Q: Discuss convergence and limit of following sequence.

$$\left\{ \frac{1}{n^2+1} \right\}$$

let $\varepsilon > 0$ and $n \in \mathbb{N}$.

Then,

$$\frac{1}{n} < \varepsilon$$

$$\Rightarrow \quad \frac{1}{n^2+1} < \frac{1}{n} < \varepsilon$$

$$\Rightarrow \quad -\varepsilon < \frac{1}{n^2+1} < \varepsilon$$

So, we can write it as:

$$\left| \frac{1}{n^2+1} \right| < \varepsilon$$

So,

$$\Rightarrow \quad \frac{1}{n^2+1} \longrightarrow 0$$

as $n \rightarrow \infty$

Therefore : $\frac{1}{n^2+1}$ converges to 0.

Limit:

$$\lim_{n \to \infty} \left(\frac{1}{n^2+1} \right)$$

Applying limits.

$$\lim_{n \to \infty} \left(\frac{1}{n^2+1} \right) = \frac{1}{\infty + 1}$$

$$= \frac{1}{\infty}$$

$$\boxed{\lim_{n \to \infty} \left(\frac{1}{n^2+1} \right) = 0}$$

The limit of the given function is 0.

Q: If $\sin(A+iB) = x+iy$, show that:

$$\frac{x^2}{\cosh^2 A} + \frac{y^2}{\sinh^2 A} = 1.$$

Proof:

$$\sin(A+iB) = x+iy.$$

$\because \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\therefore$

$$\sin(A+iB) = \sin A\cos iB + \cos A\sin iB$$

$\because \cos(iz) = \cosh z$

$\because \sin(iz) = i\sinh z.$

$$\sin(A+iB) = \sin A\cosh B + (\cos A\sinh B)i$$

Now,

$x = \sin A\cosh B \Rightarrow \left|\sin A = \dfrac{x}{\cosh B}\right|$

$y = \cos A\sinh B \Rightarrow \left|\cos A = \dfrac{y}{\sinh B}\right|$

As we know that,

$$\because \cos^2 A + \sin^2 A = 1$$

$$\therefore \left(\frac{y}{\sinh B}\right)^2 + \left(\frac{x}{\cosh B}\right)^2 = 1$$

$$\frac{y^2}{\sinh^2 B} + \frac{x^2}{\cosh^2 B} = 1$$

or

$$\boxed{\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1}$$

Proved!

Q: Find the following limit,

$$\lim_{x\to\infty} \frac{e^{1/x} - 4}{1/x}$$

We have been given;

$$\lim_{x\to\infty} \left(\frac{e^{1/x} - 4}{1/x}\right)$$

Applying limits,

$$\lim_{x\to\infty} \left(\frac{e^{1/x} - 4}{1/x}\right) = \lim_{x\to\infty} \left(\frac{e^{1/\infty} - 4}{1/\infty}\right)$$

$$= \frac{e^{0} - 4}{0}$$

$$= \frac{1 - 4}{0}$$

$$= \frac{-3}{0}$$

$$\boxed{\lim_{x\to\infty} \left(\frac{e^{1/x} - 4}{1/x}\right) = -\infty}$$

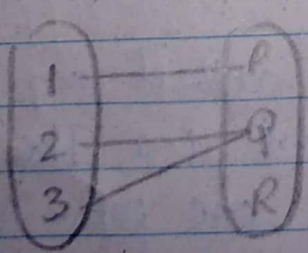
Answer

Q: Define functions and their types giving at least one example of each kind.

Functions:

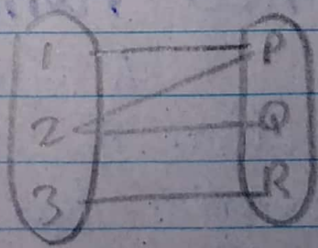
A function is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output. Let A & B be any two non-empty sets, mapping from A to B will be a function only when every element in set A has one and only ~~for~~ one image in set B.

Example:



↓

this is a function.



↓

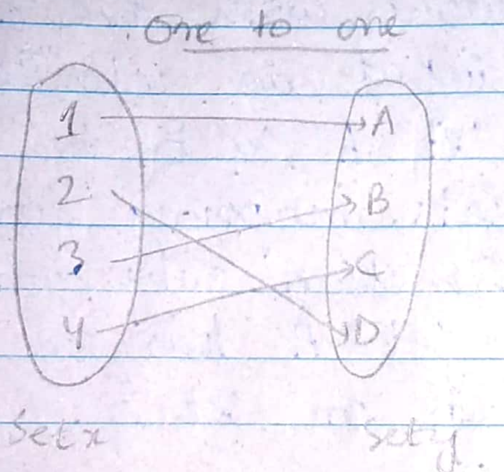
this isn't a function.

Types of Functions:

One to One Function:

If each element in the domain of a function has a distinct image in the co-domain, the function is said to be one to one function.

Example:



Onto Function:

A function is called an onto function if each element in the co-domain has at least one pre-image in the domain.

Example: let $A = \{1, 2, 3\}$, $B = \{4, 5\}$ and let $f = \{(1,4),(2,5),(3,5)\}$. Show that f is an onto function.

Sol: Domain $= \{1, 2, 3\} = A$

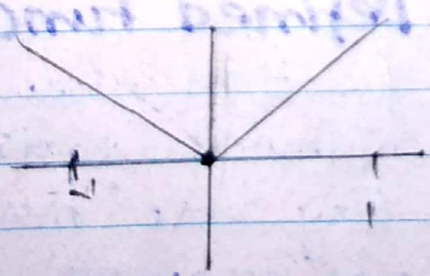
Range : { 4, 5 }

The element from A, 2 and 3 has same range 5. So $f : A \to B$ is an onto function

Into Functions:

If there exists at least one element in the co-domain which is not an image of any element in the domain then the function will be into function.

Example : let $A = \{ x : 1 < x < 1 \} = B$ be a mapping $f : A B$ find the nature of the given function? (P) $F(x) = |x|$.



Solution for $x = 1$ & -1

Hence it is many one the Range of $f(x)$ from $[-1, 1]$ is $[0, 1]$, which is not equal to co domain, Hence it is into function.

Even Function:

If $f(-x) = f(x)$, then $f(x)$ is an even function. The graph of an even function is symmetric about y-axis.

Example: $\cos x$, x^2, x^2+1.

Odd Functions:

If $f(-x) = -f(x)$, then $f(x)$ is an odd function. The graph of odd function is symmetric about the origin.

Example: $\sin x$, x^3, x.

Piece-wise Defined Function:

When functions use different formulas for different parts of its domain then they are Piecewise defined functions. An example of such functions is absolute value functions.

Example:
$$f(x) = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

Q: What is the criteria for a function to be continuous at a point in its domain? Discuss the continuity of following graphed function at $x = 4$, $x = -6$, $x = 2$ and in the interval $(2, 5)$.

Criteria for a Continuous Function:

A function f is continuous at $x = c$ if all of the following conditions are satisfied.

$f(c)$ exists, i.e $f(x)$ has a definite value at $x = c$.

$\lim_{x \to c} f(x)$ exists

$\lim_{x \to c} f(x) = f(c)$.

If one or more of the conditions above do not hold, we say the function is discontinuous at $x = c$.

Every polynomial function is continuous at all points.

A rotational function is continuous at all points except for those values of $g(x)$ where $g(x) = 0$.

Through Graph:
If we do not have to lift up the pencil while plotting graph of any function, then it shows that this is a continuous function.

Continuity:

⇒ At $x = -4$:
The function is not continuous at $x = -4$.
The graph is ~~not~~ discontinuous at $x = -4$ because it is not defined at $x = -4$.

⇒ At $x = -6$:
The function is continuous at $x = -6$.

At $x = 2$:

The function is not continuous at $x = 2$.

The ~~function is~~ graph of a function is discontinuous at $x = 2$.

Interval $(2,5)$:

The function is not continuous on the interval $(2,5)$ because the function is discontinuous at $x = 4$.