Example 1 *Show that if n is a positive integer, then*

Extra Examples

$$1+2+\cdots+n=\frac{n(n+1)}{2}.$$

Solution Let $P(n)$ be the proposition that the sum of the first n positive integers is $n(n+1)/2$. We must do two things to prove that $P(n)$ is true for $n = 1, 2, 3, \ldots$. Namely, we must show that $P(1)$ is true and that the conditional statement $P(k)$ implies $P(k+1)$ is true for $k = 1, 2, 3, \ldots$.

Basis step: $P(1)$ is true, because $1=\frac{1(1+1)}{2}$.

Inductive step: For the inductive hypothesis we assume that $P(k)$ holds for an arbitrary positive integer k. That is, we assume that

$$1+2+\cdots+k=\frac{k(k+1)}{2}.$$

Under this assumption, it must be shown that $P(k+1)$ is true, namely, that

$$1+2+\cdots+k+(k+1)=\frac{(k+1)[(k+1)+1]}{2}=\frac{(k+1)(k+2)}{2}$$

is also true. When we add $k+1$ to both sides of the equation in $P(k)$, we obtain

$$\begin{aligned}1+2+\cdots+k+(k+1)&=\frac{k(k+1)}{2}+(k+1)\\&=\frac{k(k+1)+2(k+1)}{2}\\&=\frac{(k+1)(k+2)}{2}.\end{aligned}$$

This last equation shows that $P(k+1)$ is true under the assumption that $P(k)$ is true. This completes the inductive step.

prove that ...

Example 3 *Use mathematical induction to show that*

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

for all nonnegative integers n.

Solution Let $P(n)$ be the proposition that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for the integer n.

Basis step: $P(0)$ is true because $2^0 = 1 = 2^1 - 1$. This completes the basis step.

Inductive step: For the inductive hypothesis, we assume that $P(k)$ is true. That is, we assume that

$$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1.$$

To carry out the inductive step using this assumption, we must show that when we assume that $P(k)$ is true, then $P(k+1)$ is also true. That is, we must show that

$$1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1 = 2^{k+2} - 1$$

assuming the inductive hypothesis $P(k)$. Under the assumption of $P(k)$, we see that

$$\begin{aligned} 1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} &= (1 + 2 + 2^2 + \cdots + 2^k) + 2^{k+1} \\ &= (2^{k+1} - 1) + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+2} - 1. \end{aligned}$$

Note that we used the inductive hypothesis in the second equation in this string of equalities to replace $1 + 2 + 2^2 + \cdots + 2^k$ by $2^{k+1} - 1$. We have completed the inductive step.

Because we have completed the basis step and the inductive step, by mathematical induction we know that $P(n)$ is true for all nonnegative integers n. That is, $1 + 2 + \cdots + 2^n = 2^{n+1} - 1$ for all nonnegative integers n.

Example 4 *Sums of Geometric Progressions* *Use mathematical induction to prove this formula for the sum of a finite number of terms of a geometric progression:*

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^k = \frac{ar^{n+1} - a}{r-1} \quad \text{when } r \neq 1,$$

where n is a non-negative integer.

Solution To prove this formula using mathematical induction, let $P(n)$ be the statement that the sum of the first $n+1$ terms of a geometric progression in this formula is correct.

Basis step: $P(0)$ is true, because

$$\frac{ar^{0+1} - a}{r-1} = \frac{ar - a}{r-1} = \frac{a(r-1)}{r-1} = a.$$

Inductive step: The inductive hypothesis is the statement that $P(k)$ is true, where k is a nonnegative integer. That is, $P(k)$ is the statement that

$$a + ar + ar^2 + \quad + ar^k = \frac{ar^{k+1} - a}{r-1}.$$

To complete the inductive step we must show that if $P(k)$ is true, then $P(k+1)$ is also true. To show that this is the case, we first add ar^{k+1} to both sides of the equality asserted by $P(k)$. We find that

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1} = \frac{ar^{k+1} - a}{r-1} + ar^{k+1}.$$

Rewriting the right-hand side of this equation shows that

$$\frac{ar^{k+1} - a}{r-1} + ar^{k+1} = \frac{ar^{k+1} - a}{r-1} + \frac{ar^{k+2} - ar^{k+1}}{r-1}$$

$$= \frac{ar^{k+2} - a}{r-1}.$$

Combining these last two equations gives

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1} = \frac{ar^{k+2} - a}{r-1}$$

This shows that if the inductive hypothesis $P(k)$ is true, then $P(k+1)$ must also be true. This completes the inductive argument.

We have completed the basis step and the inductive step, so by mathematical induction $P(n)$ is true for all nonnegative integers n. This shows that the formula for the sum of the terms of a geometric series is correct. ◀

As previously mentioned, the formula in Example 3 is the case of the formula in Example 4 with $a = 1$ and $r = 2$. The reader should verify that putting these values for a and r into the general formula gives the same formula as in Example 3.

Proving Inequalities Mathematical induction can be used to prove a variety of inequalities that hold for all positive integers greater than a particular positive integer, as Examples 5–7 illustrate.

Example 5 *Use mathematical induction to prove the inequality*

$$n < 2^n$$

for all positive integers n.

Extra Examples

Solution Let $P(n)$ be the proposition that $n < 2^n$.

Basis step: $P(1)$ is true, because $1 < 2^1 = 2$. This completes the basis step.

Inductive step: We first assume the inductive hypothesis that $P(k)$ is true for the positive integer k. That is, the inductive hypothesis $P(k)$ is the statement that $k < 2^k$. To complete the inductive step, we need to show that if $P(k)$ is true, then $P(k + 1)$, which is the statement that $k + 1 < 2^{k+1}$, is true. That is, we need to show that if $k < 2^k$, then $k + 1 < 2^{k+1}$. To show that this conditional statement is true for the positive integer k, we first add 1 to both sides of $k < 2^k$, and then note that $1 \leq 2^k$. This tells us that

$$k + 1 < 2^k + 1 \leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}.$$

This shows that $P(k + 1)$ is true, namely, that $k + 1 < 2^{k+1}$, based on the assumption that $P(k)$ is true. The induction step is complete.

Therefore, because we have completed both the basis step and the inductive step, by the principle of mathematical induction we have shown that $n < 2^n$ is true for all positive integers n. ◀

✓Example 6 *Use mathematical induction to prove that $2^n < n!$ for every positive integer n with $n \geq 4$. (Note that this inequality is false for $n = 1, 2,$ and 3.)*

Solution Let $P(n)$ be the proposition that $2^n < n!$.

Basis step: To prove the inequality for $n \geq 4$ requires that the basis step be $P(4)$. Note that $P(4)$ is true, because $2^4 = 16 < 24 = 4!$

Inductive step: For the inductive step, we assume that $P(k)$ is true for the positive integer k with $k \geq 4$. That is, we assume that $2^k < k!$ for the positive integer k with $k \geq 4$. We must show that under this hypothesis, $P(k+ 1)$ is also true. That is, we must show that if $2^k < k!$ for the positive integer k where $k \geq 4$, then $2^{k+1} < (k + 1)!$. We have

$$\begin{aligned}
2^{k+1} &= 2 \cdot 2^k && \text{by definition of exponent} \\
&< 2 \cdot k! && \text{by the inductive hypothesis} \\
&< (k + 1)\, k! && \text{because } 2 < k + 1 \\
&= (k + 1)! && \text{by definition of factorial function.}
\end{aligned}$$

This shows that $P(k + 1)$ is true when $P(k)$ is true. This completes the inductive step of the proof.

We have completed the basis step and the inductive step. Hence, by mathematical induction $P(n)$ is true for all integers n with $n \geq 4$. That is, we have proved that $2^n < n!$ is true for all integers n with $n \geq 4$. ◀

Proving Divisibility Results Mathematical induction can be used to prove divisibility results about integers. Although such results are often easier to prove using basic results in number theory, it is instructive to see how to prove such results using mathematical induction, as Example 8 illustrates.

Example 8 *Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer.*

Extra Examples

Solution To construct the proof, let $P(n)$ denote the proposition: "$n^3 - n$ is divisible by 3."

Basis step: The statement $P(1)$ is true because $1^3 - 1 = 0$ is divisible by 3. This completes the basis step.

Inductive step: For the inductive hypothesis we assume that $P(k)$ is true; that is, we assume that $k^3 - k$ is divisible by 3. To complete the inductive step, we must show that when we assume the inductive hypothesis, it follows that $P(k+1)$, the statement that $(k+1)^3 - (k+1)$ is divisible by 3, is also true. That is, we must show that $(k+1)^3 - (k+1)$ is divisible by 3. Note that

$$\begin{aligned}(k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k+1)\\ &= (k^3 - k) + 3(k^2 + k).\end{aligned}$$

Because both terms in this sum are divisible by 3 (the first by the inductive hypothesis, and the second because it is 3 times an integer), it follows that $(k+1)^3 - (k+1)$ is also divisible by 3. This completes the inductive step.

Because we have completed both the basis step and the inductive step, by the principle of mathematical induction we know that $n^3 - n$ is divisible by 3 whenever n is a positive integer. ◀