Statistics for Management and Economics

OPRE 6301 (Statistics and Data Analysis)
by
Ayşegül Toptal Bilhan

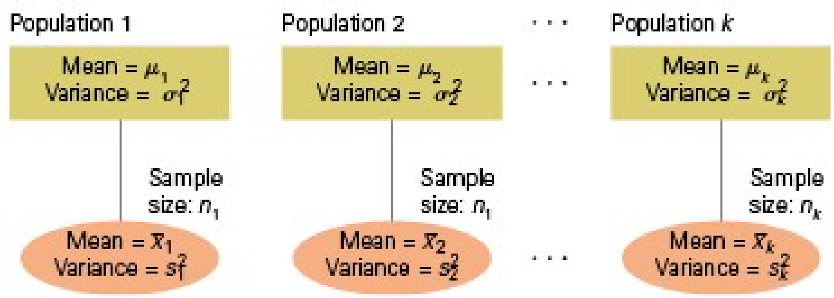
Chapter 14
(Analysis of Variance)

Analysis of Variance

- Analysis of Variance and Completely Randomized Design
- Multiple Comparison Procedures

Introduction

Determines whether differences exist between two or more population means.



Introduction (Cont'd)

Example: Three adhesives are being analyzed for their impact on the bonding strength of paper in a pulp and paper mill. The adhesives are each randomly applied to four batches. The data is shown in the below table. Is there a difference among the adhesives in terms of the mean bonding strength?

Adhesive Type	Bonding Strength (kg)			
Adhesive 1	10.2	11.8	9.6	12.4
Adhesive 2	12.8	14.7	13.3	15.4
Adhesive 3	7.2	9.8	8.7	9.2

Introduction (Cont'd)

-We conduct experiments to

- support or refute a hypothesis/conjecture
- discover new information about a product, process, service.

-Different parameters effect the output. In mathematical terms, we have

$$\eta = f\left(x_1, x_2, \ldots, x_n, \theta_1, \theta_2, \ldots, \theta_k\right),$$

where

η: mean response,

$x_1, x_2, \ldots, x_n$: input variables under control,

$\theta_1, \theta_2, \ldots, \theta_k$: uncontrollable parameters.

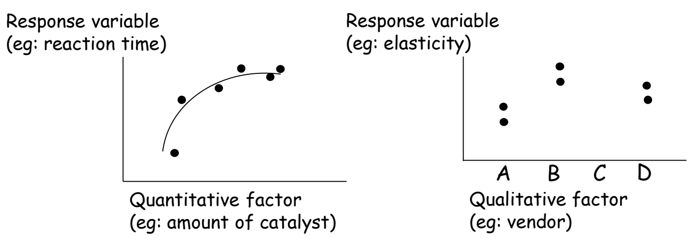
Introduction (Cont'd)

Concepts and Definitions:

Response variable: Output

Factor : Controllable parameter that effects the output

Levels : Alternative values of a factor considered in experimentation
(Continuous or discrete)



Introduction (Cont'd)

Concepts and Definitions (Cont'd):

Treatment : Combination of factor levels

Experimental unit: Quantity of material to which a single treatment is applied

Sampling unit: Fraction of experimental unit chosen for analysis

Completely Randomized Design:

- Treatments are randomly assigned to the experimental units.
- Simplest and least restrictive design
- When all treatments are replicated equal number of times, the experiment is *balanced*; otherwise it is *unbalanced*.

Introduction (Cont'd)

Notation for the One-Way Analysis of Variance

TREATMENT

	1	2	…	j	…	k
	x_{11}	x_{12}	…	x_{1j}	…	x_{1k}
	x_{21}	x_{22}	…	x_{2j}	…	x_{2k}
	.	.	.	.	.	.
	$x_{n_1 1}$	$x_{n_2 2}$	.	$x_{n_j j}$	.	$x_{n_k k}$
Sample size	n_1	n_2	.	n_j	.	n_k
Sample mean	$\bar{x}_1$	$\bar{x}_2$	…	$\bar{x}_j$		$\bar{x}_k$

k: number of populations

x_{ij}: i^{th} observation of the sample taken from the j^{th} population

n_j: number of observations in the sample taken from the j^{th} population

$\bar{x}_j$: mean of the sample taken from the j^{th} population

$$\bar{x}_j = \frac{\sum_{i=1}^{n_j} x_{ij}}{n_j}$$

$\bar{\bar{x}}$: grand mean of all the observations

$$\bar{\bar{x}} = \frac{\sum_{j=1}^{k}\sum_{i=1}^{n_j} x_{ij}}{n}$$

$$n = n_1 + n_2 + \cdots + n_k$$

Analysis of Variance

H_0: $\mu_1 = \mu_2 = \mu_3 = \cdots = \mu_k$

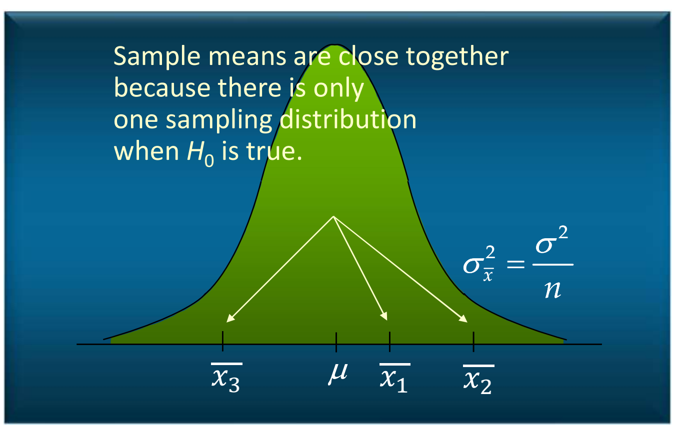
H_a: At least one of the means is different

k = the number of populations we are comparing
we call these population "treatments"

Note: A required condition is population variances are equal

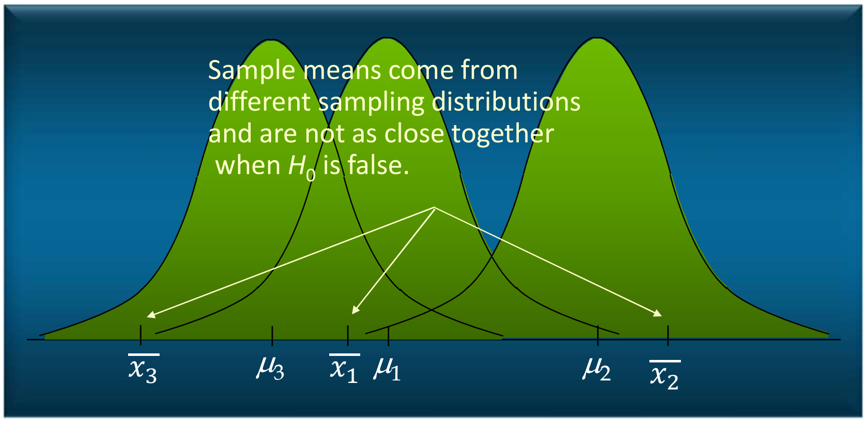
Analysis of Variance (Cont'd)

Sampling Distribution of $\overline{x}$ given H_0 is True



Analysis of Variance (Cont'd)

Sampling Distribution of $\overline{x}$ given H_0 is False



Analysis of Variance (Cont'd)

Test for Difference Among the Treatment Means

$$H_0\!: \mu_1 = \mu_2 = \mu_3 = \cdots = \mu_k$$
$$H_a\!: \text{At least one of the means is different}$$

The test statistic is

$$F = \frac{MST}{MSE},$$

where MST: the mean squares for treatment & MSE: the mean square error

For a chosen level of significance α, if

$$F > F_{\alpha,(k-1),(n-k)}$$

then the null hypothesis is rejected.

Analysis of Variance (Cont'd)

Test for Difference Among the Treatment Means (Cont'd)

How do we find MST and MSE?

$$MST = \frac{SST}{k-1} \quad \text{and} \quad MSE = \frac{SSE}{(n-k)}$$

where SST: Sum of Squares for Treatments, SSE: Sum of Squares for Error

$$SST = \sum_{j=1}^{k} n_j(\bar{x}_j - \bar{\bar{x}})^2 \quad \text{and} \quad SSE = \sum_{j=1}^{k}\sum_{i=1}^{n_j}(x_{ij} - \bar{x}_j)^2 \quad \left(or \quad SSE = \sum_{j=1}^{k}(n_j - 1)s_j^2 \right)$$

Multiple Comparison Procedures

- Fisher's Least Significant Difference Method (LSD)

$$LSD = t_{\alpha/2,\nu}\sqrt{MSE\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}$$

Here, v=n-k.

- We conclude that μ_i and μ_j differ if: $|\bar{x}_i - \bar{x}_j| > LSD$

Example (Cont'd)

Returning back to our earlier example, we are ready to answer the following question:

Is there a difference among the adhesives in terms of the mean bonding strength? Test at the 5% level of significance.

Adhesive Type	Bonding Strength (kg)				Mean
Adhesive 1	10.2	11.8	9.6	12.4	$\bar{x}_1 = 11.000$
Adhesive 2	12.8	14.7	13.3	15.4	$\bar{x}_2 = 14.050$
Adhesive 3	7.2	9.8	8.7	9.2	$\bar{x}_3 = 8.725$

$\bar{\bar{x}} = \frac{\sum_{j=1}^{3}\sum_{i=1}^{4} x_{ij}}{12} = \frac{135.1}{12}$=11.258

Example (Cont'd)

Returning back to our earlier example, we are ready to answer the following question:

Is there a difference among the adhesives in terms of the mean bonding strength? Test at the 5% level of significance.

Adhesive Type	Bonding Strength (kg)				Mean
Adhesive 1	10.2	11.8	9.6	12.4	$\bar{x}_1 = 11.000$
Adhesive 2	12.8	14.7	13.3	15.4	$\bar{x}_2 = 14.050$
Adhesive 3	7.2	9.8	8.7	9.2	$\bar{x}_3 = 8.725$

$\bar{\bar{x}} = \frac{\sum_{j=1}^{3}\sum_{i=1}^{4} x_{ij}}{12} = \frac{135.1}{12}$=11.258

Example (Cont'd)

$SST = \sum_{j=1}^{k} n_j \left(\bar{x}_j - \bar{\bar{x}}\right)^2 = 4(11 - 11.258)^2 + 4(14.05 - 11.258)^2 + 4(8.725 - 11.258)^2$=57.11

$$SSE = \sum_{j=1}^{k}\sum_{i=1}^{n_j}\left(x_{ij} - \bar{x}_j\right)^2 = \sum_{j=1}^{3}\sum_{i=1}^{4}\left(x_{ij} - \bar{x}_j\right)^2$$

$$= \sum_{i=1}^{4}(x_{i1} - \bar{x}_1)^2 + \sum_{i=1}^{4}(x_{i2} - \bar{x}_2)^2 + \sum_{i=1}^{4}(x_{i3} - \bar{x}_3)^2$$

$$= (10.2 - 11)^2 + \cdots + (12.4 - 11)^2 + (12.8 - 14.05)^2 + \cdots + (9.2 - 8.725)^2$$

$$= 13.2775$$

Example (Cont'd)

$MST = \frac{SST}{k-1} = \frac{57.11}{3-1} = 28.555$

$MSE = \frac{SSE}{(n-k)} = \frac{13.2775}{(12-3)}$=1.4753

The test statistic is: $F = \dfrac{MST}{MSE} = \dfrac{28.555}{1.4753} = 19.356$

Our critical value is $F_{\alpha,(k-1),(n-k)} = F_{0.05,2,9} = 4.26$.

Since F=19.356> $F_{\alpha,(k-1),(n-k)} = F_{0.05,2,9} = 4.26$, we reject the null hypothesis.