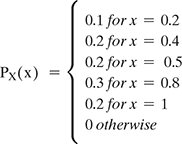
1. Given X be a discrete random variable with the following PMF



1. Find the range RX of the random variable X.

2. Find P(X ≤ 0.5)

3. Find P(0.25<X<0.75)

4. P(X = 0.2|X<0.6)

ANS:-

1. The range of X𝑋 can be found from the PMF. The range of X𝑋 consists of possible values for X𝑋. Here we have

RX={0.2,0.4,0.5,0.8,1}.𝑅𝑋={0.2,0.4,0.5,0.8,1}.

1. The event X≤0.5𝑋≤0.5 can happen only if X𝑋 is 0.2,0.4,0.2,0.4, or 0.50.5. Thus,

|  |  |
| --- | --- |
| P(X≤0.5)𝑃(𝑋≤0.5) | =P(X∈{0.2,0.4,0.5})=𝑃(𝑋∈{0.2,0.4,0.5}) |
|  | =P(X=0.2)+P(X=0.4)+P(X=0.5)=𝑃(𝑋=0.2)+𝑃(𝑋=0.4)+𝑃(𝑋=0.5) |
|  | =PX(0.2)+PX(0.4)+PX(0.5)=𝑃𝑋(0.2)+𝑃𝑋(0.4)+𝑃𝑋(0.5) |
|  | =0.1+0.2+0.2=0.5=0.1+0.2+0.2=0.5 |

1. Similarly, we have

|  |  |
| --- | --- |
| P(0.25<X<0.75)𝑃(0.25<𝑋<0.75) | =P(X∈{0.4,0.5})=𝑃(𝑋∈{0.4,0.5}) |
|  | =P(X=0.4)+P(X=0.5)=𝑃(𝑋=0.4)+𝑃(𝑋=0.5) |
|  | =PX(0.4)+PX(0.5)=𝑃𝑋(0.4)+𝑃𝑋(0.5) |
|  | =0.2+0.2=0.4=0.2+0.2=0.4 |

1. This is a conditional probability problem, so we can use our famous formula P(A|B)=P(A∩B)P(B)𝑃(𝐴|𝐵)=𝑃(𝐴∩𝐵)𝑃(𝐵). We have

|  |  |
| --- | --- |
| P(X=0.2|X<0.6)𝑃(𝑋=0.2|𝑋<0.6) | =P((X=0.2) and (X<0.6))P(X<0.6)=𝑃((𝑋=0.2) and (𝑋<0.6))𝑃(𝑋<0.6) |
|  | =P(X=0.2)P(X<0.6)=𝑃(𝑋=0.2)𝑃(𝑋<0.6) |
|  | =PX(0.2)PX(0.2)+PX(0.4)+PX(0.5)=𝑃𝑋(0.2)𝑃𝑋(0.2)+𝑃𝑋(0.4)+𝑃𝑋(0.5) |
|  | =0.10.1+0.2+0.2=0.2=0.10.1+0.2+0.2=0.2 |

2. Two equal and fair dice are rolled, and we observed two numbers X and Y.

1. Find RX, RY, and the PMFs of X and Y.

2. Find P(X = 2,Y = 6).

3. Find P(X>3|Y = 2).

4. If Z = X + Y. Find the range and PMF of Z.

5. Find P(X = 4|Z = 8).

* + 1. We have RX=RY={1,2,3,4,5,6}𝑅𝑋=𝑅𝑌={1,2,3,4,5,6}. Assuming the dice are fair, all values are equally likely so

PX(k)={160for k=1,2,3,4,5,6otherwise𝑃𝑋(𝑘)={16for 𝑘=1,2,3,4,5,60otherwise

Similarly for Y𝑌,

PY(k)={160for k=1,2,3,4,5,6otherwise𝑃𝑌(𝑘)={16for 𝑘=1,2,3,4,5,60otherwise

* + 1. Since X𝑋 and Y𝑌 are independent random variables, we can write

|  |  |
| --- | --- |
| P(X=2,Y=6)𝑃(𝑋=2,𝑌=6) | =P(X=2)P(Y=6)=𝑃(𝑋=2)𝑃(𝑌=6) |
|  | =16⋅16=136=16⋅16=136. |

* + 1. Since X𝑋 and Y𝑌 are independent, knowing the value of Y𝑌 does not impact the probabilities for X𝑋,

|  |  |
| --- | --- |
| P(X>3|Y=2)𝑃(𝑋>3|𝑌=2) | =P(X>3)=𝑃(𝑋>3) |
|  | =PX(4)+PX(5)+PX(6)=𝑃𝑋(4)+𝑃𝑋(5)+𝑃𝑋(6) |
|  | =16+16+16=12=16+16+16=12. |

* + 1. First, we have RZ={2,3,4,...,12}𝑅𝑍={2,3,4,...,12}. Thus, we need to find PZ(k)𝑃𝑍(𝑘) for k=2,3,...,12𝑘=2,3,...,12. We have

|  |  |
| --- | --- |
| PZ(2)𝑃𝑍(2) | =P(Z=2)=P(X=1,Y=1)=𝑃(𝑍=2)=𝑃(𝑋=1,𝑌=1) |
|  | =P(X=1)P(Y=1) (since X and Y are independent)=𝑃(𝑋=1)𝑃(𝑌=1) (since 𝑋 and 𝑌 are independent) |
|  | =16⋅16=136=16⋅16=136; |
| PZ(3)𝑃𝑍(3) | =P(Z=3)=P(X=1,Y=2)+P(X=2,Y=1)=𝑃(𝑍=3)=𝑃(𝑋=1,𝑌=2)+𝑃(𝑋=2,𝑌=1) |
|  | =P(X=1)P(Y=2)+P(X=2)P(Y=1)=𝑃(𝑋=1)𝑃(𝑌=2)+𝑃(𝑋=2)𝑃(𝑌=1) |
|  | =16⋅16+16⋅16=118=16⋅16+16⋅16=118; |
| PZ(4)𝑃𝑍(4) | =P(Z=4)=P(X=1,Y=3)+P(X=2,Y=2)+P(X=3,Y=1)=𝑃(𝑍=4)=𝑃(𝑋=1,𝑌=3)+𝑃(𝑋=2,𝑌=2)+𝑃(𝑋=3,𝑌=1) |
|  | =3⋅136=112=3⋅136=112. |

* + 1. We can continue similarly:

|  |  |
| --- | --- |
| PZ(5)𝑃𝑍(5) | =436=19=436=19; |
| PZ(6)𝑃𝑍(6) | =536=536; |
| PZ(7)𝑃𝑍(7) | =636=16=636=16; |
| PZ(8)𝑃𝑍(8) | =536=536; |
| PZ(9)𝑃𝑍(9) | =436=19=436=19; |
| PZ(10)𝑃𝑍(10) | =336=112=336=112; |
| PZ(11)𝑃𝑍(11) | =236=118=236=118; |
| PZ(12)𝑃𝑍(12) | =136=136. |

* + 1. It is always a good idea to check our answers by verifying that ∑z∈RZPZ(z)=1∑𝑧∈𝑅𝑍𝑃𝑍(𝑧)=1. Here, we have

|  |  |
| --- | --- |
| ∑z∈RZPZ(z)∑𝑧∈𝑅𝑍𝑃𝑍(𝑧) | =136+236+336+436+536+636=136+236+336+436+536+636 |
|  | +536+436+336+236+136+536+436+336+236+136 |
|  | =1=1. |

* + 1. Note that here we cannot argue that X𝑋 and Z𝑍 are independent. Indeed, Z𝑍 seems to completely depend on X𝑋, Z=X+Y𝑍=𝑋+𝑌. To find the conditional probability P(X=4|Z=8)𝑃(𝑋=4|𝑍=8), we use the formula for conditional probability

|  |  |
| --- | --- |
| P(X=4|Z=8)𝑃(𝑋=4|𝑍=8) | =P(X=4,Z=8)P(Z=8)=𝑃(𝑋=4,𝑍=8)𝑃(𝑍=8) |
|  | =P(X=4,Y=4)P(Z=8)=𝑃(𝑋=4,𝑌=4)𝑃(𝑍=8) |
|  | =P(X=4)P(Y=4)P(Z=8) (since X and Y are independent)=𝑃(𝑋=4)𝑃(𝑌=4)𝑃(𝑍=8) (since 𝑋 and 𝑌 are independent) |
|  | 16⋅1653616⋅16536 |
|  | =15=15. |

1. In an exam, there were 20 multiple-choice questions. Each question had 44 possible options. A student knew the answer to 10 questions, but the other 10 questions were unknown to him, and he chose answers randomly. If the student X's score is equal to the total number of correct answers, then find out the PMF of X. What is P(X>15)?
2. Let's define the random variable Y𝑌 as the number of your correct answers to the 1010 questions you answer randomly. Then your total score will be X=Y+10𝑋=𝑌+10. First, let's find the PMF of Y𝑌. For each question your success probability is 1414. Hence, you perform 1010 independent Bernoulli(14)𝐵𝑒𝑟𝑛𝑜𝑢𝑙𝑙𝑖(14) trials and Y𝑌 is the number of successes. Thus, we conclude Y∼Binomial(10,14)𝑌∼𝐵𝑖𝑛𝑜𝑚𝑖𝑎𝑙(10,14), so

PY(y)={(10y)(14)y(34)10−y0for y=0,1,2,3,...,10otherwise𝑃𝑌(𝑦)={(10𝑦)(14)𝑦(34)10−𝑦for 𝑦=0,1,2,3,...,100otherwise

Now we need to find the PMF of X=Y+10𝑋=𝑌+10. First note that RX={10,11,12,...,20}𝑅𝑋={10,11,12,...,20}. We can write

|  |  |
| --- | --- |
| PX(10)𝑃𝑋(10) | =P(X=10)=P(Y+10=10)=𝑃(𝑋=10)=𝑃(𝑌+10=10) |
|  | =P(Y=0)=(100)(14)0(34)10−0=(34)10=𝑃(𝑌=0)=(100)(14)0(34)10−0=(34)10; |
| PX(11)𝑃𝑋(11) | =P(X=11)=P(Y+10=11)=𝑃(𝑋=11)=𝑃(𝑌+10=11) |
|  | =P(Y=1)=(101)(14)1(34)10−1=10(14)(34)9=𝑃(𝑌=1)=(101)(14)1(34)10−1=10(14)(34)9. |

So, you get the idea. In general for k∈RX={10,11,12,...,20}𝑘∈𝑅𝑋={10,11,12,...,20},

|  |  |
| --- | --- |
| PX(k)𝑃𝑋(𝑘) | =P(X=k)=P(Y+10=k)=𝑃(𝑋=𝑘)=𝑃(𝑌+10=𝑘) |
|  | =P(Y=k−10)=(10k−10)(14)k−10(34)20−k=𝑃(𝑌=𝑘−10)=(10𝑘−10)(14)𝑘−10(34)20−𝑘. |

To summarize,

PX(k)={(10k−10)(14)k−10(34)20−k0for k=10,11,12,...,20otherwise𝑃𝑋(𝑘)={(10𝑘−10)(14)𝑘−10(34)20−𝑘for 𝑘=10,11,12,...,200otherwise

In order to calculate P(X>15)𝑃(𝑋>15), we know we should consider y=6,7,8,9,10𝑦=6,7,8,9,10

PY(y)={(10y)(14)y(34)10−y0for y=6,7,8,9,10otherwise𝑃𝑌(𝑦)={(10𝑦)(14)𝑦(34)10−𝑦for 𝑦=6,7,8,9,100otherwise

PX(k)={(10k−10)(14)k−10(34)20−k0for k=16,17,...,20otherwise𝑃𝑋(𝑘)={(10𝑘−10)(14)𝑘−10(34)20−𝑘for 𝑘=16,17,...,200otherwise

P(X>15)=PX(16)+PX(17)+PX(18)+PX(19)+PX(20)=(106)(14)6(34)4+(107)(14)7(34)3+(108)(14)8(34)2+(109)(14)9(34)1+(1010)(14)10(34)0.

1. The number of students arriving at a college between a time interval is a Poisson random variable. On average, 10 students arrive per hour. Let Y be the number of students arriving from 10 am to 11:30 am. What is P(10<Y≤15)?

**Part (a) Expectation and Variance of the Number of Students by 9:15 a.m.**

* **Time Interval:** From 9:00 a.m. to 9:15 a.m., we have a duration of 15 minutes.
* **Poisson Process Parameter:** Given *λ*=2 students per minute, the total number of students expected in 15 minutes is calculated as *λ*×*t*=2×15=30.

For a Poisson distribution, the expectation *E*[*X*] and variance Var(*X*) are both equal to *λt*.

* **Expectation (E[X]):** 30 students
* **Variance (Var(*X*)):** 30

**Part (b) Probability of at Least 10 Students by 9:05 a.m.**

* **Time Interval:** From 9:00 a.m. to 9:05 a.m., the duration is 5 minutes.
* **Poisson Process Parameter:** Given *λ*=2, the parameter for this interval is *λt*=2×5=10.

To find *P*(*X*≥10) for *X*∼Poisson(10), we use:

*P*(*X*≥10)=1−*P*(*X*≤9)

**Part (c) Probability that the Last Student is Late**

The event that the last student arrives late (after 9:15 a.m.) corresponds to the event that fewer than 30 students have arrived by 9:15 a.m., given the total student count is 30. The probability can be expressed as:

*P*(*X*<30) for *X*∼Poisson(30)

**Part (d) Probability That Exactly 15 Students Arrived by 9:10 a.m., Given Six Students are Late**

* **Time Interval:** From 9:00 a.m. to 9:10 a.m., the duration is 10 minutes.
* **Poisson Process Parameter:** Given *λ*=2, the parameter for this interval is *λt*=2×10=20.

Given that 6 students are late, it implies that 24 students arrived by 9:15 a.m., not 30. The probability that exactly 15 out of these 24 students arrived by 9:10 a.m. can be explored using conditional probability and understanding the arrival patterns.

**Part (e) Expected Time of Arrival of the Seventh Student**

The arrival times in a Poisson process can be modeled using the exponential distribution. The expected time of arrival for the *k*-th student in a Poisson process with parameter *λ* can be calculated using the formula for the sum of *k* exponential random variables, which simplifies to *λk*​ minutes after the start time. For the seventh student (*k*=7) and *λ*=2:

*E*[*T*7​]=27​

minutes after 9:00 a.m.

Now, let's calculate the required probabilities and expected time using the provided calculations.

The solutions are:

* **(a)** Expectation: 30, Variance: 30
* **(b)** Probability of at least 10 students by 9:05 a.m.: 0.5421
* **(c)** Probability that the last student is late: 0.5243
* **(d)** The calculation regarding the probability given six students are late and exactly 15 arrived by 9:10 a.m. needs correction. The provided solution does not account for the conditional nature properly and misapplies distributions.
* **(e)** Expected time of arrival of the seventh student is **9:03:30 a.m.**

For part (d), the original answer did not correctly apply principles related to conditional probability and the Poisson process. A correct approach would consider the distribution of student arrivals up to 9:10 and 9:15, respectively, and then compute the conditional probability given the total arrivals up to 9:15. This involves more complex Poisson process properties and potentially the use of the binomial distribution, which was incorrectly applied in the original solution.

5.Two independent random variables, X and Y,are given such that X~Poisson(α) and Y~Poisson(β). State a new random variable as Z = X + Y. Find out the PMF of Z.

The solution is as follows:

Since I get 0.2 emails per minute, λ=0.2×5=1𝜆=0.2×5=1. Then P(X=0)=e−λλkk!=1e𝑃(𝑋=0)=𝑒−𝜆𝜆𝑘𝑘!=1𝑒

Okay now let's get to the question I got stuck.

Let X∼Poisson(α)𝑋∼Poisson(𝛼) and Y∼Poisson(β)𝑌∼Poisson(𝛽) be two independent random variables. Define a new random variable as Z=X+Y𝑍=𝑋+𝑌. Find the PMF of Z𝑍.

The problem is solved as follows:

PZ(k)=P(X+Y=k)𝑃𝑍(𝑘)=𝑃(𝑋+𝑌=𝑘)

=∑i=0kP(X+Y=k∣X=i)P(X=i)=∑𝑖=0𝑘𝑃(𝑋+𝑌=𝑘∣𝑋=𝑖)𝑃(𝑋=𝑖)

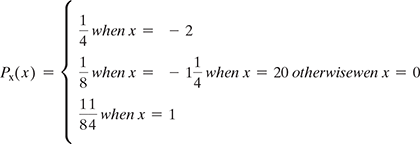
=∑i=0kP(Y=k−i)P(X=i)=∑𝑖=0𝑘𝑃(𝑌=𝑘−𝑖)𝑃(𝑋=𝑖)

then from the Binomial theorem it ends up being:

=e−(α+β)k!(α+β)k=𝑒−(𝛼+𝛽)𝑘!(𝛼+𝛽)𝑘

I have no problems following the solution and understanding why the first statement ends to the last one. What I don't understand is how can we just write PZ(k)=P(X+Y=k)𝑃𝑍(𝑘)=𝑃(𝑋+𝑌=𝑘). In the first example to calculate λ𝜆, we multipled 0.2 with 5 since we receive 0.2 emails per minute and we are asked about a 5 minute interval. What if this α𝛼 and β𝛽 parameters belong to different time framed distributions? If α𝛼 is for per minute and β𝛽 is for per hour don't we have to do something like PZ(k)=P(60X+Y=k)𝑃𝑍(𝑘)=𝑃(60𝑋+𝑌=𝑘)?

6. There is a discrete random variable X with the pmf.



If we define a new random variable Y = (X + 1)2 then

1. Find the range of Y.

2. Find the pmf of Y.

 the random variable Y𝑌 is a function of the random variable X𝑋. This means that we perform the random experiment and obtain X=x𝑋=𝑥, and then the value of Y𝑌 is determined as Y=(x+1)2𝑌=(𝑥+1)2. Since X𝑋 is a random variable, Y𝑌 is also a random variable.

1. To find RY𝑅𝑌, we note that RX={−2,−1,0,1,2}𝑅𝑋={−2,−1,0,1,2}, and

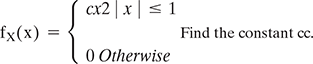
|  |  |
| --- | --- |
| RY𝑅𝑌 | ={y=(x+1)2|x∈RX}={𝑦=(𝑥+1)2|𝑥∈𝑅𝑋} |
|  | ={0,1,4,9}={0,1,4,9}. |

1. Now that we have found RY={0,1,4,9}𝑅𝑌={0,1,4,9}, to find the PMF of Y𝑌 we need to find PY(0),PY(1),PY(4)𝑃𝑌(0),𝑃𝑌(1),𝑃𝑌(4), and PY(9)𝑃𝑌(9):

|  |  |
| --- | --- |
| PY(0)𝑃𝑌(0) | =P(Y=0)=P((X+1)2=0)=𝑃(𝑌=0)=𝑃((𝑋+1)2=0) |
|  | =P(X=−1)=18=𝑃(𝑋=−1)=18; |
| PY(1)𝑃𝑌(1) | =P(Y=1)=P((X+1)2=1)=𝑃(𝑌=1)=𝑃((𝑋+1)2=1) |
|  | =P((X=−2) or (X=0))=𝑃((𝑋=−2) or (𝑋=0)); |
|  | PX(−2)+PX(0)=14+18=38𝑃𝑋(−2)+𝑃𝑋(0)=14+18=38; |
| PY(4)𝑃𝑌(4) | =P(Y=4)=P((X+1)2=4)=𝑃(𝑌=4)=𝑃((𝑋+1)2=4) |
|  | =P(X=1)=14=𝑃(𝑋=1)=14; |
| PY(9)𝑃𝑌(9) | =P(Y=9)=P((X+1)2=9)=𝑃(𝑌=9)=𝑃((𝑋+1)2=9) |
|  | =P(X=2)=14=𝑃(𝑋=2)=14. |

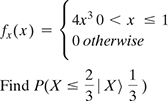
1. Again, it is always a good idea to check that ∑y∈RYPY(y)=1∑𝑦∈𝑅𝑌𝑃𝑌(𝑦)=1. We have
2. ∑y∈RYPY(y)=18+38+14+14=1.

2.Assuming X is a continuous random variable with PDF



* + 1. Find EX and Var(X).
    2. Find *P*(*X* ≥ img).

1. If *X* is a continuous random variable with pdf



Ans:-

|  |  |
| --- | --- |
| P(X≤23|X>13)𝑃(𝑋≤23|𝑋>13) | =P(13<X≤23)P(X>13)=𝑃(13<𝑋≤23)𝑃(𝑋>13) |
|  | =∫23134x3dx∫1134x3dx=∫13234𝑥3𝑑𝑥∫1314𝑥3𝑑𝑥 |
|  | =316. |

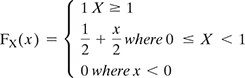
1. If *X*~Uniformimg and *Y* = sin(*X*), then find *fY*(*y*).
2. Here Y=g(X)𝑌=𝑔(𝑋), where g𝑔 is a differentiable function. Although g𝑔 is not monotone, it can be divided to a finite number of regions in which it is monotone. Thus, we can use Equation 4.6. We note that since RX=[−π2,π]𝑅𝑋=[−𝜋2,𝜋], RY=[−1,1]𝑅𝑌=[−1,1]. By looking at the plot of g(x)=sin(x)𝑔(𝑥)=sin⁡(𝑥) over [−π2,π][−𝜋2,𝜋], we notice that for y∈(0,1)𝑦∈(0,1) there are two solutions to y=g(x)𝑦=𝑔(𝑥), while for y∈(−1,0)𝑦∈(−1,0), there is only one solution. In particular, if y∈(0,1)𝑦∈(0,1), we have two solutions: x1=arcsin(y)𝑥1=arcsin⁡(𝑦), and x2=π−arcsin(y)𝑥2=𝜋−arcsin⁡(𝑦). If y∈(−1,0)𝑦∈(−1,0) we have one solution, x1=arcsin(y)𝑥1=arcsin⁡(𝑦). Thus, for y∈(−1,0)𝑦∈(−1,0), we have

|  |  |
| --- | --- |
| fY(y)𝑓𝑌(𝑦) | =fX(x1)|g′(x1)|=𝑓𝑋(𝑥1)|𝑔′(𝑥1)| |
|  | =fX(arcsin(y))|cos(arcsin(y))|=𝑓𝑋(arcsin⁡(𝑦))|cos⁡(arcsin⁡(𝑦))| |
|  | =23π1−y2√.=23𝜋1−𝑦2. |

1. For y∈(0,1)𝑦∈(0,1), we have

|  |  |
| --- | --- |
| fY(y)𝑓𝑌(𝑦) | =fX(x1)|g′(x1)|+fX(x2)|g′(x2)|=𝑓𝑋(𝑥1)|𝑔′(𝑥1)|+𝑓𝑋(𝑥2)|𝑔′(𝑥2)| |
|  | =fX(arcsin(y))|cos(arcsin(y))|+fX(π−arcsin(y))|cos(π−arcsin(y))|=𝑓𝑋(arcsin⁡(𝑦))|cos⁡(arcsin⁡(𝑦))|+𝑓𝑋(𝜋−arcsin⁡(𝑦))|cos⁡(𝜋−arcsin⁡(𝑦))| |
|  | =23π1−y2√+23π1−y2√=23𝜋1−𝑦2+23𝜋1−𝑦2 |
|  | =43π1−y2√.=43𝜋1−𝑦2. |

1. To summarize, we can write
2. fY(y)=⎧⎩⎨⎪⎪⎪⎪⎪⎪23π1−y2√43π1−y2√0−1<y<00<y<1otherwise
3. If X is a random variable with CDF



* + 1. What kind of random variable is *X*: discrete, continuous, or mixed?
    2. Find the PDF of *X*, f*X*(*x*).
    3. Find E(eX).
    4. Find P(*X* = 0|X≤0.5).

ANS:-

1. What kind of random variable is X: discrete, continuous, or mixed? We note that the CDF has a discontinuity at x=0𝑥=0, and it is continuous at other points. Since FX(x)𝐹𝑋(𝑥) is not flat in other locations, we conclude X𝑋 is a mixed random variable. Indeed, we can write

FX(x)=12u(x)+12FY(x),𝐹𝑋(𝑥)=12𝑢(𝑥)+12𝐹𝑌(𝑥),

where Y𝑌 is a Uniform(0,1)𝑈𝑛𝑖𝑓𝑜𝑟𝑚(0,1) random variable. If we use the interpretation of Problem 1, we can say the following. We toss a fair coin. If it lands heads then X=0𝑋=0, otherwise X𝑋 is obtained according the a Uniform(0,1)𝑈𝑛𝑖𝑓𝑜𝑟𝑚(0,1) distribution.

1. Find the PDF of X, fX(x)𝑓𝑋(𝑥): By differentiating the CDF, we obtain

fX(x)=12δ(x)+12fY(x),𝑓𝑋(𝑥)=12𝛿(𝑥)+12𝑓𝑌(𝑥),

where fY(x)𝑓𝑌(𝑥) is the PDF of Uniform(0,1)𝑈𝑛𝑖𝑓𝑜𝑟𝑚(0,1), i.e.,

fY(x)={100<x<1otherwise𝑓𝑌(𝑥)={10<𝑥<10otherwise

1. Find E(eX)𝐸(𝑒𝑋): We can use LOTUS to write

|  |  |
| --- | --- |
| E(eX)𝐸(𝑒𝑋) | =∫∞−∞exfX(x)dx=∫−∞∞𝑒𝑥𝑓𝑋(𝑥)𝑑𝑥 |
|  | =12∫∞−∞exδ(x)dx+12∫∞−∞exfY(x)dx=12∫−∞∞𝑒𝑥𝛿(𝑥)𝑑𝑥+12∫−∞∞𝑒𝑥𝑓𝑌(𝑥)𝑑𝑥 |
|  | =12e0+12∫10exdx=12𝑒0+12∫01𝑒𝑥𝑑𝑥 |
|  | =12+12(e−1)=12+12(𝑒−1) |
|  | =12e=12𝑒. |

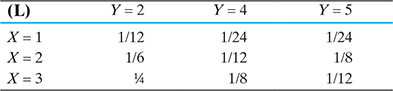
1. Here is another way to think about this part: similar to part (c) of Problem 1, we can write

|  |  |
| --- | --- |
| E(eX)𝐸(𝑒𝑋) | =12×e0+12E[eY]=12×𝑒0+12𝐸[𝑒𝑌] |
|  | =12+12∫10eydy=12+12∫01𝑒𝑦𝑑𝑦 |
|  | =12e=12𝑒. |

1. Find P(X=0|X≤0.5)𝑃(𝑋=0|𝑋≤0.5): We have

|  |  |
| --- | --- |
| P(X=0|X≤0.5)𝑃(𝑋=0|𝑋≤0.5) | =P(X=0,X≤0.5)P(X≤0.5)=𝑃(𝑋=0,𝑋≤0.5)𝑃(𝑋≤0.5) |
|  | =P(X=0)P(X≤0.5)=𝑃(𝑋=0)𝑃(𝑋≤0.5) |
|  | =0.5∫0.50fX(x)dx=0.5∫00.5𝑓𝑋(𝑥)𝑑𝑥 |
|  | =0.50.75=23=0.50.75=23. |

1. There are two random variables *X* and *Y* with joint PMF given in Table below
   * 1. Find *P*(*X*≤2, *Y*≤4).
     2. Find the marginal PMFs of *X* and *Y*.
     3. Find *P*(*Y* = 2|*X* = 1).
     4. Are *X* and *Y* independent?



ANS:-

1. Find P(X≤2,Y≤4)𝑃(𝑋≤2,𝑌≤4), we can write

P(X≤2,Y≤4)=PXY(1,2)+PXY(1,4)+PXY(2,2)+PXY(2,4)=112+124+16+112=38.𝑃(𝑋≤2,𝑌≤4)=𝑃𝑋𝑌(1,2)+𝑃𝑋𝑌(1,4)+𝑃𝑋𝑌(2,2)+𝑃𝑋𝑌(2,4)=112+124+16+112=38.

1. Note from the table that

RX={1,2,3} and RY={2,4,5}.𝑅𝑋={1,2,3} and 𝑅𝑌={2,4,5}.

Now we can use Equation 5.1 to find the marginal PMFs:

PX(x)=⎧⎩⎨⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪163811240x=1x=2x=3otherwise𝑃𝑋(𝑥)={16𝑥=138𝑥=21124𝑥=30otherwise

PY(y)=⎧⎩⎨⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪⎪1214140y=2y=4y=5otherwise𝑃𝑌(𝑦)={12𝑦=214𝑦=414𝑦=50otherwise

1. Using the formula for conditional probability, we have

P(Y=2|X=1)=P(X=1,Y=2)P(X=1)=PXY(1,2)PX(1)=11216=12.𝑃(𝑌=2|𝑋=1)=𝑃(𝑋=1,𝑌=2)𝑃(𝑋=1)=𝑃𝑋𝑌(1,2)𝑃𝑋(1)=11216=12.

1. Are X𝑋 and Y𝑌 independent? To check whether X𝑋 and Y𝑌 are independent, we need to check that P(X=xi,Y=yj)=P(X=xi)P(Y=yj)𝑃(𝑋=𝑥𝑖,𝑌=𝑦𝑗)=𝑃(𝑋=𝑥𝑖)𝑃(𝑌=𝑦𝑗), for all xi∈RX𝑥𝑖∈𝑅𝑋 and all yj∈RY𝑦𝑗∈𝑅𝑌. Looking at the table and the results from previous parts, we find

P(X=2,Y=2)=16≠P(X=2)P(Y=2)=316.𝑃(𝑋=2,𝑌=2)=16≠𝑃(𝑋=2)𝑃(𝑌=2)=316.

Thus, we conclude that X𝑋 and Y𝑌 are not independent.

6.A box containing 40 white shirts and 60 black shirts. If we choose 10 shirts (without replacement) at random, find the joint PMF of X and Y, where X is the number of white shirts and Y is the number of black shirts.

Let EWW𝐸𝑊𝑊 be the event that two balls drawn from the first bag are both white. Likewise, EWR𝐸𝑊𝑅 be the event that one white ball and one red ball are drawn from the first bag, and ERR𝐸𝑅𝑅 be the event that two balls drawn from the first bag are both red. Let W𝑊 be the event that a white ball is drawn from the second bag.

By Law of total probability,

Pr(W)=Pr(EWW)Pr(W|EWW)+Pr(EWR)Pr(W|EWR)+Pr(ERR)Pr(W|ERR)=(102)(132)×510+(101)(31)(132)×410+(32)(132)×310=4578×510+3078×410+378×310=354780=59130≈0.4538

7.If A and B are two jointly continuous random variables with joint PDF

images

a. Find fX(a) and fY(b).

b. Are A and B independent of each other?

c. Find the conditional PDF of A given B = b, fA|B(a|b).

d. Find E[A|B = b], for 0 ≤ y ≤ 1.

e. Find Var(A|B = b), for 0 ≤ y ≤ 1.

1. o find c𝑐, we use

∫∞−∞∫∞−∞fXY(x,y)dxdy=1.∫−∞∞∫−∞∞𝑓𝑋𝑌(𝑥,𝑦)𝑑𝑥𝑑𝑦=1.

Thus, we have

1=∫∞−∞∫∞−∞fXY(x,y)dxdy=∫10∫10x+cy2dxdy=∫10[12x2+cy2x]x=1x=0dy=∫1012+cy2dy=[12y+13cy3]y=1y=0=12+13c.1=∫−∞∞∫−∞∞𝑓𝑋𝑌(𝑥,𝑦)𝑑𝑥𝑑𝑦=∫01∫01𝑥+𝑐𝑦2𝑑𝑥𝑑𝑦=∫01[12𝑥2+𝑐𝑦2𝑥]𝑥=0𝑥=1𝑑𝑦=∫0112+𝑐𝑦2𝑑𝑦=[12𝑦+13𝑐𝑦3]𝑦=0𝑦=1=12+13𝑐.

Therefore, we obtain c=32𝑐=32.

1. To find P(0≤X≤12,0≤Y≤12)𝑃(0≤𝑋≤12,0≤𝑌≤12), we can write

P((X,Y)∈A)=∬AfXY(x,y)dxdy,for A={(x,y)|0≤x,y≤1}.𝑃((𝑋,𝑌)∈𝐴)=∬𝐴𝑓𝑋𝑌(𝑥,𝑦)𝑑𝑥𝑑𝑦,for 𝐴={(𝑥,𝑦)|0≤𝑥,𝑦≤1}.

Thus,

P(0≤X≤12,0≤Y≤12)=∫120∫120(x+32y2)dxdy=∫120[12x2+32y2x]120dy=∫120(18+34y2)dy=332.

8.There are 100 men on a ship. If Xi is the ith man's weight on the ship and Xi's are independent and identically distributed and EXi = μ = 170 and σXi = σ = 30. Find the probability that the men's total weight on the ship exceeds 18,000.

W𝑊 is the total weight, then W=X1+X2+⋯+Xn𝑊=𝑋1+𝑋2+⋯+𝑋𝑛, where n=100𝑛=100. We have

EWVar(W)=nμ=(100)(170)=17000,=100Var(Xi)=(100)(30)2=90000.𝐸𝑊=𝑛𝜇=(100)(170)=17000,Var(𝑊)=100Var(𝑋𝑖)=(100)(30)2=90000.

Thus, σW=300𝜎𝑊=300. We have

P(W>18000)=P(W−17000300>18000−17000300)=P(W−17000300>103)=1−Φ(103)(by CLT)≈4.3×10−4.

9.Let X1, X2, ……, X25 are independent and identically distributed. And have the following PMF

We have

EXi=(0.6)(1)+(0.4)(−1)=15,𝐸𝑋𝑖=(0.6)(1)+(0.4)(−1)=15,

EX2i=0.6+0.4=1.𝐸𝑋𝑖2=0.6+0.4=1.

Therefore,

Var(Xi)thus,σXi=1−125=2425;=26–√5.Var(𝑋𝑖)=1−125=2425;thus,𝜎𝑋𝑖=265.

Therefore,

EY=25×15=5,𝐸𝑌=25×15=5,

Var(Y)thus,σY=25×2425=24;=26–√.Var(𝑌)=25×2425=24;thus,𝜎𝑌=26.

P(4≤Y≤6)=P(3.5≤Y≤6.5)(continuity correction)=P(3.5−526–√≤Y−526–√≤6.5−526–√)=P(−0.3062≤Y−526–√≤+0.3062)≈Φ(0.3062)−Φ(−0.3062)(by the CLT)=2Φ(0.3062)−1≈0.2405