Aarhus university

Computer-Science

Introduction to probability and statistics

Handin 10

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November 10, 2025

Let $X_1, \ldots, X_n$ be a sample where $X_i \sim \text{Poisson}(\theta)$ with an unknown parameter $\theta > 0$.

Compute the likelihood function $L(\theta) = L(x_1, \ldots, x_n; \theta)$ and find the log-likelihood function $l(\theta)$.

Let's get some facts straight up in this `[Censored]!`.
From chapter 8.2.3 [1] we have the definition of the fancy likelihood function and since the Poisson distribution is a discrete probability, we'll use the definition for discrete random variables:

$$L(x_1, x_2, \ldots, x_n; \theta) = P_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n; \theta)$$

The likelihood function is the joint probability of all the random variables in our sample. We note that each X is independant, which allows us to multiply the marginal PMFs to get the joint probability, so we remember the PMF of the Poisson distribution is $P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$

$$L(\theta) = \prod_{i=1}^{n} \frac{\theta^x e^{-\theta}}{x!}$$

Now we should be ready to take the log-likelihood, but let's simplify the likelihood function first. We take each part of the product seperately:

$$\prod_{i=1}^{n} \theta^{x_i} = \theta^{\sum_{i=1}^{n} x_i}$$

$$\prod_{i=1}^{n} e^{-\theta} = e^{-n\theta}$$

$$\prod_{i=1}^{n} \frac{1}{x_i!} = \frac{1}{\prod_{i=1}^{n} x_i!}$$

Okay...Bear(rawr) with me here, this might not be pretty, but you just gotta trust me on this one:

$$L(\theta) = \frac{\theta^{\sum_{i=1}^{n} x_i} e^{-n\theta}}{\prod_{i=1}^{n} x_i!}$$

We did this to make it easier since Mr.Sørensen don't know how to take the log of a product...Anyway!
Now we can take the log-likelihood and since we are only intereseted in the parts with θ (which conveniently doesn't include the product in the denominator hehe...) Note that we can do this since $\prod_{i=1}^{n} x_i!$ is a constant in respect to θ and will be zero when we differentiate later:

$$l(\theta) = \log L(\theta) = \log \left(\theta^{\sum_{i=1}^{n} x_i} \cdot e^{-n\theta} \right)$$

Using the 'Product Rule' for log, $\log(a \cdot b) = \log(a) + \log(b)$, we split it up:

$$l(\theta) = \log\left(\theta^{\sum_{i=1}^{n} x_i}\right) + \log\left(e^{-n\theta}\right)$$

Next is the 'Power Rule', $\log(a^c) = c \cdot \log(a)$, on the first fish and the 'fancy log rule', $\log(e^a) = a$, on the second fish:

$$l(\theta) = \left(\sum_{i=1}^{n} x_i\right) \log\theta - n\theta$$

There we have it folks or folk?, the log-likelihood function!

> Set up the likelihood equation and find the maximum likelihood estimator $\hat{\theta}_{ML}$.

Now for for the easy part! Click for epic intro music
To find the MLE we take the derivative of the log-likelihood function and set it equal to zero:

$$\frac{d}{d\theta} l(\theta) = \frac{d}{d\theta}\left(\left(\sum_{i=1}^{n} x_i\right) \log\theta - n\theta\right) = 0$$

To solve this, we first use the 'Sum Rule', $\frac{d}{d\theta}(f(\theta) - g(\theta)) = f'(\theta) - g'(\theta)$, which means we can split the fishies up:

$$\frac{d}{d\theta}\left(\left(\sum_{i=1}^{n} x_i\right) \log\theta\right) - \frac{d}{d\theta}(n\theta) = 0$$

Then for both fishies, we use the 'Constant Rule', $\frac{d}{d\theta}(c \cdot f(\theta)) = c \cdot f'(\theta)$, (treating $\sum x_i$ and n as a constant) taking them our of the derivative

$$\left(\sum_{i=1}^{n} x_i\right) \frac{d}{d\theta}(\log\theta) - n\frac{d}{d\theta}(\theta) = 0$$

The first fish becomes $\frac{1}{\theta}$ using the 'Log Rule', $\frac{d}{d\theta}(\log\theta) = \frac{1}{\theta}$, and the second fish becomes 1 using the 'Power Rule', $\frac{d}{d\theta}(\theta) = 1$ Which gives us the beautiful fishy:

$$\left(\sum_{i=1}^{n} x_i\right) \frac{1}{\theta} - n = 0$$

Now we isolate θ to find the MLE:

$$\left(\sum_{i=1}^{n} x_i\right) \frac{1}{\theta} = n$$

$$\sum_{i=1}^{n} x_i = n\theta$$

Finally we divide by n:

$$\frac{\sum_{i=1}^{n} x_i}{n} = \theta$$

And there we have it... folkies? (idk, i know numbers not words), the MLE for θ that is:

$$\hat{\theta}_{ML} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Interpret what the problem states about the MLE for θ.

Well, if we looke reeeal careful at the MLE we can see it's just the sample mean Def 7.1 [1]:

$$\hat{\theta}_{ML} = \overline{X}$$

This means that the MLE for the Poisson with θ is simply the average of all the data in our sample

References

[1] Hossein Pishro-Nik. *Introduction to Probability ,Statistics and Random Processes*. Kappa Research, LLC, 2014.