# Fluid mechanics of turbulent flows

## Fluidmechanik turbulenter Strömungen

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Lecture 4

# LECTURE 4

# Free shear flows

# Questions to be answered in the present lecture

How does a turbulent flow develop away from solid boundaries?

How can the equations be simplified for slow spatial evolution?

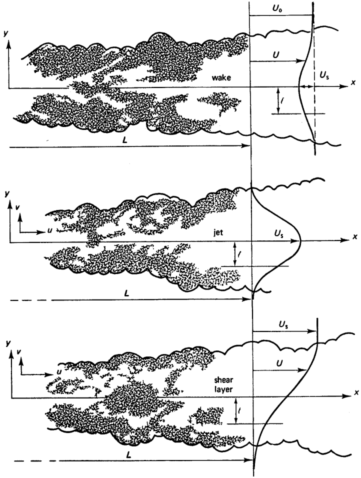
What is the evolution in the self-similar region?

What is the turbulence structure in a plane jet?

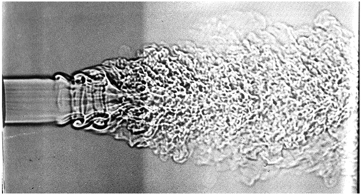
# Types of free shear flows

## Flows developing far from solid boundaries

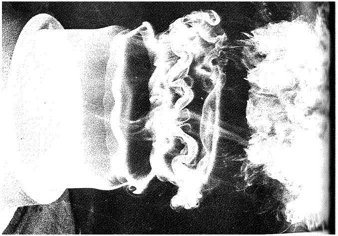
- wake
- jet
- mixing layer



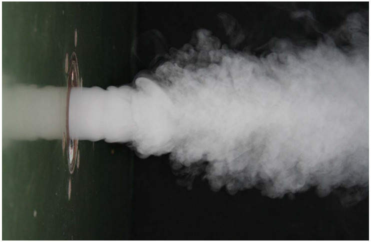
# The round jet



$Re = 3000$ (van Dyke)



$Re = 13000$ (van Dyke)



$Re \approx 5000$ (Deutsch & Haneka)

(movie)

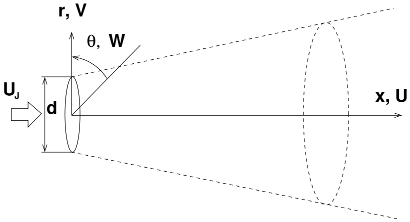
# Configuration and coordinate system

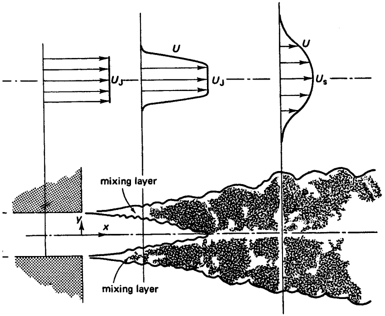
## Steady round jet

- ambient surroundings
- nozzle diameter $d$
- exit velocity $U_J$

$\rightarrow$ Reynolds: $Re = U_J d/\nu$

- statistically stationary, stat. axisymmetric flow
- initial development $(20d)$ then self-similar



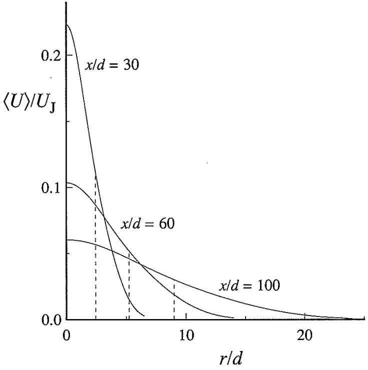
# Mean velocity and self-similarity

## Observations from experiments

- mean axial velocity $\langle u(x,r)\rangle$

$\rightarrow$ velocity decays on axis
& jet spreads radially

- profile shape remains similar
- define centerline velocity:
  $U_0(x) \equiv \langle u(x,0)\rangle$
- define jet half width $r_{1/2}(x)$:
  $\langle u(x,r_{1/2})\rangle = U_0(x)/2$




($Re = 95500$, Hussein et al. 1994)

# Mean velocity and self-similarity (2)

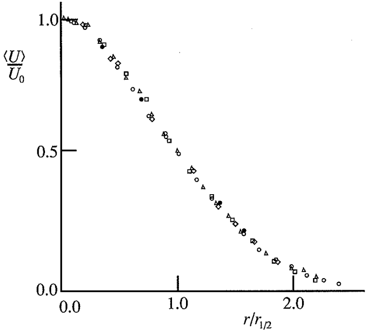
## Observations from experiments

- scaling of radial coordinate: $\xi \equiv r/r_{1/2}$
- scaling of velocity: $f(\xi) \equiv \langle u \rangle(x,r)/U_0(x)$

$\rightarrow$ profiles collapse

- spreading rate $S \approx 0.094$: $r_{1/2}(x) = S(x - x_0)$
- velocity decay ($B \approx 5.8$): $U_0(x) = B \cdot d \cdot U_J/(x - x_0)$

$\rightarrow$ local Reynolds is constant! $Re_0 = r_{1/2}(x)U_0(x)/\nu$




($Re = 10^5$, Wygnanski & Fiedler 1969)

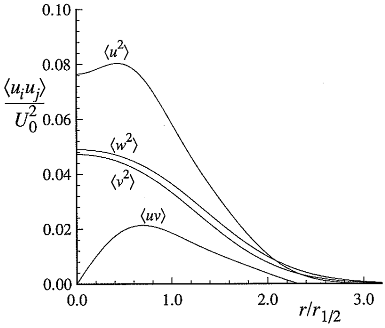
# Reynolds stresses

## Observations from experiments

- coordinate system:
  $\mathbf{x} = (x, r, \theta)$, $\mathbf{u} = (u, v, w)$
- due to symmetry:
  $\langle w \rangle = \langle u'w' \rangle = \langle v'w' \rangle = 0$

$$\begin{bmatrix} \langle u'u' \rangle & \langle u'v' \rangle & 0 \\ \langle u'v' \rangle & \langle v'v' \rangle & 0 \\ 0 & 0 & \langle w'w' \rangle \end{bmatrix}$$

- self-similarity of $\langle u_i'u_j' \rangle / U_0^2$
- significant anisotropy




(Hussein et al. 1994)

# Reynolds stresses

## Observations from experiments

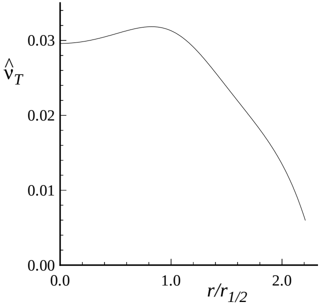
- supposing stress-strain relation, shear stress can be written:

$$\langle u'v' \rangle = -\nu_T \frac{\partial \langle u \rangle}{\partial r}$$

- positive eddy viscosity $\nu_T$
- self-similarity:

$$\hat{\nu}_T(r) \equiv \frac{\nu_T(x,r)}{U_0(x)\, r_{1/2}(x)}$$

$\Rightarrow$ plateau across jet




(Hussein et al. 1994)

# Navier-Stokes equations in cylindrical coordinates

## Instantaneous equations

$$
\begin{aligned}
\partial_x u + \frac{1}{r}\partial_r(rv) + \frac{1}{r}\partial_\theta w &= 0 \\
\partial_t u + u\partial_x u + v\partial_r u + \frac{w}{r}\partial_\theta u &= -\frac{1}{\rho}\partial_x p + \nu\nabla^2 u \\
\partial_t v + u\partial_x v + v\partial_r v + \frac{w}{r}\partial_\theta v - \frac{w^2}{r} &= -\frac{1}{\rho}\partial_r p + \nu\left(\nabla^2 v - \frac{v}{r^2} - \frac{2}{r^2}\partial_\theta w\right) \\
\partial_t w + u\partial_x w + v\partial_r w + \frac{w}{r}\partial_\theta w + \frac{vw}{r} &= -\frac{1}{r\rho}\partial_\theta p + \nu\left(\nabla^2 w - \frac{w}{r^2} + \frac{2}{r^2}\partial_\theta v\right)
\end{aligned}
$$

with:

$$
\nabla^2 = \partial_{xx} + \frac{1}{r}\partial_r r\partial_r + \frac{1}{r^2}\partial_{\theta\theta}
$$

# Reynolds-averaged Navier-Stokes in cylindrical coordinates

## Averaged equations

- non-swirling, statistically stationary & axisymmetric

$$
\begin{aligned}
0 &= \partial_x \langle u \rangle + \frac{1}{r}\partial_r (r\langle v \rangle) \\
\frac{\bar{\mathsf{D}}\langle u \rangle}{\bar{\mathsf{D}}t} &= -\frac{1}{\rho}\partial_x \langle p \rangle - \partial_x \langle u'u' \rangle - \frac{1}{r}\partial_r (r\langle u'v' \rangle) + \nu \nabla^2 \langle u \rangle \\
\frac{\bar{\mathsf{D}}\langle v \rangle}{\bar{\mathsf{D}}t} &= -\frac{1}{\rho}\partial_r \langle p \rangle - \partial_x \langle u'v' \rangle - \frac{1}{r}\partial_r (r\langle v'v' \rangle) + \frac{\langle w'w' \rangle}{r} \\
&\quad +\nu \left( \nabla^2 \langle v \rangle - \frac{\langle v \rangle}{r^2} \right)
\end{aligned}
$$

with: $\frac{\bar{\mathsf{D}}}{\bar{\mathsf{D}}t} = \partial_t + \langle u \rangle \partial_x + \langle v \rangle \partial_r$

# Boundary layer approximation

## Applicable to flows with slow spatial evolution

- small streamwise gradients: $\partial_x \sim 1/L$
- large cross-stream variation: $\partial_r \sim 1/\delta$

  with $L \gg \delta$

$\Rightarrow$ simplification of Navier-Stokes equations

## Examples:

- free shear flows (jet, wake, mixing layer)
- wall-bounded flows (boundary layer)

# Boundary layer equations for round jet

## Perform order-of-magnitude analysis

- $U$, $V$ are reference velocities
- continuity equation:

$$\underbrace{\partial_x \langle u \rangle}_{\mathcal{O}\left(\frac{U}{L}\right)} + \underbrace{\frac{1}{r}\partial_r (r\langle v \rangle)}_{\mathcal{O}\left(\frac{V}{\delta}\right)} = 0$$

- balance of the terms:

$$V \sim \frac{U\delta}{L}$$

$\Rightarrow$ reference velocity in radial direction: $\dfrac{U\delta}{L}$

# Boundary layer equations for round jet (2)

## Radial momentum equation

| term | order $\mathcal{O}(\cdot)$ | $\mathcal{O}(\cdot)$, divided by $U^2\delta/L^2$ | dominant |
|---|---|---|---|
| $\partial_t\langle v\rangle$ | $0$ (stationary) | $0$ | |
| $+\langle u\rangle\partial_x\langle v\rangle$ | $UV/L = U^2\delta/L^2$ | $1$ | |
| $+\langle v\rangle\partial_r\langle v\rangle$ | $V^2/\delta = U^2\delta/L^2$ | $1$ | |
| $= -\frac{1}{\rho}\partial_r\langle p\rangle$ | $\Delta p_r/(\rho\delta)$ | $\Delta p_r L^2/(\rho U^2\delta^2)$ | • |
| $-\partial_x\langle u'v'\rangle$ | $R_{12}/L$ | $R_{12}L/(U^2\delta)$ | |
| $-\frac{1}{r}\partial_r(r\langle v'v'\rangle)$ | $R_{22}/\delta$ | $R_{22}L^2/(U^2\delta^2)$ | • |
| $+\frac{1}{r}\langle w'w'\rangle$ | $R_{33}/\delta$ | $R_{33}L^2/(U^2\delta^2)$ | • |
| $+\nu\partial_{xx}\langle v\rangle$ | $\nu V/L^2 = \nu U\delta/L^3$ | $1/Re$ | |
| $+\nu\frac{1}{r}\partial_r(r\partial_r\langle v\rangle)$ | $\nu V/\delta^2 = \nu U/(\delta L)$ | $\frac{1}{Re}\frac{L^2}{\delta^2}$ | |
| $+\nu\frac{1}{r^2}\partial_{\theta\theta}\langle v\rangle$ | $0$ (axisymmetric) | $0$ | |
| $-\nu\frac{\langle v\rangle}{r^2}$ | $\nu V/\delta^2 = \nu U/(L\delta)$ | $\frac{1}{Re}\frac{L^2}{\delta^2}$ | |

$$\boxed{\Rightarrow \quad \frac{1}{\rho}\partial_r\langle p\rangle + \frac{1}{r}\partial_r(r\langle v'v'\rangle) - \frac{1}{r}\langle w'w'\rangle = 0}$$

# Boundary layer equations for round jet (3)

## Obtaining the streamwise pressure gradient

- integrate the approximate radial momentum equation:

$$\frac{\langle p \rangle}{\rho} = \frac{p_0(x)}{\rho} - \langle v'v' \rangle + \int_r^\infty \left( \frac{\langle v'v' \rangle}{r'} - \frac{\langle w'w' \rangle}{r'} \right) \mathrm{d}r'$$

- 

$$\frac{1}{\rho} \partial_x \langle p \rangle = \frac{1}{\rho} \frac{\mathrm{d}p_0}{\mathrm{d}x} - \partial_x \langle v'v' \rangle + \partial_x \int_r^\infty \left( \frac{\langle v'v' \rangle}{r'} - \frac{\langle w'w' \rangle}{r'} \right) \mathrm{d}r'$$

- neglecting streamwise gradients of Reynolds stresses:

$$\boxed{\partial_x \langle p \rangle \approx \frac{\mathrm{d}p_0}{\mathrm{d}x}}$$

# Boundary layer equations for round jet (4)

## Streamwise momentum equation

| term | order $\mathcal{O}(\cdot)$ | $\mathcal{O}(\cdot)$, divided by $U^2/L$ | dominant |
|---|---|---|---|
| $\partial_t \langle u \rangle$ | $0$ (stationary) | $0$ | |
| $+\langle u \rangle \partial_x \langle u \rangle$ | $U^2/L$ | $1$ | • |
| $+\langle v \rangle \partial_r \langle u \rangle$ | $VU/\delta = U^2/L$ | $1$ | • |
| $= -\frac{1}{\rho}\partial_x \langle p \rangle$ | $\Delta p_x/(\rho L)$ | $\Delta p_x/(\rho U^2)$ | • |
| $-\partial_x \langle u'u' \rangle$ | $R_{11}/L$ | $R_{11}/U^2$ | |
| $-\frac{1}{r}\partial_r (r\langle u'v' \rangle)$ | $R_{12}/\delta$ | $R_{12}L/(U^2\delta)$ | • |
| $+\nu \partial_{xx} \langle u \rangle$ | $\nu U/L^2$ | $1/Re$ | |
| $+\nu \frac{1}{r}\partial_r (r\partial_r \langle u \rangle)$ | $\nu U/\delta^2$ | $\frac{1}{Re\delta^2}$ | |
| $+\nu \frac{1}{r^2}\partial_{\theta\theta} \langle u \rangle$ | $0$ (axisymmetric) | $0$ | |

$$\boxed{\Rightarrow \quad \langle u \rangle \partial_x \langle u \rangle + \langle v \rangle \partial_r \langle u \rangle = -\frac{1}{\rho}\partial_x \langle p \rangle - \frac{1}{r}\partial_r (r\langle u'v' \rangle)}$$

# Boundary layer equations for round jet – final result

$$\begin{aligned}
\partial_x\langle u\rangle + \frac{1}{r}\partial_r(r\langle v\rangle) &= 0 \\
\langle u\rangle\partial_x\langle u\rangle + \langle v\rangle\partial_r\langle u\rangle &= -\cancel{\frac{1}{\rho}\frac{\mathrm{d}p_0}{\mathrm{d}x}} - \frac{1}{r}\partial_r(r\langle u'v'\rangle)
\end{aligned}$$

Using the BL approximation:

- it can be shown that self-similarity implies:

$$\begin{aligned}
U_0 &\sim \frac{1}{(x-x_0)} \\
r_{1/2} &\sim x - x_0
\end{aligned}$$

$\rightarrow$ consistent with experimental observations (cf. above)

# Closure of equations in BL approximation

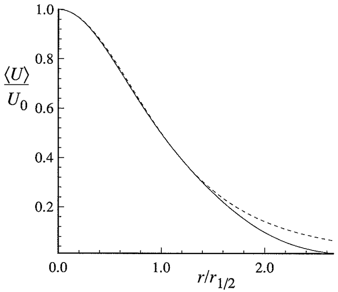
## Assuming a uniform turbulent viscosity

- define $-\langle u'v'\rangle = \nu_T \partial_r \langle u\rangle$
- experiments show similarity: $\nu_T(x,r) = r_{1/2}(x) U_0(x) \hat{\nu}_T(r)$
- $\hat{\nu}_T \approx 0.028$ across jet

$\Rightarrow$ assume $\hat{\nu}_T = cst.$

- BL equations closed

$\Rightarrow$ reasonable prediction of $\langle u\rangle$ (disagreement near edge)




$\nu_T = cst$ approx. vs. Hussein et al. 1994

# Kinetic energy equation

## Instantaneous kinetic energy

- definition: $E_k(\mathbf{x},t) \equiv \frac{1}{2}\mathbf{u}\cdot\mathbf{u}$
- transport equation (cf. lecture 2):

$$\frac{\mathrm{D}E_k}{\mathrm{D}t} = -\left(u_j\, p/\rho\right)_{,j} + \left(2\nu u_i S_{ij}\right)_{,j} - 2\nu S_{ij}S_{ij}$$

## Mean kinetic energy

- decomposition:

$$\langle E \rangle = \underbrace{\frac{1}{2}\langle \mathbf{u}\rangle \cdot \langle \mathbf{u}\rangle}_{\substack{\equiv \bar{E} \\ \text{(due to mean flow)}}} + \underbrace{\frac{1}{2}\langle \mathbf{u}' \cdot \mathbf{u}'\rangle}_{\substack{\equiv k \\ \text{(due to turbulence)}}}$$

# Kinetic energy of mean flow and turbulence

## Transport equations

$$
\begin{aligned}
\partial_t \bar{E} + \left(\langle u_j\rangle \bar{E} + \langle u_i\rangle\langle u_i' u_j'\rangle + \langle u_j\rangle\langle p\rangle/\rho - 2\nu\langle u_i\rangle \bar{S}_{ij}\right)_{,j} &= -\mathcal{P} - \bar{\varepsilon} \\
\partial_t k + \left(\langle u_j\rangle k + \frac{1}{2}\langle u_i' u_i' u_j'\rangle + \langle u_j' p'\rangle/\rho - 2\nu\langle u_i' S_{ij}'\rangle\right)_{,j} &= +\mathcal{P} - \varepsilon
\end{aligned}
$$

- production term: $\mathcal{P} \equiv -\langle u_i' u_j'\rangle \langle u_i\rangle_{,j}$

$\Rightarrow$ sink for $\bar{E}$, source for $k$ $\rightarrow$ exchange term!

- dissipation due to mean flow: $\bar{\varepsilon} \equiv 2\nu \bar{S}_{ij}\bar{S}_{ij}$
- dissipation due to turbulence: $\varepsilon \equiv 2\nu\langle S_{ij}' S_{ij}'\rangle$

# Scaling of dissipation

## Round jet flow

- considering self-similarity of $\langle u \rangle$, $\langle u'_i u'_j \rangle$
- production term (BL approximation) scales as:

$$\mathcal{P}/(U_0^3/r_{1/2}) \approx -\frac{\langle u'v' \rangle}{U_0^2}\frac{r_{1/2}}{U_0}\partial_r \langle u \rangle$$

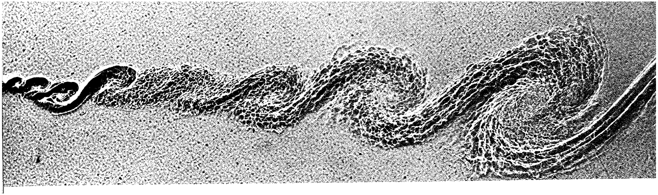
$\rightarrow$ turbulent dissipation will also scale as:

$$\hat{\varepsilon} = \varepsilon/(U_0^3/r_{1/2})$$

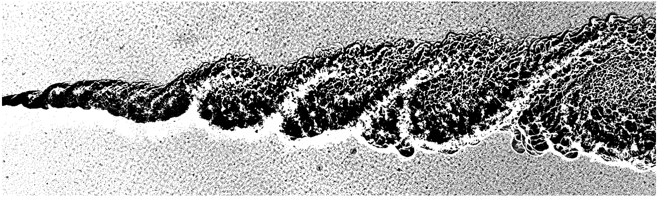
$\Rightarrow$ dissipation defined as $2\nu\langle s'_{ij}s'_{ij} \rangle$, but found independent of $\nu$!

- finer scales at higher $Re$ $\rightarrow$ higher gradients $s'_{ij}$

# Finer scales with increasing Reynolds



$Re = 10^5$



$Re = 2 \cdot 10^5$ (Brown & Roshko 1974)

# Kolmogorov scales

## Why is dissipation independent of the value for viscosity?

- define characteristic scales of smallest turbulent motion:

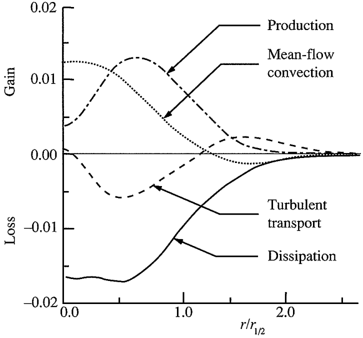
$$\eta \equiv \left(\nu^3/\varepsilon\right)^{1/4} \ , \quad \tau_\eta \equiv (\nu/\varepsilon)^{1/2} \ , \quad u_\eta \equiv (\nu\varepsilon)^{1/4}$$

- Kolmogorov-scale Reynolds number: $Re_\eta \equiv \eta \, u_\eta/\nu = 1$
- length scale decreases with Reynolds: $\eta/r_{1/2} = Re_0^{-3/4} \, \hat{\varepsilon}^{-1/4}$
  $\rightarrow$ compensates changes in viscosity
- more details on scaling in lecture 5

# Turbulent kinetic energy budget

## Observations from experiments

- mean-flow convection:
  $-\left(\langle u_j \rangle k\right)_{,j}$
- turbulent transport:
  $$-\left(\frac{1}{2}\langle u_i' u_i' u_j' \rangle + \langle u_j' p' \rangle / \rho - 2\nu \langle u_i' S_{ij}' \rangle\right)_{,j}$$
- production: $\mathcal{P}$
- dissipation: $-\varepsilon$




(Panchapakesan & Lumley 1993)

# Comparison of time scales in turbulent free shear flow

## Round jet data

- reference scale:
  $\tau_0 \equiv r_{1/2}/U_0$ $\qquad\tau_0/\tau_0 = 1$
- mean shear:
  $\tau_{\mathcal{S}} \equiv 1/\partial_r\langle U\rangle$ $\qquad\tau_{\mathcal{S}}/\tau_0 \approx 1.7$
- turbulence production:
  $\tau_{\mathcal{P}} \equiv k/\mathcal{P}$ $\qquad\tau_{\mathcal{P}}/\tau_0 \approx 6$
- turbulence dissipation:
  $\tau_\varepsilon \equiv k/\varepsilon$ $\qquad\tau_\varepsilon/\tau_0 \approx 4.5$
- mean "flight time" from virtual origin:
  $\tau_J \equiv \frac{1}{2}x/U_0$ $\qquad\tau_J/\tau_0 \approx 5.3$

$\rightarrow$ turbulence is long-lived

## Other canonical free shear flows

Plane jet

Mixing layer

Plane or axisymmetric wake

Homogeneous shear flow

Homogeneous-isotropic flow

# Summary

## Main questions of the present lecture

- How does a turbulent flow develop away from solid boundaries?
- How can the equations be simplified for slow spatial evolution?
    - boundary layer approximation
- What is the evolution in the self-similar region?
    - round jet: linear spreading, mean velocities $\sim 1/x$
- Turbulence structure in the round jet:
    - turbulent kinetic energy budget
    - crude approximation with uniform turbulent viscosity

# Problem

*Derive the formula for the mean "flight time" of a fluid particle on the round jet axis, measured from the virtual origin $x_0$ to a position $x$. (Answer: $\tau_J = \frac{1}{2}x/U_0$)*

# Outlook on next lecture: The scales of turbulence

How are energy and anisotropy distributed among scales?

Which physical processes occur on each scale?

# Further reading

- S. Pope, *Turbulent flows*, 2000
  $\rightarrow$ chapter 5
- H. Tennekes and J.L. Lumley, *First Course in Turbulence*, 1972
  $\rightarrow$ chapter 5