Large-scale structure of complex networks (Part 2)

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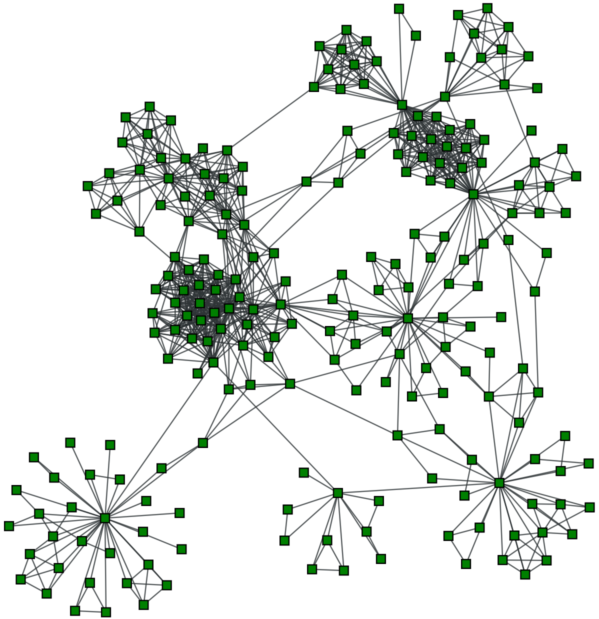
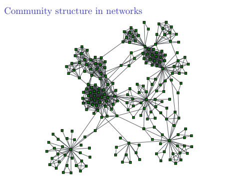
Hello

Community structure in networks





Network of coauthorships in a university department

Community structure in networks

What are communities?

- **Traditional definition**: Groups of nodes with a high internal link density
- **Modern definition**: Nodes with similar connection probabilities to the rest of the network

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Communities in the real-world networks

- **Social networks**:
  - Friend-circles
  - Research communities
  - Co-workers
- **World Wide Web**:
  - Pages with similar contents
  - Webpages under the same domain (e.g. Wikipedia)
- **Biological networks**:
  - Proteins with similar roles in protein interaction networks
  - Chemicals together taking part in chemical reactions in metabolic networks
  - Communities in neuronal networks

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Community detection

Detecting communities is important!

- Communities are building blocks of networks
- Communities allow us to see “the big picture”
- Functional/Autonomous units
- Non-trivial effects on the processes on networks

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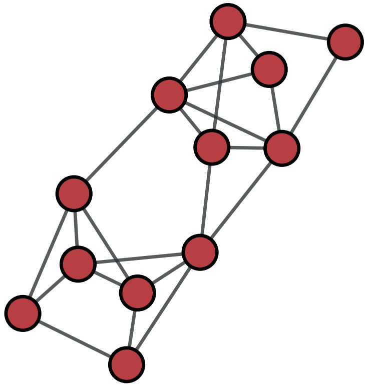
Graph partitioning

Problem of dividing a graph in a given number of groups of given sizes such that the number of links between the groups (**cut size**) is minimized



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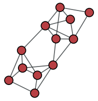

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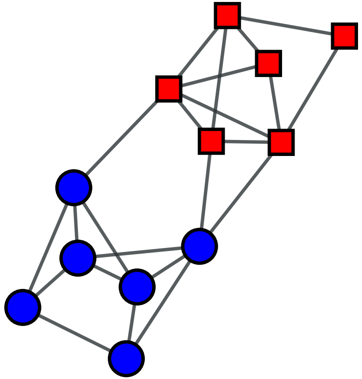

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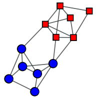

Partitioning is hard!

- Graph with n vertices
- Find two groups with sizes n_1 and n_2 such that the cut size is minimum
- Number of ways: $\frac{n!}{n_1!n_2!} \approx \frac{2^{n+1}}{\sqrt{n}}$

Heuristics are needed!

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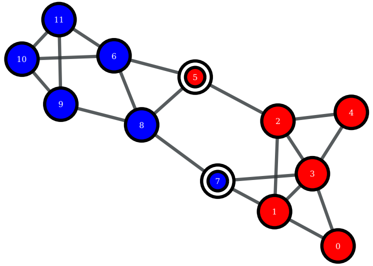
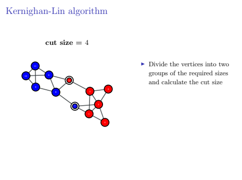
Kernighan-Lin algorithm

cut size $= 4$



- Divide the vertices into two groups of the required sizes and calculate the cut size



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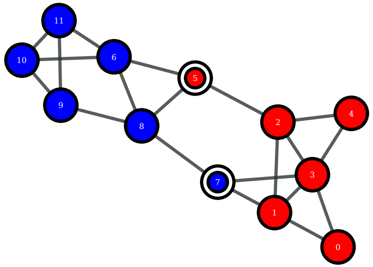
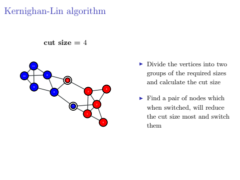


- Divide the vertices into two groups of the required sizes and calculate the cut size
- Find a pair of nodes which when switched, will reduce the cut size most and switch them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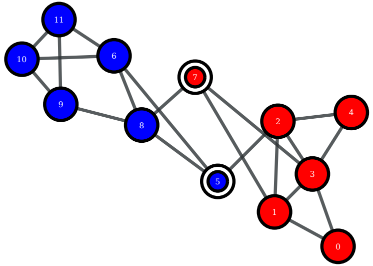

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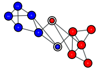

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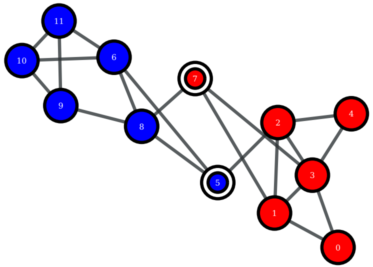

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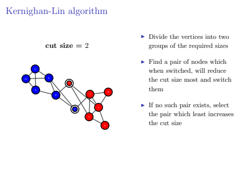

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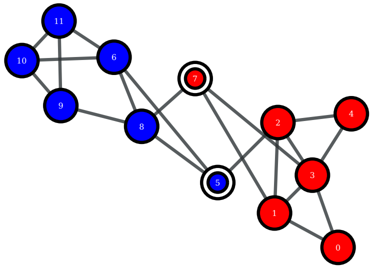

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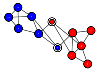

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Kernighan-Lin algorithm

- Go through all the states and select the one with the least cut size
- Start with this state and repeat the whole procedure
- Continue till the cut size no longer becomes smaller
- Starting with many random initial conditions is better

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Group sizes remain constant

Spectral partitioning

- Faster algorithm than Kernighan-Lin
- Uses properties of the graph Laplacian
- More complex to implement than Kernighan-Lin

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Spectral partitioning

Cut size:

$$R = \frac{1}{2} \sum_{\substack{i,j \text{ in} \\ \text{different} \\ \text{groups}}} A_{ij}$$

Define

$$s_i = \begin{cases} +1 & \text{if vertex } i \text{ belongs to group 1} \\ -1 & \text{if vertex } i \text{ belongs to group 2} \end{cases}$$

Then

$$\frac{1}{2}(1 - s_i s_j) = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are in different groups,} \\ 0 & \text{if } i \text{ and } j \text{ are in the same group} \end{cases}$$

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$$R = \frac{1}{4} \sum_{ij} A_{ij} (1 - s_i s_j)$$

First term,

$$\sum_{ij} A_{ij} = \sum_i k_i = \sum_i k_i s_i^2 = \sum_{ij} k_i \delta_{ij} s_i s_j$$

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L is so imp that we have a name for it! Laplacian

.

s is a columnvector

.

L: structure, s: division

.

find s that minimizes R

.

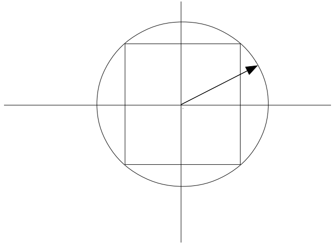
Problem is hard, s takes only integer values

.

Relaxation method

Two constraints:

- s_i can be only ± 1
- $\sum_i s_i = n_1 - n_2 \Rightarrow \mathbf{1}^T \mathbf{s} = n_1 - n_2$



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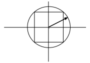

hypercube

.

continuous s, differentiate

Spectral partitioning

Minimization with constraints ⇒ Lagrange multipliers

$$\frac{\partial}{\partial s_i}\left[\sum_{jk} L_{jk} s_j s_k + \lambda\left(n - \sum_j s_j^2\right) + 2\mu\left((n_1 - n_2) - \sum_j s_j\right)\right] = 0$$

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$$\mathbf{L}\mathbf{s} = \lambda\mathbf{s} + \mu\mathbf{1} = \lambda\left(\mathbf{s} + \frac{\mu}{\lambda}\mathbf{1}\right)$$

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$\mathbf{1}$ is an eigenvector of the Laplacian with eigenvalue 0

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Spectral partitioning

$\mathbf{x}$ is an eigenvector of the Laplacian with eigenvalue λ

Which eigenvector to choose?

$$\mathbf{x} \text{ cannot be the eigenvector } \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{pmatrix}$$

$$\mathbf{1}^T\mathbf{x} = \mathbf{1}\left(\mathbf{s} + \frac{\mu}{\lambda}\mathbf{1}\right) = (n_1 - n_2) + \frac{\mu}{\lambda}n = 0$$

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x is orthogonal to 1

.

x is eigenvector but not 1

Spectral partitioning

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$$R = \frac{1}{4}\mathbf{s}^T\mathbf{L}\mathbf{s} = \frac{1}{4}\mathbf{x}^T\mathbf{x} = \frac{n_1 n_2}{n}\lambda$$

Choose the eigenvector with smallest possible eigenvalue!

Eigenvalues of the Laplacian are non-negative and smallest is always 0

$\mathbf{v}_1 = \mathbf{1}$ is ruled out already. So choose $\mathbf{v}_2$ with the smallest positive eigenvalue

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$$\mathbf{s} = x + \frac{n_1 - n_2}{n}\mathbf{1}$$

OR

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But s_i can be only ± 1

Thus, we want $\mathbf{x}$ to be as close as possible to $\mathbf{s}$

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Maximize:

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Equivalently, maximize:

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Simply put the n_1 vertices with most positive elements in group 1 and the rest ones in group 2

Group assignments are arbitrary

Spectral partitioning
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Choose the assignment with the smaller cut size

Spectral partitioning

- Calculate $\mathbf{v}_2$ of the Laplacian
- Put vertices corresponding to largest n_1 elements in group 1 and others in group 2. Calculate the cut size
- Put vertices corresponding to smallest n_1 elements in group 1 and others in group 2. Calculate the cut size
- Choose the division with the smallest cut size among the two

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Community detection is harder!

- **Graph partitioning**
  - well defined
  - Number of groups is fixed
  - Sizes of the groups are fixed
  - Divide even if no good division exists

- **Community detection**
  - ill-defined
  - Number of groups depends on the structure of the network
  - Sizes of the groups depend on the structure of the network
  - Discover natural fault lines

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Many definitions.. many algorithms!

- Girvan-Newman algorithm
- Kernighan-Lin-Newman algorithm
- Spectral decomposition
- Clique-percolation
- Radom walk methods
- Statistical inference
- Label propagation
- Hierarchical clustering

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I can go on.. These algorithms use different definitions/views of communities

Broad classification

- **Agglomerative algorithms:**
  - Hierarchical clustering
  - Louvain method
  - CNM algorithm
- **Divisive algorithms:**
  - Girvan-Newman algorithm
  - Radichhi algorithm
- **Assignment algorithms:**
  - Label propagation
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“The” simplest community detection problem

- Bisecting a graph with n nodes
- Group sizes are not fixed
- Minimum cut size?

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Empty group

"The" simplest community detection problem

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A different measure of the quality of division is required..

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Different measure

Quantification of community structure

- Fewer than expected edges between the groups

Quantification of community structure

- Fewer than expected edges between the groups

few edges = expected edges = not a good division

Quantification of community structure

- Fewer than expected edges between the groups
- Equivalently, more than expected edges inside the groups

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Remember assortativity

Quantification of community structure

- Fewer than expected edges between the groups
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- Assortativity mixing and modularity

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Divide network using modularity

Quantification of community structure

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- Look for divisions with high modularity

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Newman-Girvan algorithm

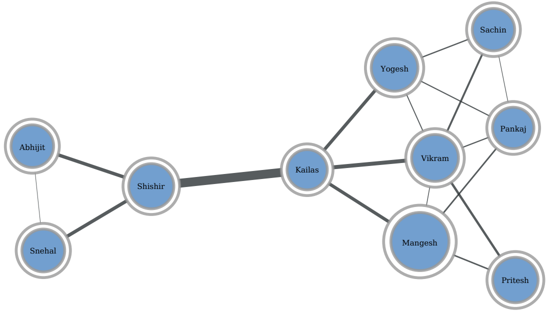
- Look for edges between the communities
- Edge betweenness

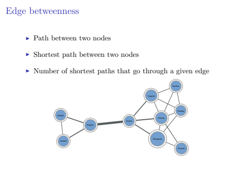
Newman-Girvan algorithm
- Look for edges between the communities
- Edge betweenness

Let's have a look at the edge betweenness

Edge betweenness

- Path between two nodes
- Shortest path between two nodes
- Number of shortest paths that go through a given edge




Edge betweenness
Path between two nodes
Shortest path between two nodes
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The algorithm

- Calculate betweenness for all edges
- Remove the edge with the highest betweenness
- Recalculate betweenness for all edges
- Repeat

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