First, we need to prove that $Fib(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}$

Base case:

$$Fib(0) = \frac{\phi^0 - \psi^0}{\sqrt{5}} \tag{1}$$

$$= \frac{0}{\sqrt{5}} = 0 \tag{2}$$

$$Fib(1) = \frac{\phi^1 - \psi^1}{\sqrt{5}} \tag{3}$$

$$= \frac{\frac{1}{2}\left(1 + \sqrt{5} - 1 + \sqrt{5}\right)}{\sqrt{5}} = 1 \tag{4}$$

Assuming $n = k$ such that $Fib(0), Fib(1), ..., Fib(k) = \frac{\phi^k - \psi^k}{\sqrt{5}}$ true when $k \geq 1$, we want to show that $n = k + 1$ such that
$Fib(0), Fib(1), ..., Fib(k+1) = \frac{\phi^{k+1} - \psi^{k+1}}{\sqrt{5}}$ is also true when $k \geq 1$.

$$Fib(k+1) = \frac{\phi^{k+1} - \psi^{k+1}}{\sqrt{5}} \tag{5}$$

$$Fib(k+1) = Fib(k) + Fib(k-1) = \frac{\phi^k - \psi^k + \phi^{k-1} - \psi^{k-1}}{\sqrt{5}} \tag{6}$$

$$= \frac{\phi^{k-1}(\phi + 1) - \psi^{k-1}(\psi + 1)}{\sqrt{5}} \tag{7}$$

$$= \frac{\phi^{k-1}(\frac{1 + \sqrt{5} + 2}{2}) - \psi^{k-1}(\frac{1 - \sqrt{5} + 2}{2})}{\sqrt{5}} \tag{8}$$

$$= \frac{\phi^{k-1}(\frac{6 + 2\sqrt{5}}{4}) - \psi^{k-1}(\frac{6 - 2\sqrt{5}}{4})}{\sqrt{5}} \tag{9}$$

$$= \frac{\phi^{k-1}(\phi^2) - \psi^{k-1}(\psi^2)}{\sqrt{5}} \tag{10}$$

$$= \frac{\phi^{k+1} - \psi^{k+1}}{\sqrt{5}} \tag{11}$$

So the statement is proven true.

Finally, we are going to prove that $Fib(n)$ is the closest integer to $\frac{\phi^n}{\sqrt{5}}$.

Notice that since $|\psi| < 1$ and $\sqrt{5} > 2$, one has $\left|\frac{\psi^n}{\sqrt{5}}\right| < \frac{1}{2}$

Thus the integer closest to $Fib(n) + \frac{\psi^n}{\sqrt{5}} = \frac{\phi^n}{\sqrt{5}}$ is $Fib(n)$.