Aarhus university

Computer-Science

Introduction to probability and statistics

Handin 7

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Let X be a continuous random variable with the probability density functuion (PDF) f_X

f_X is given by:

$$f_X(x) = \begin{cases} 3x^2 & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

The set $Y = -\frac{1}{2}\log(X)$. Find the PDF f_Y for Y. As always, indicate the distribution of Y if it is a known distribution.

For this we shal utilize Theorem 4.1' from 'Uge Seddel 6'

Theorem 4.1'. Lad X betegne en *kontinuert* stokastisk variabel med en PDF f_X og range R_X. Lad $g : R_X \to \mathbb{R}$ være en differentiabel funktion, der enten er *strengt voksende* eller *strengt aftagende*. Lad h betegne den inverse funktion (=omvendte funktion) til g.

Så gælder der at $Y = g(X)$ er en kontinuert stokastisk variabel med en PDF f_Y givet ved

$$f_Y(y) = \begin{cases} |h'(y)| f_X(h(y)) & \text{for } y \in R_Y, \\ 0 & \text{ellers,} \end{cases}$$

hvor mængden R_Y er givet ved $R_Y = \{g(x) : x \in R_X\}$.

We start by defining a function $g(x) = -\frac{1}{2}\log(x)$, then let $Y = g(X)$.
Now we show its monotonic and differentiable by taking the derivative of g:

$$g'(x) = -\frac{1}{2} \cdot \frac{1}{x} = -\frac{1}{2x}$$

We can see that g is monotonic (strongly decreasing) and differentiable on the interval $(0, 1)$, our range R_X, so we can use Theorem 4.1'.

The Inverse

We define the inverse function $g^{-1}(y) = h(y)$:

$$h(y) = e^{-2y}$$

we can check this by plugging h into g, it's important to note the fact that $\log(e) = 1$ and $e^{\log(y)} = y$:

$$g(h(y)) = -\frac{1}{2}\log(e^{-2y}) = -\frac{1}{2} \cdot -2y \cdot \log(e) = -\frac{1}{2} \cdot -2y = y$$

And then plugging g into h:

$$h(g(y)) = e^{-2\cdot(-\frac{1}{2})\log(y)} = e^{\log(y)} = y$$

Since they both equal y, we can say that they are inverses (Source? WebMatematik).

The derivative of the Inverse

Let's plug it in:

$$h'(y) = \frac{d}{dy}e^{-2y} = -2e^{-2y}$$

The PDF of X at $h(y)$?

That is just $f_X(h(y))$ where we have the PDF of $f_X(x) = 3x^2$:

$$f_X(h(y)) = f_X(e^{-2y}) = 3(e^{-2y})^2 = 3e^{-4y}$$

For this we used a small trick of $(a^b)^c = a^b \cdot a^b \cdot a^b \ldots = a^{b+b+\cdots+b} = a^{b\cdot c}$ where c is the number of times we multiply a^b with itself (more of a note to myself OwO).

PDF of Y + Conclusion

Now we have all the parts we need to find $f_Y(y)$:

$$f_Y(y) = |h'(y)| \cdot f_X(h(y)) = |-2e^{-2y}| \cdot 3e^{-4y} = 6e^{-6y}$$

Now we need to find the range R_Y of Y. We can do this by finding the minimum and maximum of Y:

$$\min(Y) = g(1) = -\frac{1}{2}\log(1) = 0$$

$$\max(Y) = g(0) = -\frac{1}{2}\log(0) = \infty$$

So we have $R_Y = (0, \infty)$.
So now we have the final PDF of Y:

$$f_Y(y) = \begin{cases} 6e^{-6y} & \text{for } 0 < y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

This PDF has the form of an exponential distribution with parameter $\lambda = 6$
or $Y \sim \text{Exponential}(6)$

$$f_X(x) = \lambda e^{-\lambda x} \text{ for } x > 0 \quad \leftarrow \quad \textit{PDF for an exponential distribution}$$

Thanks for reading my TED-talk on **Math fueled by coffee and need for money!**