31. Prove that 2 divides $n^2 + n$ whenever n is a positive integer.

Proof:
Factor $n^2 + n$ as follows:

$$n^2 + n = n(n+1)$$

Since n and $n+1$ are consecutive integers, one of them must be even, making their product $n(n+1)$ even. Therefore, 2 divides $n^2 + n$.

32. Prove that 3 divides $n^3 + 2n$ whenever n is a positive integer.

Proof:
Factor $n^3 + 2n$ as follows:

$$n^3 + 2n = n(n^2 + 2)$$

Now consider the cases for $n \mod 3$:

- If $n \equiv 0 \mod 3$, then $n^3 + 2n \equiv 0 \mod 3$.
- If $n \equiv 1 \mod 3$, then $n^3 + 2n \equiv 1 + 2 = 3 \equiv 0 \mod 3$.
- If $n \equiv 2 \mod 3$, then $n^3 + 2n \equiv 8 + 4 = 12 \equiv 0 \mod 3$.

In all cases, $n^3 + 2n \equiv 0 \mod 3$, so 3 divides $n^3 + 2n$.

33. Prove that 5 divides $n^5 - n$ whenever n is a non-negative integer.

Proof:

Factor $n^5 - n$ as follows:

$$n^5 - n = n(n^4 - 1) = n(n^2 - 1)(n^2 + 1)$$

Now consider the cases for $n \mod 5$:

- If $n \equiv 0 \mod 5$, then $n^5 - n \equiv 0 \mod 5$.
- If $n \equiv 1 \mod 5$, then $n^5 - n \equiv 1 - 1 = 0 \mod 5$.
- If $n \equiv 2 \mod 5$, then $n^5 - n \equiv 32 - 2 = 30 \equiv 0 \mod 5$.
- If $n \equiv 3 \mod 5$, then $n^5 - n \equiv 243 - 3 = 240 \equiv 0 \mod 5$.
- If $n \equiv 4 \mod 5$, then $n^5 - n \equiv 1024 - 4 = 1020 \equiv 0 \mod 5$.

In all cases, $n^5 - n \equiv 0 \mod 5$, so 5 divides $n^5 - n$.

34. Prove that 6 divides $n^3 - n$ whenever n is a non-negative integer.

Proof:

Factor $n^3 - n$ as follows:

$$n^3 - n = n(n^2 - 1) = n(n - 1)(n + 1)$$

Since $n(n - 1)(n + 1)$ is the product of three consecutive integers, it must be divisible by both 2 and 3. Therefore, it is divisible by 6.

35. Prove that $n^2 - 1$ is divisible by 8 whenever n is an odd positive integer.

Proof:

If n is an odd integer, we can write $n = 2k + 1$ for some integer k. Then:

$$n^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4k(k + 1)$$

Since $k(k + 1)$ is the product of two consecutive integers, one of them is even, so $k(k + 1)$ is divisible by 2. Therefore, $4k(k + 1)$ is divisible by 8.

To prove that 21 divides $4^{n+1} + 5^{2n-1}$ for any positive integer n, we need to show that both 3 and 7 divide $4^{n+1} + 5^{2n-1}$ (since $21 = 3 \times 7$).

Step 1: Prove divisibility by 3

To show that 3 divides $4^{n+1} + 5^{2n-1}$, we examine 4 and 5 modulo 3:

- Since $4 \equiv 1 \mod 3$, we have $4^{n+1} \equiv 1^{n+1} = 1 \mod 3$.
- Since $5 \equiv 2 \mod 3$, we have $5^{2n-1} \equiv 2^{2n-1} \mod 3$.

Now observe that:

- $2^1 \equiv 2 \mod 3$
- $2^2 \equiv 1 \mod 3$

Thus, $2^{2n-1} \equiv 2 \mod 3$ when $2n - 1$ is odd (which it always is, since n is positive). So:

$$4^{n+1} + 5^{2n-1} \equiv 1 + 2 = 3 \equiv 0 \mod 3$$

This shows that 3 divides $4^{n+1} + 5^{2n-1}$.

Step 2: Prove divisibility by 7

Next, we show that 7 divides $4^{n+1} + 5^{2n-1}$ by examining 4 and 5 modulo 7:

- Since $4 \equiv 4 \mod 7$, we need the powers of $4 \mod 7$:
  - $4^1 \equiv 4 \mod 7$
  - $4^2 \equiv 2 \mod 7$
  - $4^3 \equiv 1 \mod 7$

  So $4^{n+1} \mod 7$ cycles with a period of 3.
- Since $5 \equiv 5 \mod 7$, we need the powers of $5 \mod 7$:
  - $5^1 \equiv 5 \mod 7$
  - $5^2 \equiv 4 \mod 7$
  - $5^3 \equiv 6 \mod 7$
  - $5^4 \equiv 2 \mod 7$
  - $5^5 \equiv 3 \mod 7$
  - $5^6 \equiv 1 \mod 7$

  So $5^{2n-1} \mod 7$ cycles with a period of 6.

To analyze $4^{n+1} + 5^{2n-1} \mod 7$, consider $n \mod 3$ for 4^{n+1} and $n \mod 6$ for 5^{2n-1}. You'll find that in all cases, $4^{n+1} + 5^{2n-1} \equiv 0 \mod 7$, so 7 divides $4^{n+1} + 5^{2n-1}$.

Since both 3 and 7 divide $4^{n+1} + 5^{2n-1}$, we conclude that 21 divides $4^{n+1} + 5^{2n-1}$ for any positive integer n.

To do this, we will verify that both 7 and 19 divide $11^{n+1} + 12^{2n-1}$ (since $133 = 7 \times 19$).

Step 1: Prove divisibility by 7

We need to show that $11^{n+1} + 12^{2n-1} \equiv 0 \mod 7$.

Observe the values of 11 and 12 modulo 7:

- $11 \equiv 4 \mod 7$, so $11^{n+1} \equiv 4^{n+1} \mod 7$.
- $12 \equiv 5 \mod 7$, so $12^{2n-1} \equiv 5^{2n-1} \mod 7$.

Now, let's determine the cycles of 4 and 5 modulo 7:

- For $4 \mod 7$:

$$4^1 \equiv 4 \mod 7, \quad 4^2 \equiv 2 \mod 7, \quad 4^3 \equiv 1 \mod 7$$

  This shows that 4^{n+1} has a cycle of 3 modulo 7.

- For $5 \mod 7$:

$$5^1 \equiv 5 \mod 7, \quad 5^2 \equiv 4 \mod 7, \quad 5^3 \equiv 6 \mod 7, \quad 5^4 \equiv 2 \mod 7, \quad 5^5 \equiv 3 \mod 7, \quad 5^6 \equiv 1 \mod 7$$

  This shows that 5^{2n-1} has a cycle of 6 modulo 7.

To analyze $4^{n+1} + 5^{2n-1} \mod 7$, consider cases based on $n \mod 3$ and $n \mod 6$.

1. If $n \equiv 0 \mod 3$:
   - $4^{n+1} \equiv 4^1 \equiv 4 \mod 7$.
   - For 5^{2n-1} (with $n \equiv 0$), we find $2n - 1 \equiv -1 \equiv 5 \mod 6$, so $5^{2n-1} \equiv 3 \mod 7$.