Aarhus university

Computer-Science

Introduction to probability and statistics

Handin 5

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October 6, 2025

Game Strategy with Dice!

We consider the following strategy:

$$W = \begin{cases} X, & \text{if } X \geq 4, \\ Y, & \text{if } X < 4. \end{cases}$$

That is, the strategy consists of stopping the game if the number of eyes (die roll) is 4, 5, or 6, and rolling a new die if the number of eyes is 1, 2, or 3.

(a) What is the expected payoff of the game if you choose to never roll a new die? That is, you win X kr.

The formula for the expected value from Def 3.11 is:

$$E(X) = \sum_{x_k \in R_X} x_k \cdot P_X(x_k) \tag{1}$$

For this little case our range is $R_X = \{1, 2, 3, 4, 5, 6\}$ and the probability is $P_X(x_k) = \frac{1}{6}$ for all $x_k \in R_X$ since we are using a die.
This gives us:

$$E(X) = \sum_{x_k \in R_X} x_k \cdot P_X(x_k) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$

This simplifies to:

$$E(X) = \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = \underline{3.5}$$

(b) Interpret strategy W in relation to subquestion (a).

Strategy (a) takes the first outcome from a single die roll, which just gives a uniform distribution over the die's (dies?) range. Strategy (b) introduces a conditional roll, if the first roll was less than four it rolls again and takes the outcome, this creates a bias towards higher outcomes compared to (a), which should give a higher expected value.

(c) Find the range R_W and the PMF P_W for W. Also, compute the expected payoff when using strategy W.

Let's take it from the top jesus!
The range is the same as before in (a):

$$R_W = \{1, 2, 3, 4, 5, 6\}$$

now for the PMF.
The probability of getting a 1, 2 or 3 is rolling 1, 2 or 3 twice in a row.
The first roll is X and the second roll is Y and they are independent events.

$$P_W(w) = P(X < 4) \cdot P(Y = w) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12} \quad \text{for } w = 1, 2, 3$$

And the probability of getting a 4, 5 or 6 is either getting them on the first role or getting a 1, 2 or 3 and then rolling a 4, 5 or 6.
Note that the events $P(X = w) + P(X < 4)$ are disjoint so we can add the probabilities of multiple events using the 'Inclusion-Exclusion Principle' since their intersection is always zero according to chapter 1.2.2 [1].

$$P_W(w) = P(X = w) + P(X < 4) \cdot P(Y = w) = \frac{1}{6} + \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{4} \quad \text{for } w = 4, 5, 6$$

$$P_W(W = w) = \begin{cases} \frac{1}{12}, & \text{if } w = 1 \\ \frac{1}{12}, & \text{if } w = 2 \\ \frac{1}{12}, & \text{if } w = 3 \\ \frac{1}{4}, & \text{if } w = 4 \\ \frac{1}{4}, & \text{if } w = 5 \\ \frac{1}{4}, & \text{if } w = 6 \end{cases} \tag{2}$$

We can do a sanity check with axiom 2 of probability from chapter 1.3.2 [1] to make sure our PMF is correct:

$$\sum_{w_k \in R_W} P_W(w_k) = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$$

Now for the expected value of W:

$$E(W) = \sum_{w_k \in R_W} w_k \cdot P_W(w_k) = 1 \cdot \frac{1}{12} + 2 \cdot \frac{1}{12} + 3 \cdot \frac{1}{12} + 4 \cdot \frac{1}{4} + 5 \cdot \frac{1}{4} + 6 \cdot \frac{1}{4}$$

This simplifies to:

$$E(W) = \frac{1+2+3}{12} + \frac{4+5+6}{4} = \frac{6}{12} + \frac{15}{4} = \frac{51}{12} = \underline{4.25}$$

Which gives us a higher expected value than strategy (a) as expected.

$$\underline{\text{E(W)} = 4.25 > \text{E(X)} = 3.5}$$