Module 3 : Combinatorics and Mathematical Inductions

Ex.4. In how many ways can 4 physics books, 3 mathematics book, 3 chemistry books and 2 biology books be arranged on a shelf so that all books of the same subject are together?

Solution. 'Physics', 'Mathematics', 'Chemistry' and 'Biology' books can be arranged among themselves in ${}^4P_4 = 4! = 24$ ways.

Now 'Physics' contains 4 books which can be arranged among themselves in $4! = 24$ ways. Similarly the 3 mathematics books can be arranged among themselves in $3! = 6$ ways. 3 Chemistry and 2 biology books can be arranged among themselves in $3! = 6$ and $2! = 2$ ways respectively. So by Product Rule Principle the required number of arrangement

$$= 24 \times 24 \times 6 \times 6 \times 2 = 41472$$

Ex.5. 6 boys and 6 girls are to be seated in a row. How many ways can they be seated if

(i) all boys are to be seated together and all girls are to be seated together.

(ii) no two girls should be seated together.

(iii) the boys occupy extreme positions.

Ex. 7. How many ways can the letters in the word MISSISSIPPI can be arranged? If P's are to be sepeated then find the number of arrangements.

Solution. Total ways of arrangement of all the letter

$$= \frac{11!}{4!4!2!} = 34650$$

Keeping P together two number of arrangements

$$= \frac{10!}{4!4!} = 6300$$

$\therefore$ Number of arrangement where the Ps' are separated

$$= 34650 - 6300 = 28350$$

Ex. 8. How many different signals, each consisting of 8 flags hung in a vertical line, can be formed from a set of 4 identical red flags, 3 identical white flags and a blue flag?

Ex.9. If repetitions are not permitted, how many four digit numbers can be formed from the digits 1, 2, 3, 7, 8 and 5 that contain both the digits 3 and 5.

Solution. After inclusion on 3 and 5 other two digits can be selected from 1, 2, 7, 8 in $^4C_2$ ways. After selecting the two digits all the four digits can be arranged among themselves in $^4P_4 =!$ ways.

$\therefore$ total number of required numbers $= {}^4C_4 \times 4! = 6 \times 24 = 144$

Ex.10. Find the number of different words of six letters taken from 10 different letters such that in each of which at least one letter is repeated.

Solution. Number of words of six different letters taken from 10 different letters $= {}^{10}P_6$.

Number of words of six letters where the letters may repeat any number of times $= 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^6$.

$\therefore$ Required number of words $= 10^6 - {}^{10}P_6 = 848800$.

Ex.11. 15 different balls are to be placed in 15 different holes one in each hole. If 6 holes are too small for 8 of the balls, how many ways the placement of all the balls in the hole is possible?

13. Find the number of permutations made with the letters of the word MISSISSIPPI taken all together. In how many of these will the vowels occupying the even places?

14. How many different words containing all the letters of the word TRIANGLE can be formed so that

(i) consonants are never separated

(ii) consonants never come together

(iii) vowels occupy odd places.

15. How many different words can be formed with the letters of the word DOGMATIC? How many of these

(i) letters G will occupy odd places?

(ii) letters D, O, G occupy only first three places?

16. If all the permutations of the letters of the word

(i) CHALK be written down as in a dictionary, what is the rank of this word?

(ii) MODESTY be written down as in a disctionary, what is the rank of this word?

17. (i) In how many ways can three 1st year students and two second year students sit in a row?

(ii) In how many ways can they sit in a row if the 1st year students and 2nd year students are each to sit together?

(iii) In how many ways can the students sit in a row if just the 2nd year students sit together?

18. Find the number of ways that 4 digit number can be formed with 0, 4, 5, 6, 7, 8, 9; no digit being repeated. How many of them are not divisible by 5?

19. How many number not more than 5 digits taken from the digits 1, 2, 3, 4, 5, 6, 7 in each of which digits may repeat any number of times.

20. How many seven digit telephone numbers are possible, if-

(i) only odd digits may be used.

(ii) The number is multiple of 100.

(iii) the first three digits are 481.

Ex.4. Using Principle of Inclusion and Exclusion show that for any three sets A, B and C,

$n(A \cup B \cup C) = n(A) + n(B) + n(C)$ if they are pairwise mutually disjoint. [W.B.U.T.2013]

Ex.3. Find the total number of integers lying between 1 and 1000 that are divisible by at least one of 2, 3 and 7.

Ex.1. Of 32 students who play football or cricket (or both), 30 play football and 14 play cricket. Find the number of students who (i) play both the game (ii) play only football and (iii) play only cricket.

Ex.2. Let A, B, C, D denote, respectively English, Japanese, French and Russian language courses. Let 12 take A, 20 take B, 5 take A and B, 7 take A and C, 3 take A, B and C, 2 take A, B and D; 20 take C, 8 take D, 4 take A and D, 16 take B and C, 4 take B and D, 3 take C and D, 2 take B, C, D, 3 take A, C, D, 2 take all the four and 71 take none.

Find the number of candidates who take at least one courses. Hence find the total number of candidates on which the survey is done.

Example 1 *Show that if n is a positive integer, then*

Extra Examples

$$1+2+\cdots+n=\frac{n(n+1)}{2}.$$

Example 3 *Use mathematical induction to show that*

$$1+2+2^2+\cdots+2^n=2^{n+1}-1$$

for all nonnegative integers n.

Example 5 *Use mathematical induction to prove the inequality*

Extra Examples

$$n<2^n$$

for all positive integers n.

Example 6 *Use mathematical induction to prove that $2^n < n!$ for every positive integer n with $n \geq 4$. (Note that this inequality is false for $n = 1, 2,$ and 3.)*

Example 8 *Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer.*

31. Prove that 2 divides $n^2 + n$ whenever n is a positive integer.

32. Prove that 3 divides $n^3 + 2n$ whenever n is a positive integer.

33. Prove that 5 divides $n^5 - n$ whenever n is a nonnegative integer.

34. Prove that 6 divides $n^3 - n$ whenever n is a nonnegative integer.

*35. Prove that $n^2 - 1$ is divisible by 8 whenever n is an odd positive integer.

*36. Prove that 21 divides $4^{n+1} + 5^{2n-1}$ whenever n is a positive integer.

*37. Prove that if n is a positive integer, then 133 divides $11^{n+1} + 12^{2n-1}$.