Date: ________

Name : Nashra Ghaffar

Roll No : CT - 032

Subject : Differential Equation

Section 1

Date: ____

Q8:

The no. of people adapted innovation = $x(t)$
t = time.

let the no. of people who don't adapt innovation is $y(t)$

Hence, the rate of spread of innovation is assumed to be similar to the rate of spread of disease. i.e

$$\frac{dx}{dt} \propto xy$$

$$\frac{dx}{dt} = Kxy \quad \text{—— eq (i)}$$

where K is the constant of proportionality

As 'n' is the total no. of people in the community,

so 'n' is given by

$$n = x(t) + y(t)$$
$$y = n - x$$

, Substituting in eq (i)

$$\boxed{\frac{dx}{dt} = kx(n-x)}$$

Teacher's Signature: ____

Date:

Initially, at time $t = 0$,

$$\boxed{x(0) = 1}$$

at least one person has adapted innovation, as $x \neq 0$.

Q9:

Initial volume = $S = 300$ gal
Initial amount of salt = $A_0 = 50$ lb
Inflow rate = $r_1 = 3$ gal/min
Outflow rate = Inflow rate.

Differential equation of the rate of change of $A(t)$ is given by:

$$\frac{dA}{dt} = R_{in} - R_{out} \qquad \text{——— eq (1)}$$

where,

R_{in} = Input rate
R_{out} = Output rate

$$R_{in} = \left(\begin{array}{c}\text{Concentration of} \\ \text{inflow solution}\end{array}\right) \times \left(\begin{array}{c}\text{Inflow rate} \\ \text{of solution}\end{array}\right)$$

$$R_{in} = 0 \times 3 = 0 \text{ lb/min}$$

$$R_{out} = \begin{pmatrix} \text{Concentration of} \\ \text{Outflow solution} \end{pmatrix} \times \begin{pmatrix} \text{Outflow rate} \\ \text{of solution} \end{pmatrix}$$

$$R_{out} = \frac{A}{300} \times 3 = \frac{A}{100} \text{ lb/min}$$

The net rate of change of $x(t)$ is given by

$$eq(i) \Rightarrow \quad \frac{dA}{dt} = 0 - \frac{A}{100}$$

$$\frac{dA}{dt} = \frac{-A}{100}$$

-ve sign shows that the concentration is decreasing with time.

Q19:

The force acting on the mass is along vertical direction, then the net force will be given by,

$$F_{net} = F_g - F_r$$

where,

$F_g = mg$ (weight)
$F_r = -kx$ (restoring force)

and -ve sign shows that F_r is upward.

So, $F_{net} = mg - kx$

as mass is moving about the equilibrium position hence the distance is given by

$$x = x + s.$$

$$F_{net} = mg - k(x+s)$$

and A/c to Newton's 2^{nd} law of motion,

$$F = ma$$

$$ma = mg - k(x+s)$$

As a is the double derivative $\frac{d^2x}{dt^2}$ (w.r.t t)

$$m\frac{d^2x}{dt^2} = mg - kx - ks$$

At equilibrium, $mg - ks = 0$

$$\boxed{m\frac{d^2x}{dt^2} = -kx}$$

Hence this is the required differential equation at any time (t)

Section 2

Date:

Q1:

Initial amount of salt = x_0 = 50lb
Initial volume of solution = S_0 = 300gal
Inflow rate = r_1 = 3gal/min
Outflow rate = Inflow rate
Inflow concentration = x_1 = 2lb/gal

Rate of change of concentration,

$$\frac{dx}{dt} = R_1 - R_2$$

where,

R_1 = Input rate
R_2 = Output rate

$$R_1 = x_1 r_1 = 2\,lb/gal \times 3\,gal/min = 6\,lb/min$$

$$R_2 = \frac{x}{S_0} \times r_2 = \frac{x}{300}\,lb/gal \times 3\,gal/min = \frac{x}{100}\,lb/min$$

Hence,

$$\frac{dx}{dt} = 6 - \frac{x}{100}$$

$$\frac{dx}{dt} = \frac{600-x}{100}$$

$$\int \frac{1}{600-x}\,dx = \int \frac{1}{100}\,dt$$

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$$-\ln(600-x) = \frac{t}{100} - \ln c$$

$$\ln(600-x) - \ln c = -\frac{t}{100}$$

$$\ln\left|\frac{600-x}{c}\right| = -\frac{t}{100}$$

$$\frac{600-x}{c} = e^{-t/100}$$

$$600-x = ce^{-t/100}$$

$$x = 600 - ce^{-t/100} \quad \text{--- eq (i)}$$

As initially, $x_0 = 50$, $t = 0$

$$50 = 600 - ce^{0}$$

$$50 = 600 - c$$

$$c = 600 - 50$$

$$\boxed{c = 550}$$

Therefore, eq (i) becomes:

$$\boxed{x = 600 - 550e^{-t/100}}$$

(At time (t))

Date:

After 50 mins,

$$x = 600 - 550e^{-50/100}$$

$$\boxed{x = 266.408 \text{ lb}}$$

After long time,

when $t \to \infty$

$$x = 600 - 550e^{-\infty}$$

$$\boxed{x = 600 \text{ lb}}$$

Q2:

Initial volume of solution = S_0 = 50gal
Initial amount of salt = x_0 = 10 lb
Inflow concentration of solution = x_1 = 2 lb
Inflow rate = r_1 = 5gal/min
Outflow rate = r_2 = 3 gal/min

Rate of change of concentration,

$$\frac{dx}{dt} = R_1 - R_2$$

$R_1 = x_1 r_1 = 2 \times 5 = 10$ lb/min

As $r_1 > r_2$, the volume is increasing at a net rate $(r_1 - r_2) \Rightarrow (5-3)t$

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$$R_2 = \frac{x}{50+(5-3)t} \times 3 = \frac{3x}{50+2t} \text{ lb/min}$$

Hence,

$$\frac{dx}{dt} = 10 - \frac{3x}{50+2t}$$

$$\frac{dx}{dt} + \frac{3x}{50+2t} = 10$$

This is a linear equation, so:

$$P = \frac{3}{50+2t}, \quad Q = 10$$

$$I.F = e^{\int \frac{3}{50+2t} dt} = e^{\ln(50+2t)^{3/2}} = (50+2t)^{3/2}$$

$$x \times I.F = \int Q \times I.F \, dt$$

$$x(50+2t)^{3/2} = \int 10(50+2t)^{3/2} dt$$

$$x(50+2t)^{3/2} = 10 \times \frac{2}{5}(50+2t)^{5/2} + C$$

$$x = \frac{2(50+2t)^{5/2}}{(50+2t)^{3/2}} + \frac{C}{(50+2t)^{3/2}}$$

$$x = 2(50+2t) + \frac{C}{(50+2t)^{3/2}}$$

$$x = 100 + 4t + \frac{C}{(50+2t)^{3/2}}$$

Date:

Initially, $x = 10$ lb , $t = 0$

$$10 = 100 + \frac{c}{(50)^{3/2}}$$

$$10 = \frac{100(50)^{3/2} + c}{(50)^{3/2}}$$

$$c = 10(50)^{3/2} - 100(50)^{3/2} = -90(50)^{3/2}$$

$$c = -90\left[5\sqrt{2}\right]^3 = 22500\sqrt{2}$$

So,

$$\boxed{x = 100 + 4t + \frac{22500\sqrt{2}}{(50+2t)^{3/2}}}$$