2.1.10. Principle of Inclusion and Exclusion

[W.B.U.T.2014]

Let A_1 and A_2 be two sets. $n(A_1)$, $n(A_2)$ are cardinal number of A_1 and A_2 respectively (that is the number of elements in the set).

Then $n(A_1 \cup A_2) = n(A_1) + n(A_2) - n(A_1 \cap A_2)$

For three sets A_1, A_2, A_3,

$$n(A_1 \cup A_2 \cup A_3) = n(A_1) + n(A_2) + n(A_3) - n(A_1 \cap A_2) - n(A_2 \cap A_3)$$
$$-n(A_3 \cap A_1) + n(A_1 \cap A_2 \cap A_3)$$

In general for m number of sets $A_1, A_2, A_3 \cdots A_m$

$$n(A_1 \cup A_2 \cup .... \cup A_m) = \sum_{i=1}^{m} n(Ai) - \sum_{\substack{i,i=1 \\ i \neq j}}^{m} n(A_i \cap A_j)$$
$$+ \sum_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{m} n(A_i \cap A_j \cap A_k) \; .... + (-1)^{m-1} n(A_1 \cap A_2 \cap .... A_m)$$

Remark. If $A_1, A_2, ... A_m$ are mutually disjoint finite set then

$$n(A_1 \cup A_2 \cup .... \cup A_m) = \sum_{i=1}^{m} n(A_i)$$

Illustrations

(i) Let $U = \{1, 2, 3, \cdots 1000\}$. Then find $n(S)$ where S = set of such integers of U which are not divisible by 3,5 or 7.

Let A = set of integers of U which are divisible by 3;

B = set of integers of U which are divisible by 5 and

C = set of integers U which are divisible by 7.

Then $S = A^c \cap B^c \cap C^c$.

Since $\dfrac{1000}{3} = 333\frac{1}{3}$, so $n(A) = 333$

Again since $\dfrac{1000}{5} = 200$ and $\dfrac{1000}{7} = 142\frac{6}{7}$,

so $n(B) = 200$ and $n(C) = 142$

Now, $A \cap B =$ set of all integers of U which are divisible by 3 and 5 both.

= set of all integers of U which are divisible by 15 (l.c.m of 3 and 5)

Since $\dfrac{1000}{15} = 66\frac{2}{3}$, so $n(A \cap B) = 66$

Similarly, since $\dfrac{1000}{35} = 28\frac{4}{7}$, so $n(B \cap C) = 28$

and also since $\dfrac{1000}{21} = 47\frac{13}{21}$, so $n(C \cap A) = 47$

Now, $A \cap B \cap C =$ Set of integers of U which are multiple of 105 (l.c.m of 3, 5 and 7).

Since $\dfrac{1000}{105} = 9\frac{11}{21}$, so $n(A \cap B \cap C) = 9$

By Principle of Inclusion and Exclusion

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

$$= 333 + 200 + 142 - 66 - 28 - 47 + 9 = 543$$

By D' Morgan's law, $A^c \cap B^c \cap C^c = (A \cup B \cup C)^c$

$$\therefore \; n(S) = n(A \cup B \cup C)^c = n(U - (A \cup B \cup C))$$

$$= n(U) - n(A \cup B \cup C) \quad [\because \; A \cup B \cup C \subset U]$$

$$= 1000 - 543 = 457$$

2.1.11. Illustrative Examples.

Ex.1. Of 32 students who play football or cricket (or both), 30 play football and 14 play cricket. Find the number of students who (i) play both the game (ii) play only football and (iii) play only cricket.

Solution. Let F = Set of students playing football

C = Set of students playing cricket

$\therefore \; n(F \cup C) = 32, n(F) = 30, n(C) = 14.$

(i) $F \cap C$ = Set of students playing both the game. By principle of Inclusion and Exclusion,

$$n(F \cup C) = n(F) + n(C) - n(F \cap C)$$

or, $32 = 30 + 14 - n(F \cap C) = 44 - n(F \cap C)$

$\therefore \quad n(F \cap C) = 44 - 32 = 12$

$\therefore$ No. of students playing both the game = 12.

(ii) Now, $F - F \cap C$ = Set of students playing only football.

$\therefore$ Required number $= n(F - F \cap C)$

$= n(F) - n(F \cap C) \quad [\because \; F \cap C \subset F]$

$= 30 - 12 = 18$

(iii) Similarly, No. of students playing only cricket

$$= n(C - F \cap C) = n(C) - n(F \cap C) = 14 - 12 = 2$$

Ex.2. Let A, B, C, D denote, respectively English, Japanese, French and Russian language courses. Let 12 take A, 20 take B, 5 take A and B, 7 take A and C, 3 take A, B and C, 2 take A, B and D; 20 take C, 8 take D, 4 take A and D, 16 take B and C, 4 take B and D, 3 take C and D, 2 take B, C, D, 3 take A, C, D, 2 take all the four and 71 take none.

Find the number of candidates who take at least one courses. Hence find the total number of candidates on which the survey is done.

Solution.

Let A = Set of candidates taking language-course A

B = Set of candidates taking language-course B

C = Set of candidates taking language-course C

D = Set of candidates taking language-course D

$\therefore$ $A \cup B \cup C \cup D$ = set of all candidates taking at least one course.

By Principle of Inclusion and Exclusion,

$$n(A \cup B \cup C \cup D)$$

$$= n(A) + n(B) + n(C) + n(D) - n(A \cap B)$$
$$- n(A \cap C) - n(A \cap D) - n(B \cap C) - n(B \cap D) - n(C \cap D)$$
$$+ n(A \cap B \cap C) + n(A \cap B \cap D) + n(B \cap C \cap D)$$
$$+ n(C \cap D \cap A) - n(A \cap B \cap C \cap D)$$

$$= (12 + 20 + 20 + 8) - (5 + 7 + 4 + 16 + 4 + 3)$$
$$+ (3 + 2 + 2 + 3) - 2$$

$$= 60 - 39 + 10 - 2 = 29$$

Total number of candidates = 29+71=100

Ex.3. Find the total number of integers lying between 1 and 1000 that are divisible by at least one of 2, 3 and 7.

Solution. Let A = set of integers, lying between 1 and 1000, which are divisible by 2.

B = set of such integers divisible by 3

C = set of such integers divisible by 7

Now $\frac{1000}{2} = 500 \quad \therefore \; n(A) = 500$

$\frac{1000}{3} = 333\frac{1}{3} \quad \therefore \; n(B) = 333$

$\frac{1000}{7} = 142\frac{6}{7} \quad \therefore \; n(C) = 142$

Now l.c.m of 2 and 3 = 6

l.c.m of 3 and 7 = 21

l.c.m of 7 and 2 = 14

l.c.m of 2, 3 and 7 = 42

Now, $\frac{1000}{6}=166\frac{2}{3}$, $\frac{1000}{21}=47\frac{13}{21}$, $\frac{1000}{14}=71\frac{3}{7}$

and $\frac{1000}{42}=23\frac{17}{21}$

$\therefore\ n(A\cap B)=166,\ n(B\cap C)=47,\ n(C\cap A)=71,$

$n(A\cap B\cap C)=23.$

$\therefore$ Required number of integers $=n(A\cup B\cup C)$

$$=n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)-n(C\cap A)+n(A\cap B\cap C)$$

$$=500+333+142-166-47-71+23=714$$

Ex.4. Using Principle of Inclusion and Exclusion show that for any three sets A, B and C,

$n(A\cup B\cup C)=n(A)+n(B)+n(C)$ if they are pairwise mutually disjoint. *[W.B.U.T.2013]*

Solution. If A, B, C are pairwise mutually disjoint then $A\cap B=B\cap C=C\cap A=A\cap B\cap C=\phi$, the null set,

Then $n(A\cap B)=n(B\cap C)=n(C\cap A)=n(A\cap B\cap C)=0$

Then by Principle of Inclusion and Exclusion,

$n(A\cup B\cup C)$

$$=n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)-n(C\cap A)+n(A\cap B\cap C)$$

$$=n(A)+n(B)+n(C)$$