ANSWER SCRIPT COVER PAGE

NED University of Engineering & Technology

Online Fall Semester Examinations - 2021

Seat No. CT- 20032 Batch 2020

Course Title Differential & Integral Calculus. Course Code MT-171

Enrol No. Provisionally Allowed. Date 04-02-2021.

Roll No: CT-032

PLEASE READ THESE INSTRUCTIONS CAREFULLY

1) Download and print this cover page (separately for each exam).
2) Fill the above mentioned particulars before attempting the questions.
3) Students are not allowed to use red or green ink. Solve the questions on A4 size paper using blue or black pen ONLY.

Question No.	Award	
	First Examiner/ Internal	Second/ External Examiner/ ERC
1.		
2.		
3.		
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7.		
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12.		
Total in figures		
Total in words		

First / Internal Examiner's Signature

Second / External Examiner's / ERC Signature

Q1 (a)

Find the Taylor series for $\ln x$ about $x = 1$.

Solution:

We have been given a function,

$$f(x) = \ln x$$

Now, we have to find the Taylor series,

First we find the derivatives,

at $x = 1$

$f(x) = \ln x$; $f(1) = 0$

$f'(x) = \frac{1}{x}$; $f'(1) = 1$

$f''(x) = -\frac{1}{x^2}$; $f''(1) = -1$

$f'''(x) = \frac{2}{x^3}$; $f'''(1) = 2$.

$\vdots$ ..... $\longrightarrow f^n(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$; $f^n(1) = (-1)^{n+1}(n-1)!$

A/c to the formula,

$$\because P_n(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \dots\dots + \frac{f^{(n)}(a)(x-a)^n}{n!}.$$

$$P_n(x) = 0 + 1(x-1) + \frac{(-1)(x-1)^2}{2} + \cdots + \frac{(-1)^{n+1}(n-1)!(x-1)^n}{n!}$$

$$\boxed{P_n(x) = (x-1) - \frac{(x-1)^2}{2} + \cdots + \frac{(-1)^{n+1}(n-1)!(x-1)^n}{n!}}$$

Answer.

Q1 (b):

Find the curvature and radius of curvature of $y = 4\cos x + \sin x$ at $x = \frac{\pi}{2}$.

Solution:

The given function is:

$$y = 4\cos x + \sin x.$$

First we find the first & second derivative,

$$\boxed{y' = \cos x - 4\sin x}$$

and

$$\boxed{y'' = -\sin x - 4\cos x}$$

According to the formula,

$$\because K = \frac{|y''(x)|}{[1 + (y'(x))^2]^{3/2}}$$

Putting values,

$$K = \frac{|-(\sin x + 4\cos x)|}{[1 + (\cos x - 4\sin x)^2]^{3/2}}$$

$$K = \frac{\sin x + 4\cos x}{\left[1 + (\cos x - 4\sin x)^2\right]^{3/2}}$$

Now, putting $x = \frac{\pi}{2}$.

$$K = \frac{\sin\frac{\pi}{2} + 4\cos\frac{\pi}{2}}{\left(1 + \left(\cos\frac{\pi}{2} - 4\sin\frac{\pi}{2}\right)^2\right)^{3/2}}$$

$$K = \frac{1 + 4(0)}{\left[1 + \{0 - 4(1)\}^2\right]^{3/2}}$$

$$K = \frac{1}{\left[1 + (-4)^2\right]^{3/2}}$$

$$K = \frac{1}{17^{3/2}}$$

$$\boxed{K = 0.014 \text{ units}}$$

Now, we find ρ

$$\because \rho = \frac{1}{K}$$

So,

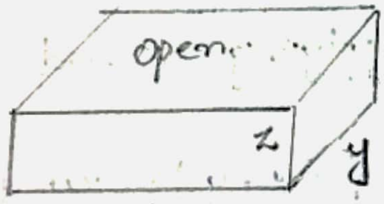
$$\rho = \frac{1}{0.014}$$

$$\boxed{\rho = 71.42 \text{ units}}$$

Q2 (a):

Determine the dimensions of a rectangular box, open at the top, having the Volume V and requiring the least amount of material for its construction.

Solution:



let,

x = length of the box (in feet)

y = width of the box (in feet).

z = height of the box (in feet).

S = Total surface area of the box (in sq. feet).

We may reasonably assume that the box with least surface area requires the least amount of material, so our objective is to minimize the surface area.

$$S = xy + 2xz + 2yz \quad \text{—— eq (i)}$$

According to the figure, subject to the volume requirement

$$xyz = V \quad \text{—— eq (ii)}$$

From eq(ii), we obtain $z = \frac{V}{xy}$, so substitute the value of z in eq (i)

$eq(i) \Rightarrow S = xy + 2xz + 2yz$

$$S = xy + \frac{2xV}{xy} + \frac{2yV}{xy}$$

$$S = xy + \frac{2V}{y} + \frac{2V}{x} \quad \text{—— eq (iii)}$$

Applying second partial derivative test,

$$f(x,y) = xy + \frac{2V}{y} + \frac{2V}{x}.$$

$$\Rightarrow f_x(x,y) = y - \frac{2V}{x^2}$$

$$\Rightarrow f_y(x,y) = x - \frac{2V}{y^2}$$

Now, the coordinates of critical points of S satisfy

$$y - \frac{2V}{x^2} = 0 \quad \text{and} \quad x - \frac{2V}{y^2} = 0.$$

Solving the first eq. for y yields,

$$y = \frac{2V}{x^2}$$

Substituting this expression in the second eq. yields

$$x = \frac{2V}{\left(\frac{2V}{x^2}\right)^2}$$

$\Rightarrow 1 - \frac{x^3}{2V} = 0$

The solution of the equation ais $x^2 y = \sqrt[3]{2V}$.

By theorem 13.8.6, let f be a function of two variables with continuous second-order partial derivative in some disk centered at a critical point (x_0, y_0) and let,

$$D = f_{xx}(x_0, y_0) \times f_{yy}(x_0, y_0) - f^2_{xy}(x_0, y_0)$$

$f_{xx}(x, y) = \frac{4V}{x^3}$, $f_{xx}(\sqrt[3]{2V}, \sqrt[3]{2V}) = 2$

$f_{xy}(x, y) = 1$, $f_{xy}(\sqrt[3]{2V}, \sqrt[3]{2V}) = 1$

$f_{yy}(x, y) = \frac{4V}{y^3}$, $f_{yy}(\sqrt[3]{2V}, \sqrt[3]{2V}) = 2$.

$$D = (2)(2) - 1$$

$$D = 3 > 0$$

Since $D > 0$ and $f_{xx}(\sqrt[3]{2V}, \sqrt[3]{2V}) > 0$, so f has a relative minimum at $(\sqrt[3]{2V}, \sqrt[3]{2V})$

Now, $\frac{V}{(\sqrt[3]{2V})(\sqrt[3]{2V})} = \sqrt[3]{2V/2}$.

The result is:

The length and width are

$$\boxed{x = y = \sqrt[3]{2V}}$$

and height is $\boxed{\sqrt[3]{2V}/2.}$ Ans

Q2 (b):

let $g(x) = \frac{5}{x} - \frac{2x}{x+4}$. Find those values of x, if any, where g is not continuous.

Solution:

The given function is:

$$g(x) = \frac{5}{x} - \frac{2x}{x+4}.$$

$$g(x) = \frac{5(x+4) - 2x(x)}{x(x+4)}$$

$$g(x) = \frac{5x + 20 - 2x^2}{x^2 + 4x}$$

For a continuous function, denominator $\neq 0$.

So,

$$x^2 + 4x \neq 0$$

$$x(x+4) \neq 0.$$

$$\boxed{x \neq 0} \text{ and } \boxed{x \neq -4}$$

Ans: The function $g(x)$ is not continuous at $\boxed{x=0}$ and $\boxed{x=-4.}$

Q3 (a):

Use reduction formula to evaluate $\int \sec^6 x \, dx$.

Solution:

$$\int \sec^6 x \, dx$$

Using reduction formula,

$$\because \int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx.$$

$$\int \sec^6 x \, dx = \frac{\sec^{6-2} x \tan x}{6-1} + \frac{6-2}{6-1} \int \sec^{6-2} x \, dx.$$

$$\int \sec^6 x \, dx = \frac{\sec^4 x \tan x}{5} + \frac{4}{5} \int \sec^4 x \, dx.$$

Here again apply this formula,

$$\int \sec^6 x \, dx = \frac{\sec^4 x \tan x}{5} + \frac{4}{5} \left[\frac{\sec^2 x \tan x}{3} + \frac{2}{3} \int \sec^2 x \, dx \right]$$

Here again apply reduction formula,

$$\int \sec^6 x\,dx = \frac{1}{5}\sec^4 x\tan x + \frac{4}{5}\left[\frac{1}{3}\sec^2 x\tan x + \frac{2}{3}\tan x\right]$$

$$\boxed{\int \sec^6 x\,dx = \frac{1}{5}\sec^4 x\tan x + \frac{4}{15}\sec^2 x\tan x + \frac{8}{15}\tan x + C}$$

Ans

Q3(b):

Evaluate $\int_{-1}^{\infty} \frac{x}{x^2+1} \, dx$.

$$\int_{-1}^{\infty} \frac{x}{x^2+1} \, dx$$

$$\cancel{\lim} \int_{-1}^{\infty} \frac{x}{x^2+1} \, dx = \lim_{b \to \infty} \int_{-1}^{b} \frac{x}{x^2+1} \, dx.$$

let,

$$u = x^2 + 1$$

$$du = 2x \, dx$$

$$\frac{du}{2} = x \, dx.$$

$$\int_{-1}^{\infty} \frac{x}{x^2+1} \, dx = \lim_{b \to \infty} \int_{-1}^{b} \frac{du/2}{u}$$

$$= \lim_{b \to \infty} \frac{1}{2} \int_{-1}^{b} \frac{du}{u}$$

$$= \lim_{b \to \infty} \frac{1}{2} \left[\ln u \right]_{-1}^{b}$$

$$= \lim_{b \to \infty} \frac{1}{2} \left(\ln b - \ln(-1) \right)$$

$$\int_{-1}^{\infty} \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \frac{1}{2}\left[\ln b + \ln 1\right]$$

$$= \lim_{b\to\infty} \frac{1}{2}\left[\ln b + 0\right]$$

Applying limit,

$$\int_{-1}^{\infty} \frac{x}{x^2+1} = \frac{1}{2}\left(\ln \infty\right)$$

$$\boxed{\int_{-1}^{\infty} \frac{x}{x^2+1} = \infty}$$

Thus, the given improper integral is divergent.

Q4 (a):

Let $f(x,y) = y^2 + xy + 3y + 2x + 3$. Locate all relative maxima, relative minima and saddle points, if any.

Solution:

The given function is:

$$f(x,y) = y^2 + xy + 3y + 2x + 3.$$

First, we find the derivatives,

$f_x = y + 2$, $f_y = 2y + x + 3$

$f_{xx} = 0$, $f_{yy} = 2$

$f_{xy} = 1$, $f_{yx} = 1$

Now, we find the critical points,

Consider,

$f_x = y + 2$

$y + 2 = 0$

$\boxed{y = -2}$

And,

$f_y = 2y + 3 + x$

$2(-2) + 3 + x = 0$

$-4 + 3 + x = 0$

$\boxed{x = 1}$

So, the critical point is $(x,y) = (1,-2)$.

According to the formula,

$\because D = f_{xx} \times f_{yy} - f_{xy}^2$

$D = (0) \times (2) - (1)^2$

$\boxed{D = -1}$

Since $D < 0$, so f has a suddle point at $(1, -2)$.

There's no relative extrema existed.

Q4 (b):

Find the asymptotes of the function

$$f(x) = \frac{x^2-3x+2}{x^2+3x+2}.$$

Solution:

The given function is:

$$f(x) = \frac{x^2-3x+2}{x^2+3x+2}.$$

$$f(x) = \frac{x^2-3x+2}{x^2+2x+x+2}$$

$$f(x) = \frac{x^2-3x+2}{x(x+2)+1(x+2)}$$

$$f(x) = \frac{x^2-3x+2}{(x+2)(x+1)}$$

<u>Vertical</u> <u>Asymptotes:</u>

$$(x+2)(x+1) = 0$$

$$\boxed{x=-2}, \boxed{x=-1}.$$

Verification:

First we verify for $x = -2$

$$\lim_{x \to -2} \frac{x^2 - 3x + 2}{(x+2)(x+1)}$$

$$= \frac{(-2)^2 - 3(-2) + 2}{(-2+2)(-2+1)}$$

$$= \infty \cdot \quad \text{Verified!}$$

So, $\boxed{x = -2}$ is the first vertical asymptote.

Now, we verify for $x = -1$.

$$\lim_{x \to -1} \frac{x^2 - 3x + 2}{(x+2)(x+1)}$$

Applying limits,

$$= \frac{(-1)^2 - 3(-1) + 2}{(-1+2)(-1+1)}$$

$$= \infty \quad \text{Verified.}$$

So, $\boxed{x = -1}$ is the second vertical asymptote.

⇒ Horizontal Asymptote:

$$\lim_{x \to \infty} \frac{x^2 - 3x + 2}{(x+2)(x+1)}$$

If we directly apply limit, we will get $\frac{\infty}{\infty}$, which is indeterminate form.

So, first we apply L'Hopital rule,

$$\lim_{x \to \infty} \frac{\frac{d}{dx}(x^2 - 3x + 2)}{\frac{d}{dx}(x+2)(x+1)}$$

$$\lim_{x \to \infty} \frac{2x - 3}{2x + 3}$$

Still getting indeterminate form, again applying L'Hopital rule,

$$\lim_{x \to \infty} \frac{\frac{d}{dx}(2x-3)}{\frac{d}{dx}(2x+3)}$$

$$\lim_{x \to \infty} \frac{\cancel{2}^{1}}{\cancel{2}}$$

$$\lim_{x \to \infty} 1$$

$$= 1$$

So, the horizontal asymptote is:

$\boxed{y=1}$.

=> Oblique Asymptote:

Since the degree of numerator is not greater than degree of denominator, so oblique asymptote does not exist.

=> Overall Result:

Vertical Asymptotes:

$\boxed{x=-1}$ and $\boxed{x=-2}$

Horizontal Asymptotes:

$\boxed{y=1.}$

Q5(a):

Evaluate $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)$

Solution:

$$\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right)$$

If we directly apply limits, we will get the indeterminate form.

So first apply L'Hopital rule,

$$\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1} \right) = \lim_{x \to 0} \frac{d}{dx} \left(\frac{e^x - 1 - x}{x(e^x - 1)} \right)$$

$$= \lim_{x \to 0} \left[\frac{d/dx\,(e^x - 1 - x)}{d/dx\,\{x(e^x - 1)\}} \right]$$

$$= \lim_{x \to 0} \frac{e^x - 1}{e^x + xe^x - 1}$$

Again, we are having the same situation, applying L'Hopital rule one more time,

$$= \lim_{x \to 0} \frac{d/dx\,(e^x - 1)}{d/dx\,(e^x + xe^x - 1)}$$

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\lim_{x\to 0}\frac{e^x}{2e^x+xe^x}$$

$$=\lim_{x\to 0}\frac{e^x}{2e^x+xe^x}$$

Applying limits.

$$=\frac{e^0}{2e^0+0(e^0)}$$

$$=\frac{1}{2(1)+0(1)}$$

$$\boxed{\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\frac{1}{2}}$$

Ans

Q5 (b):

Find the n^{th} order derivative of the function $h(x) = x^2 \cos x$.

Solution:

The given function is:

$$h(x) = x^2 \cos x.$$

let,

$u = \cos x$ and $v = x^2$.

A/c to the formula,

$$\because y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right) \rightarrow y = \cos(ax+b).$$

Similarly,

$a = 1$

$$u_n = \cos\left(x + \frac{n\pi}{2}\right)$$

$$v_1 = 2x,\ v_2 = 2,\ v_3 = v_4 = \ldots\ldots v_n = 0.$$

$$\therefore D^n(x^2\cos x) = D^n(u).v + {}^nC_1 D^{n-1}(u).D(v) + {}^nC_2 D^{n-2}(u)D^2(v)$$
$$+ 0 + 0 + \ldots\ldots$$

$$D^n(x^2\cos x) = \cos\left(x + \frac{n\pi}{2}\right)x^2 + n\left[\cos\left\{x + (n-1)\frac{\pi}{2}\right\}\right]2x + \frac{n(n-1)}{\cancel{2}}$$
$$\left[\cos\left\{x + (n-2)\pi/2\right\}\right].\cancel{2} + 0 + 0 + \ldots\ldots$$

$$D^n(x^2\cos x) = x^2\cos\left(x+\frac{n\pi}{2}\right) + 2nx\cos\left\{x+(n-1)\frac{\pi}{2}\right\} + n(n-1)\cos\left\{x+(n-2)\frac{\pi}{2}\right\} + 0 + 0 + \ldots$$

$$\boxed{D^n(x^2\cos x) = x^2\cos x\left(x+\frac{n\pi}{2}\right) + 2nx\cos x\left[x+(n-1)\frac{\pi}{2}\right] + n(n-1)\cos\left[x+(n-2)\frac{\pi}{2}\right]}$$

Ans