigma(t) &= \int_0^t \sigma_V^2(s)ds - \lambda^2(0)\int_0^t e^{-2rs}\sigma_Z^2(s)ds \\ &= \Sigma(0) + \Gamma(0) - \Gamma(t) - \lambda^2(0)(\Phi(0) - \Phi(t))\end{aligned}$$

Hence,

$$\begin{aligned}\Psi(t) &= \Sigma(t) - \Omega(t) \\ &= \Sigma(0) + \Gamma(0) - \Gamma(t) - \lambda^2(0)(\Phi(0) - \Phi(t)) - \Omega(t)\end{aligned}$$

Next, using the $\lambda$ equation, the trading rate satisfies

$$\beta(t) = \frac{\lambda(t)\sigma_Z^2(t)}{\Psi(t)}.$$

Finally, integrating the $\gamma$ differential equation yields

$$\gamma(t) = e^{rt}\left(\int_t^\infty \alpha(s)\chi(s)\Omega^2(s)ds + \int_t^\infty \alpha(s)\lambda^2(s)\sigma_Z^2(s)ds - \int_t^\infty c\left(\chi(s)\right)ds + C\right)$$

with $C$ a constant to be determined.

We have now specified all functions $(\lambda, \beta, \eta; \Omega, \Psi; \alpha, \gamma)$, up to the constants $\lambda(0)$ and $C$. To determine the values of these constants, one must enforce the transversality condition for the trader

$$\lim_{t\to\infty} \mathbb{E}\left[e^{-rt}J(t, M_t)\right] = \lim_{t\to\infty} \mathbb{E}\left[e^{-rt}\left(\alpha(t)M_t^2 + \gamma(t)\right)\right].$$

One immediately concludes $C = 0$. Now, note that given the above expression for $\alpha(t)$, $\lim_{t\to\infty} e^{-rt}\alpha(t)\mathbb{E}\left[M_t^2\right] = 0$ if and only if $\mathbb{E}\left[M_t^2\right] \to 0$. Hence we must demonstrate $\mathbb{E}\left[M_t^2\right] \to 0$ under the trading strategy $\beta(t) = \frac{\lambda(t)\sigma_Z^2(t)}{\Psi(t)}$ specified above. Under an arbitrary strategy $\beta(t)$, we have

$$
\begin{aligned}
dM_t &= -\lambda(t)\beta(t)M_t dt - \lambda(t)\sigma_Z(t)dW_{Zt} + \sqrt{\chi}\Omega(t)d\widehat{W}_{vt} \\
\Rightarrow d(M_t^2) &= -2\lambda(t)\beta(t)M_t^2 dt - 2M_t\sigma_Z(t)dW_{Zt} + 2M_t\sqrt{\chi}\Omega(t)d\widehat{W}_{vt} \\
&\quad + \left(\lambda^2(t)\sigma_Z^2(t) + \chi^2(t)\Omega^2(t)\right) dt \\
\Rightarrow d\left(e^{\int_0^t 2\lambda(s)\beta(s)ds} M_t^2\right) &= e^{\int_0^t 2\lambda(s)\beta(s)ds}\left(-2M_t\sigma_Z(t)dW_{Zt} + 2M_t\sqrt{\chi}\Omega(t)d\widehat{W}_{vt}\right) \\
&\quad + e^{\int_0^t 2\lambda(s)\beta(s)ds}\left(\lambda^2(t)\sigma_Z^2(t) + \chi^2(t)\Omega^2(t)\right) dt \\
\Rightarrow \mathbb{E}\left[M_t^2\right] &= e^{-\int_0^t 2\lambda(s)\beta(s)ds} M_0^2 + \int_0^t e^{-\int_s^t 2\lambda(u)\beta(u)du}\left(\lambda^2(s)\sigma_Z^2(s) + \chi^2(s)\Omega^2(s)\right) dt.
\end{aligned}
$$

For this to tend to zero we require $\int_0^t 2\lambda(s)\beta(s)ds \to \infty$. Under the posited trading strategy $\beta(t)$, and the function $\lambda(t)$ derived above, we need

$$
\lambda^2(0)\int_0^t e^{-2rs}\frac{\sigma_Z^2(s)}{\Psi(s)}ds \to \infty.
$$

Since $\Phi_0 = \int_0^t e^{-2rs}\sigma_Z^2(s)ds < \infty$ by assumption, this hold if and only if $\Psi(t) \to 0$. Therefore, returning to the earlier expression for $\Psi(t)$, using $\Gamma(t) \to 0$, $\Phi(t) \to 0$, and supposing for the moment that $\Omega(t)$ has a well-defined limit:

$$
\lambda(0) = \sqrt{\frac{\Sigma(0) + \Gamma(0) - \lim_{t\to\infty}\Omega(t)}{\Phi(0)}}. \tag{40}
$$

We have now characterized the equilibrium up to (i) the initial value problem (39) for $\Omega(t)$, which depends on $\lambda(0)$, and (ii) the expression for $\lambda(0)$ in (40), which depends on the limiting behavior of $\Omega(t)$. If we can demonstrate existence of a unique, positive solution $\Omega$ to the initial value problem 39 and show that $\lim_{t\to\infty}\Omega(t) = 0$, regardless of the value of $\lambda(0)$, the proof is complete since this will (i) establish $\lambda(0) = \sqrt{\frac{\Sigma(0)+\Gamma(0)}{\Phi(0)}}$ and (ii) guarantee a solution $\Omega(t)$ corresponding to that particular $\lambda(0)$. Plugging $\lambda(0) = \sqrt{\frac{\Sigma(0)+\Gamma(0)}{\Phi(0)}}$ back into the expressions above for the various functions and simplifying then delivers the explicit expressions in the Proposition in the text.

The following Lemma characterizes $\Omega(t)$ for an arbitrary value of $\lambda(0)$ and completes the proof:

**Lemma 4.** *Suppose $\Gamma_0 = \int_0^\infty \sigma_V^2(s)ds < \infty$. Fix any $C > 0$. There exists a unique, (weakly) positive solution to the initial value problem $\frac{d\Omega}{dt} = \sigma_V^2(t) - f\left(\frac{1}{2C}e^{rt}\Omega^2(t)\right)\Omega(t)^2$, $\Omega(0) = var(V_0|S_0) \geq 0$. This solution satisfies $\lim_{t\to\infty}\Omega(t) = 0$.*

*Proof.* Clearly any solution, if one exists, is positive since $\Omega(0) \geq 0$ and at any point at which $\Omega(t) = 0$ we have $\frac{d\Omega}{dt} = \sigma_V^2(t) \geq 0$ so that $\Omega(t)$ cannot become strictly negative. Define the function $F(t,x) = \sigma_V^2(t) - f\left(\frac{1}{2C}e^{rt}x^2\right)x^2$ and write the differential equation as $\frac{d\Omega}{dt} = F(t,\Omega)$. Pick any $\varepsilon > 0$. On the open set $D = (-\varepsilon,\infty) \times (-\varepsilon,\infty)$, the function $F$ is locally Lipschitz continuous with respect to its second argument since its derivative with respect to $x$ is continuous. Continuity of the derivative holds since the function $f = (c')^{-1}$ has derivative $f'(x) = 1/c'(c^{-1}(x))$, which is continuous by our assumptions on the cost function $c$. Since our initial condition $(t_0, x_0) = (0, \operatorname{var}(V_0|S_0))$ lies in $D$, it follows from Walter (1998) (Ch. 2, Thm. VII) that there exists a unique solution to the initial value problem on an interval $[0,b)$ where $0 < b \leq \infty$. To ensure that $b = \infty$ and therefore a solution exists at all times, we must rule out solutions that blow up in finite time. To do so, we proceed by contradiction. Suppose that there exists $0 < b < \infty$ such that the solution $\Omega(t)$ to the initial value problem has $\limsup_{t\uparrow b}|\Omega(t)| = \infty$. Given that $\Gamma_0 < 0$ we know that $\sigma_V^2(t)$ is bounded in any neighborhood of $b$. However, since $\Omega(t)$ blows up at $b$, by continuity of $\Omega$ there exists a neighborhood $B = (b-\kappa, b)$ for $\kappa > 0$ sufficiently small such that $\frac{d\Omega}{dt} = \sigma_V^2(t) - f\left(\frac{1}{2C}e^{rt}\Omega^2(t)\right)\Omega(t)^2 < 0$ in $B$. This implies that $\Omega(t)$ is strictly decreasing in $B$, which contradicts $\limsup_{t\uparrow b}|\Omega(t)| = \infty$. Hence, $b = \infty$ and there exists a unique, positive solution $\Omega(t)$ to the initial value problem, defined for all $t \geq 0$.

It remains to show that $\lim_{t\to\infty}\Omega(t) = 0$. Let $M \equiv \limsup \Omega(t)$ and $m = \liminf \Omega(t)$. Note that we must have $m = 0$. If $m > 0$, then for all sufficiently large $t$ we would have $\Omega(t) > m > 0$ and differential equation would imply that $\Omega$ eventually becomes strictly negative: for sufficiently large $t$ we would have

$$\begin{aligned}
\frac{\partial\Omega}{\partial t} &= \sigma_V^2(t) - f\left(\frac{1}{2C}e^{rt}\Omega^2(t)\right)\Omega(t)^2 \\
&\leq \sigma_V^2(t) - f\left(\frac{1}{2C}m^2\right)m^2,
\end{aligned}$$

which, for $t' > t$ would then yield

$$\begin{aligned}
\Rightarrow \Omega(t') &\leq \Omega(t) + \int_t^{t'} \sigma_V^2(s)ds - \int_t^{t'} f\left(\frac{1}{2C}m^2\right)m^2 ds \\
&< \Omega(t) + \Gamma_0 - \int_t^{t'} f\left(\frac{1}{2C}m^2\right)m^2 ds \\
&= \Omega(t) + \Gamma_0 - (t'-t)f\left(\frac{1}{2C}m^2\right)m^2 \\
&< 0 \qquad \text{for } t' \text{ sufficiently large.}
\end{aligned}$$

This contradicts the positivity of $\Omega$. Hence, we conclude $m = 0$.

It remains to show that $M = 0$. Suppose instead that $M > 0$. Note that $\Gamma_0 < \infty$ implies that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for $t > \delta$ we have $\Gamma_t = \int_t^\infty \sigma_V^2(s)ds < \varepsilon$. Furthermore, for $t' > t > \delta$ we have

$$\begin{aligned} \frac{\partial \Omega}{\partial t} &= \sigma_V^2(t) - f\left(\frac{1}{2C}e^{rt}\Omega^2(t)\right)\Omega(t)^2 \leq \sigma_V^2(t) \\ \Rightarrow \Omega(t') &\leq \Omega(t) + \int_t^{t'} \sigma_V^2(s)ds < \Omega(t) + \varepsilon \end{aligned} \tag{41}$$

Pick $\varepsilon' \in (0, M/2)$ and let $\varepsilon \in (0, M/2 - \varepsilon')$. There exists $\delta > 0$ such that $\Gamma_t < \varepsilon$ for all $t > \delta$. Furthermore, because $m = 0$ we can choose such a $t > \delta$ so that $\Omega(t) < \varepsilon'$. Applying eq. (41) now implies that for all $t' > t$

$$\Omega(t') \leq \Omega(t) + \varepsilon < \varepsilon' + \varepsilon = M/2.$$

This implies $\limsup_{t' \to \infty} \Omega(t') \leq M/2 < M$, which is a contradiction. We conclude that $M = m = 0$, which establishes $\lim_{t\to\infty} \Omega(t) = 0$. $\square$