Aarhus university

Computer-Science

Introduction to probability and statistics

Handin 8

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Consider the set of points in the set C:

$$C = \{(x, y) | x, y \in \mathbb{Z}, x^2 + |y| \leq 2\}$$

Suppose that we pick a point (X, Y) from this set completely at random. Thus, each point has a probability $\frac{1}{11}$ of being chosen.

Find the joint and marginal PMFs of X and Y

from the set C we can see that there are 11 points that satisfy the condition $x^2 + |y| \leq 2$:

$$(0,0),(1,0),(-1,0),(0,1),(0,-1),(1,1),(1,-1),(-1,1),(-1,-1),(0,2),(0,-2)$$

So the joint PMF is given by:

$$P_{X,Y}(x, y) = \begin{cases} \frac{1}{11} & (x, y) \in C \\ 0 & \text{otherwise} \end{cases}$$

or it can also be written as a table like example 5.1 [1]:

$P_{X,Y}(x, y)$	$y = -2$	$y = -1$	$y = 0$	$y = 1$	$y = 2$
$x = -1$	0	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	0
$x = 0$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$
$x = 1$	0	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{1}{11}$	0

To find the marginal PMF of X we sum over all possible values of Y like in chapter 5 *marginal PMFs* [1]:

$$P_X(x) = \sum_{y_j \in \mathbb{R}_Y} P_{X,Y}(x, y_j)$$

This gives us:

$$P_X(x) = \begin{cases} \frac{5}{11} & x = 0 \\ \frac{3}{11} & x = 1 \\ \frac{3}{11} & x = -1 \\ 0 & \text{otherwise} \end{cases}$$

Similarly, for the marginal PMF of Y we sum over all possible values of X:

$$P_Y(y) = \sum_{x_i \in \mathbb{R}_X} P_{X,Y}(x_i, y)$$

This gives us:

$$P_Y(y) = \begin{cases} \frac{1}{11} & y = -2 \\ \frac{3}{11} & y = -1 \\ \frac{3}{11} & y = 0 \\ \frac{3}{11} & y = 1 \\ \frac{1}{11} & y = 2 \\ 0 & \text{otherwise} \end{cases}$$

Note: Since we are in $\mathbb{Z}$ we can't include decimals when we draw C in a coordinate system, so the correct would only be to plot the points in C.

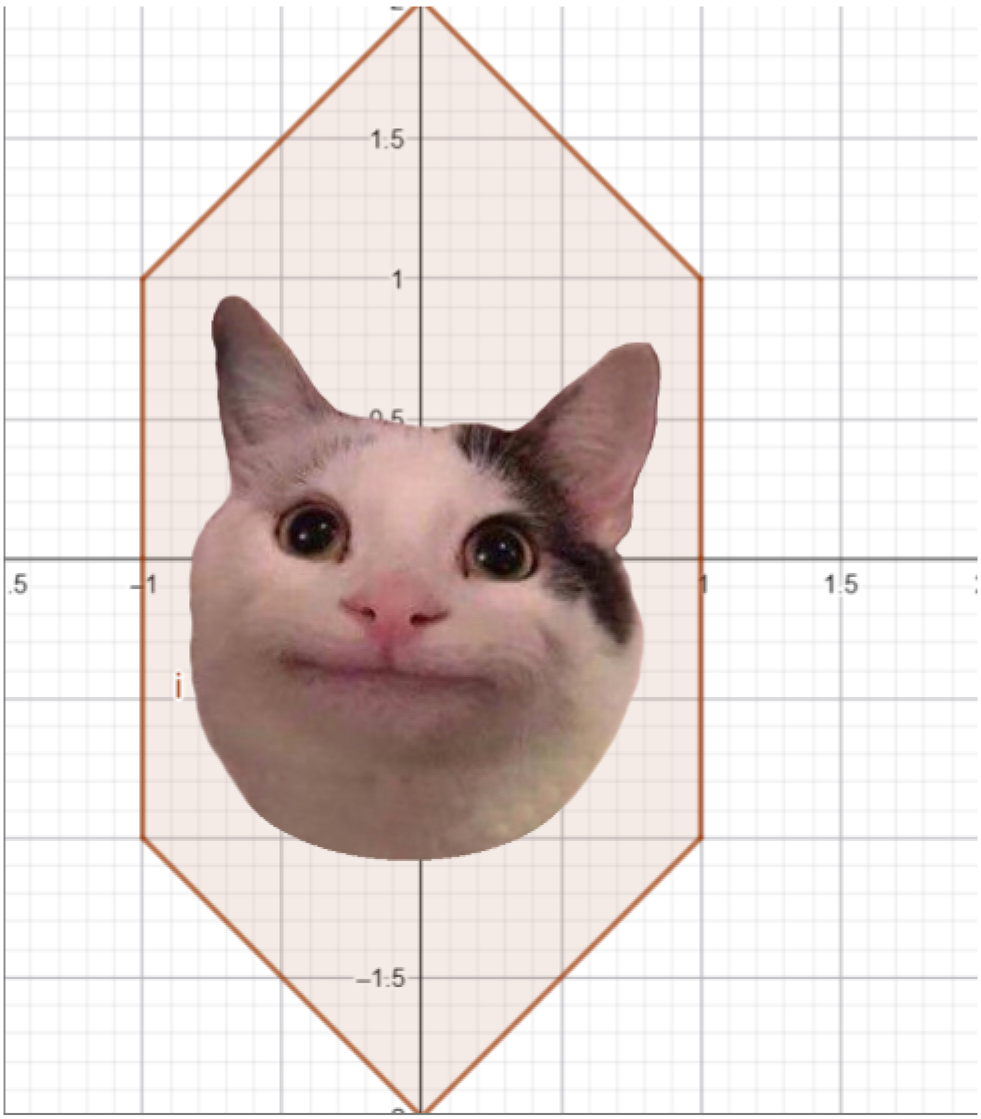

Figure 1: The set C plotted in a coordinate system.

Find the conditional PMF of X given $Y = 1$.

The formula for finding the conditional PMF is given in chapter 5.1.3 [1]:

$$P_{X|Y}(x_i|y_i) = \frac{P_{XY}(x_i, y_j)}{P_Y(y_i)}$$

Then we sum all values of X where $Y = 1$:

$$\sum_{x_i \in R_X} P_{X|Y}(x_i|1)$$

This gets us:

$$P_{X|Y}(-1|1) + P_{X|Y}(0|1) + P_{X|Y}(1,1)$$

And we insert the numbers from the joint PMF:

$$\frac{\frac{1}{11}}{\frac{3}{11}} + \frac{\frac{1}{11}}{\frac{3}{11}} + \frac{\frac{1}{11}}{\frac{3}{11}} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

Which gives us the conditional PMF:

$$P_{X|Y}(x|1) = \begin{cases} \frac{1}{3} & x = -1 \\ \frac{1}{3} & x = 0 \\ \frac{1}{3} & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

It all sums up to 1 which validates our findings of the conditional PMF according to axiom 2 of probability [1].

Are X and Y independent?

Here we take the time machine back to chapter 1.4.1 [1] where independence was born. it is said in this chapter that when two variables are independant the following holds true:

$$P(A|B) = P(A)$$

This is applicable in our case, sinc we just found the conditional PMF of X given $Y = 1$. So we just need to check if $P_{X|Y}(x|1) = P_X(x)$.
We can see that this is not the case since:

$$P_{X|Y}(0|1) = \frac{1}{3} \neq P_X(0) = \frac{5}{11}$$

hence X and Y are not independent. Womp Womp *Cry Emoji*.

Find $E[XY^2]$

By the way of LOTUS and a little bit of equation 5.5 [1] we can find $E[XY^2]$ like so:

$$E[XY^2] = \sum_{(x_i,y_j)\in R_X} x_i y_j^2 \cdot P_{X,Y}(x_i, y_j)$$

Before we insert the values we note that every point in C where $x = 0$ and $y = 0$ will result in 0 when calculating $E[XY^2]$, so we exclude those points:

$$E[XY^2] = 1 \cdot 1^2 \frac{1}{11} + (-1) \cdot 1^2 \frac{1}{11} + 1 \cdot (-1)^2 \frac{1}{11} + (-1) \cdot (-1)^2 \frac{1}{11}$$

$$E[XY^2] = \frac{1}{11} + \frac{1}{11} - \frac{1}{11} - \frac{1}{11} = 0$$

Hence $E[XY^2] = 0$.

References

[1] Hossein Pishro-Nik. *Introduction to Probability ,Statistics and Random Processes.* Kappa Research, LLC, 2014.