# Large-scale structure of complex networks (Part 2)

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2018-01-01

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Hello

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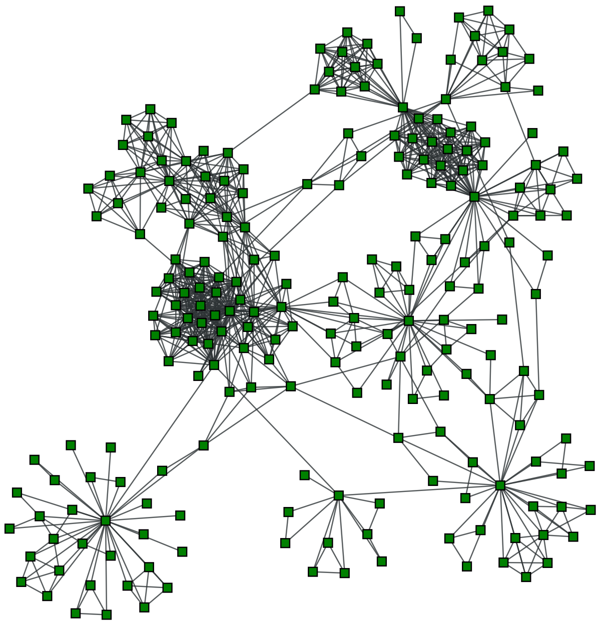
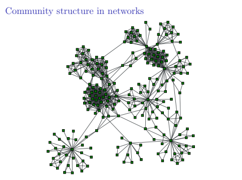
# Community structure in networks








Network of coauthorships in a university department

# Community structure in networks

**What are communities?**

- **Traditional definition**: Groups of nodes with a high internal link density
- **Modern definition**: Nodes with similar connection probabilities to the rest of the network



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# Communities in the real-world networks

- **Social networks**:
  - Friend-circles
  - Research communities
  - Co-workers
- **World Wide Web**:
  - Pages with similar contents
  - Webpages under the same domain (e.g. Wikipedia)
- **Biological networks**:
  - Proteins with similar roles in protein interaction networks
  - Chemicals together taking part in chemical reactions in metabolic networks
  - Communities in neuronal networks



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# Community detection

**Detecting communities is important!**

- Communities are building blocks of networks
- Communities allow us to see "the big picture"
- Functional/Autonomous units
- Non-trivial effects on the processes on networks



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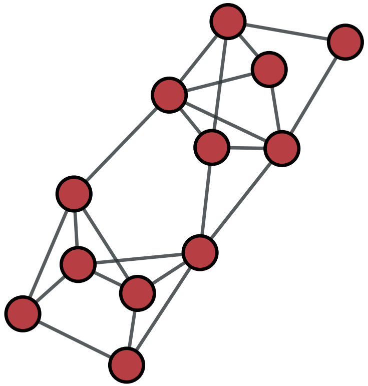
# Graph partitioning

Problem of dividing a graph in a given number of groups of given sizes such that the number of links between the groups (**cut size**) is minimized





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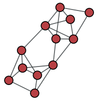

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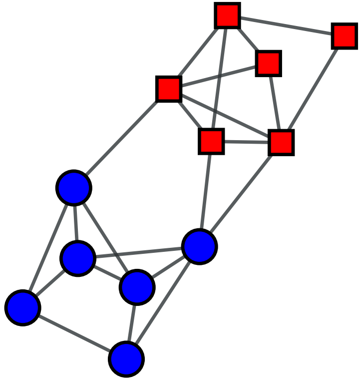



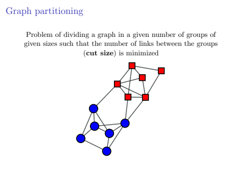

# Partitioning is hard!

- Graph with $n$ vertices
- Find two groups with sizes $n_1$ and $n_2$ such that the cut size is minimum
- Number of ways: $\frac{n!}{n_1!n_2!} \approx \frac{2^{n+1}}{\sqrt{n}}$

**Heuristics are needed!**



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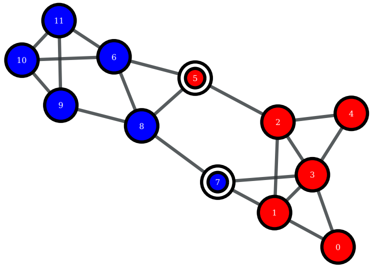
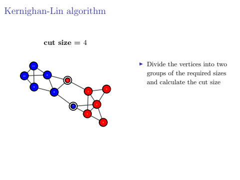
# Kernighan-Lin algorithm

**cut size** $= 4$




- Divide the vertices into two groups of the required sizes and calculate the cut size





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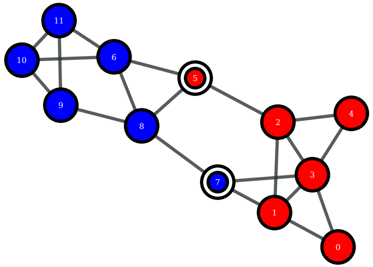
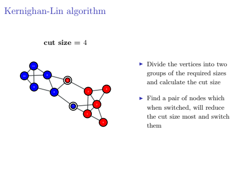



- Divide the vertices into two groups of the required sizes and calculate the cut size
- Find a pair of nodes which when switched, will reduce the cut size most and switch them





# Kernighan-Lin algorithm

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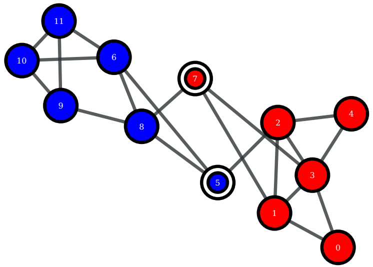


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Kernighan-Lin algorithm

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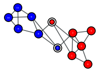


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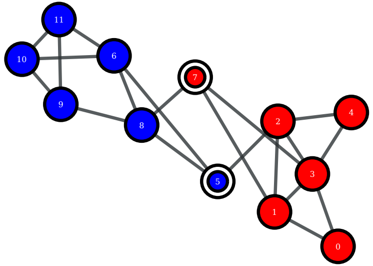


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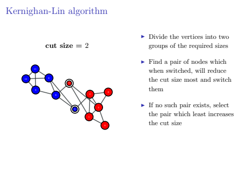

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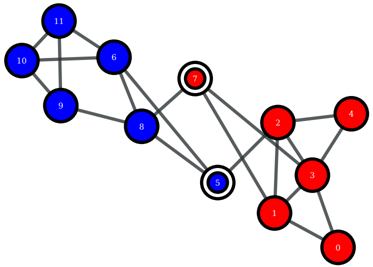


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Kernighan-Lin algorithm
cut size = 2
Divide the vertices into two groups of the required sizes
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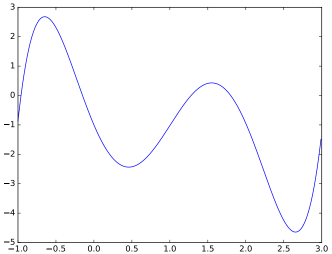


- Go through all the states and select the one with the least cut size
- Start with this state and repeat the whole procedure
- Continue till the cut size no longer becomes smaller
- Starting with many random initial conditions is better




Kernighan-Lin algorithm

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Group sizes remain constant

# Spectral partitioning

- Faster algorithm than Kernighan-Lin
- Uses properties of the graph Laplacian
- More complex to implement than Kernighan-Lin



Spectral partitioning

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# Spectral partitioning

Cut size:

$$R = \frac{1}{2} \sum_{\substack{i,j \text{ in} \\ \text{different} \\ \text{groups}}} A_{ij}$$

Define

$$s_i = \begin{cases} +1 & \text{if vertex } i \text{ belongs to group 1} \\ -1 & \text{if vertex } i \text{ belongs to group 2} \end{cases}$$

Then

$$\frac{1}{2}(1 - s_i s_j) = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are in different groups,} \\ 0 & \text{if } i \text{ and } j \text{ are in the same group} \end{cases}$$



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$$R = \frac{1}{4}\sum_{ij} A_{ij}(1 - s_i s_j)$$

First term,

$$\sum_{ij} A_{ij} = \sum_i k_i = \sum_i k_i s_i^2 = \sum_{ij} k_i \delta_{ij} s_i s_j$$

$$R = \frac{1}{4}\sum_{ij}(k_i \delta_{ij} - A_{ij}) s_i s_j = \frac{1}{4}\sum_{ij} L_{ij} s_i s_j$$

$$R = \frac{1}{4}\mathbf{s}^T \mathbf{L} \mathbf{s}$$



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$$R = \frac{1}{4}\mathbf{s}^T \mathbf{L} \mathbf{s}$$

L is so imp that we have a name for it! Laplacian

.

s is a columnvector

.

L: structure, s: division

.

find s that minimizes R

.

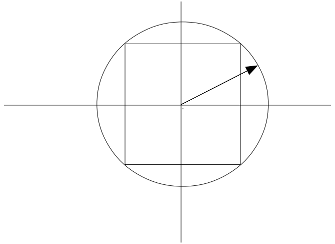
Problem is hard, s takes only integer values

.

# Relaxation method

**Two constraints**:

- $s_i$ can be only $\pm 1$
- $\sum_i s_i = n_1 - n_2 \Rightarrow \mathbf{1}^T\mathbf{s} = n_1 - n_2$



Relax the first constraint



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Relax the first constraint

hypercube

.

continuous s, differentiate

# Spectral partitioning

Minimization with constraints $\Rightarrow$ Lagrange multipliers

$$\frac{\partial}{\partial s_i}\left[\sum_{jk} L_{jk} s_j s_k + \lambda\left(n - \sum_j s_j^2\right) + 2\mu\left((n_1 - n_2) - \sum_j s_j\right)\right] = 0$$



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$$\mathbf{L}\mathbf{s} = \lambda\mathbf{s} + \mu\mathbf{1} = \lambda\left(\mathbf{s} + \frac{\mu}{\lambda}\mathbf{1}\right)$$

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$\mathbf{1}$ is an eigenvector of the Laplacian with eigenvalue 0

$$\mathbf{L}\mathbf{x} = \lambda\mathbf{x}$$



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$$\mathbf{L}\mathbf{x} = \lambda\mathbf{x}$$

# Spectral partitioning

$\mathbf{x}$ is an eigenvector of the Laplacian with eigenvalue $\lambda$

Which eigenvector to choose?

$$\mathbf{x} \text{ cannot be the eigenvector } \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{pmatrix}$$

$$\mathbf{1}^T\mathbf{x} = \mathbf{1}\left(\mathbf{s} + \frac{\mu}{\lambda}\mathbf{1}\right) = (n_1 - n_2) + \frac{\mu}{\lambda}n = 0$$



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x is orthogonal to 1

.

x is eigenvector but not 1

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Which eigenvector to choose?

$$R = \frac{1}{4}\mathbf{s}^T\mathbf{L}\mathbf{s} = \frac{1}{4}\mathbf{x}^T\mathbf{x} = \frac{n_1 n_2}{n}\lambda$$

Choose the eigenvector with smallest possible eigenvalue!

Eigenvalues of the Laplacian are non-negative and smallest is always 0

$\mathbf{v}_1 = \mathbf{1}$ is ruled out already. So choose $\mathbf{v}_2$ with the smallest positive eigenvalue



Spectral partitioning

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$$\mathbf{s} = x + \frac{n_1 - n_2}{n}\mathbf{1}$$

**OR**

$$s_i = x_i + \frac{n_1 - n_2}{n}$$

But $s_i$ can be only $\pm 1$

Thus, we want $\mathbf{x}$ to be as close as possible to $\mathbf{s}$




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Maximize:

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Equivalently, maximize:

$$\sum_i s_i x_i$$

Simply put the $n_1$ vertices with most positive elements in group 1 and the rest ones in group 2

Group assignments are arbitrary



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Group assignments are arbitrary

Choose the assignment with the smaller cut size

# Spectral partitioning

- Calculate $\mathbf{v}_2$ of the Laplacian
- Put vertices corresponding to largest $n_1$ elements in group 1 and others in group 2. Calculate the cut size
- Put vertices corresponding to smallest $n_1$ elements in group 1 and others in group 2. Calculate the cut size
- Choose the division with the smallest cut size among the two



Spectral partitioning

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# Community detection is harder!

- **Graph partitioning**
  - well defined
  - Number of groups is fixed
  - Sizes of the groups are fixed
  - Divide even if no good division exists
- **Community detection**
  - ill-defined
  - Number of groups depends on the structure of the network
  - Sizes of the groups depend on the structure of the network
  - Discover natural fault lines



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# Many definitions.. many algorithms!

- Girvan-Newman algorithm
- Kernighan-Lin-Newman algorithm
- Spectral decomposition
- Clique-percolation
- Radom walk methods
- Statistical inference
- Label propagation
- Hierarchical clustering



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I can go on.. These algorithms use different definitions/views of communities

# Broad classification

- **Agglomerative algorithms:**
  - Hierarchical clustering
  - Louvain method
  - CNM algorithm
- **Divisive algorithms:**
  - Girvan-Newman algorithm
  - Radichhi algorithm
- **Assignment algorithms:**
  - Label propagation
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# “The” simplest community detection problem

- Bisecting a graph with $n$ nodes
- Group sizes are not fixed
- Minimum cut size?



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Empty group

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**A different measure of the quality of division is required..**



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Different measure

# Quantification of community structure

- Fewer than expected edges between the groups



Quantification of community structure

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few edges = expected edges = not a good division

# Quantification of community structure

- Fewer than expected edges between the groups
- Equivalently, more than expected edges inside the groups



Quantification of community structure
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Remember assortativity

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Divide network using modularity

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# Kernighan-Lin-Newman algorithm

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Kernighan-Lin-Newman algorithm

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Variation of KL algorithm

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Sizes of the groups are not fixed

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Kernighan-Lin-Newman algorithm

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- Change in modularity for shifting each vertex to the other group
- Choose vertex whose shift makes maximum modularity change
- If no such vertex exists, choose the one resulting in the least decrease in the modularity
- Repeat so that the vertex once moved is not moved again

Variation of KL algorithm
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Sizes of the groups are not fixed
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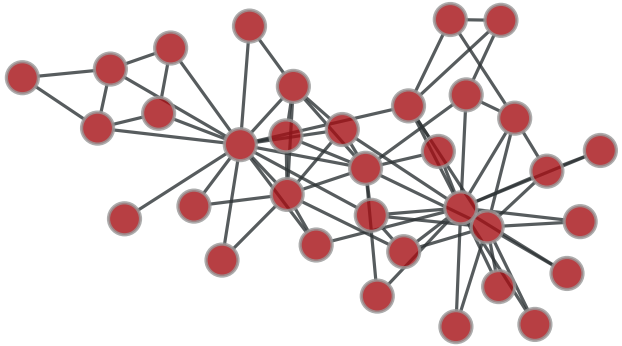
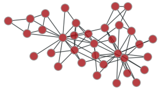


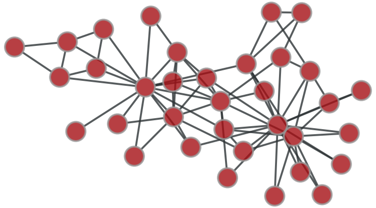


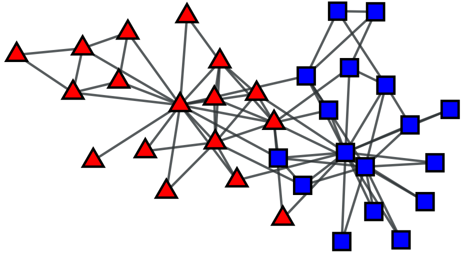
# Large-scale structure of complex networks (Part 2)



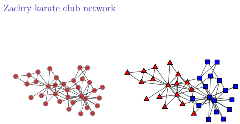
Does somebody know this network?

# Zachry karate club network



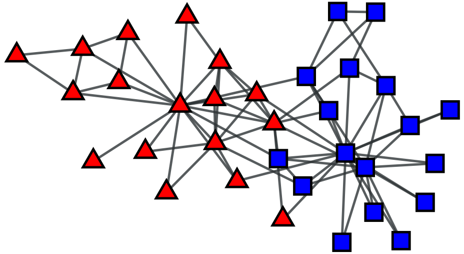






# Application to Zachry karate club

Actual division





Application to Zachry karate club
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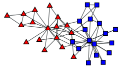
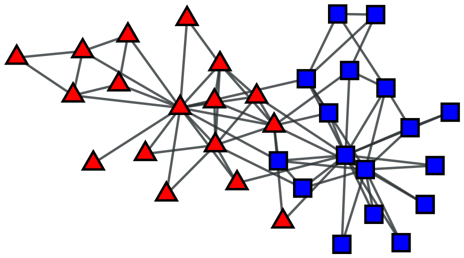
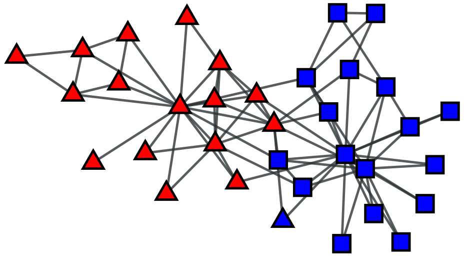


# Application to Zachry karate club

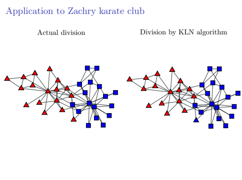
Actual division



Division by KLN algorithm







# Spectral modularity maximization

$$Q = \frac{1}{2m}\sum_{ij}\left(A_{ij} - \frac{k_i k_j}{2m}\right) = \frac{1}{2m}\sum_{ij} B_{ij}\delta(c_i, c_j)$$

Note that:

$$\sum_j B_j = \sum_j A_{ij} - \frac{k_i}{2m}\sum_j k_j = k_i - \frac{k_i}{2m}2m = 0$$



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spectral partitioning: cut size

.

analogous algorithm exists

# Spectral modularity maximization

$$s_i = \begin{cases} +1 & \text{if vertex } i \text{ belongs to group 1} \\ -1 & \text{if vertex } i \text{ belongs to group 2} \end{cases}$$



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$$\frac{1}{2}(1 + s_i s_j) = \begin{cases} 1 & \text{if } i \text{ and } j \text{ belong to the same group} \\ 0 & \text{Otherwise} \end{cases}$$



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$$Q = \frac{1}{4m} \mathbf{s}^T \mathbf{B} \mathbf{s}$$



Spectral modularity maximization

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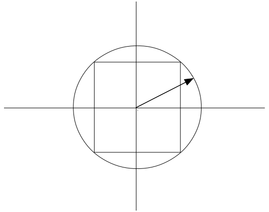
# Spectral modularity maximization

Relaxation method

- Numbers of elements with values $+1$ and $-1$ are not fixed
- Only constraint: $\mathbf{s}^T\mathbf{s} = \sum_i s_i^2 = n$



$$\frac{\partial}{\partial s_i}\left[\sum_{ij} B_{jk} s_j s_k + \beta\left(n - \sum_j s_j^2\right)\right] = 0$$

$$\sum_j B_{ij} s_j = \beta s_i$$

$$\mathbf{B}\mathbf{s} = \beta\mathbf{s}$$



Spectral modularity maximization
Relaxation method
Numbers of elements with values +1 and −1 are not fixed
Only constraint: $\mathbf{s}^T\mathbf{s} = \sum_i s_i^2 = n$
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s is eigenvector of modularity matrix

# Spectral modularity maximization

$$Q = \frac{1}{4m}\beta \mathbf{s}^T \mathbf{B} \mathbf{s} = \frac{1}{4m}\beta \mathbf{s}^T \mathbf{s} = \frac{n}{4m}\beta$$

Thus, choose $\mathbf{s}$ to be the eigenvector $\mathbf{u}_1$ corresponding to the largest eigenvalue of the modularity matrix

Maximize:

$$\mathbf{s}^T \mathbf{u}_1 = \sum_i s_i u_{1i}$$

Maximum is achieved when each term is non-negative $\Rightarrow$ Use signs of $u_{1i}$!



Spectral modularity maximization

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# Spectral modularity maximization

- Calculate the modularity matrix
- Calculate its eigenvector corresponding to the largest eigenvalue
- Assign nodes to communities based on the signs of elements



Spectral modularity maximization

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# Newman-Girvan algorithm

- Look for edges between the communities
- Edge betweenness



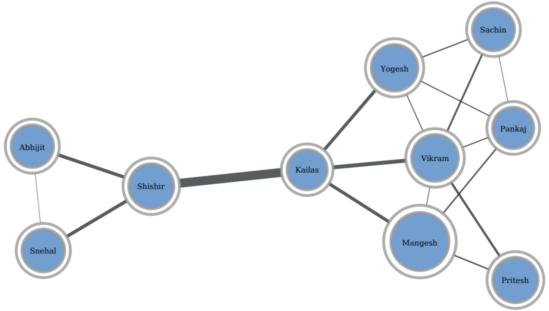
Newman-Girvan algorithm

- Look for edges between the communities
- Edge betweenness

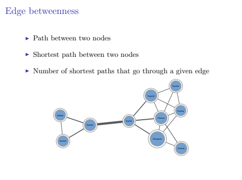
Let's have a look at the edge betweenness

# Edge betweenness

- Path between two nodes
- Shortest path between two nodes
- Number of shortest paths that go through a given edge





# The algorithm

- Calculate betweenness for all edges
- Remove the edge with the highest betweenness
- Recalculate betweenness for all edges
- Repeat



The algorithm

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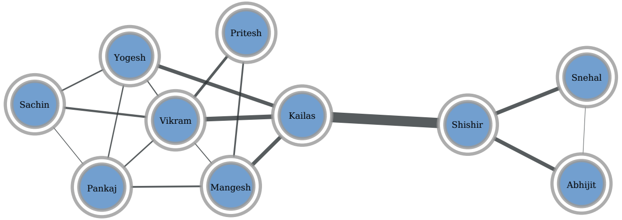
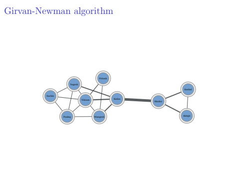
# Girvan-Newman algorithm





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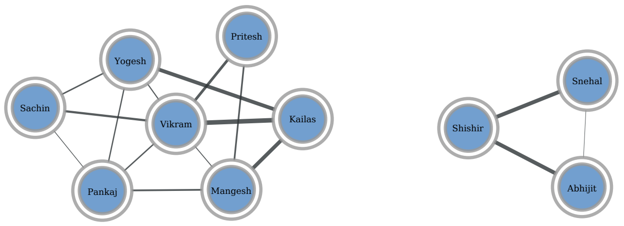




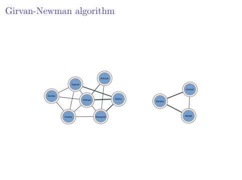

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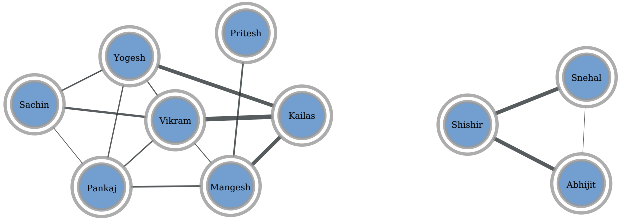


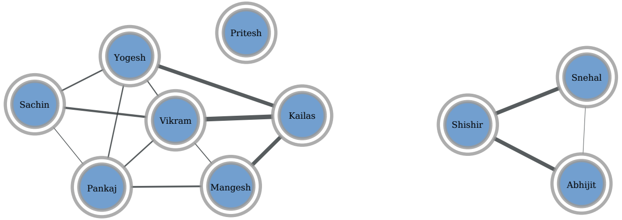

2018-01-01

Large-scale structure of complex networks (Part 2)

└─Girvan-Newman algorithm

Girvan-Newman algorithm

# Girvan-Newman algorithm






Girvan-Newman algorithm

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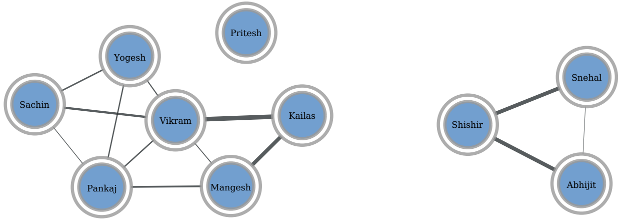


2018-01-01

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└─Girvan-Newman algorithm



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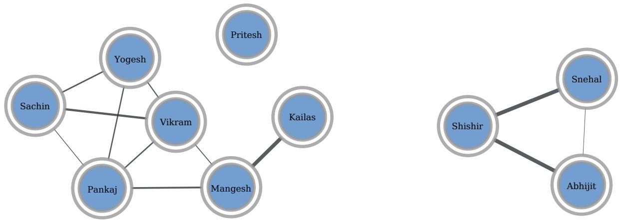




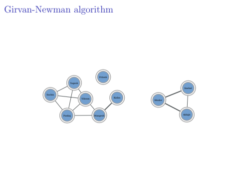

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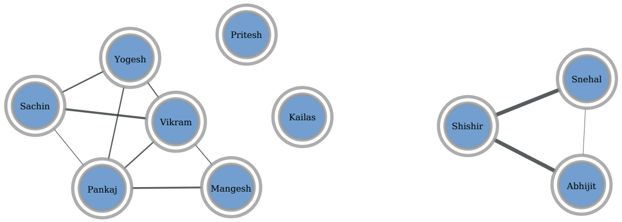




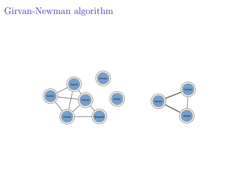

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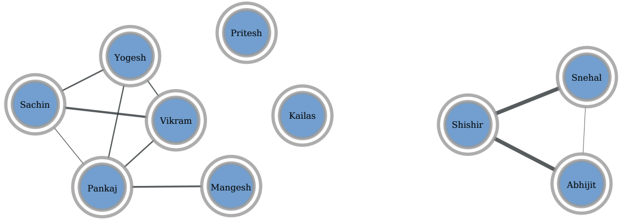




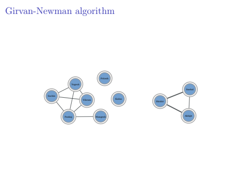

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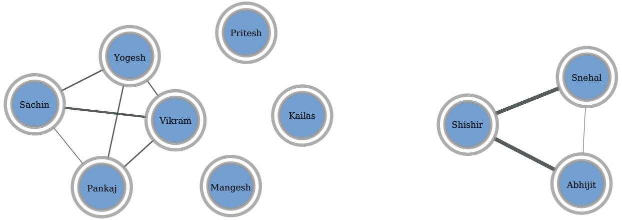




Girvan-Newman algorithm

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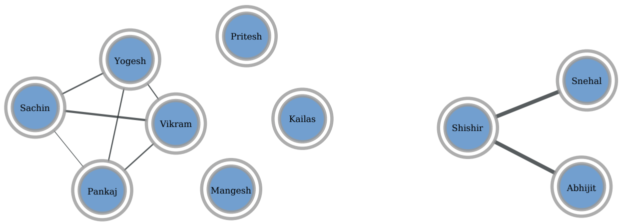




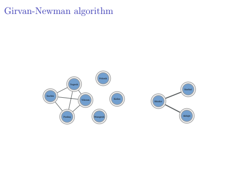

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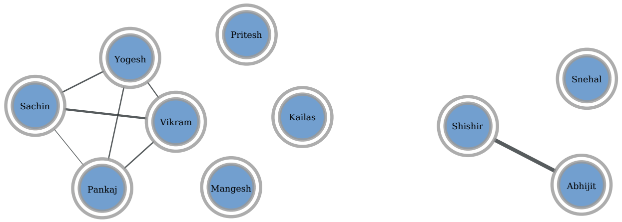




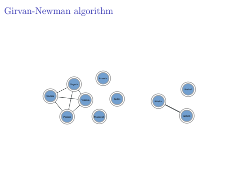

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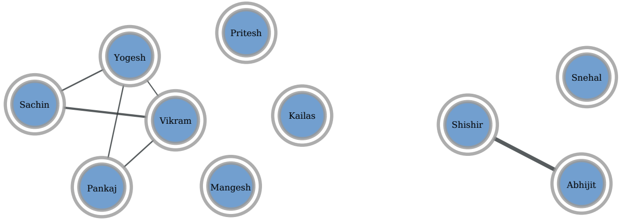




Girvan-Newman algorithm

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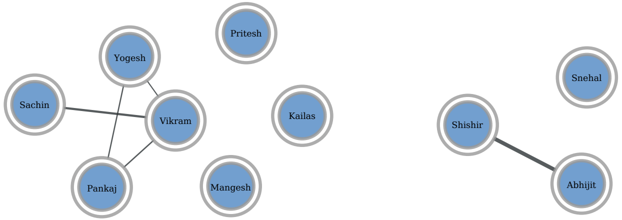




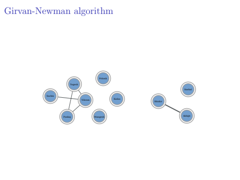

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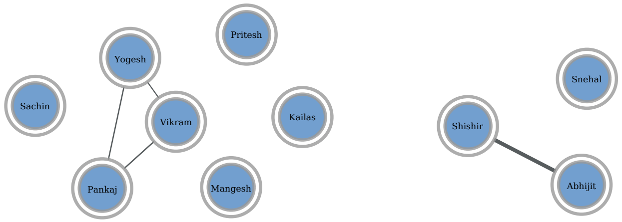




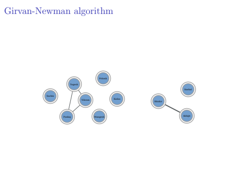

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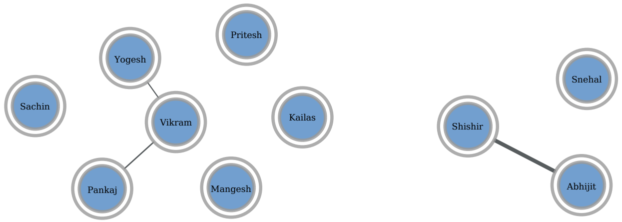








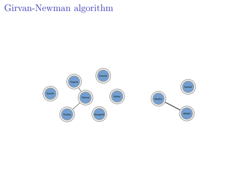

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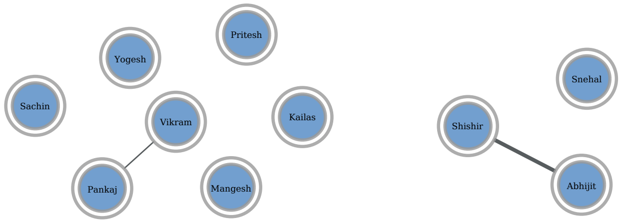




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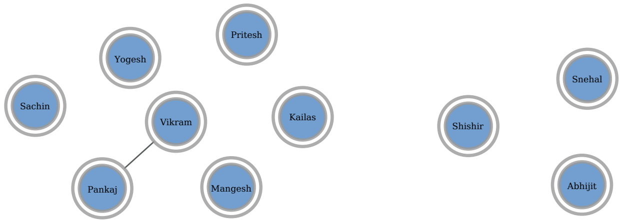




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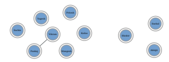

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