Fisher and Linear Discriminant Analysis

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Data and Knowledge Modeling and Analysis (ECE 657A)

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Dataset Notations

$$\text{training dataset: } \{\boldsymbol{x}_i \in \mathbb{R}^d\}_{i=1}^n, \boldsymbol{X} = [\boldsymbol{x}_1, \dots, \boldsymbol{x}_n] \in \mathbb{R}^{d \times n} \tag{1}$$

$$j\text{-th training class: } \{\boldsymbol{x}_i^{(j)} \in \mathbb{R}^d\}_{i=1}^{n_j}, \quad \forall j \in \{1, \dots, c\} \tag{2}$$

$$\text{mean of training data: } \mathbb{R}^d \ni \boldsymbol{\mu}_x := \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{x}_i \tag{3}$$

$$\text{test dataset: } \{\boldsymbol{x}_{t,i} \in \mathbb{R}^d\}_{i=1}^{n_t}, \boldsymbol{X}_t = [\boldsymbol{x}_{t,1}, \dots, \boldsymbol{x}_{t,n_t}] \in \mathbb{R}^{d \times n_t} \tag{4}$$

Lecture Outline

FDA: Inter- and Intra-Class Scatters

$$\mathbb{R}^d \ni \boldsymbol{\mu}_j := \frac{1}{n_j} \sum_{i=1}^{n_j} \boldsymbol{x}_i^{(j)} \tag{5}$$

$$\mathbb{R}^d \ni \boldsymbol{\mu} := \frac{1}{\sum_{k=1}^{c} n_k} \sum_{j=1}^{c} n_j \, \boldsymbol{\mu}_j = \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{x}_i \tag{6}$$

$$\mathbb{R}^{d \times d} \ni \boldsymbol{S}_B := \sum_{j=1}^{c} n_j (\boldsymbol{\mu}_j - \boldsymbol{\mu})(\boldsymbol{\mu}_j - \boldsymbol{\mu})^\top \tag{7}$$

$$\mathbb{R}^{d \times d} \ni \boldsymbol{S}_W := \sum_{j=1}^{c} \boldsymbol{S}_j = \sum_{j=1}^{c} \sum_{i=1}^{n_j} (\boldsymbol{x}_i^{(j)} - \boldsymbol{\mu}_j)(\boldsymbol{x}_i^{(j)} - \boldsymbol{\mu}_j)^\top \tag{8}$$

Lecture Outline

Scatters of Projections

$$\widehat{\boldsymbol{x}} = \boldsymbol{u}\boldsymbol{u}^\top \breve{\boldsymbol{x}} \tag{9}$$

$$\begin{aligned} ||\widehat{\boldsymbol{x}}||_2^2 &= ||\boldsymbol{u}\boldsymbol{u}^\top \breve{\boldsymbol{x}}||_2^2 = (\boldsymbol{u}\boldsymbol{u}^\top \breve{\boldsymbol{x}})^\top (\boldsymbol{u}\boldsymbol{u}^\top \breve{\boldsymbol{x}}) \\ &= \breve{\boldsymbol{x}}^\top \boldsymbol{u} \underbrace{\boldsymbol{u}^\top \boldsymbol{u}}_{1} \boldsymbol{u}^\top \breve{\boldsymbol{x}} = \breve{\boldsymbol{x}}^\top \boldsymbol{u}\, \boldsymbol{u}^\top \breve{\boldsymbol{x}} = \boldsymbol{u}^\top \breve{\boldsymbol{x}}\, \breve{\boldsymbol{x}}^\top \boldsymbol{u} \end{aligned} \tag{10}$$

$$\sum_{i=1}^{n} ||\widehat{\boldsymbol{x}}_i||_2^2 = \sum_{i=1}^{n} \boldsymbol{u}^\top \breve{\boldsymbol{x}}_i\, \breve{\boldsymbol{x}}_i^\top \boldsymbol{u} = \boldsymbol{u}^\top \Big(\sum_{i=1}^{n} \breve{\boldsymbol{x}}_i\, \breve{\boldsymbol{x}}_i^\top \Big) \boldsymbol{u} \tag{11}$$

covariance matrix: $\mathbb{R}^{d \times d} \ni \boldsymbol{S} := \sum_{i=1}^{n} \breve{\boldsymbol{x}}_i\, \breve{\boldsymbol{x}}_i^\top = \breve{\boldsymbol{X}}\breve{\boldsymbol{X}}^\top$

$$||\widehat{\boldsymbol{X}}||_F^2 = \boldsymbol{u}^\top \boldsymbol{S}\boldsymbol{u}, \quad ||\widehat{\boldsymbol{X}}||_F^2 = \mathbf{tr}(\boldsymbol{U}^\top \boldsymbol{S}\, \boldsymbol{U}). \tag{12}$$

Projection & Reconstruction

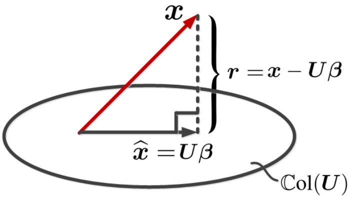
$$\text{projection: } \mathbb{R}^p \ni \widetilde{\boldsymbol{x}} := \boldsymbol{U}^\top \boldsymbol{x}, \widetilde{\boldsymbol{X}} = [\widetilde{\boldsymbol{x}}_1, \dots, \widetilde{\boldsymbol{x}}_n] \in \mathbb{R}^{p \times n} \tag{13}$$

$$\text{reconstruction: } \mathbb{R}^d \ni \widehat{\boldsymbol{x}} := \boldsymbol{U}\boldsymbol{U}^\top \boldsymbol{x} + \boldsymbol{\mu}_x = \boldsymbol{U}\widetilde{\boldsymbol{x}} + \boldsymbol{\mu}_x \tag{14}$$

$$\widehat{\boldsymbol{X}} = [\widehat{\boldsymbol{x}}_1, \dots, \widehat{\boldsymbol{x}}_n] \in \mathbb{R}^{d \times n}$$

$$\text{test projection: } \mathbb{R}^{p \times n_t} \ni \widetilde{\boldsymbol{X}}_t = \boldsymbol{U}^\top \boldsymbol{X}_t, \tag{15}$$

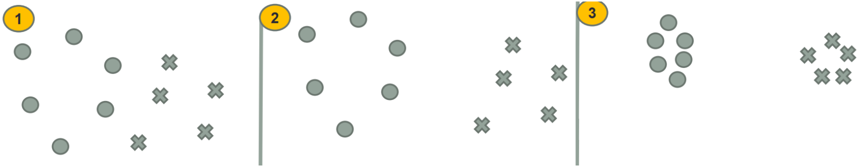
$$\text{test reconst.: } \mathbb{R}^{d \times n_t} \ni \widehat{\boldsymbol{X}}_t = \boldsymbol{U}\boldsymbol{U}^\top \boldsymbol{X}_t + \boldsymbol{\mu}_x = \boldsymbol{U}\widetilde{\boldsymbol{X}}_t + \boldsymbol{\mu}_x, \tag{16}$$



FDA Goal

For better **separation** of classes in some data (Fig 1):

- Make **inter-class scatter** larger (Fig 2)
- Make **intra-class scatter** smaller (Fig 3)



FDA does this for **supervised** subspace learning.
The first FDA paper: [1] by Ronald A. Fisher (1890 – 1962)

FDA Optimization

$$\underset{\boldsymbol{U}}{\text{maximize}} \quad f(\boldsymbol{U}) := \frac{\mathbf{tr}(\boldsymbol{U}^\top \boldsymbol{S}_B\, \boldsymbol{U})}{\mathbf{tr}(\boldsymbol{U}^\top \boldsymbol{S}_W\, \boldsymbol{U})}. \tag{17}$$

$$\begin{aligned} \underset{\boldsymbol{U}}{\text{maximize}} \quad & \mathbf{tr}(\boldsymbol{U}^\top \boldsymbol{S}_B\, \boldsymbol{U}) \\ \text{subject to} \quad & \boldsymbol{U}^\top \boldsymbol{S}_W\, \boldsymbol{U} = \boldsymbol{I}. \end{aligned} \tag{18}$$

$$\mathcal{L} = \mathbf{tr}(\boldsymbol{U}^\top \boldsymbol{S}_B\, \boldsymbol{U}) - \mathbf{tr}\big(\Lambda^\top (\boldsymbol{U}^\top \boldsymbol{S}_W\, \boldsymbol{U} - \boldsymbol{I})\big) \tag{19}$$

$$\mathbb{R}^{d \times p} \ni \frac{\partial \mathcal{L}}{\partial \boldsymbol{U}} = 2\, \boldsymbol{S}_B\, \boldsymbol{U} - 2\, \boldsymbol{S}_W\, \boldsymbol{U}\Lambda \overset{\text{set}}{=} 0$$

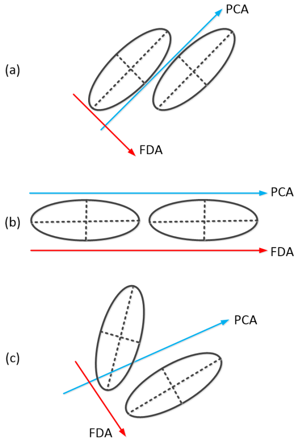
$$\implies 2\, \boldsymbol{S}_B\, \boldsymbol{U} = 2\, \boldsymbol{S}_W\, \boldsymbol{U}\Lambda \implies \boxed{\boldsymbol{S}_B\, \boldsymbol{U} = \boldsymbol{S}_W\, \boldsymbol{U}\Lambda} \tag{20}$$

FDA: Rank and Dimensionality of Subspace

$$\boldsymbol{S}_B\,\boldsymbol{U} = \boldsymbol{S}_W\,\boldsymbol{U}\Lambda \implies \boxed{\boldsymbol{S}_W^{-1}\boldsymbol{S}_B\,\boldsymbol{U} = \boldsymbol{U}\Lambda} \tag{21}$$

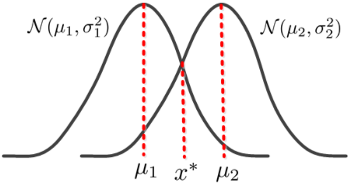
$$\begin{aligned}
&\textbf{rank}(\boldsymbol{S}_W^{-1}\boldsymbol{S}_B) \leq \min\big(\textbf{rank}(\boldsymbol{S}_W^{-1}), \textbf{rank}(\boldsymbol{S}_B)\big) \\
&\leq \min\big(\min(d, n-1), \min(d, c-1)\big) \\
&= \min(d, n-1, c-1) = \boxed{c-1}
\end{aligned} \tag{22}$$

FDA vs PCA



Lecture Outline

LDA: Optimization for Boundary of Classes



$$\begin{aligned}\mathbb{P}(\text{error}) = \;& \mathbb{P}(x > x^* \,|\, x \in \mathcal{C}_1)\, \mathbb{P}(x \in \mathcal{C}_1) \\ & + \mathbb{P}(x < x^* \,|\, x \in \mathcal{C}_2)\, \mathbb{P}(x \in \mathcal{C}_2)\end{aligned}$$

$$\mathbb{P}(x < c, x \in \mathcal{C}_1) = F_1(c) \implies \mathbb{P}(x > x^*, x \in \mathcal{C}_1) = 1 - F_1(x^*)$$

$$\mathbb{P}(x < x^*, x \in \mathcal{C}_2) = F_2(x^*)$$

$$\mathbb{P}(x \in \mathcal{C}_1) = f_1(x) = \pi_1, \mathbb{P}(x \in \mathcal{C}_2) = f_2(x) = \pi_2$$

$$\underset{x^*}{\text{minimize}} \quad \mathbb{P}(\text{error}) \implies \underset{x^*}{\text{minimize}} \quad \big(1 - F_1(x^*)\big)\, \pi_1 + F_2(x^*)\, \pi_2$$

LDA: Optimization for Boundary of Classes

$$\underset{x^*}{\text{minimize}} \quad \mathbb{P}(\text{error}) \implies \underset{x^*}{\text{minimize}} \quad \big(1 - F_1(x^*)\big)\,\pi_1 + F_2(x^*)\,\pi_2$$

$$\frac{\partial\,\mathbb{P}(\text{error})}{\partial x^*} = -f_1(x^*)\,\pi_1 + f_2(x^*)\,\pi_2 \overset{\text{set}}{=} 0$$

$$\implies \boxed{f_1(x^*)\,\pi_1 = f_2(x^*)\,\pi_2} \tag{23}$$

$$f(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^d|\Sigma|}} \exp\Big(-\frac{(\boldsymbol{x}-\boldsymbol{\mu})^\top\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}{2}\Big) \tag{24}$$

$$\frac{1}{\sqrt{(2\pi)^d|\Sigma_1|}} \exp\Big(-\frac{(\boldsymbol{x}-\boldsymbol{\mu}_1)^\top\Sigma_1^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_1)}{2}\Big)\,\pi_1$$

$$= \frac{1}{\sqrt{(2\pi)^d|\Sigma_2|}} \exp\Big(-\frac{(\boldsymbol{x}-\boldsymbol{\mu}_2)^\top\Sigma_2^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_2)}{2}\Big)\,\pi_2,$$

Lecture Outline

LDA: Binary Classes

$$\Sigma_1 = \Sigma_2 = \Sigma \tag{25}$$

$$\frac{1}{\sqrt{(2\pi)^d |\Sigma|}} \exp\left(-\frac{(\boldsymbol{x}-\boldsymbol{\mu}_1)^\top \Sigma^{-1} (\boldsymbol{x}-\boldsymbol{\mu}_1)}{2}\right) \pi_1$$

$$= \frac{1}{\sqrt{(2\pi)^d |\Sigma|}} \exp\left(-\frac{(\boldsymbol{x}-\boldsymbol{\mu}_2)^\top \Sigma^{-1} (\boldsymbol{x}-\boldsymbol{\mu}_2)}{2}\right) \pi_2, \tag{26}$$

$$\implies \exp\left(-\frac{(\boldsymbol{x}-\boldsymbol{\mu}_1)^\top \Sigma^{-1} (\boldsymbol{x}-\boldsymbol{\mu}_1)}{2}\right) \pi_1$$

$$= \exp\left(-\frac{(\boldsymbol{x}-\boldsymbol{\mu}_2)^\top \Sigma^{-1} (\boldsymbol{x}-\boldsymbol{\mu}_2)}{2}\right) \pi_2 \tag{27}$$

LDA: Binary Classes

$$\exp\Big(-\frac{(\boldsymbol{x}-\boldsymbol{\mu}_1)^\top\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_1)}{2}\Big)\,\pi_1$$

$$=\exp\Big(-\frac{(\boldsymbol{x}-\boldsymbol{\mu}_2)^\top\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_2)}{2}\Big)\,\pi_2, \tag{28}$$

$$\implies -\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_1)^\top\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_1)+\ln(\pi_1) \tag{29}$$

$$=-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_2)^\top\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_2)+\ln(\pi_2), \tag{30}$$

$$(\boldsymbol{x}-\boldsymbol{\mu}_1)^\top\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_1)=(\boldsymbol{x}^\top-\boldsymbol{\mu}_1^\top)\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_1)$$

$$=\boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{x}-\boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{\mu}_1-\boldsymbol{\mu}_1^\top\Sigma^{-1}\boldsymbol{x}+\boldsymbol{\mu}_1^\top\Sigma^{-1}\boldsymbol{\mu}_1$$

$$=\boldsymbol{x}^\top\Sigma^{-1}\boldsymbol{x}+\boldsymbol{\mu}_1^\top\Sigma^{-1}\boldsymbol{\mu}_1-2\,\boldsymbol{\mu}_1^\top\Sigma^{-1}\boldsymbol{x}, \tag{31}$$

LDA: Binary Classes

$$
\begin{aligned}
(\boldsymbol{x}-\boldsymbol{\mu}_1)^\top \Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_1) &= \boldsymbol{x}^\top \Sigma^{-1}\boldsymbol{x} + \boldsymbol{\mu}_1^\top \Sigma^{-1}\boldsymbol{\mu}_1 - 2\,\boldsymbol{\mu}_1^\top \Sigma^{-1}\boldsymbol{x}, \\
\implies -\frac{1}{2}\boldsymbol{x}^\top \Sigma^{-1}\boldsymbol{x} - \frac{1}{2}\boldsymbol{\mu}_1^\top \Sigma^{-1}\boldsymbol{\mu}_1 &+ \boldsymbol{\mu}_1^\top \Sigma^{-1}\boldsymbol{x} + \ln(\pi_1) \\
= -\frac{1}{2}\boldsymbol{x}^\top \Sigma^{-1}\boldsymbol{x} - \frac{1}{2}\boldsymbol{\mu}_2^\top \Sigma^{-1}\boldsymbol{\mu}_2 &+ \boldsymbol{\mu}_2^\top \Sigma^{-1}\boldsymbol{x} + \ln(\pi_2)
\end{aligned}
\tag{32}
$$

$\implies$

$$
\boxed{
\begin{aligned}
&2\left(\Sigma^{-1}(\boldsymbol{\mu}_2-\boldsymbol{\mu}_1)\right)^\top \boldsymbol{x} \\
&\qquad + \left(\boldsymbol{\mu}_1+\boldsymbol{\mu}_2\right)^\top \Sigma^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)\big) + 2\,\ln\left(\frac{\pi_2}{\pi_1}\right) = 0
\end{aligned}
}
\tag{33}
$$

LDA: Binary Classes

$$\delta(\boldsymbol{x}) := 2\left(\Sigma^{-1}(\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)\right)^\top \boldsymbol{x} + \left(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\right)^\top \Sigma^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)\big) + 2\ln(\frac{\pi_2}{\pi_1}) \tag{34}$$

$$\widehat{\mathcal{C}}(x) = \begin{cases} 1, & \text{if } \delta(\boldsymbol{x}) < 0, \\ 2, & \text{if } \delta(\boldsymbol{x}) > 0. \end{cases} \tag{35}$$

LDA: Multiple Classes

$$f_k(\boldsymbol{x})\,\pi_k = \frac{1}{\sqrt{(2\pi)^d|\Sigma_k|}}\exp\Big(-\frac{(\boldsymbol{x}-\boldsymbol{\mu}_k)^\top\Sigma_k^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_k)}{2}\Big)\pi_k. \tag{36}$$

$$\begin{aligned}\ln(f_k(\boldsymbol{x})\,\pi_k) = &-\frac{d}{2}\ln(2\pi) - \frac{1}{2}\ln(|\Sigma_k|)\\ &-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_k)^\top\Sigma_k^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_k) + \ln(\pi_k)\end{aligned} \tag{37}$$

$$\delta_k(\boldsymbol{x}) := -\frac{1}{2}\ln(|\Sigma_k|) - \frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_k)^\top\Sigma_k^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_k) + \ln(\pi_k) \tag{38}$$

$$\widehat{\mathcal{C}}(\boldsymbol{x}) = \arg\max_k\ \delta_k(\boldsymbol{x}) \tag{39}$$

Lecture Outline

Relation of FDA and LDA

$$d^2_{\boldsymbol{A}}(\boldsymbol{x}, \boldsymbol{\mu}_k) := ||\boldsymbol{x} - \boldsymbol{\mu}_k||^2_{\boldsymbol{A}} = (\boldsymbol{x} - \boldsymbol{\mu}_k)^\top \boldsymbol{A} (\boldsymbol{x} - \boldsymbol{\mu}_k) \tag{40}$$

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{U}^\top \succeq 0 \tag{41}$$

$$\begin{aligned} ||\boldsymbol{x} - \boldsymbol{\mu}_k||^2_{\boldsymbol{A}} &= (\boldsymbol{x} - \boldsymbol{\mu}_k)^\top \boldsymbol{U}\boldsymbol{U}^\top (\boldsymbol{x} - \boldsymbol{\mu}_k) \\ &= (\boldsymbol{U}^\top \boldsymbol{x} - \boldsymbol{U}^\top \boldsymbol{\mu}_k)^\top (\boldsymbol{U}^\top \boldsymbol{x} - \boldsymbol{U}^\top \boldsymbol{\mu}_k) \end{aligned} \tag{42}$$

$$\boldsymbol{x} \mapsto \boldsymbol{u}^\top \boldsymbol{x}, \quad \boldsymbol{\mu} \mapsto \boldsymbol{u}^\top \boldsymbol{\mu}, \quad \Sigma \mapsto \boldsymbol{u}^\top \Sigma\, \boldsymbol{u} \tag{43}$$

$$f := \frac{\sigma_b^2}{\sigma_w^2} = \frac{(\boldsymbol{u}^\top \boldsymbol{\mu}_2 - \boldsymbol{u}^\top \boldsymbol{\mu}_1)^2}{\boldsymbol{u}^\top \Sigma_2\, \boldsymbol{u} + \boldsymbol{u}^\top \Sigma_1\, \boldsymbol{u}} = \frac{\big(\boldsymbol{u}^\top (\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)\big)^2}{\boldsymbol{u}^\top (\Sigma_2 + \Sigma_1)\, \boldsymbol{u}}. \tag{44}$$

Relation of FDA and LDA

$$\underset{\boldsymbol{u}}{\text{maximize}} \quad \frac{\big(\boldsymbol{u}^\top(\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)\big)^2}{\boldsymbol{u}^\top(\Sigma_2 + \Sigma_1)\,\boldsymbol{u}}, \tag{45}$$

$$\begin{aligned} &\underset{\boldsymbol{u}}{\text{maximize}} && \big(\boldsymbol{u}^\top(\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)\big)^2, \\ &\text{subject to} && \boldsymbol{u}^\top(\Sigma_2 + \Sigma_1)\,\boldsymbol{u} = 1, \end{aligned} \tag{46}$$

$$\mathcal{L} = \big(\boldsymbol{u}^\top(\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)\big)^2 - \lambda\big(\boldsymbol{u}^\top(\Sigma_2 + \Sigma_1)\,\boldsymbol{u} - 1\big) \tag{47}$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{u}} = 2\,(\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)(\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)^\top \boldsymbol{u} - 2\,\lambda\,(\Sigma_2 + \Sigma_1)\,\boldsymbol{u} \overset{\text{set}}{=} 0$$

$$\implies (\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)(\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)^\top\,\boldsymbol{u} = \lambda\,(\Sigma_2 + \Sigma_1)\,\boldsymbol{u}, \tag{48}$$

$$\boldsymbol{u} \propto (\Sigma_2 + \Sigma_1)^{-1}(\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)(\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)^\top \tag{49}$$

Relation of FDA and LDA

$$\boldsymbol{u} \propto (\Sigma_2 + \Sigma_1)^{-1}(\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)(\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)^\top \tag{50}$$

$$\text{In LDA: } \Sigma_1 = \Sigma_2 \tag{51}$$

$$\boldsymbol{u} \propto (2\,\Sigma)^{-1}(\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)(\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)^\top \propto \Sigma^{-1}(\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)(\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)^\top \tag{52}$$

$$\boldsymbol{u}^\top \boldsymbol{x} \propto \big(\Sigma^{-1}(\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)(\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)^\top\big)^\top \boldsymbol{x} \tag{53}$$

Compare with Eq.(34) $\implies$ Upto a scaling factor:

$$\boxed{\text{LDA} \equiv \text{FDA}} \tag{54}$$

Note that FDA is subspace learning but LDA is classifier (but can be seen as metric learning).

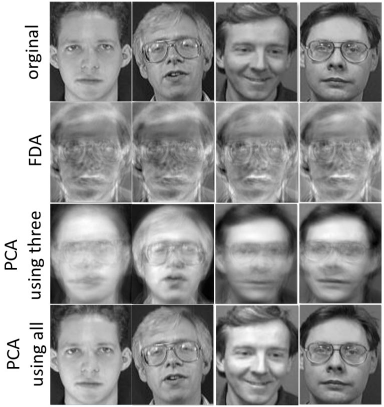
Lecture Outline

Fisherfaces

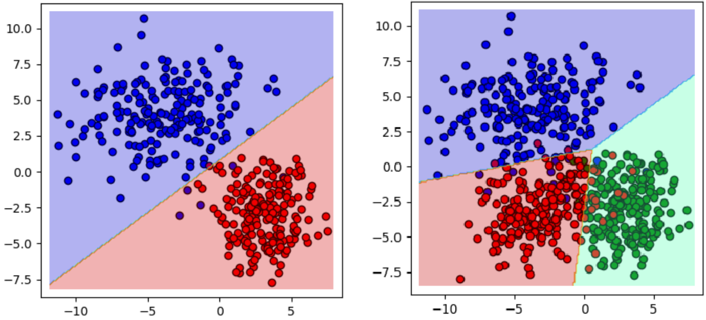
Fisherfaces vs. Eigenfaces on ORL face dataset:

Reconstruction: FDA vs. PCA

Reconstructions by FDA and PCA:

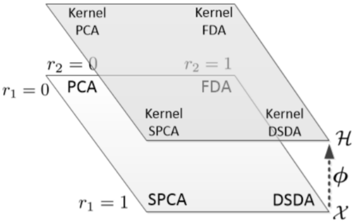


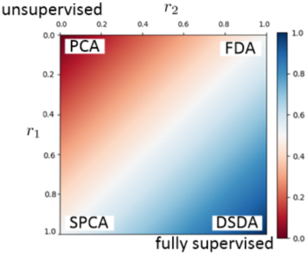
LDA Examples



Additional Note: Roweis Discriminant Analysis

- A **generalized** subspace learning method based on **generalized eigenvalue problem**.
- The **Roweis map** generalizes PCA, supervised PCA, and FDA:
- Paper: [2, 3]





Useful Resources To Read

- Tutorial paper: "Linear and Quadratic Discriminant Analysis: Tutorial" [4]
- Tutorial paper: "Fisher and Kernel Fisher Discriminant Analysis: Tutorial" [5]
- Tutorial paper: "Eigenvalue and generalized eigenvalue problems: Tutorial" [6]
- Tutorial YouTube videos by Prof. Ali Ghodsi at University of Waterloo: [Click here] and [Click here]

References

[1] R. A. Fisher, "The use of multiple measurements in taxonomic problems," *Annals of eugenics*, vol. 7, no. 2, pp. 179–188, 1936.

[2] B. Ghojogh, F. Karray, and M. Crowley, "Roweis discriminant analysis: A generalized subspace learning method," *arXiv preprint arXiv:1910.05437*, 2019.

[3] B. Ghojogh, F. Karray, and M. Crowley, "Generalized subspace learning by Roweis discriminant analysis," in *International Conference on Image Analysis and Recognition*, pp. 328–342, Springer, 2020.

[4] B. Ghojogh and M. Crowley, "Linear and quadratic discriminant analysis: Tutorial," *arXiv preprint arXiv:1906.02590*, 2019.

[5] B. Ghojogh, F. Karray, and M. Crowley, "Fisher and kernel Fisher discriminant analysis: Tutorial," *arXiv preprint arXiv:1906.09436*, 2019.

[6] B. Ghojogh, F. Karray, and M. Crowley, "Eigenvalue and generalized eigenvalue problems: Tutorial," *arXiv preprint arXiv:1903.11240*, 2019.