CS 453/698: Software and Systems Security

Module: Bug Finding Tools and Practices
Lecture: Abstract interpretation

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Outline

1. Introduction to abstraction interpretation
2. Reaching fixedpoint: joining, widening, and narrowing
3. Comparison with other techniques

Why this topic?

A significant portion of software security research is related to program analysis:

- derive properties which hold for program P (i.e., inference)
- prove that some property holds for program P (i.e., verification)
- given a program P, generate a program P' which is
  - in most ways equivalent to P
  - behaves better than P w.r.t some criteria

  (i.e., transformation)

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  - in most ways equivalent to P
  - behaves better than P w.r.t some criteria

  (i.e., transformation)

Abstract interpretation provides a formal framework for developing program analysis tools.

What is abstract interpretation?

Consider detecting that one branch will not be taken in:

int $x, y, z; \quad y := read(file); \quad x := y * y;$

if $x \ \geq \ 0$ **then** $z := 1$ **else** $z := 0$

What is abstract interpretation?

Consider detecting that one branch will not be taken in:

$\textbf{int}\ x, y, z; \quad y := read(file); \quad x := y * y;$

if $x \ \geq \ 0$ **then** $z := 1$ **else** $z := 0$

- Exhaustive analysis in the standard domain: non-termination
- Human reasoning about programs – uses abstractions: signs, order of magnitude, odd/even, ...

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Consider detecting that one branch will not be taken in:
int $x, y, z; \quad y := read(file); \quad x := y * y;$
if $x \ \geq \ 0$ **then** $z := 1$ **else** $z := 0$

- Exhaustive analysis in the standard domain: non-termination
- Human reasoning about programs – uses abstractions: signs, order of magnitude, odd/even, ...

Basic idea: use approximate (generally finite) representations of computational objects to make the problem of program dataflow analysis tractable.

What is abstract interpretation?

Abstract interpretation is a formalization of the above procedure:

- define a non-standard semantics which can approximate the meaning (or behaviour) of the program in a finite way
- expressions are computed over an approximate (abstract) domain rather than the concrete domain (i.e., meaning of operators has to be reconsidered w.r.t. this new domain)

Example: integer sign arithmetic

Consider the domain $D = Z$ (integers)
and the multiplication operator: $* : Z^2 \to Z$

We define an "abstract domain:" $D_\alpha = \{[-], [+]\}$
and abstract multiplication: $*_\alpha : D_\alpha^2 \to D_\alpha$ defined by:

$*_\alpha$	$[-]$	$[+]$
$[-]$	$[+]$	$[-]$
$[+]$	$[-]$	$[+]$

Example: integer sign arithmetic

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$*_\alpha$	$[-]$	$[+]$
$[-]$	$[+]$	$[-]$
$[+]$	$[-]$	$[+]$

This allows us to conclude, for example, that $y = x^2 = x * x$ is never negative.

Some observations

- The basis is that whenever we have $z = x * y$ then:
  if $x, y \in Z$ are approximated by $x_\alpha, y_\alpha \in D_\alpha$
  then $z \in Z$ is approximated by $z_\alpha = x_\alpha *_\alpha y_\alpha$
  - Essentially, we map from an unbounded domain to a finite domain.
- It is important to formalize this notion of approximation, in order to be able to reason/prove that the analysis is correct.
- Approximate computation is generally less precise but faster (hence the tradeoff).

Example: integer sign arithmetic (refined)

Again, $D = Z$ (integers)
and: $* : Z^2 \to Z$

We can define a more refined "abstract domain"
$D'_\alpha = \{[-], [0], [+]\}$

and the corresponding abstract multiplication: $*_\alpha : {D'}_\alpha^2 \to D'_\alpha$

$*_\alpha$	$[-]$	$[0]$	$[+]$
$[-]$	$[+]$	$[0]$	$[-]$
$[0]$	$[0]$	$[0]$	$[0]$
$[+]$	$[-]$	$[0]$	$[+]$

Example: integer sign arithmetic (refined)

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We can define a more refined "abstract domain"
$D'_\alpha = \{[-], [0], [+]\}$

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$*_\alpha$	$[-]$	$[0]$	$[+]$
$[-]$	$[+]$	$[0]$	$[-]$
$[0]$	$[0]$	$[0]$	$[0]$
$[+]$	$[-]$	$[0]$	$[+]$

This allows us to conclude, for example, that $z = y * (0 * x)$ is zero.

More observations

- There is a degree of freedom in defining different abstract operators and domains.
- The minimal requirement is that they be “safe” or “correct”.
- Different “safe” definitions result in different kinds of analysis.

Example: integer sign arithmetic (with addition)

Again, $D = Z$ (integers)
and now we want to define the *addition* operator $+ : Z^2 \to Z$

Example: integer sign arithmetic (with addition)

Again, $D = Z$ (integers)
and now we want to define the *addition* operator $+ : Z^2 \rightarrow Z$

We cannot use $D'_\alpha = \{[-], [0], [+]\}$ because we wouldn't know how to represent the result of $[+] +_\alpha [-]$, (i.e., the abstract addition would not be closed).

Example: integer sign arithmetic (with addition)

Again, $D = Z$ (integers)
and now we want to define the *addition* operator $+ : Z^2 \rightarrow Z$

We cannot use $D'_\alpha = \{[-], [0], [+]\}$ because we wouldn't know how to represent the result of $[+] +_\alpha [-]$, (i.e., the abstract addition would not be closed).

Solution: introduce a new element "$\top$" in the abstract domain as an approximation of any integer.

Example: integer sign arithmetic (with addition)

New "abstract domain": $D'_\alpha = \{[-], [0], [+], \top\}$

Abstract $+_\alpha : D'^2_\alpha \to D'_\alpha$

$+_\alpha$	$[-]$	$[0]$	$[+]$	$\top$
$[-]$	$[-]$	$[-]$	$\top$	$\top$
$[0]$	$[-]$	$[0]$	$[+]$	$\top$
$[+]$	$\top$	$[+]$	$[+]$	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$

Abstract $*_\alpha : D'^2_\alpha \to D'_\alpha$

$*_\alpha$	$[-]$	$[0]$	$[+]$	$\top$
$[-]$	$[+]$	$[0]$	$[-]$	$\top$
$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$[+]$	$[-]$	$[0]$	$[+]$	$\top$
$\top$	$\top$	$[0]$	$\top$	$\top$

Example: integer sign arithmetic (with addition)

New "abstract domain": $D'_\alpha = \{[-], [0], [+], \top\}$

Abstract $+_\alpha : D'^2_\alpha \to D'_\alpha$

$+_\alpha$	$[-]$	$[0]$	$[+]$	$\top$
$[-]$	$[-]$	$[-]$	$\top$	$\top$
$[0]$	$[-]$	$[0]$	$[+]$	$\top$
$[+]$	$\top$	$[+]$	$[+]$	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$

Abstract $*_\alpha : D'^2_\alpha \to D'_\alpha$

$*_\alpha$	$[-]$	$[0]$	$[+]$	$\top$
$[-]$	$[+]$	$[0]$	$[-]$	$\top$
$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$[+]$	$[-]$	$[0]$	$[+]$	$\top$
$\top$	$\top$	$[0]$	$\top$	$\top$

We can now reason that $z = x^2 + y^2$ is never negative

More observations

- In addition to the imprecision due to the coarseness of D_α, the abstract versions of the operations (dependent on D_α) may introduce further imprecision
- Thus, the choice of *abstract domain* and the definition of the *abstract operators* are crucial.

Concerns in abstract interpretation

- Required:
  - Correctness – safe approximations: the analysis should be "conservative" and errs on the "safe side"
  - Termination – compilation should definitely terminate

  (note: not always the case in everyday program analysis tools!)

- Desirable – "practicality":
  - Efficiency – in practice finite analysis time is not enough: finite *and* small is the requirement.
  - Accuracy – too many false alarms is harmful to the adoption of the analysis tool ("the boy who cried wolf").
  - Usefulness – determines which information is worth collecting.

Outline

Abstract domain example: intervals

Consider the following abstract domain for $x \in Z$ (integers):
$x = [a, b]$ where

- a can be either a constant or $-\infty$ and
- b can be either a constant or ∞.

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- b can be either a constant or ∞.

Example:

$\{x^{\#} = [0, 3],\ y^{\#} = [0, 2]\}$

```
z = 2 * x + 4 * y
```

$\{z^{\#} = 2 \times^{\#} [0, 3] +^{\#} 4 \times^{\#} [0, 2] = [0, 14]\}$

Abstract domain example: intervals

Consider the following abstract domain for $x \in Z$ (integers):
$x = [a, b]$ where

- a can be either a constant or $-\infty$ and
- b can be either a constant or ∞.

Example:

$\{x^{\#} = [0, 3],\ y^{\#} = [0, 2]\}$

```
z = 2 * x + 4 * y
```

$\{z^{\#} = 2 \times^{\#} [0, 3] +^{\#} 4 \times^{\#} [0, 2] = [0, 14]\}$

Q: Why $z^{\#}$ is an abstraction of z?

Join operator

The join operator $\sqcup$ merges two or more abstract states into one abstract state.

Joining operator example

$\{x^{\#} = [0, 10]\}$

```
if (x < 0) then
  s := -1
else if (x > 0) then
  s := 1
else
  s := 0
```

Joining operator example

$\{x^{\#} = [0, 10]\}$

```
if (x < 0) then
```

$\{x^{\#} = \emptyset\}$

```
  s := -1
```

$\{x^{\#} = \emptyset,\ s^{\#} = \emptyset\}$

```
else if (x > 0) then

  s := 1

else

  s := 0
```

Joining operator example

$\{x^{\#} = [0, 10]\}$

```
if (x < 0) then
```

$\{x^{\#} = \emptyset\}$

```
s := -1
```

$\{x^{\#} = \emptyset,\ s^{\#} = \emptyset\}$

```
else if (x > 0) then
```

$\{x^{\#} = [1, 10]\}$

```
s := 1
```

$\{x^{\#} = [1, 10],\ s^{\#} = [1, 1]\}$

```
else

    s := 0
```

Joining operator example

$\{x^{\#} = [0, 10]\}$

```
if (x < 0) then
```
$\{x^{\#} = \emptyset\}$
```
  s := -1
```
$\{x^{\#} = \emptyset,\ s^{\#} = \emptyset\}$
```
else if (x > 0) then
```
$\{x^{\#} = [1, 10]\}$
```
  s := 1
```
$\{x^{\#} = [1, 10],\ s^{\#} = [1, 1]\}$
```
else
```
$\{x^{\#} = [0, 0]\}$
```
  s := 0
```
$\{x^{\#} = [0, 0],\ s^{\#} = [0, 0]\}$

Joining operator example

$\{x^\# = [0, 10]\}$

```
if (x < 0) then
```
$\{x^\# = \emptyset\}$
```
    s := -1
```
$\{x^\# = \emptyset,\ s^\# = \emptyset\}$
```
else if (x > 0) then
```
$\{x^\# = [1, 10]\}$
```
    s := 1
```
$\{x^\# = [1, 10],\ s^\# = [1, 1]\}$
```
else
```
$\{x^\# = [0, 0]\}$
```
    s := 0
```
$\{x^\# = [0, 0],\ s^\# = [0, 0]\}$

$\{x^\# = \emptyset \sqcup [1, 10] \sqcup [0, 0] = [0, 10],\ s^\# = \emptyset \sqcup [1, 1] \sqcup [0, 0] = [0, 1]\}$

What about loops?

$\{x^{\#} = \emptyset\}$

```
x := 0

while (x < 100) {

  x := x + 2

}
```

What about loops?

$\{x^{\#} = \emptyset\}$

```
x := 0
```

$\{x^{\#} = \langle even \rangle\}$

```
while (x < 100) {

  x := x + 2

}
```

What about loops?

$\{x^{\#} = \emptyset\}$

```
x := 0
```

$\{x^{\#} = \langle even \rangle\}$

```
while (x < 100) {
```

$\{x^{\#} = \langle even \rangle\}_1$

```
  x := x + 2
```

$\{x^{\#} = \langle even \rangle\}_1$

```
}
```

What about loops?

$\{x^{\#} = \emptyset\}$

```
x := 0
```
$\{x^{\#} = \langle even \rangle\}$
```
while (x < 100) {
```
$\quad \{x^{\#} = \langle even \rangle\}_1 \qquad \{x^{\#} = \langle even \rangle \sqcup \langle even \rangle = \langle even \rangle\}_2$
```
  x := x + 2
```
$\quad \{x^{\#} = \langle even \rangle\}_1$
```
}
```

What about loops?

$\{x^{\#} = \emptyset\}$

```
x := 0
```
$\{x^{\#} = \langle even \rangle\}$
```
while (x < 100) {
```
$\quad\{x^{\#} = \langle even \rangle\}_1 \qquad \{x^{\#} = \langle even \rangle \sqcup \langle even \rangle = \langle even \rangle\}_2$
```
  x := x + 2
```
$\quad\{x^{\#} = \langle even \rangle\}_1$
```
}
```
$\{x^{\#} = \langle even \rangle\}$

Two iterations to reach fixedpoint (i.e., none of the abstract states changes).

Collecting semantics

$\{x^{\#} = \emptyset\}$

```
x := 0

while (x < 100) {

   x := x + 2

}
```

Collecting semantics

$\{x^\# = \emptyset\}$

```
x := 0
```

$\{x^\# = [0,0]\}$

```
while (x < 100) {

   x := x + 2

}
```

Collecting semantics

$\{x^{\#} = \emptyset\}$

```
x := 0
```
$\{x^{\#} = [0, 0]\}$
```
while (x < 100) {
```
$\quad \{x^{\#} = [0, 0]\}_1$
```
    x := x + 2
```
$\quad \{x^{\#} = [2, 2]\}_1$
```
}
```

Collecting semantics

$\{x^{\#} = \emptyset\}$

```
x := 0
```

$\{x^{\#} = [0,0]\}$

```
while (x < 100) {
```

$\{x^{\#} = [0,0]\}_1$ $\quad$ $\{x^{\#} = [0,0] \sqcup [2,2] = [0,2]\}_2$

```
    x := x + 2
```

$\{x^{\#} = [2,2]\}_1$ $\quad$ $\{x^{\#} = [2,2] \sqcup [2,4] = [2,4]\}_2$

```
}
```

Collecting semantics

$\{x^{\#} = \emptyset\}$

```
x := 0
```
$\{x^{\#} = [0,0]\}$
```
while (x < 100) {
```
$\{x^{\#} = [0,0]\}_1$ $\quad\{x^{\#} = [0,2] \sqcup [2,4] = [0,4]\}_3$
```
  x := x + 2
```
$\{x^{\#} = [2,2]\}_1$ $\quad\{x^{\#} = [2,4] \sqcup [2,6] = [2,6]\}_3$
```
}
```

Collecting semantics

$\{x^{\#} = \emptyset\}$

```
x := 0
```
$\{x^{\#} = [0,0]\}$
```
while (x < 100) {
```
$\quad \{x^{\#} = [0,0]\}_1 \qquad \{\cdots\}_4, \{\cdots\}_5, \cdots$
```
    x := x + 2
```
$\quad \{x^{\#} = [2,2]\}_1 \qquad \{\cdots\}_4, \{\cdots\}_5, \cdots$
```
}
```

Collecting semantics

$\{x^\# = \emptyset\}$

```
x := 0
```

$\{x^\# = [0,0]\}$

```
while (x < 100) {
```

$\{x^\# = [0,0]\}_1$ $\{x^\# = [0,96] \sqcup [2,98] = [0,98]\}_{50}$

```
    x := x + 2
```

$\{x^\# = [2,2]\}_1$ $\{x^\# = [2,98] \sqcup [2,100] = [2,100]\}_{50}$

```
}
```

Collecting semantics

$\{x^\# = \emptyset\}$

```
x := 0
```
$\{x^\# = [0,0]\}$
```
while (x < 100) {
```
$\{x^\# = [0,0]\}_1$ $\{x^\# = [0,96] \sqcup [2,98] = [0,98]\}_{50}$
```
  x := x + 2
```
$\{x^\# = [2,2]\}_1$ $\{x^\# = [2,98] \sqcup [2,100] = [2,100]\}_{50}$
```
}
```
$\{x^\# = [100,100]\}$

50 iterations to reach fixedpoint (i.e., none of the abstract states changes).

Collecting semantics

$\{x^{\#} = \emptyset\}$

```
x := 0
```
$\{x^{\#} = [0, 0]\}$
```
while (x < 100) {
```
$\{x^{\#} = [0, 0]\}_1$ $\{x^{\#} = [0, 96] \sqcup [2, 98] = [0, 98]\}_{50}$
```
  x := x + 2
```
$\{x^{\#} = [2, 2]\}_1$ $\{x^{\#} = [2, 98] \sqcup [2, 100] = [2, 100]\}_{50}$
```
}
```
$\{x^{\#} = [100, 100]\}$

50 iterations to reach fixedpoint (i.e., none of the abstract states changes).

Q: can we reach the fixedpoint faster?

Widening operator

We compute the limit of the following sequence:

$$X_0 = \perp$$

$$X_{i+1} = X_i \nabla F^{\#}(X_i)$$

where ∇ denotes the widening operator.

Widening operator example

$\{x^\# = \emptyset\}$

```
x := 0

while (x < 100) {

   x := x + 2

}
```

Widening operator example

$\{x^\# = \emptyset\}$

```
x := 0
```

$\{x^\# = [0, 0]\}$

```
while (x < 100) {

   x := x + 2

}
```

Widening operator example

$\{x^\# = \emptyset\}$

```
x := 0
```
$\{x^\# = [0, 0]\}$
```
while (x < 100) {
```
$\quad\{x^\# = [0, 0]\}_1$
```
   x := x + 2
```
$\quad\{x^\# = [2, 2]\}_1$
```
}
```

Widening operator example

$\{x^\# = \emptyset\}$

```
x := 0
```
$\{x^\# = [0,0]\}$
```
while (x < 100) {
```
$\{x^\# = [0,0]\}_1$ $\quad$ $\{x^\# = [0,0] \triangledown [2,2] = [0,+\infty]\}_2$
```
    x := x + 2
```
$\{x^\# = [2,2]\}_1$ $\quad$ $\{x^\# = [2,+\infty]\}_2$
```
}
```

Widening operator example

$\{x^\# = \emptyset\}$

```
x := 0
```

$\{x^\# = [0,0]\}$

```
while (x < 100) {
```

$\{x^\# = [0,0]\}_1$ $\{x^\# = [0,+\infty]\nabla[2,+\infty] = [0,+\infty]\}_3$

```
    x := x + 2
```

$\{x^\# = [2,2]\}_1$ $\{x^\# = [2,+\infty]\}_3$

```
}
```

Widening operator example

$\{x^{\#} = \emptyset\}$

```
x := 0
```
$\{x^{\#} = [0, 0]\}$
```
while (x < 100) {
```
$\{x^{\#} = [0, 0]\}_1$ $\quad$ $\{x^{\#} = [0, +\infty] \nabla [2, +\infty] = [0, +\infty]\}_3$
```
  x := x + 2
```
$\{x^{\#} = [2, 2]\}_1$ $\quad$ $\{x^{\#} = [2, +\infty]\}_3$
```
}
```
$\{x^{\#} = [100, +\infty]\}$

3 iterations to reach fixedpoint (i.e., none of the abstract states changes).

Narrowing operator

We compute the limit of the following sequence:

$$X_0 = \perp$$
$$X_{i+1} = X_i \triangle F^{\#}(X_i)$$

where $\triangle$ denotes the narrowing operator.

Narrowing operator example

$\{x^{\#} = \emptyset\}$

```
x := 0
```

$\{x^{\#} = [0, 0]\}$

```
while (x < 100) {
```

$\{x^{\#} = [0, +\infty]\}$

```
    x := x + 2
```

$\{x^{\#} = [2, +\infty]\}$

```
}
```

$\{x^{\#} = [100, 101]\}$

Narrowing operator example

$\{x^\# = \emptyset\}$

```
x := 0
```
$\{x^\# = [0, 0]\}$
```
while (x < 100) {
```
$\{x^\# = [0, +\infty]\}$ $\quad$ $\{x^\# = [0, +\infty] \triangle [0, 99] = [0, 99]\}_1$
```
  x := x + 2
```
$\{x^\# = [2, +\infty]\}$ $\quad$ $\{x^\# = [2, 101]\}_1$
```
}
```
$\{x^\# = [100, 101]\}$

Narrowing operator example

$\{x^\# = \emptyset\}$

```
x := 0
```

$\{x^\# = [0, 0]\}$

```
while (x < 100) {
```

$\{x^\# = [0, +\infty]\}$ $\{x^\# = [2, 101] \triangle [0, 99] = [0, 99]\}_2$

```
  x := x + 2
```

$\{x^\# = [2, +\infty]\}$ $\{x^\# = [2, 101]\}_2$

```
}
```

$\{x^\# = [100, 101]\}$

Narrowing operator example

$\{x^\# = \emptyset\}$

```
x := 0
```
$\{x^\# = [0, 0]\}$
```
while (x < 100) {
```
$\{x^\# = [0, +\infty]\}$ $\{x^\# = [2, 101] \triangle [0, 99] = [0, 99]\}_2$
```
  x := x + 2
```
$\{x^\# = [2, +\infty]\}$ $\{x^\# = [2, 101]\}_2$
```
}
```
$\{x^\# = [100, 101]\}$

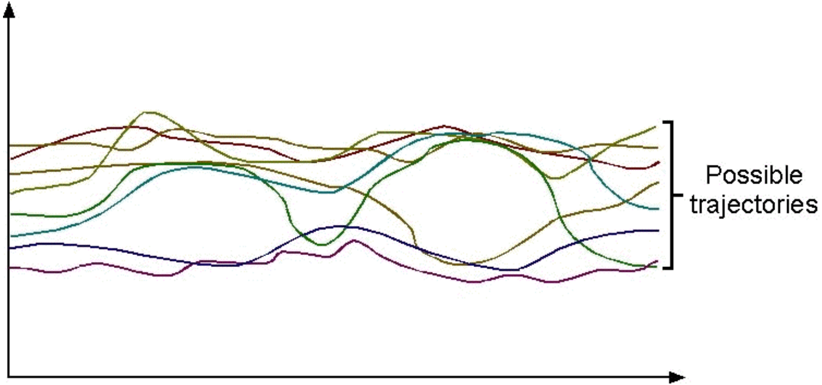
2 iterations to reach fixedpoint (i.e., none of the abstract states changes).

Outline

Abstract interpretation in a nutshell

Acknowledgement: the illustrations in this section is borrowed from Prof. Patrick Cousot's webpage Abstract Interpretation in a Nutshell.

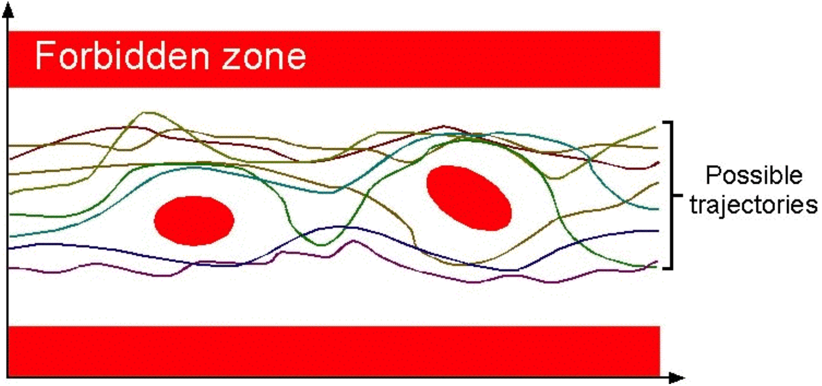
Program analysis: concrete semantics



The concrete semantics of a program is formalized by the set of all possible executions of this program under all possible inputs.

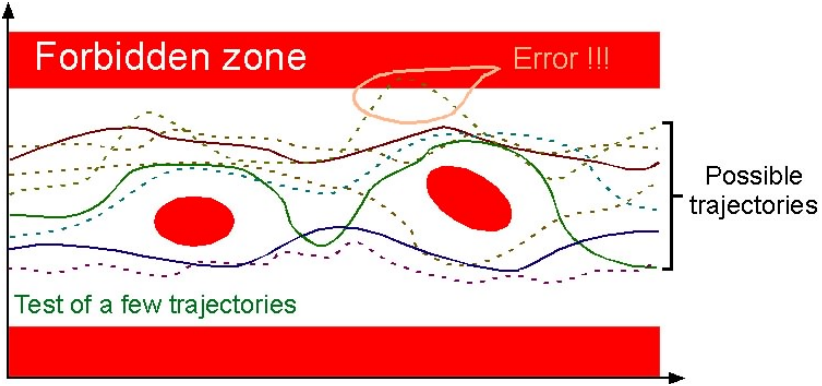
The concrete semantics of a program can be a *close to infinite* mathematical object / sequence which is impractical to enumerate.

Program analysis: safety properties



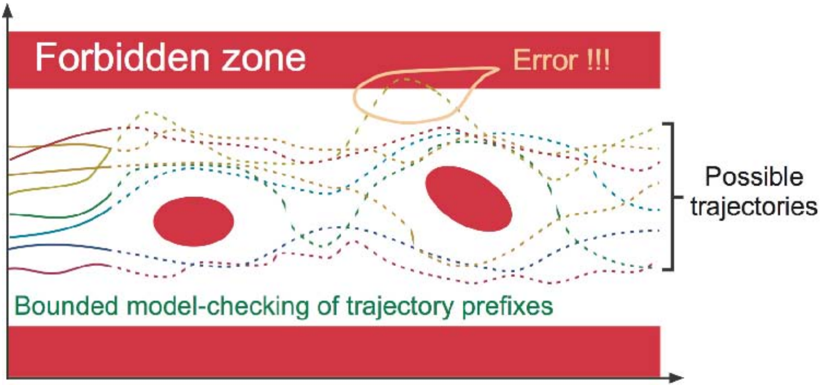
Safety properties of a program express that no possible execution of the program, when considering all possible execution environments, can reach an erroneous state.

Program analysis: testing



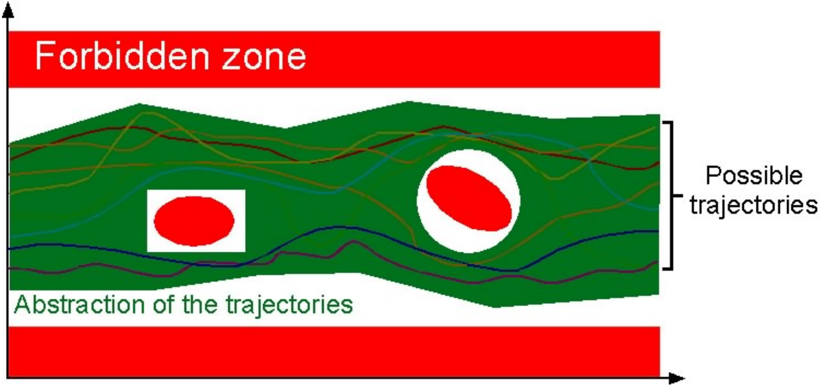
Testing consists in considering a subset of the possible executions.

Program analysis: bounded model checking



Bounded model checking consists in exploring the prefixes of the possible executions.

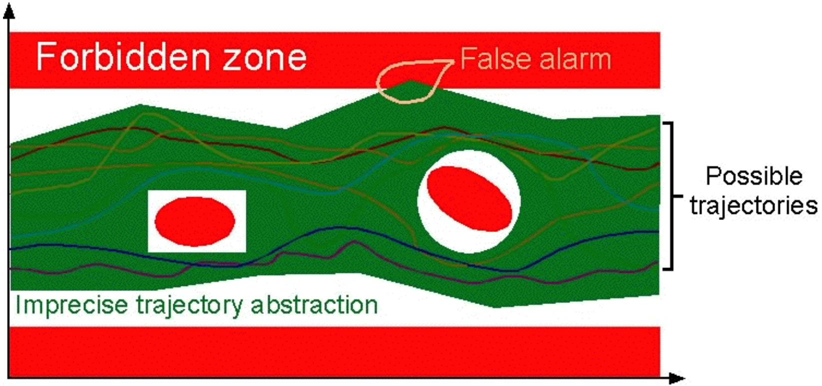
Program analysis: abstract interpretation



Abstract interpretation consists in considering an abstract semantics, that is a superset of the concrete program semantics.

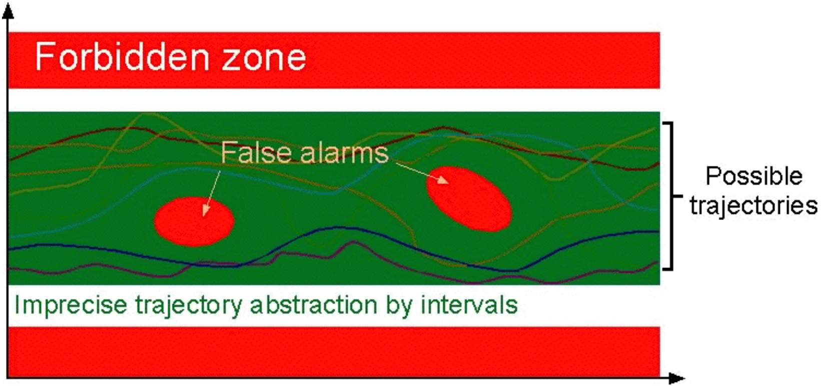
The abstract semantics covers all possible cases
$\Longrightarrow$ if the abstract semantics is safe (i.e. does not intersect the forbidden zone) then so is the concrete semantics.

Program analysis: abstract interpretation false alarm 1



False alarms caused by widening during execution.

Program analysis: abstract interpretation false alarm 2



False alarms caused by abstract domains.

$\langle$ **End** $\rangle$