Statistics for Management and Economics

OPRE 6301 (Statistics and Data Analysis)
by
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Chapters 11, 12, and 13
(Introduction to Hypothesis Testing,
Inference About a Population,
Inference About Comparing Two Populations)

Confidence Intervals and Hypothesis Testing

- Developing Null and Alternative Hypotheses
- Type I and Type II Errors
- Inferences About A Population
  - Population Mean: When σ Known
  - Population Mean: When σ Unknown
  - Population Proportion
- Inferences About Two Populations
  - Difference Between Two Means: Variances Known
  - Difference Between Two Means: Variances Unknown

Introduction (Cont'd)

Hypothesis testing can be used to determine whether a statement about the value of a population parameter should or should not be rejected.

- **Null Hypothesis:** denoted by H_0, is a tentative assumption about a population parameter
- **Alternative Hypothesis:** denoted by H_a, is the opposite of what is stated in the null hypothesis

The hypothesis testing procedure uses data from a sample to test the two competing statements indicated by H_0 and H_a, which are *mutually exclusive and exhaustive*.

Introduction (Cont'd)

Null hypothesis: Claim we try to find evidence against. It usually reflects the present situation.

Alternative hypothesis: Claim that we hope or suspect to be true instead of the null hypothesis. It reflects the anticipated change, hence, it is called the *research* hypothesis.

- Hypothesis testing is designed to assess the strength of the evidence against the null hypothesis.
- The null hypothesis is rejected in favor of the alternative only if sample evidence strongly suggests so. Otherwise we continue to believe the null hypothesis.

Developing Null and Alternative Hypotheses

Alternative Hypothesis as a Research Hypothesis

- Example:
  A new teaching method is developed that is believed to be better than the current method.

- Null Hypothesis:
  The new method is no better than the old method.

- Alternative Hypothesis:
  The new teaching method is better.

Developing Null and Alternative Hypotheses (Cont'd)

Alternative Hypothesis as a Research Hypothesis

- Example:
  A new sales force bonus plan is developed in an attempt to increase sales.

- Null Hypothesis:
  The new bonus plan does not increase sales.

- Alternative Hypothesis:
  The new bonus plan increases sales.

Developing Null and Alternative Hypotheses (Cont'd)

Alternative Hypothesis as a Research Hypothesis

- Example:
  A new drug is developed with the goal of lowering blood pressure more than the existing drug.

- Null Hypothesis:
  The new drug does not lower blood pressure more than the existing drug.

- Alternative Hypothesis:
  The new drug lowers blood pressure more than the existing drug.

Type I and Type II Errors
(Type I Error)

Because hypothesis tests are based on sample data, we must allow for the possibility of errors.

- A Type I error is rejecting H_0 when it is true.

  α=P(Type I error)=P(reject H_0 | H_0 is true)

  α is also referred to as significance level of the test.

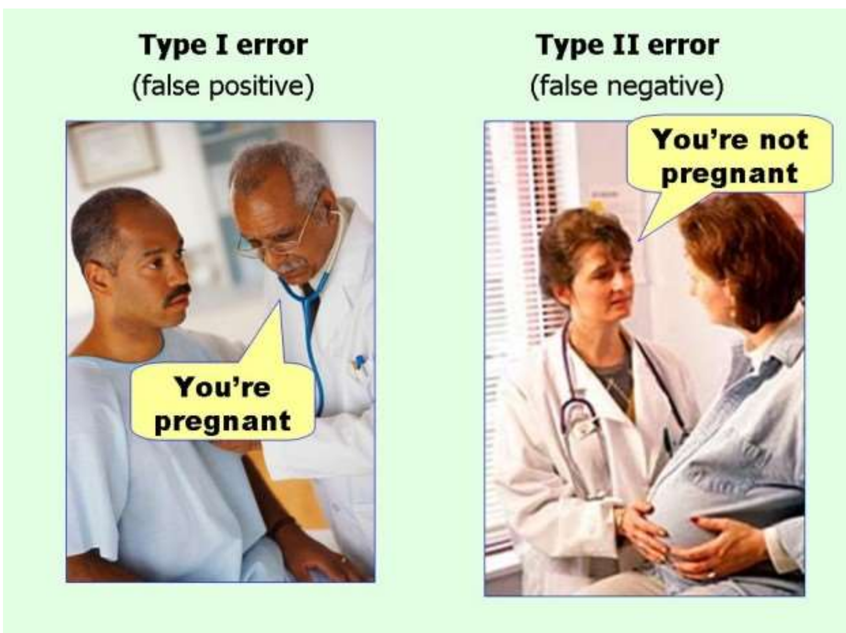
Type I and Type II Errors
(Type II Error)

Because hypothesis tests are based on sample data, we must allow for the possibility of errors.

- A Type II error is failing to reject H_0 when it is false.

$$\beta = P(\text{Type II error}) = P(\text{fail to reject } H_0 \mid H_0 \text{ is false})$$

(1- β) is also referred to as the power of the test.



Type I and Type II Errors (Cont'd)

Example:

Let's consider a non-statistical example:

A person is accused of a crime and faces a trial.

H_0: The defendant is innocent

H_a: The defendant is guilty

Type I and Type II Errors (Cont'd)

Example (Cont'd):

Jury Decision	Truth about the Defendant: H_0 True (Defendant is Innocent)	H_0 False (Defendant is Guilty)
Reject H_0 (Convict Defendant)	Type I Error α	Correct Decision
Do Not Reject H_0 (Acquit Defendant)	Correct Decision	Type II Error β

Hypothesis Testing of the Population Mean

The equality part of the hypotheses always appears in the null hypothesis.

- A hypothesis test about the value of a population mean μ must take one of the following three forms (where μ_0 is the hypothesized value of the population mean).

One-tailed (lower-tail)	One-tailed (upper-tail)	Two-tailed
$H_0: \mu \geq \mu_0$ $H_a: \mu < \mu_0$	$H_0: \mu \leq \mu_0$ $H_a: \mu > \mu_0$	$H_0: \mu = \mu_0$ $H_a: \mu \neq \mu_0$

Hypothesis Testing of the Population Mean (Cont'd)

Steps in Hypothesis Testing

1. State the null (H_0) and the alternative (H_a) hypothesis.
2. Determine α=P(Type I Error)=P(reject H_0 | H_0 is true)
   Traditional levels are α=0.01, 0.05.
3. Determine the test statistic and the distribution of the test statistic T under H_0.
4. Collect data and calculate T.
5. Compare the value of T to the reference value obtained from the null distribution of T. If the calculated value of T falls in the critical region, reject H_0. Otherwise we say that there is not enough evidence in the data to reject H_0.

Hypothesis Testing of the Population Mean (Variance Known)

Hypothesis Testing of the Mean (σ Known):

Defining $P(Z > z_\alpha)=\alpha$

	$H_0: \mu \geq \mu_0$ $H_a: \mu < \mu_0$	$H_0: \mu \leq \mu_0$ $H_a: \mu > \mu_0$	$H_0: \mu = \mu_0$ $H_a: \mu \neq \mu_0$
Rejection Region	$z_0 < -z_\alpha$	$z_0 > z_\alpha$	$\lvert z_0\rvert > z_{\alpha/2}$

Test Statistic $z_0 = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$

Hypothesis Testing of the Population Mean (Variance Known, Cont'd)

Example: Metro EMS

A major west coast city provides one of the most comprehensive emergency medical services in the world. Operating in a multiple hospital system with approximately 20 mobile medical units, the service goal is to respond to medical emergencies with a mean time of 12 minutes or less.

The response times for a random sample of 40 medical emergencies were tabulated. The sample mean is 13.25 minutes. The population standard deviation is believed to be 3.2 minutes.

The EMS director wants to perform a hypothesis test, with a .05 level of significance, to determine whether the service goal of 12 minutes or less is being achieved.

Hypothesis Testing of the Population Mean (Variance Known, Cont'd)

Example (Cont'd):

H_0: $\mu \leq 12$

The emergency service is meeting the response goal; no follow-up action is necessary.

H_a: $\mu > 12$

The emergency service is not meeting the response goal; appropriate follow-up action is necessary.

where: μ = mean response time for the population of medical emergency requests

Hypothesis Testing of the Population Mean (Variance Known, Cont'd)

Example (Cont'd):

Note that level of significance α is given as follows:

$$\alpha = .05$$

Let's compute the value of the test statistic.

$$z_0 = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{13.25 - 12}{3.2/\sqrt{40}} = 2.47$$

Rejection region is $z_0 > z_\alpha$. Since $z_{.05}$ = 1.645 and $z_0 = 2.47 > z_\alpha = 1.645$, we reject the null hypothesis.

Confidence Intervals and Hypothesis Testing
(Hypothesis Testing, Cont'd)

Solution (Cont'd):

Solution by P-Value Approach

- The P-value is the probability that the test statistic will take on a value that is at least as extreme as the observed value of the statistic when the null hypothesis H_0 is true.
- The P-value is the smallest level of significance that would lead to rejection of the null hypothesis H_0.

p-value= $P(\bar{X} \geq 13.25) = 1 - P(\bar{X} < 13.25)$

$$=1 - P\left(Z < \frac{13.25-}{\frac{3.2}{\sqrt{40}}}\right) = 1 - P(Z < 2.47) = 1 - 0.9932 = 0.0068$$

Since 0.0068<0.05, we reject H_0.

Hypothesis Testing of the Population Mean
(Variance Unknown)

Hypothesis Testing of the Mean (σ Unknown):

	$H_0: \mu \geq \mu_0$ $H_a: \mu < \mu_0$	$H_0: \mu \leq \mu_0$ $H_a: \mu > \mu_0$	$H_0: \mu = \mu_0$ $H_a: \mu \neq \mu_0$
Rejection Region	$t_0 < -t_{\alpha,n-1}$	$t_0 > t_{\alpha,n-1}$	$\lvert t_0 \rvert > t_{\alpha/2,n-1}$

Test Statistic $t_0 = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$

Hypothesis Testing of the Population Mean (Variance Unknown, Cont'd)

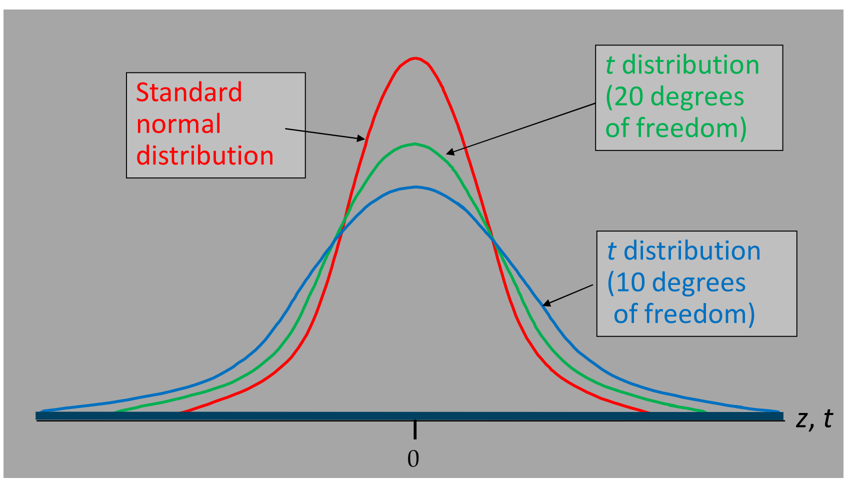
t Distribution:

- The t distribution is a family of similar probability distributions.
- A specific t distribution depends on a parameter known as the degrees of freedom.
- A t distribution with more degrees of freedom has less dispersion.

As the degrees of freedom increases, the difference between the t distribution and the standard normal probability distribution becomes smaller and smaller.

Hypothesis Testing of the Population Mean (Variance Unknown, Cont'd)

t Distribution (Cont'd):



For more than 100 degrees of freedom, the standard normal z value provides a good approximation to the t value.

t-Distribution Table (Portion)

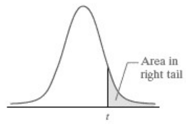


TABLE III

t-Distribution

Area in Right Tail

df	0.25	0.20	0.15	0.10	0.05	0.025	0.02	0.01	0.005
1	1.000	1.376	1.963	3.078	6.314	12.706	15.894	31.821	63.657
2	0.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925
3	0.765	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841
4	0.741	0.941	1.190	1.533	2.132	2.776	2.999	3.747	4.604
5	0.727	0.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032
6	0.718	0.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707
7	0.711	0.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499
8	0.706	0.889	1.108	1.397	1.860	2.306	2.449	2.896	3.355
9	0.703	0.883	1.100	1.383	1.833	2.262	2.398	2.821	3.250
10	0.700	0.879	1.093	1.372	1.812	2.228	2.359	2.764	3.169
11	0.697	0.876	1.088	1.363	1.796	2.201	2.328	2.718	3.106
12	0.695	0.873	1.083	1.356	1.782	2.179	2.303	2.681	3.055
13	0.694	0.870	1.079	1.350	1.771	2.160	2.282	2.650	3.012
14	0.692	0.868	1.076	1.345	1.761	2.145	2.264	2.624	2.977
15	0.691	0.866	1.074	1.341	1.753	2.131	2.249	2.602	2.947
16	0.690	0.865	1.071	1.337	1.746	2.120	2.235	2.583	2.921
17	0.689	0.863	1.069	1.333	1.740	2.110	2.224	2.567	2.898
18	0.688	0.862	1.067	1.330	1.734	2.101	2.214	2.552	2.878
19	0.688	0.861	1.066	1.328	1.729	2.093	2.205	2.539	2.861
20	0.687	0.860	1.064	1.325	1.725	2.086	2.197	2.528	2.845
21	0.686	0.859	1.063	1.323	1.721	2.080	2.189	2.518	2.831
22	0.686	0.858	1.061	1.321	1.717	2.074	2.183	2.508	2.819
23	0.685	0.858	1.060	1.319	1.714	2.069	2.177	2.500	2.807
24	0.685	0.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797
25	0.684	0.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787

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Hypothesis Testing of the Population Mean (Variance Unknown, Cont'd)

***t* Distribution (Cont'd):**

Degrees of Freedom	Area in Upper Tail .20	.10	.05	.025	.01	.005
.	.	.	.	.	.	.
50	.849	1.299	1.676	2.009	2.403	2.678
60	.848	1.296	1.671	2.000	2.390	2.660
80	.846	1.292	1.664	1.990	2.374	2.639
100	.845	1.290	1.660	1.984	2.364	2.626
∞	.842	1.282	1.645	1.960	2.326	2.576

Standard normal z values

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Inferences About Population Mean
(Interval Estimate of a Population Mean: σ Unknown)

Interval Estimate of μ:

$$\bar{x} \pm t_{\alpha/2} \frac{s}{\sqrt{n}}$$

where: $1 - \alpha$ is the confidence coefficient

$t_{\alpha/2}$ is the t value providing an area of $\alpha/2$ in the upper tail of a t distribution with $n - 1$ degree of freedom

s is the sample standard deviation

n is the sample size

Hypothesis Testing of the Population Mean
(Variance Unknown, Cont'd)

Example:

A water quality regulation in an effluent required a minimum dissolved oxygen concentration of three parts per million (3ppm). If a sample of five measurements resulted in an average value of 2.8 ppm and standard deviation of s=0.32 ppm, could it be concluded that the effluent meets the water quality regulation? Use α = 0.05.

$\mu = 3$, $s = 0.32$, $\overline{X} = 2.8$, $\alpha=0.05$, $n=5$

Hypothesis Testing of the Population Mean (Variance Unknown, Cont'd)

Example (Cont'd):

Our null and alternative hypotheses are as follows:

$$H_0 : \mu \geq 3$$
$$H_a : \mu < 3$$

Next, we will compute the value of $t_0 = \dfrac{\overline{X} - \mu_0}{s/\sqrt{n}}$.

$$t_0 = \frac{2.8 - 3}{0.32/\sqrt{5}} = -1.398$$

Hypothesis Testing of the Population Mean (Variance Unknown, Cont'd)

Example (Cont'd):

Our rejection region is defined by:

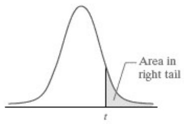
$$t_0 < -t_{\alpha,n-1} = -t_{0.05,4}$$

It turns out that

$$t_{0.05,4} = 2.132$$

Since t_0=-1.398 > -2.132, we conclude that there is no statistical basis for rejecting the null hypothesis. Hence the effluent meets the water quality regulation, even though the sample mean is less than the standard of 3 ppm.

t-Distribution Table (Portion)



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25	0.684	0.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787

Inferences About A Population Proportion

One-tailed (lower tail)	One-tailed (upper tail)	Two-tailed
$H_0: p \geq p_0$ $H_a: p < p_0$	$H_0: p \leq p_0$ $H_a: p > p_0$	$H_0: p = p_0$ $H_a: p \neq p_0$

Rejection Region: $z_0 < -z_\alpha$ | $z_0 > z_\alpha$ | $|z_0| > z_{\alpha/2}$

Test Statistic $$z_0 = \frac{\bar{p} - p_0}{\sqrt{p_0(1-p_0)/n}}$$

Inferences About A Population Proportion (Cont'd)

Confidence Interval for A Population Proportion:

The assumption here is $np \geq 5$ and $n(1-p) \geq 5$ so that the distribution of sample proportions approximately follow a Normal distribution.

$$\bar{p} - z_{\alpha/2}\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} \leq p \leq \bar{p} + z_{\alpha/2}\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

Inferences About A Population Proportion (Cont'd)

Example: National Safety Council (NSC)

For a Christmas and New Year's week, the National Safety Council estimated that 500 people would be killed and 25,000 injured on the nation's roads. The NSC claimed that 50% of the accidents would be caused by drunk driving.

A sample of 120 accidents showed that 67 were caused by drunk driving. Use these data to test the NSC's claim with $\alpha = .05$.

Inferences About A Population Proportion (Cont'd)

Example (Cont'd):

Our null and alternative hypotheses are as follows:

$$H_0: p = .5 \text{ and } H_a: p \neq .5$$

Next, we will compute the value of $z_0 = \frac{\bar{p}-p_0}{\sqrt{p_0(1-p_0)/n}}$.

$\bar{p}=\frac{67}{120} = 0.5583$. Therefore,

$$z_0 = \frac{\bar{p}-p_0}{\sqrt{p_0(1-p_0)/n}} = \frac{0.5583-0.5}{\sqrt{0.5\,(1-0.5)/120}} = 1.278$$

Rejection region is $|z_0| > z_{\alpha/2}$. We have $z_{0.025}$=1.96. Since the test statistic does not fall into the rejection region, we continue believing in the null hypothesis.

Inferences the Difference Between Two Means (Variances Known)

	$H_0: \mu_1 - \mu_2 \geq \mu_0$ $H_a: \mu_1 - \mu_2 < \mu_0$	$H_0: \mu_1 - \mu_2 \leq \mu_0$ $H_a: \mu_1 - \mu_2 > \mu_0$	$H_0: \mu_1 - \mu_2 = \mu_0$ $H_a: \mu_1 - \mu_2 \neq \mu_0$
Rejection Region	$z_0 < -z_\alpha$	$z_0 > z_\alpha$	$\lvert z_0 \rvert > z_{\alpha/2}$

Test Statistic $z_0 = \dfrac{(\bar{x}_1-\bar{x}_2) - \mu_0}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$

Inferences the Difference Between Two Means (Variances Known, Cont'd)

Confidence Interval for the Difference Between Two Population Means:

The assumption here is that n_1 and n_2 are so large that Central Limit Theorem holds.

$$(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Inferences the Difference Between Two Means (Variances Known, Cont'd)

Example: Unloading Times of Logs

The owner of a local logging operation wants to examine the average unloading time of logs. Two methods are used for unloading. A random sample of size 40 for the first sample gives an average unloading time $\bar{x}_1$ of 20.5 min. A random sample of size 50 for the second method yields an average unloading time $\bar{x}_2$ of 17.6 min. We know that the variance of the unloading times using the first method is 3, while that for the second method is 4. At a significance level $\alpha = .05$, can we conclude that there is a difference in the mean unloading times for the two methods?

Inferences the Difference Between Two Means (Variances Known, Cont'd)

Example (Cont'd):

Our null and alternative hypotheses are as follows:

$$H_0 : \mu_1 - \mu_2 = 0$$
$$H_a : \mu_1 - \mu_2 \neq 0$$

Next, we will compute the value of $z_0 = \dfrac{(\bar{x}_1 - \bar{x}_2) - \mu_0}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$.

Inferences the Difference Between Two Means (Variances Known, Cont'd)

Example (Cont'd):

It turns out that

$$z_0 = \frac{(20.5 - 17.6) - 0}{\sqrt{\frac{3}{40} + \frac{4}{50}}} = 7.366.$$

We have $z_{0.025}$=1.96. Rejection region is $|z_0|$>1.96. Since the test statistic lies in the rejection region, we reject the null hypothesis, and conclude that there is a difference in the mean unloading times for the two methods.

Inferences the Difference Between Two Means (Variances UnKnown)

Variances Unknown But Equal ($\sigma_1^2 = \sigma_2^2$):

	$H_0: \mu_1 - \mu_2 \geq \mu_0$ $H_a: \mu_1 - \mu_2 < \mu_0$	$H_0: \mu_1 - \mu_2 \leq \mu_0$ $H_a: \mu_1 - \mu_2 > \mu_0$	$H_0: \mu_1 - \mu_2 = \mu_0$ $H_a: \mu_1 - \mu_2 \neq \mu_0$
Rejection Region	$t_0 < -t_{\alpha, n_1+n_2-2}$	$t_0 > t_{\alpha, n_1+n_2-2}$	$\lvert t_0 \rvert > t_{\alpha/2, n_1+n_2-2}$

Test Statistic $t_0 = \dfrac{(\bar{x}_1 - \bar{x}_2) - \mu_0}{s_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$ where $s_p^2 = \dfrac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}$

Inferences the Difference Between Two Means (Variances Unknown, Cont'd)

Confidence Interval for the Difference Between Two Population Means:

The assumption here is that n_1 and n_2 are so large that Central Limit Theorem holds, and variances are equal.

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\frac{\alpha}{2}, n_1+n_2-2}\, s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$