CS486/686: Introduction to Artificial Intelligence
Lecture 3a - States and Uninformed Search

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Readings: Poole & Mackworth Chap. 3.1-3.5

Searching

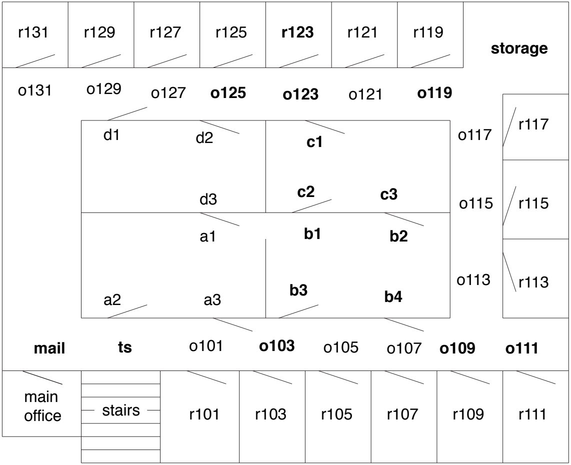
- Often we are not given an algorithm to solve a problem, but only a specification of **what is a solution**—we have to **search** for a solution
- We can do so by exploring a **directed graph** that represents the **state space** of our problem
- Sometimes the graph is literal—nodes may represent actual locations in space, and edges represent the distance between them
- Sometimes the graph is implicit—nodes represent **states**, and edges represent legal **state transitions**

Directed Graphs

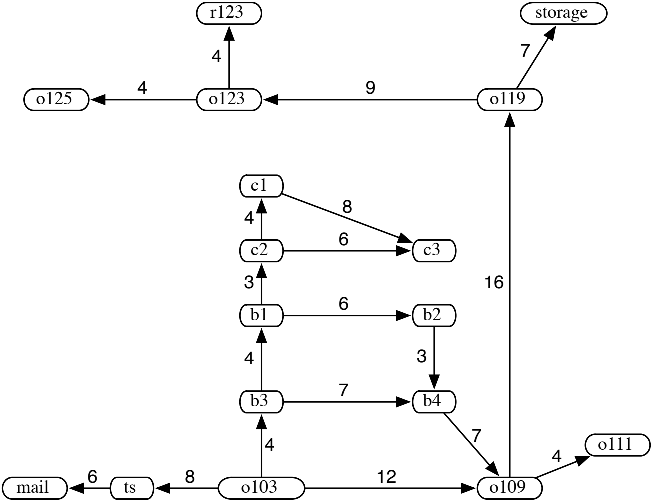
- A **graph** consists of a set N of **nodes** and a set A of ordered pairs of nodes, called **arcs** (or **edges**)
- Node n_2 is a **neighbor** of n_1 if there is an arc from n_1 to n_2 (i.e., $\langle n_1, n_2 \rangle \in A$)
- A **path** is a sequence of nodes $\langle n_0, n_1, \ldots, n_k \rangle$ such that $\langle n_{i-1}, n_i \rangle \in A$
- Often there is a **cost** associated with arcs and the cost of a path is the sum of the costs of the arcs in the path

Example Problem for Delivery Robot

The robot wants to get from outside room 103 to the inside of room 123.



Graph for the Delivery Robot



cost = distance travelled

A Search Problem

Definition (Search Problem)

A **search problem** is defined by:

- A set of **states**
- An **initial state**
- **Goal states** or a **goal test**
  - a boolean function which tells whether a given state is a goal state
- A **successor (neighbour) function**
  - an action which takes us from one state to other states
- (Optionally) a **cost** associated with each action

A solution to this problem is a path from the start state to a goal state (optionally with the smallest total cost)

Example: 8-Puzzle

Initial State

5	3	
8	7	6
2	4	1

Goal State

1	2	3
4	5	6
7	8	

Formulating 8-Puzzle as a Search Problem

- State: $x_{00}x_{01}x_{02}, x_{10}x_{11}x_{12}, x_{20}x_{21}x_{22}$
  x_{ij} is the number in row i and column j, $i, j \in \{0, 1, 2\}$
  $x_{ij} \in \{0, \ldots, 8\}$. $x_{ij} = 0$ denotes the empty square

- Initial state: $530, 876, 241$

- Goal states: $123, 456, 780$

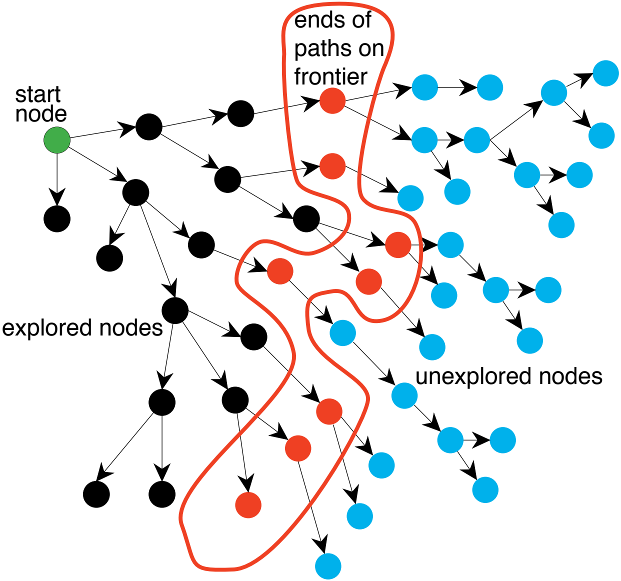
- Successor function: Consider the empty square as a tile. State B is a successor of state A if and only if we can convert A to B by moving the empty tile up, down, left, or right by one step

- Cost function: Each move has a cost of 1

Graph Searching

- Generic search algorithm: given a graph, start nodes, and goal nodes, **incrementally explore paths** from the start nodes
- Maintain a **frontier** of paths from the start node that have been explored
- As search proceeds, the frontier **expands** into the unexplored nodes until a goal node is encountered
- The way in which the frontier is expanded defines the **search strategy**

Problem Solving by Graph Searching



Graph Search Algorithm

Input: A graph, a set of start nodes, Boolean procedure `goal(n)` that tests if n is a goal node

frontier $\leftarrow \{\langle s \rangle : s$ is a start node$\}$
while *frontier* is not empty **do**
    select and **remove** path $\langle n_0, \ldots, n_k \rangle$ from *frontier*
    if $goal(n_k)$ **then**
        return $\langle n_0, \ldots, n_k \rangle$
    for each neighbor n of n_k **do**
        add $\langle n_0, \ldots, n_k, n \rangle$ to *frontier*

Graph Search Algorithm

- We assume that after the search algorithm returns an answer, it can be asked for more answers and the **procedure continues**
- The **neighbors** define the graph structure
- Which value is **selected** from the frontier (and how the new values are **added** to the frontier) at each stage defines the search strategy
- *Goal* defines what is a **solution**

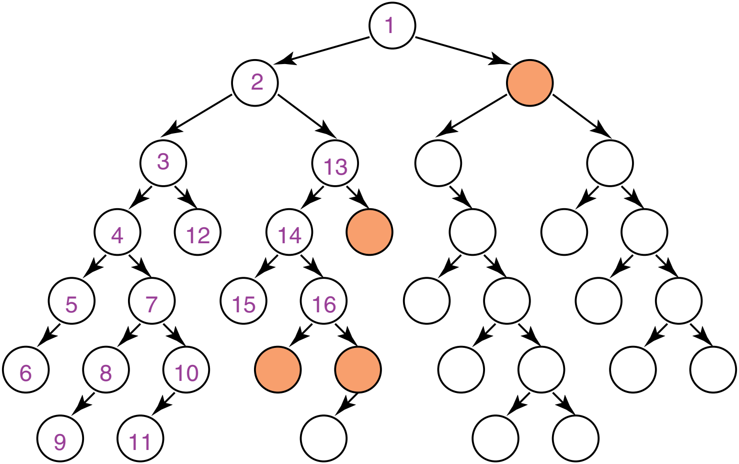
Types of Search

- **Uninformed (blind)**
- Heuristic
- More sophisticated “hacks”

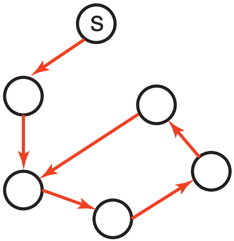
Depth-First Search

- **Depth-first search** treats the frontier as a **stack**
- It always selects the **last element** added to the frontier
- If the list of paths on the frontier is $[p_1, p_2, \ldots]$
  - p_1 is selected, then paths that extend p_1 are added to the front of the stack (in front of p_2)
  - p_2 is only selected when all paths from p_1 have been explored

Illustrative Graph: Depth-First Search

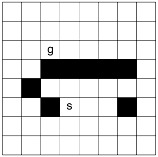


Depth-First Search: Cycle Checking



- A searcher can **prune** a path that ends in a node **already on the path**
- We will assume that this check can be done via hashing in constant time

Graph Search Algorithm with Cycle Check



- Use **depth-first search** to get from **s** to **g**
- Number the nodes as they are removed
- Add neighbors CW from top U,R,D,L
- Use cycle check

Input: A graph, a set of start nodes, Boolean procedure `goal(n)` that tests if n is a goal node

$frontier \leftarrow \{\langle s \rangle : s \text{ is a start node}\}$

while *frontier* is not empty **do**

  select and **remove** path $\langle n_0, \ldots, n_k \rangle$ from *frontier*

  if $goal(n_k)$ **then**

    return $\langle n_0, \ldots, n_k \rangle$

  for each neighbor n of n_k **do**

    if $n \notin \langle n_0, \ldots, n_k \rangle$ **then**

      add $\langle n_0, \ldots, n_k, n \rangle$ to *frontier*

Recursive Implementation of DFS

An equivalent recursive implementation of DFS:

Input: A graph, a set of start nodes, Boolean procedure `goal(n)` that tests if n is a goal node

function $\mathrm{DFS}(\langle n_0, \ldots, n_k \rangle)$
    if $goal(n_k)$ **then**
        return $\langle n_0, \ldots, n_k \rangle$
    for each neighbor n of n_k **do**
        if $n \notin \langle n_0, \ldots, n_k \rangle$ **then**
            $\mathrm{DFS}(\langle n_0, \ldots, n_k, n \rangle)$
end function

for each start node s **do**
    $\mathrm{DFS}(\langle s \rangle)$

The frontier is "implicitly" stored in the call stack

Properties of DFS

Properties

- *Space Complexity:* size of frontier in worst case
- *Time Complexity:* # nodes visited in worst case
- *Completeness:* does it find a solution when one exists?
- *Optimality:* if solution found, is it the one with the least cost?

Useful Quantities

- b is the branching factor
- m is the maximum depth of the search tree
- d is the depth of the shallowest goal node

Properties of DFS: Space Complexity

$\mathcal{O}(bm)$

- b is the branching factor (max number of children of any node)
  m is the max depth of the search tree
- Linear in m
- Remembers m nodes on current path and at most b siblings for each node

Properties of DFS: Time Complexity

$\mathcal{O}(b^m)$

- b is the branching factor (max number of children of any node)
  m is the max depth of the search tree
- Exponential in m
- Visit the entire search tree in the worst case

Properties of DFS: Completeness

Is DFS guaranteed to find a solution if a solution exists?

- No
- Will get stuck in an infinite path
- An infinite path may or may not be a cycle

Properties of DFS: Optimality

Is DFS guaranteed to return an optimal solution if it terminates?

- No
- It pays no attention to the costs and makes no guarantee on the solution's quality

When should we use DFS?

DFS is useful when:

- Space is restricted
- Many solutions exist, perhaps with long paths

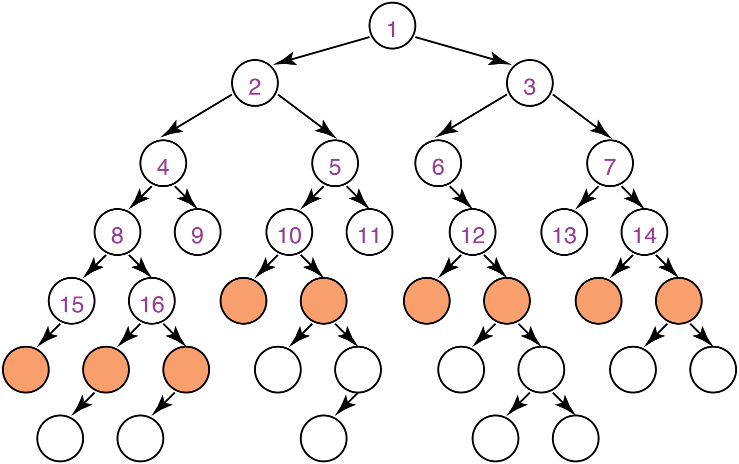
DFS is a poor method when:

- There are infinite paths
- (Optimal) solutions are shallow
- There are multiple paths to a node

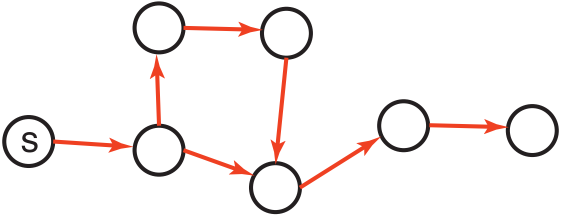
Breadth-First Search

- **Breadth-first search** treats the frontier as a **queue**
- It always selects the **earliest** element added to the frontier
- If the list of paths on the frontier is $[p_1, p_2, \ldots, p_r]$:
  - p_1 is selected, then its neighbors are added to the end of the queue, after p_r
  - p_2 is selected next

Illustrative Graph: Breadth-First Search

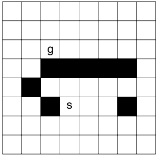


Multiple-Path Pruning



- Multiple path pruning: **prune** a path to node *n* that **if any previously-found path terminates in *n***
- Multiple-path pruning **subsumes a cycle check** (because the current path is a path to the node)
- This entails **storing all nodes** it has found paths to
- Want to **guarantee that an optimal solution** can still be found

Graph Search Algorithm with Multiple Path Pruning



- Use **breadth first search** to get from **s** to **g**
- Number the nodes as they are removed
- Add neighbours CW from top U,R,D,L
- Use multiple path pruning

Input: A graph, a set of start nodes, Boolean procedure `goal(n)` that tests if n is a goal node

$\quad$ *frontier* $\leftarrow \{\langle s \rangle : s$ is a start node$\}$
$\quad$ *explored* $\leftarrow \emptyset$
$\quad$ **while** *frontier* is not empty **do**
$\quad\quad$ **select** and **remove** path $\langle n_0, \ldots, n_k \rangle$ from *frontier*
$\quad\quad$ **if** $n_k \notin$ *explored* **then**
$\quad\quad\quad$ *explored* $\leftarrow$ *explored* $\cup \{n_k\}$
$\quad\quad\quad$ **if** $goal(n_k)$ **then**
$\quad\quad\quad\quad$ **return** $\langle n_0, \ldots, n_k \rangle$
$\quad\quad\quad$ **for each** neighbor n of n_k **do**
$\quad\quad\quad\quad$ **add** $\langle n_0, \ldots, n_k, n \rangle$ to *frontier*

Properties of BFS: Space Complexity

$\mathcal{O}(b^d)$

- b is the branching factor
  d is the depth of the shallowest goal node
- Exponential in d
- Must visit the top d levels Size of frontier is dominated by the size of level d

Properties of BFS: Time Complexity

$\mathcal{O}(b^d)$

- Exponential in d
- Visit the entire search tree in the worst case

Properties of BFS: Completeness

Is BFS guaranteed to find a solution if a solution exists?

- Yes
- Explores the tree level by level until it finds a goal

Properties of BFS: Optimality

Is BFS guaranteed to return an optimal solution if it terminates?

- No
- Guaranteed to find the shallowest goal node

When should we use BFS?

BFS is useful when:

- Space is not a concern
- We would like a solution with the fewest arcs

BFS is a poor method when:

- All the solutions are deep in the tree
- The problem is large and the graph is dynamically generated

Combining the Best of BFS and DFS

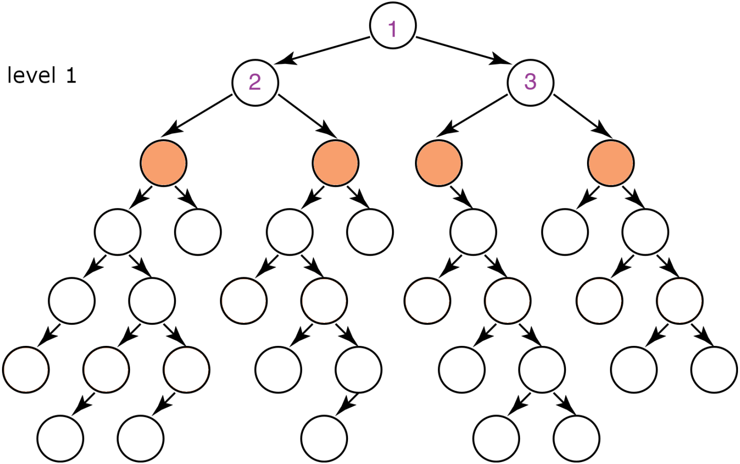
Can we create a search algorithm that combines the best of BFS and DFS?

BFS	**DFS**
$\mathcal{O}(b^d)$ exponential space	$\mathcal{O}(bm)$ linear space
Guaranteed to find a solution if one exists	May get stuck on infinite paths

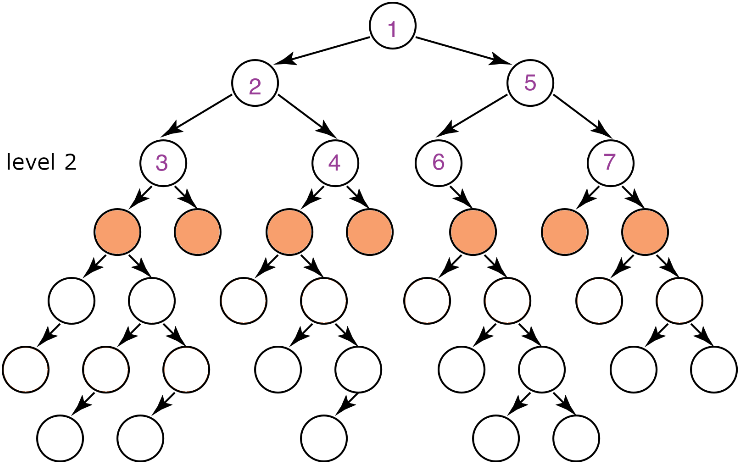
Iterative-Deepening Search:
For every depth limit, perform depth-first search until the depth limit is reached

Illustrative Graph: Iterative Deepening

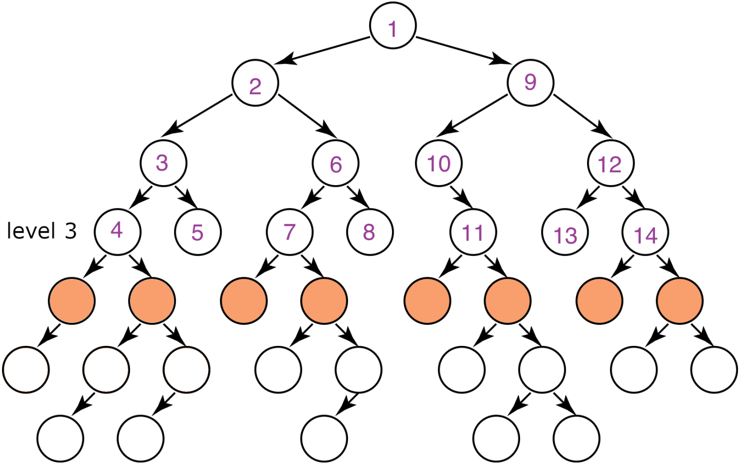
level 1



Illustrative Graph: Iterative Deepening



Illustrative Graph: Iterative Deepening



Properties of IDS: Space Complexity

$\mathcal{O}(bd)$

- b is the branching factor d is the depth of the shallowest goal node
- Linear in d Similar to DFS
- Executes DFS for each depth limit Guaranteed to terminate at depth d

Properties of IDS: Time Complexity

$\mathcal{O}(b^d)$

- b is the branching factor d is the depth of the shallowest goal node
- Exponential in d Similar to BFS

Properties of IDS - Time Complexity

Complexity with solution at depth d & branching factor b:

	# times each node is expanded		
level	breadth-first	iterative deepening	# nodes
1	1	d	b
2	1	$d-1$	b^2
...	...	...	...
$d-1$	1	2	b^{d-1}
d	1	1	b^d
	$\geq b^d$	$\leq b^d\left(\frac{b}{b-1}\right)^2$	

$$
\begin{aligned}
b^d + 2b^{d-1} + 3b^{d-2} + \ldots &= b^d \sum_{n=1}^{d} n \left(\frac{1}{b}\right)^{n-1} && \text{rewrite} && (1)\\
&< b^d \sum_{n=1}^{\infty} n \left(\frac{1}{b}\right)^{n-1} && \text{extend to infinity} && (2)\\
&= b^d \left(\frac{b}{1-b}\right)^2 && \text{derivative of the geometric series} && (3)
\end{aligned}
$$

Properties of IDS - Completeness

Is IDS guaranteed to find a solution if a solution exists?

- Yes
  Same as BFS
- Explores the tree level by level until it finds a goal

Properties of IDS - Optimality

Is IDS guaranteed to return an optimal solution if it terminates?

- No
- Guaranteed to find the shallowest goal node
  Same as BFS

A Summary of IDS Properties

- Space Complexity: $\mathcal{O}(bd)$, linear in d
  Similar to DFS
- Time Complexity: $\mathcal{O}(b^d)$, exponential in d
  Same as BFS
- Completeness: Yes
  Same as BFS
- Optimality: No, but guaranteed to find the shallowest goal node
  Same as BFS

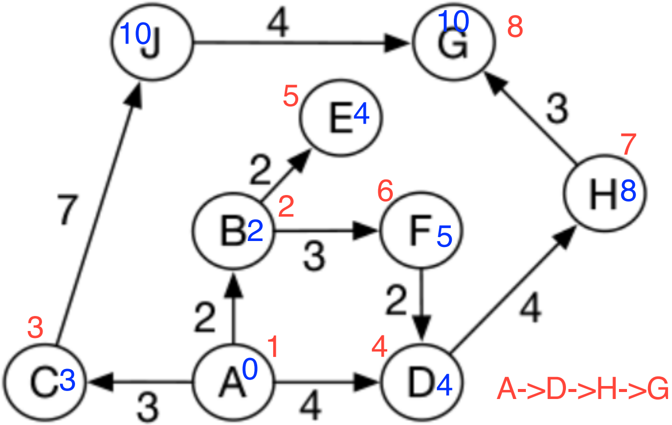
Lowest-Cost-First Search

- Sometimes there are **costs** associated with arcs
  The cost of a path is the **sum of the costs** of its arcs

$$cost(\langle n_0, \ldots, n_k \rangle) = \sum_{i=1}^{k} |\langle n_{i-1}, n_i \rangle|$$

- At each stage, **lowest-cost-first search** selects a path on the frontier with lowest cost
- The frontier is a **priority queue** ordered by path cost
- It finds a **least-cost path** to a goal node
- When arc costs are **equal** $\Rightarrow$ breadth-first search
- **Uninformed/blind** search (in that it does not take the goal into account)

Trace LCFS



How does LCFS find a path from A to G?

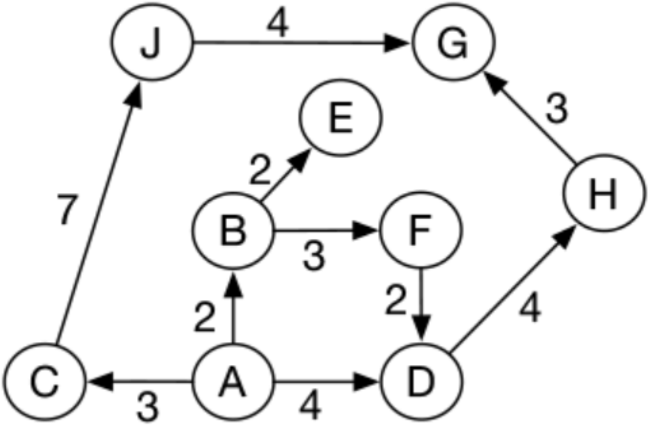
Properties of LCFS

- Space and Time Complexities
  Both complexities are exponential
  LCFS examines a lot of paths to ensure that it returns the optimal solution first
- Completeness and Optimality
  Yes and yes under mild conditions:
  (1) The branching factor is finite
  (2) The cost of every edge is **strictly positive**

Dijkstra's Algorithm

- Dijkstra's algorithm is a variant of LCFS with a kind of multiple-path pruning
- Like LCFS, the frontier is stored in a priority queue, sorted by cost
- For every node in the graph, we keep track of the lowest cost to reach it so far
- If we find a lower cost path to a node, we update that value, which may require re-sorting the priority queue
- Dijkstra's algorithm is an example of **dynamic programming** because it trades space (we must store a value for every node) for time (we may find the shortest path faster)

Trace Dijkstra's Algorithm



How does Dijkstra's algorithm find a path from A to G?

Next

- Informed/Heuristic Search (Poole & Mackworth Chap. 3.6-3.8)