〈국문초록〉

q-급수와 모듈러 형식

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요약문

정수론 분야에서 Ramanujan의 업적은 대부분은 q-series와 theta 함수에서 나타났습니다. 그러나 Ramanujan은 자신이 발견한 수학적 결과를 노트북에 증명 없이 기록하였습니다. 따라서 많은 정수론 학자들은 Ramanujan이 기록한 결과를 증명하는데 관심이 있었습니다. 본 연구에서는 Ramanujan의 노트에 기록된 23개의 theta function identities와 유사한 특정 유형의 모듈 방정식에 대해 연구를 하였습니다. 그들 중 일부는 모듈 형태로 증명되었습니다.

우리는 새로운 모듈 방정식을 발견하고 발견 과정을 일반화했습니다. 간단한 대수 계산을 사용하여 새롭게 발견한 모듈 방정식을 증명하였습니다. 또한 투영 점으로 원뿔 곡선에서 the reverse function과 the reverse invariant을 정의하고 투영 점이 있는 두 원뿔 곡선의 등가성에 대해 연구하였습니다.

주제어 : q-series, modular equation, invariant, conic, projective geometry

Ⅰ. Introduction

Ramanujan had recorded many modular equations in his lost notebooks. In this study, we use elementary algebraic formulas for the theta functions φ, ψ and f and basic theta functions identities to find some new modular equations.

We begin this section by introducing the standard notation of q-series which is used throughout this paper.

$$(a)_0 := (a;q)_0 := 1,$$
$$(a)_n := (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k) \text{ for } n=1,2,\dots,$$

and

$$(a)_\infty := (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k) \text{ for } |q|<1.$$

Note that we often omit q from the notation if the identification of q is clear from the context.

We now give Ramanujan's definition of his general theta function $f(a,b)$. It is defined by, for $|ab|<1$,

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

Note that three special cases of $f(a,b)$ are defined by, for $|q|<1$,

$$\varphi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty,$$
$$\psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

and

$$f(-q)=f(-q,\,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}=(q;q)_\infty\,.$$

Let a, b, and c be arbitrary complex numbers except that c cannot be a non-positive integer. Then, for $|z|<1$, the Gaussian or ordinary hyper-geometric function ${}_2F_1(a,b;c;z)$ is defined by

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n,$$

where $(a)_0=1$ and $(a)_n=a(a+1)(a+2)\cdots(a+n-1)$ for each positive integer n. Note that this notation is the same as $(a)_n:=(a;q)_n$. However, they are rarely used in the same context, and so no confusion should arise.

Now the complete elliptic integral of the first kind $K(k)$ is defined by

$$K(k)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=\frac{\pi}{2}\,{}_2F_1\!\left(\frac{1}{2},\frac{1}{2};1;k^2\right)=\frac{\pi}{2}\varphi^2\left(e^{-\pi K'/K}\right),$$

where $0<k<1$, $K'=K(k')$, and $k'=\sqrt{1-k^2}$. The number k is called the modulus of K and k' is called the complementary modulus.

Let K, K', L, and L' denote complete elliptic integrals of the first kind associated with the moduli k, k', l, and l', respectively, where $0<k<1$ and $0<l<1$. Suppose that

$$\frac{L'}{L}=n\frac{K'}{K}$$

holds for some positive integer n. A relation between k and l induced by $\frac{L'}{L}=n\frac{K'}{K}$ is called a modular equation of degree n. Following Ramanujan, set $\alpha=k^2$ and $\beta=l^2$. We say that β has degree n over α.

Now let k be the modulus as mentioned before. Set $x=k^2$ and also set

$$x=k^2=1-\frac{\varphi^4(-q)}{\varphi^4(q)}.$$

Then we have

$$z=\varphi^2(q)={}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x\right),$$

where

$$q=e^{-y}=\exp\left(-\pi\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-x\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x\right)}\right)=\exp\left(-\pi\frac{K(k')}{K(k)}\right).$$

If we set

$$q=\exp\left(-\pi\frac{K'}{K}\right) \text{ and } q'=\exp\left(-\pi\frac{L'}{L}\right),$$

then we see that $\frac{L'}{L}=n\frac{K'}{K}$ is equivalent to the relation $q^n=q'$. Hence a modular equation can be viewed as an identity involving theta functions at the arguments q and q^n. Let $z_n:=\varphi^2(q^n)$. Then the multiplier m for degree n is defined by

$$m=\frac{\varphi^2(q)}{\varphi^2(q^n)}=\frac{z_1}{z_n}.$$

In this research, we proved existing modular equations in new way and found new modular equation related to theta function.

Sometimes modular equation can be found by using the term 'modular form'. Modular form is very important to treat some functions in number theory. This is defined with *upper half plane* $\mathbb{H}:=\{z\in\mathbb{C}:\Im(z)>0\}$, *special linear group* $\mathrm{SL}_2(F):=\{M\in M_{nn}(F):\det M=1\}$, and *fractional linear transformation*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}(z):=\frac{az+b}{cz+d}$$

where $a,b,c,d \in \mathbb{C}$. If $f:\mathbb{H} \to \mathbb{C}$ satisfies following conditions, we call the function as *modular form of weight* k.

1. $f(\gamma(z)) = (cz+d)^k f(z)$ where $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$

2. f is holomorphic on $\mathbb{H} \cup \{i\infty\}$.

At this point, differentiablity at point of infinity $\{i\infty\}$ is defined as differentiability of $f(1/z)$ at $z=0$. If f goes 0 as the point goes further from the origin, the function f is called as *cusp form*. Similarly, if $f:\mathbb{H} \to \mathbb{C}$ satisfies followings, then f is called *modular function*.

1. $f(\gamma(z)) = f(z)$ where $\gamma \in \mathrm{SL}_2(\mathbb{Z})$

2. f is meromorphic.

Ler G be a (multiplicative) group and X be a set. We call G act on X with binary operation $\times$: $G \times X \to X$ which satisfies

1. for all $a,b \in G$ and $x \in X$, $(ab)x = a(bx)$

2. for all $x \in X$, $1x = x$.

Then we can think fractional linear transformation as action of $\mathrm{SL}_2(\mathbb{Z})$ on $\hat{\mathbb{C}}$ or $\mathbb{H}$.

Ⅱ. Preliminary Results and Some New Formulas for Theta Functions

In this section, we introduce some theta function identities that will play key roles in the proofs of new modular equations.

Lemma 2.1. [2, Theorem 5.4.1] *We have*

(i). $$\varphi(q) = \sqrt{z}$$

(ii). $$\varphi(-q) = \sqrt{z}(1-x)^{1/4}$$

(iii). $$\varphi(q^2) = \sqrt{z}\sqrt{(1+\sqrt{1-x})/2}$$

(iv). $$\varphi(-q^2) = \sqrt{z}\,(1-x)^{1/8}$$

(v). $$\varphi(q^4) = \sqrt{z}\sqrt{1+(1-x)^{1/4}}/2$$

Lemma 2.2. [2, Theorem 5.4.3] *We have*

(i). $$f(-q) = \sqrt{z}\, 2^{-1/6}(1-x)^{1/6}(x/q)^{1/24}$$

(ii). $$f(-q^2) = \sqrt{z}\, 2^{-1/3}\{x(1-x)/q\}^{1/12}$$

(iii). $$f(-q^4) = \sqrt{z}\, 4^{-1/3}(1-x)^{1/24}(x/q)^{1/6}$$

(iv). $$f(-q^8) = \frac{f(-q^4)^2}{\varphi(-q^4)} = \sqrt{z}\, 2^{-13/12}(1-x)^{1/48}\left(\frac{x}{q}\right)^{1/3}(1+\sqrt{1-x})^{-1/4}$$

Lemma 2.3. [1, Chapter 17, Entry 11, 12] *We have*

$$f(-q^2) = \frac{f(-q)^2}{\varphi(-q)};\ \varphi(-q^2)^2 = \varphi(q)\varphi(-q);\ \varphi(q)^2+\varphi(-q)^2 = 2\varphi(q^2)^2$$

Definition 2.4. (Elementary Algebraic Function) *We call f is elementry algebraic function if f can be represented by finite number of the four fundamental arithmetic operations and n-th root.*

Theorem 2.5. *We have*

(i). $$\varphi(-q^4) = 2^{-1/4}\sqrt{z}\,(1+\sqrt{1-x})^{1/4}(1-x)^{1/32}$$

(ii). $$\varphi(q^8) = \sqrt{(1+(1-x)^{1/4})^2 z/4+\sqrt{1+\sqrt{1-x}}\,(1-x)^{1/16}z/\sqrt{2}}\,/\sqrt{2}$$

(iii). $$\varphi(-q^8) = 2^{-5/8}\sqrt{z}\,(1+(1-x)^{1/4})^{1/2}(1+\sqrt{1-x})^{1/8}(1-x)^{1/32}$$

(iv). $$f(-q^{16}) = \sqrt{z}\, 2^{-37/24}(1-x)^{1/96}\left(\frac{x}{q}\right)^{2/3}(1+\sqrt{1-x})^{-5/8}\{1+(1-x)^{1/4}\}^{-1/2}$$

and we can represent $\varphi(q^{2^n})$ and $\varphi(-q^{2^n})$ as $\sqrt{z}\,\phi(x)$, where ϕ is an arbitrary elementry algebraic function. Also every $f(-q^{2^n})$ can be represented as $\sqrt{z}\,q^{-\frac{2^n}{24}}g(x)$ where g is arbitrary elementry algebraic function.

Proof. For (i), It is easy to calculate by Lemma 2.3, the recurrence relation and Lemma 2.1.

$$\varphi^2(-q^4) = \varphi(-q^2)\varphi(q^2) = z\sqrt{\frac{1+\sqrt{1-x}}{2}}\,(1-x)^{\frac{1}{8}}$$

By definition of φ, it is always positive. So we can take square root in both sides and get (i). We used same method to prove (ii), (iii) and (iv).

(ii) By Lemma 2.1, Lemma 2.3, and Theorem 2.5 (i),

$$2\varphi(q^8)^2 = \varphi(q^4)^2 + \varphi(-q^4)^2 = \frac{z}{4}\left(1+(1-x)^{\frac{1}{4}}\right)^2 + \frac{z}{\sqrt{2}}(1+\sqrt{1-x})^{\frac{1}{2}}(1-x)^{\frac{1}{8}},$$

$$\varphi(q^8) = \frac{\sqrt{z(1+(1-x)^{\frac{1}{4}})^2 + z2^{\frac{3}{2}}(1+\sqrt{1-x})^{\frac{1}{2}}(1-x)^{\frac{1}{8}}}}{2\sqrt{2}}.$$

(iii) By Lemma 2.1, Lemma 2.3, and Theorem 2.5 (i),

$$\varphi^2(-q^8) = \varphi(-q^4)\varphi(q^4) = \frac{z}{2}(1+(1-x)^{\frac{1}{4}})^{\frac{1}{2}}\left(\frac{1}{2}(1+\sqrt{1-x})\right)^{\frac{1}{4}}(1-x)^{\frac{1}{16}},$$

$$\varphi(-q^8) = \frac{z}{2^{5/8}}(1+(1-x)^{\frac{1}{4}})^{\frac{1}{2}}(1+\sqrt{1-x})^{\frac{1}{8}}(1-x)^{\frac{1}{32}}.$$

(iv) By Lemma 2.2, Lemma 2.3 and Theorem 2.5 (iii),

$$f(-q^{16}) = \frac{f(-q^8)^2}{\varphi(-q^8)} = \frac{z2^{-13/6}(1-x)^{1/24}(x/q)^{2/3}/(1+\sqrt{1-x})^{1/2}}{\sqrt{z}\,2^{-3/8}(1-x)^{1/32}(1+(1-x)^{1/8})^{1/2}(1+\sqrt{1-x})^{1/8}}$$

$$= \sqrt{z}\,2^{-37/24}(1-x)^{1/96}\left(\frac{x}{q}\right)^{2/3}(1+\sqrt{1-x})^{-5/8}(1+(1-x)^{1/4})^{-1/2}.$$

Lemma 2.3 guarantees that we can do this process infinitely many times. By mathematical induction, it is easy to show that every $\varphi(q^{2^n})$ and $\varphi(-q^{2^n})$ can be represented as $\sqrt{z}\,\phi(x)$, where $\phi(x)$ is an arbitrary elementry algebraic function of x. Also every $f(-q^{2^n})$ can be represented as $\sqrt{z}\,q^{-\frac{2^n}{24}}g(x)$ where $g(x)$ is arbitrary elementry algebraic function of x.

□

Ⅲ. Some New Modular Equations

In this section, we give some new modular equations and give proofs based on simple algebraic computations.

Theorem 3.1. *Let*

$$P=\frac{f(-q)}{q^{1/24}f(-q^2)}\ \text{and}\ Q=\frac{f(-q^2)}{q^{1/12}f(-q^4)},$$

then

$$P^8+\frac{16}{Q^8}=\left(\frac{Q}{P}\right)^8.$$

Proof 1. Using Lemma 2.2,

$$P=\frac{\sqrt{z}\,2^{-1/6}(1-x)^{1/6}(x/q)^{1/24}}{q^{1/24}\sqrt{z}\,2^{-1/3}(x(1-x)/q)^{1/12}}=2^{1/6}\times\frac{(1-x)^{1/12}}{x^{1/24}},$$

$$Q=\frac{\sqrt{z}\,2^{-1/3}(x(1-x)/q)^{1/12}}{q^{1/12}\sqrt{z}\,2^{-2/3}(1-x)^{1/24}(x/q)^{1/6}}=2^{1/3}\times\frac{(1-x)^{1/24}}{x^{1/12}}.$$

Also, we can find relation of P and Q.

$$\left(\frac{P}{Q}\right)^{24}=2^{-4}\times x(1-x);\ \left(\frac{P}{Q^2}\right)^{24}=2^{-12}\times x^3$$

$$\left(\frac{P}{Q^2}\right)^8=2^{-4}x$$

Thus

$$\left(\frac{P}{Q}\right)^{24}=\left(\frac{P}{Q^2}\right)^8\times\left(1-2^4\left(\frac{P}{Q^2}\right)^8\right),$$

and this implies the theorem by multiplying Q^{24}/P^{16} both sides.

□

Proof 2. Although this proof is much complicated than Proof 1, the method in this proof can be applied to making modular equations for P and Q consisted of high-degree theta functions like we will do in later.

As we did in Proof 1, P and Q can be represented by function of x.

$$P=\frac{\sqrt{z}\,2^{-1/6}(1-x)^{1/6}(x/q)^{1/24}}{q^{1/24}\sqrt{z}\,2^{-1/3}(x(1-x)/q)^{1/12}}=2^{1/6}\times\frac{(1-x)^{1/12}}{x^{1/24}}$$
$$Q=\frac{\sqrt{z}\,2^{-1/3}(x(1-x)/q)^{1/12}}{q^{1/12}\sqrt{z}\,2^{-2/3}(1-x)^{1/24}(x/q)^{1/6}}=2^{1/3}\times\frac{(1-x)^{1/24}}{x^{1/12}}.$$

So we have

$$\left(\frac{P}{Q}\right)^{24}=2^{-4}\times x(1-x)\ \text{and}\ \left(\frac{P}{Q^2}\right)^{24}=2^{-12}\times x^3.$$

Let

$$A=x(1-x)\ \text{and}\ B=x^3,$$

then

$$1, A, A^2, A^3, B, B^2, AB$$

or

$$1, x-x^2, x^2-2x^3+x^4, x^3-3x^4+3x^5-x^6, x^3, x^6, x^4-x^5.$$

These polynomials have degree less than or equal than 6 but numbers of them is 7, larger than 6. Thus there are $c_1, c_2, \cdots, c_7$ which is at least one of them is non-zero and satisfies

$$c_1+c_2A+c_3A^2+c_4A^3+c_5B+c_6B^2+c_7AB=0.$$

Having some trial, we could find that next equation holds.

$$A^3 = x^3 - 3x^4 + 3x^5 - x^6 = B - 3AB - B^2,$$

therefore

$$2^{12}\left(\frac{P}{Q}\right)^{72} = 2^{12}\left(\frac{P}{Q^2}\right)^{24} - 3 \times 2^{16}\left(\frac{P^2}{Q^3}\right)^{24} - 2^{24}\left(\frac{P}{Q^2}\right)^{48},$$

$$P^{24} = \frac{Q^{24}}{P^{24}} - 48 - \frac{2^{12}}{Q^{24}}.$$

Now we should check this implies $P^8 + \frac{16}{Q^8} = \left(\frac{Q}{P}\right)^8$ by some proper factorization. Multiply $P^{24}Q^{24}$ on both sides and let $X = P^8$ and $Y = Q^8$.

$$Y^6 - 48X^3Y^3 - 2^{12}X^3 - X^6Y^3 = 0$$

We can factorize as

$$(Y^2 - 2^4X - X^2Y)((Y^2 - 2^4X)^2 + (Y^2 - 2^4)X^2Y + X^4Y^2 + 48XY^2) = 0$$

Let $E = (Y^2 - 2^4X)^2 + (Y^2 - 2^4)X^2Y + X^4Y^2 + 48XY^2$.
If $(Y^2 - 2^4) \geqq 0$, then $E > 0$ since each term are positive.
If $Y^2 - 2^4 < 0$, we can rewrite E as

$$(Y^2 - 2^4X + X^2Y)^2 - (Y^2 - 2^4)X^2Y + 48XY^2.$$

Thus $E > 0$ for all case and $Y^2 - 2^4X - X^2Y = 0$, which implies the theorem.

$\square$

Theorem 3.2. *Let*

$$P=\frac{f(-q)}{q^{1/8}f(-q^4)}\ \text{and}\ Q=\frac{f(-q^2)}{q^{1/4}f(-q^8)}$$

then

$$P^8+\left(\frac{2}{Q}\right)^8+16=\left(\frac{Q}{P}\right)^8-16\left(\frac{P}{Q}\right)^8$$

Proof. Using Lemma 2.2,

$$P=\frac{(1-x)^{1/8}}{x^{1/8}}2^{1/2}\quad Q=\frac{(1-x)^{1/16}}{(1-\sqrt{1-x})^{1/4}}2^{3/4}.$$

Let $t=\sqrt{1-x}$, then

$$P=\frac{t^{1/4}}{(1-t^2)^{1/8}}2^{1/2}\quad Q=\frac{t^{1/8}}{(1-t)^{1/4}}2^{3/4},$$

$$P^8=\frac{t^2}{1-t^2}2^4;\ Q^8=\frac{t}{(1-t)^2}2^6.$$

Then we can get $t(2^6+2Q^8)=Q^8+Q^8t^2$. Squaring both sides and using $t^2=\frac{P^8}{16-P^8}$, this implies

$$256P^8+16P^{16}+16P^8Q^8+P^{16}Q^8-Q^{16}=0.$$

Multiplying $1/(PQ)^8$, we can get the result.

□

Theorem 3.3. *Let*

$$P=\frac{\varphi(-q)}{\varphi(-q^4)} \text{ and } Q=\frac{\varphi(-q^2)}{\varphi(-q^8)},$$

then

$$16\left(\frac{2P}{Q^2}-1\right)^4\left(1+\left(\frac{2P}{Q^2}-1\right)^4\right)=\frac{2^4}{P^{16}}\left(\frac{2P}{Q^2}-1\right)^{12}-8\left(\frac{2P}{Q^2}-1\right)^4-\left(1-\left(\frac{2P}{Q^2}-1\right)^4\right)^2.$$

Proof. Applying Lemma 2.1 and Theorem 2.5, We can get

$$P=2^{1/4}\frac{(1-x)^{3/16}}{(1+\sqrt{1-x})^{1/4}} \text{ and } Q=2^{5/8}\frac{(1-x)^{3/32}}{(1+(1-x)^{1/4})^{1/2}(1+\sqrt{1-x})^{1/8}}.$$

And these implies

$$\frac{P}{Q^2}=\frac{1+(1-x)^{1/4}}{2},$$
$$x=1-\left(\frac{2P}{Q^2}-1\right)^4.$$

However,,

$$P^{16}=2^4\frac{(1-x)^3}{(1+\sqrt{1-x})^4},$$
$$(1+\sqrt{1-x})^4=1+4\sqrt{1-x}+6(1-x)+4(1-x)\sqrt{1-x}+(1-x)^2=2^4\frac{(1-x)^3}{P^{16}},$$
$$4\sqrt{1-x}(2-x)=2^4\frac{(1-x)^3}{P^{16}}-8(1-x)-x^2 \ .$$

by putting $x=1-\left(\frac{2P}{Q^2}-1\right)^4$, this gives us the formula.

□

Theorem 3.4. *Consider the inequality $an+bm \leq 2ab$ where $a,b \in \mathbb{N}$ are fixed. Then the number of solution pairs $(n,m) \in \mathbb{N}_0^2$ is greater than $2ab+1$.*

Proof. It is easy to bound the solutions as $0 \leq n \leq 2b$ and $0 \leq m \leq 2a$ by letting $n=0$ and $m=0$ respectively. On nm plane, $an+bm=2ab$ is linear and this bisects the rectangle range $0 \leq n \leq 2b$, $0 \leq m \leq 2a$.

Therefore N, the number of lattice points (n,m) satisfying $an+bm \leq 2ab$ is larger than half of number of total lattice points in rectangle range.

$$N > \frac{(2a+1)(2b+1)}{2} = 2ab+a+b+\frac{1}{2} > 2ab+1$$

The inequality shows that $N > 2ab+1$.

□

Theorem 3.5.

$$P = \frac{f(-q)}{q^{7/24}f(-q^8)} \text{ and } Q = \frac{f(-q^2)}{q^{7/24}f(-q^{16})},$$

then

$$\begin{aligned}
&4294967296P^{168} - 49517601571415210995964968 96P^{56}Q^8 \\
&+36267774588438875241185 28P^{80}Q^8 \\
&-7747632510958011678 72P^{104}Q^8 + 45035996273704960P^{128}Q^8 \\
&-176093659136P^{152}Q^8 + 1208925819614629174706176P^{64}Q^{16} \\
&-178933417514982650 6752P^{88}Q^{16} - 18586355662158036992P^{112}Q^{16} \\
&-3113851289600P^{136}Q^{16} - 1048576P^{160}Q^{16} - P^{184}Q^{16} \\
&-2126764793255865396646091296448551 3216Q^{24} \\
&+1559717298520813455601543623894 6304P^{24}Q^{24} \\
&-3342753635709446160442233716736P^{48}Q^{24} \\
&+1968379552872121528337039 36P^{72}Q^{24} - 993903060939092000768P^{96}Q^{24} \\
&+64291365453824P^{120}Q^{24} + 51380224P^{144}Q^{24} + 7P^{168}Q^{24} \\
&-3799856950586474753802861805568P^{32}Q^{32} \\
&-1470149424572667186758287360P^{56}Q^{32} \\
&-263869502232589489930 24P^{80}Q^{32} - 4944933286838272P^{104}Q^{32} \\
&-4802478080P^{128}Q^{32} - 21P^{152}Q^{32} \\
&+3113308648203367412976 8448P^{40}Q
\end{aligned}$$

Proof. By Lemma 2.2 and Theorem 2.5,

$$
\begin{aligned}
P=\frac{f(-q)}{q^{7/24}f(-q^8)}&=\frac{\sqrt{z}\,2^{-1/6}(1-x)^{1/6}(x/q)^{1/24}}{q^{7/24}\sqrt{z}\,2^{-13/12}(1-x)^{1/48}(x/q)^{1/3}(1+\sqrt{1-x}\,)^{1/4}}\\
&=2^{11/12}\times\frac{(1-x)^{7/48}(1+\sqrt{1-x}\,)^{1/4}}{x^{7/24}},\\
Q=\frac{f(-q^2)}{q^{7/12}f(-q^{16})}&=\frac{\sqrt{z}\,2^{-1/3}(x(1-x)/q)^{1/12}}{q^{7/12}\sqrt{z}\,2^{37/24}(1-x)^{1/96}(x/q)^{2/3}(1+\sqrt{1-x}\,)^{-5/8}(1+(1-x)^{1/4})^{-1/2}}\\
&=2^{29/24}\times\frac{(1-x)^{7/96}(1+(1-x)^{1/4})^{1/2}(1+\sqrt{1-x}\,)^{5/8}}{x^{7/12}}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
P^{24}&=2^{22}\times\frac{(1-x)^{7/2}(1+\sqrt{1-x}\,)^{6}}{x^{7}},\\
Q^{24}&=2^{29}\times\frac{(1-x)^{7/4}(1+(1-x)^{1/4})^{12}(1+\sqrt{1-x}\,)^{15}}{x^{14}}.
\end{aligned}
$$

Substitute $(1-x)^{1/4}=t$ then

$$
\begin{aligned}
P^{24}&=2^{22}\times\frac{t^{14}(1+t^2)^6}{(1-t)^7(1+t)^7(1+t^2)^7}=2^{22}\times\frac{t^{14}}{(1-t)^7(1+t)^7(1+t^2)},\\
Q^{24}&=2^{29}\times\frac{t^{7}(1+t)^{12}(1+t^2)^{15}}{(1-t)^{14}(1+t)^{14}(1+t^2)^{14}}=2^{29}\times\frac{t^{7}(1+t^2)}{(1-t)^{14}(1+t)^{2}}.
\end{aligned}
$$

Using above formulas, we can get following results

$$
(PQ)^8=2^{17}\times\frac{t^7}{(1-t)^7(1+t)^3},\ \frac{Q^8}{P^{16}}=2^{-5}\times\frac{(1+t)^4(1+t^2)}{t^7}.
$$

Substitute $s=1/t$ and focus on first and second equations. Then

$$
(PQ)^8=2^{17}\times\frac{1}{(s-1)^7(1+1/s)^3},\ \frac{Q^8}{P^{16}}=2^{-5}\times s(1+s)^4(1+s^2),
$$

$$\frac{Q^{16}}{P^{56}}=\frac{1}{(PQ)^8}\times\left(\frac{Q^8}{P^{16}}\right)^3=2^{-32}\times(s-1)^7(1+s)^7(1+s^2)^3.$$

Let $A=s(1+s)^4(1+s^2)$ and $B=(s-1)^7(1+s)^7(1+s^2)^3$.

Now we want to represent 0 by linear combination of $\mathbb{R}[A,B]$ to find the modular equation. For fixed $k\in\mathbb{N}$, there are finite solutions $(n,m)\in\mathbb{N}_0^2$ for $n\deg A+m\deg B\le k$. Then each A^nB^m is vector and these vectors form a subspace of $k+1$ dimensional vector space. So we want to find k such that $n\deg A+m\deg B\le k$ and the number of solution pairs (n,m) is greater than $k+1$ and Theorem 3.4 guarantees k always exists. Thus $S=\{A^nB^m|n\deg A+m\deg B\le k\}$ is linearly dependent.

Definition. (Coefficient Matrix). Define coefficient matrix of ordered pair $(f_1,f_2,\cdots,f_n)$ where $f_i\in\mathbb{R}[x]$ by

$$M=(a_{i,j})\in M_{n,d+1}(\mathbb{R})$$

where $d=\max\{\deg f_i\}$ and $f_i=\sum_{m=0}^{d}a_{i,d+1-m}x^m$.

We can make coefficient matrix M from function set S. But we took S be linearly dependent, so $M^Tx=0$ has nonzero solution. Then $c_1f_1+c_2f_2+\cdots+c_nf_n=0$ where $x=<c_1,c_2,\cdots,c_n>$ is polynomial we want. To find this, we used next code in *Mathematica*.

```
A[s_]:=s (1+s)·4 (1+s·2)
B[s_]:=(s-1)·7 (1+s)·7 (1+s·2)·3
a:=Exponent[A[s],s]
b:=Exponent[B[s],s]

mat:={}
n:=141

Do[If[a*i+b*j<n,m:=CoefficientList[A[s]·i*B[s]·j,s];
   Do[If[Length[m]<n,m=Append[m,0]],{k,n-Length[m]}];(*Print[Length[m]]*) ;
mat=Append[mat,m];Print[{i,j}];(*Print[-1]*)],{i,0,IntegerPart[n/a]},{j,0,IntegerPart[n/b]}]

mat=Transpose[mat]
NullSpace[mat]
```

Figure 3. Finding proper linear combination using *Mathematica 9*

This proves the theorem by putting

$$A = 2^5 \times \frac{Q^8}{P^{16}} \text{ and } B = 2^{32} \times \frac{Q^{16}}{P^{56}}.$$

□

Theorem 3.6. *Let*

$$P = \frac{\varphi(q)}{\varphi(q^4)} \textit{ and } Q = \frac{\varphi(q^2)}{\varphi(q^8)},$$

then

$$4\left(1+\left(\frac{2}{P}-1\right)^2 - \frac{Q^2}{P^2}\right)^4 = Q^4\left(\frac{2}{P}-1\right)\left(1+\left(\frac{2}{P}-1\right)^2\right)^2.$$

Proof. We can know that

$$P = \frac{2}{1+(1-x)^{1/4}},$$

$$Q = \frac{\sqrt{1+\sqrt{1-x}}}{\left(\frac{(1+(1-x)^{1/4})^2}{4} + \frac{\sqrt{1+\sqrt{1-x}}\,(1-x)^{1/16}}{\sqrt{2}}\right)^{1/2}}.$$

let $(1-x)^{1/16} = t$, then

$$P = \frac{2}{1+(1-x)^{1/4}} = \frac{2}{1+t^4},$$

$$Q = \frac{\sqrt{1+\sqrt{1-x}}}{\left(\frac{(1+(1-x)^{1/4})^2}{4} + \frac{\sqrt{1+\sqrt{1-x}}\,(1-x)^{1/16}}{\sqrt{2}}\right)^{1/2}} = \frac{\sqrt{1+t^8}}{\left(\frac{(1+t^4)^2}{4} + \frac{\sqrt{1+t^8}\,t}{\sqrt{2}}\right)^{1/2}}.$$

Now we can put $2/P-1 = t^4$ on Q as following equation.

$$Q^2 = \frac{1+\left(\frac{2}{P}-1\right)^2}{\frac{1}{P^2}+\frac{t\sqrt{1+\left(\frac{2}{P}-1\right)^2}}{\sqrt{2}}}$$

Hence

$$\frac{Q^2 t\sqrt{1+\left(\frac{2}{P}-1\right)^2}}{\sqrt{2}} = 1+\left(\frac{2}{P}-1\right)^2-\left(\frac{Q}{P}\right)^2.$$

Therefore

$$\left(1+\left(\frac{2}{P}-1\right)^2-\frac{Q^2}{P^2}\right)^4 = \frac{Q^4\left(\frac{2}{P}-1\right)\left(1+\left(\frac{2}{P}-1\right)^2\right)^2}{4}.$$

And we can get the theorem by expanding above equation.

□

Remark 1. *Every expression P and Q which represented by finite number of the four fundamental arithmetic operations and n-th root can be represented by rational equation at the same time by substitution.*

Remark 2. *Every rational equation P and Q can be represented by polynomial at the same time by substitution. Or more weakly, there exists linearly independent non constant two variable polynomials p and q such that $p(P,Q)$ and $q(P,Q)$ are all polynomials by substitution.*

Theorem 3.7. *It is always able to find a modular equation for every*

$$P=\frac{f(-q^{2^a})}{q^{\frac{1}{24}(2^b-2^a)}f(-q^{2^b})} \text{ and } Q=\frac{f(-q^{2^c})}{q^{\frac{1}{24}(2^d-2^c)}f(-q^{2^d})}$$

where $a,b,c,d \in \mathbb{N}_0$.

Proof. By Theorem 2.5, we can express each theta function $f(-q^{2^n})$ by $\sqrt{z}\, q^{-\frac{2^n}{24}} g(x)$ where g is arbitrary elementry algebraic function. Then each P and Q is elementry algebraic function of x since q and z are eliminated. By Conjecture 1 and Conjecture 2, there exists linearly independent non constant two variable polynomials p and q such that $p(P,Q)$ and $q(P,Q)$ are all polynomials by proper substitution of x and let them A and B respectively.

To find the modular equation of P and Q, we should represent 0 by linear combination of $\mathbb{R}[A,B]$. For fixed $k \in \mathbb{N}$, there are finite solutions $(n,m) \in \mathbb{N}_0^2$ for $n\deg A + m\deg B \le k$. Then each $A^n B^m$ is vector and these vectors form a subspace of $k+1$ dimensional vector space. So we should find k such that $n\deg A + m\deg B \le k$ and the number of solution pairs (n,m) is greater than $k+1$. Theorem 3.4 guarantees k always exists as $2\deg A \deg B$. Thus $S = \{A^n B^m \mid n\deg A + m\deg B \le k\}$ is linearly dependent.

Consider a coefficient matrix M from function set S. But we took S be linearly dependent, so $M^T x = 0$ has nonzero solution, then $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$ where $x = < c_1, c_2, \cdots, c_n >$ is polynomial we want. Thus by finding x, nullspace of M^T and doing some re-substitution task, we can get the modular equation of P and Q we wanted.

Theorem 3.8. *It is always able to find a modular equation for every*

$$P = \frac{\varphi((-1)^{a'} q^a)}{\varphi((-1)^{b'} q^b)} \text{ and } Q = \frac{\varphi((-1)^{c'} q^c)}{\varphi((-1)^{d'} q^d)}$$

where $a,b,c,d \in \mathbb{N}_0$ *and* $a',b',c',d' \in \{0,1\}$.

Proof of Theorem 3.8 is identical to the proof of Theorem 3.7.

Lemma 3.9. [1, Chapter 19, Entry 19(ix)] *Let*

$$P = \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \text{ and } Q = \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6}$$

where β has degree 7 of α, then

$$Q+\frac{1}{Q}+7=2^{3/2}\left(P+\frac{1}{P}\right).$$

Theorem 3.10. *Let*

$$P=\frac{f(-q)}{q^{1/24}f(-q^2)} \text{ and } Q=\frac{f(-q^7)}{q^{7/24}f(-q^{14})},$$

then

$$(PQ)^3+\left(\frac{2}{PQ}\right)^3=\left(\frac{Q}{P}\right)^4-7+\left(\frac{P}{Q}\right)^4.$$

Proof. Replace $-q$ instead of q then P and Q are replaced as followings.

$$P'=\frac{f(q)}{(-q)^{1/24}f(-q^2)} \text{ and } Q'=\frac{f(q^7)}{(-q)^{7/24}}f(-q^{14})$$

And this means $(PQ)^3$ changes to $-(P'Q')^3$ and $(P/Q)^4$ changes to $-(P'/Q')^4$. Thus we want to show

$$(P'Q')^3+\left(\frac{2}{P'Q'}\right)^3=\left(\frac{Q'}{P'}\right)^4+\left(\frac{P'}{Q'}\right)^4+7.$$

Let β has degree 7 of α. Then

$$P'=\frac{\sqrt{z_1}2^{-1/6}\alpha(1-\alpha)/\mathrm{q}^{1/24}}{(-q)^{1/24}\sqrt{z_1}2^{1/3}\alpha(1-\alpha)/\mathrm{q}^{1/12}}=2^{1/6}(-1)^{1/24}\times\left(\frac{1}{\alpha(1-\alpha)}\right)^{1/24},$$

$$P'^{24}=-\frac{2^{1/6}}{\alpha(1-\alpha)}.$$

Similarly,

$$Q'^{24} = -\frac{2^{1/6}}{\beta(1-\beta)}.$$

Using Lemma 3.9 to each P' and Q', we can get

$$\left(\frac{P'}{Q'}\right)^4 + \left(\frac{Q'}{P'}\right)^4 + 7 = \left(\frac{2}{P'Q'}\right)^3 + (P'Q')^3,$$

and this implies the theorem.

□

Ⅳ. Invariant of Conic

Classical modular form is an invariant of fractional linear transformation which is a rotation, a translation, and a scaling of the Riemann sphere. More precisely, see Theorem 4.1.

Theorem 4.1. *For Riemann sphere S^2, let's say the stereographic projection on a projection point $(0,0,1)$ from the Riemann sphere on $\mathbb{C}\times\mathbb{R}$ to complex plane as $\mathbb{P}$. Then for any fractional linear transformation $\gamma\in \mathrm{SL}_2(\mathbb{R})$, $\mathbb{P}^{-1}\circ\gamma\circ\mathbb{P}$ is a rotation about the imaginary axis, a translation along the real axis, and a scaling of the Riemann sphere.*

Proof. Take a point pair (x,y,z) on Riemann sphere. The line pass through (x,y,z) and $(0,0,1)$ is $(x,y,z-1)t+(0,0,1)$ for $t\in\mathbb{R}$. Thus the intersection point of the line and xy plane is

$$\left(\frac{x}{1-z},\frac{y}{1-z},0\right).$$

For (x,y,z) on Riemann sphere, take θ-counterclockwise rotational transform about the imaginary axis. Then transformed point is $(x\cos\theta - z\sin\theta, x\sin\theta + z\cos\theta)$ and projected a point on same stereographic projection to complex plane is

$$\left(\frac{x\cos\theta - z\sin\theta}{1 - x\sin\theta - z\cos\theta}, \frac{y}{1 - x\sin\theta - z\cos\theta}, 0\right).$$

What we want to show is the transformation T which maps

$$\left(\frac{x}{1-z}, \frac{y}{1-z}\right) \xrightarrow{T} \left(\frac{x\cos\theta - z\sin\theta}{1 - x\sin\theta - z\cos\theta}, \frac{y}{1 - x\sin\theta - z\cos\theta}\right)$$

is $\gamma \in \mathrm{SL}_2(\mathbb{R})$.

Assume that $y=0$ then we get $x^2+z^2=1$. Let $t=\frac{x}{1-z}$, then $1/(1-z)=(t^2+1)/2$ and $z/(1-z)=(t^2-1)/2$. Thus

$$\begin{aligned}
\frac{x\cos\theta - z\sin\theta}{1 - x\sin\theta - z\cos\theta} &= \frac{t\cos\theta + \frac{1-t^2}{2}\sin\theta}{\frac{1+t^2}{2} - t\sin\theta + \frac{1-t^2}{2}\cos\theta} \\
&= \frac{\sin\theta}{\cos\theta - 1}\frac{t - \frac{\cos\theta - 1}{\sin\theta}}{t - \frac{1+\cos\theta}{\sin\theta}} \\
&= \frac{t\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{-t\sin\frac{\theta}{2} + \cos\frac{\theta}{2}}
\end{aligned}$$

And right hand side of previous equation is a fractional linear transformation which determinant of matrix is 1. Thus the statement for a rotation is proved for $y=0$.

This holds also on general complex number by substitution, and this implies the rotation of the Riemann sphere is a fractional linear transformation. But a scaling and translation along the real axis of the Riemann sphere are

$$\begin{bmatrix} r & 0 \\ 0 & 1/r \end{bmatrix} \text{ and } \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}$$

where r is a real number. Then it is enough to prove that

$$\mathrm{SL}_2(\mathbb{R}) = \left\langle \begin{bmatrix} r & 0 \\ 0 & 1/r \end{bmatrix}, \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \right\rangle.$$

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R})$ then

$$\begin{bmatrix} r_2 & 0 \\ 0 & 1/r_2 \end{bmatrix} \begin{bmatrix} 1 & r_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} r_2(a+cr_1) & r_2(b+dr_1) \\ c/r_2 & d/r_2 \end{bmatrix}.$$

Take $r_1 = \dfrac{\dfrac{d}{c^2+d^2} - a}{c}$ and $r_2 = \sqrt{c^2+d^2}$ then we can get

$$r_2^2(a+cr_1) = (c^2+d^2)\frac{d}{c^2+d^2} = d, \left(\frac{c}{r_2}\right)^2 + \left(\frac{d}{r_2}\right)^2 = 1.$$

And

$$c + r_2^2(b+dr_1) = c + (c^2+d^2)\left(b + \frac{\dfrac{d^2}{c^2+d^2} - ad}{c}\right) = \frac{c^2+d^2}{c}\left(\frac{c^2}{c^2+d^2} + bc + \frac{d^2}{c^2+d^2} - ad\right) = 0,$$

thus we can set $c/r_2 = \sin\theta, d/r_2 = \cos\theta$, and this means

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

This implies the theorem.

□

Thus we can think the function which is an invariant of transformation of the conic, which has a projective correspond with complex plane, or more generally, complete field $k(=\mathbb{R}$ or $\mathbb{C})$. In this paper, conic means the non-degenerated conic.

Definition 4.2. *For any conic* $\mathscr{C} \subseteq k^2$ *and any projection point* $\mathscr{O} \in \mathscr{C}$ *which is not on* k*, if* $X \in \mathscr{C}$ *is a projection of* $Y \in k$ *with a projection center* $\mathscr{O}$ *then denote this as* $\mathbb{P}_{\mathscr{O}}(X) = Y$*. If there is no projection of* X *then denote* $\mathbb{P}_{\mathscr{O}}(X) = \infty$ *and call this as the point at infinity (of* k*). Also, if there is no point* X *such that* $\mathbb{P}_{\mathscr{O}}(X) = Y$ *then add* X *which satisfies that and this is also called as a point at infinity (of the conic* $\mathscr{C}$ *).*

This definition gives us we can do analysis on the conic. And next, we want to define some transformation on the conic. But we know number of intersections between a conic and line is two if they exists (include the point at infinity). Thus we define the followings.

Definition 4.3. *We can define the transformation* $\times : \mathscr{C} \times (k^2 \setminus \mathscr{C}) \to \mathscr{C}$ *as* AX *is another intersection between the conic and line which through* A *and* X*.*

Now, we can define the reverse for the field k.

Definition 4.4. *For any conic* $\mathscr{C} \subseteq k^2$*, a projection point* $\mathscr{O} \in \mathscr{C}$*, and a reverse point* $P \in \mathscr{C}$*, define the reverse function* $\rho_P : k \to k$ *as* $\rho_P(z) := \mathbb{P}_{\mathscr{O}}(\mathbb{P}_{\mathscr{O}}^{-1}(z)P)$*. Let's denote this structure as* $(\mathscr{C}, \mathscr{O})$

Then we can define the following function.

Definition 4.5. *For any conic with the reverse structure* $(\mathscr{C}, \mathscr{O})$*, the function* $f : k \to k$ *and the function* $\chi : k \to k$*, if* f *satisfies*

$$f \circ \rho_P = \chi f$$

for some $P \in k^2 \setminus \mathscr{C}$*, then we call* f *as* P *reverse invariant with respect to* χ *(over* $\mathscr{C}$ and $\mathscr{O}$). If there are such n points for the function f, then we call f is the reverse invariant degree n *(over* $\mathscr{C}$ and $\mathscr{O}$).

We can think an isomorphism or equivalence of the reverse structure.

Definition 4.6. *For conics* $\mathscr{C}$ *and* $\mathscr{D}$ *on* k^2*, if there is bijective differentiable function* $\varphi : k \to k$ *and* $\psi : k^2 \setminus \mathscr{C} \to k^2 \setminus \mathscr{D}$ *which makes*

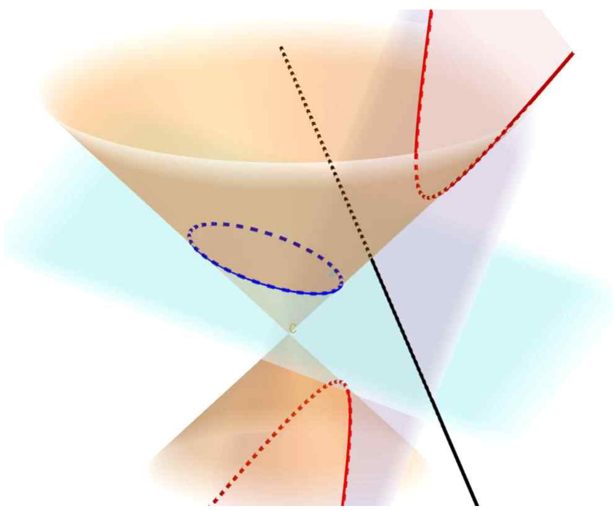
$$\begin{array}{ccc} \mathcal{C} \times (k^2 \setminus \mathcal{C}) & \xrightarrow{\mathbb{P}_{\mathcal{O}_2}^{-1} \circ \mathbb{P}_{\mathcal{O}_1} \times \psi} & \mathcal{D} \times (k^2 \setminus \mathcal{D}) \\ \downarrow \times & & \downarrow \times \\ \mathcal{C} & \xrightarrow{\mathbb{P}_{\mathcal{O}_2}^{-1} \circ \mathbb{P}_{\mathcal{O}_1}} & \mathcal{D} \end{array}$$

commutes, then we say this as $(\mathscr{C}, \mathscr{O}_1)$ *is equivalent with* $(\mathscr{D}, \mathscr{O}_2)$*, and denote this as*

$$(\mathscr{C}, \mathscr{O}_1) \cong (\mathscr{D}, \mathscr{O}_2)$$

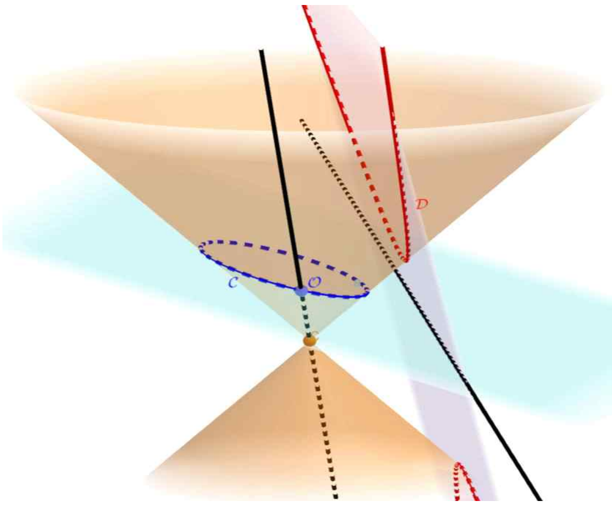
Theorem 4.7. Let $(\mathscr{C}, \mathscr{O})$ be a conic with projection point. Then there is a projective transformation $\mathbb{P}$ which preserves the x-axis line k satisfying $(\mathbb{P}(\mathscr{C}), \mathbb{P}(\mathscr{O})) \cong (\mathscr{C}, \mathscr{O})$. Specially, we can find such projective transformation $\mathbb{P}$ which sends $\mathscr{O}$ to the point at infinity.

Proof.



Since every conic is a section of a cone, we can think like the figure. Let the blue conic be $\mathscr{C}$ with projection point $\mathscr{O}$. Let $\mathbb{P}$ as a projective transformation from the plane containing $\mathscr{C}$ to the plane which contains k, which has projection center as the apex of the cone. Then $\mathbb{P}$ preserves the intersection line of blue plane and purple plane. Therefore it is sufficient to prove $(\mathbb{P}(\mathscr{C}),\mathbb{P}(\mathscr{O})) \cong (\mathscr{C},\mathscr{O})$.

Every projective transformation sends line to line. Thus the line passing $\mathscr{O}$ is transformed to the line passing $\mathbb{P}(\mathscr{O})$. Therefore $(\mathbb{P}(\mathscr{C}),\mathbb{P}(\mathscr{O})) \cong (\mathscr{C},\mathscr{O})$, ending the proof.



In addition, we can take a purple plane which is parallel with the line which passing the apex of cone and $\mathscr{O}$. Then the projective transformation sends the point $\mathscr{O}$ to the point at infinity. Thus the projection on $\mathbb{P}(\mathscr{C})$ is caused through the line which is an asymptote if $\mathbb{P}(\mathscr{C})$ is not parabola, and axis when $\mathbb{P}(\mathscr{C})$ is parabola.

□

Theorem 4.8. *For any two conics with projection points* $(\mathscr{C},\mathscr{O}_1)$ *and* $(\mathscr{D},\mathscr{O}_2)$,

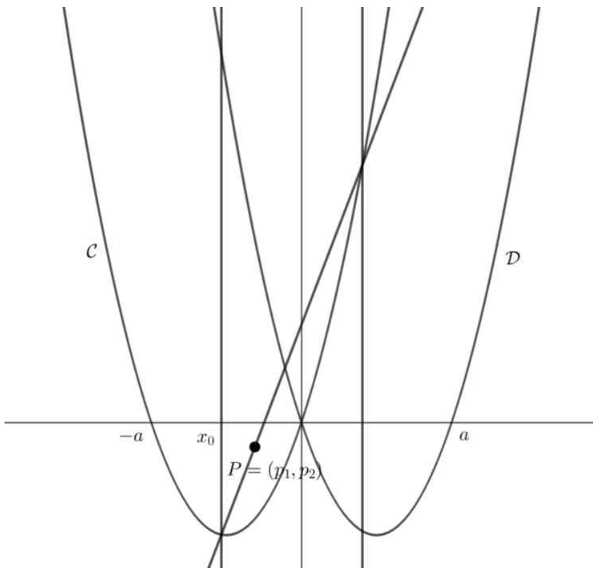
$$(\mathscr{C},\mathscr{O}_1) \cong (\mathscr{D},\mathscr{O}_2)$$

Theorem 4.8 implies that if we choose different conics, then it gives us the same result on reverse invariant. If we chose same conics $\mathscr{C}=\mathscr{D}$, this gives us that we can choose any projection point on the conic.

Proof.

First, we will prove the case of translation and scaling of parabola and a projection point on it.

Step 1. Translation and scaling of parabola with same projection point at infinity.



Case 1.1. x-axis translation

For two x-axis translated parabolas $\mathscr{C}$, $\mathscr{D}$ in the figure and let $\mathscr{O}$, $\mathscr{P}$ as projection points respectively. Think an affine transformation $(x,y)\mapsto(x,y-2ax)$ then we can know that this preserves the x coordinate, and sends $\mathscr{C}$ to $\mathscr{D}$. This implies ($\mathscr{C}$,$\mathscr{O}$) and ($\mathscr{D}$,$\mathscr{P}$) are equivalent.

Case 1.2. y-axis translation

It is enough to find the translation between the two lines. But affine transformation which is translating $\mathscr{C}$ to $\mathscr{D}$ satisfies this condition, thus ($\mathscr{C}$,$\mathscr{O}$) and ($\mathscr{D}$,$\mathscr{P}$) are equivalent.

Case 1.3. x-scaling

It is sufficient to find the transformation between two lines, but scaling which sends $\mathscr{C}$ to $\mathscr{D}$ satisfies this. Thus ($\mathscr{C}$,$\mathscr{O}$) and ($\mathscr{D}$,$\mathscr{P}$) are equivalent.

Case 1.4. y-scaling

y-scaling is also x-axis scaling and x-translation thus this case is same as the Case 3.

Step 2. Translation and scaling of hyperbola with same projection point at infinity.

It is easy that the image of $\mathscr{C}$ by affine transformation which preserves lines which passes $\mathscr{O}$ is equivalent with $\mathscr{C}$. And it is possible to send the hyperbola to parabola by using this affine transformation.

Case 2.1. Translation of hyperbola

This case is equivalent with the problem of translation of parabola, and it was solved.

Case 2.2. Scaling of hyperbole

This case is equivalent with the problem of scaling of parabola, and it was solved.

Step 3. x-axis Translation and Scaling of Circles

Case 3.1. x-axis Translation

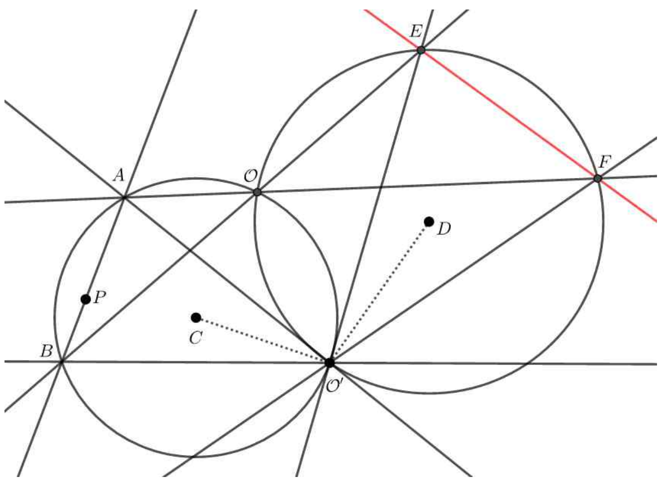
Let $\mathscr{C}$ and $\mathscr{D}$ be the x-axis translated two circles and $\mathscr{O}_1$, $\mathscr{O}_2$ be projection points respectively. We can regard each circles to be on a cone respectively. Use the special projective transformation of Theorem 4.7, $\mathbb{P}$ for each circle. Then we can remove x-axis line k since line which passes $\mathbb{P}(\mathscr{O}_1)$ and line which passes $\mathbb{P}(\mathscr{O}_2)$ are parallel. Therefore the scaling which focused on the line which passes $\mathbb{P}(\mathscr{O}_1)$, of $\mathscr{C}$ are all equivalent. But we can make the conic $\mathscr{C}$ goes to the parabola $\mathscr{C}'$ by scaling, and we can do this at $\mathscr{D}$ which goes to parabola $\mathscr{D}'$. We proved the case of the translation of the parabola in Step 1, thus this case is true.

Case 3.2. Scaling of the Circle

Let $(\mathscr{D}, \mathscr{O}_2)$ is scaling of $(\mathscr{C}, \mathscr{O}_1)$ whose center is scaling center which is on the x-axis line k. Similarly to Case 1, it is sufficient to prove the case of conic with projection through the line, and two lines $\mathbb{P}(\mathscr{O}_1)$ and $\mathbb{P}(\mathscr{O}_2)$ are parallel since it is scaling. Thus we can reduce this case into parabola, and prove is over.

Step 4. The Case of Circles

Let $(\mathscr{C}, \mathscr{O}_1)$ and $(\mathscr{D}, \mathscr{O}_2)$ are two circle with projection points.. Then by Step 1, we can make two projective points $\mathscr{O}_1$ and $\mathscr{O}_2$ are same. Suppose $\mathscr{O}_1$ and $\mathscr{O}_2$ are same with $\mathscr{O}$.



It the figure, it is enough to prove the line which passes A and B goes to line which passes E and F by constant spiral transformation. But we know two triangles are similar, and the $\angle C\mathscr{O}'D$ is constant, The spiral transformation rotates as this angles ans scales as ratio between two radius. This proves the step.

Step 5. General Case

Let $(\mathscr{C},\mathscr{O}_1)$ and $(\mathscr{D},\mathscr{O}_2)$ are two conics with projection points. We can send $\mathscr{D}$ to circle by using a proper projective transformation. Thus we can suppose $\mathscr{D}$ be a circle.

Similarly, consider a projective transformation which sends $\mathscr{C}$ to a circle, By step 4, the circle and its projection point is equivalent to the circle and arbitrary projection point on the circle. Taking the inverse of projective transformation, $\mathscr{C}$ and its projection point is equivalent to $\mathscr{C}$ and arbitrary projection point on $\mathscr{C}$. And we will put the projection point on the symmetric axis of $\mathscr{C}$.

By proper scaling, translation of $\mathscr{D}$ and Step 4, you can take the projective point $\mathscr{D}$ same as the projecton point of $\mathscr{C}$. Also I\It is able to make tangent line at projection point is same respect to each conics. Now, send the tangent line which passes the projection point to line at infinity, then $\mathscr{C}$ and $\mathscr{D}$ goes to two parabolas which two are scaling. But we know two are equivalent, and this proves the theorem.

□

Corollary 4.8. A degree of the reverse invariant is invariant on choosing the conic-projective point pair $(\mathscr{C},\mathscr{O})$.

Corollary 4.9. ψ is projective transformation.

Conjecture 3. Any projective transformation on k^n can be expressed not using higher dimension, but using the lines, conics, and intersections.

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ABSTRACT

q-Series and Modular Form

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Abstract

Most of Ramanujan's work in number theory arose out of q-series and theta functions. However, Ramanujan recorded his discoveries without proofs in notebooks. Hence there has been interest by number theorists in providing the proofs of Ramanujan's results. In this paper, we focus on 23 theta function identities which are certain types of modular equations in Ramanujan's notebooks. Some of them are proved by using modular form. We found new modular equations and generalized the finding process. We find not only elementary proofs for them, but several new modular equations with their proofs based on simple algebraic computations. In addition we give some definitions of the reverse function and the reverse invariant on the conic with projection point, and then find the equivalence for any two conics with projection points.

Key Words : q-series, modular equation, invariant, conic, projective geometry