Equity Investment

Module 68.1: Derivatives Markets

LOS 68.a: Definition and Basic Features of a Derivative

1. Definition

- A **derivative** is a financial instrument whose value is derived from the value of another asset, rate, or index — the **underlying**.
- The derivative's price depends on the performance of the underlying at a specific future date.
- Common underlyings: equities, bonds, indices, currencies, interest rates, or commodities.

Exhibit 1: Core Features of a Derivative Contract

Term	**Definition / Example**
Underlying	The asset, rate, or index upon which the derivative's payoff depends (e.g., stock, bond, interest rate).
Forward Price (Exercise / Strike Price)	Agreed price at which the underlying will be exchanged in the future (e.g., $30 per share).
Contract Size	Quantity of the underlying per contract (e.g., 100 shares).
Settlement / Maturity Date	Future date on which the transaction is executed or settled.
Value at Initiation	Typically set to zero for both parties (no upfront cost).
Settlement Type	• **Physical (Deliverable):** Actual delivery of the underlying asset. • **Cash-Settled:** Only the net gain/loss exchanged in cash.

2. Key Contract Terms

3. Forward Contract Example **Forward Contract:** Buy 100 shares of Acme at \$30 in 3 months.

Exhibit 2: Settlement Outcomes

Spot Price (\$)	Forward Price (\$)	Buyer's Gain/Loss	Seller's Gain/Loss
30	30	0	0
40	30	+1,000	-1,000
25	30	-500	+500

$$\text{Payoff to Long} = \text{Spot Price} - \text{Forward Price}$$

$$\text{Payoff to Short} = \text{Forward Price} - \text{Spot Price}$$

- **Long Position (Buyer):** Gains when price $\uparrow$.
- **Short Position (Seller):** Gains when price $\downarrow$.

Exhibit 3: Purpose of Derivative Use

Use	Description / Example
Hedging	Using a derivative to offset an existing risk exposure. Example: A farmer sells a forward on wheat to lock in the selling price.
Speculation	Using a derivative to gain exposure to price movements without owning the underlying. Example: Buying an oil forward to profit if oil prices rise.
Arbitrage	Exploiting price discrepancies across markets to earn risk-free profits.

4. Hedging vs. Speculation

5. Economic Function of Derivatives

- Transfer risk efficiently between parties.
- Provide leverage and low-cost market exposure.
- Reduce transaction costs relative to spot market trades.
- Allow customized exposures to specific risks (interest rates, FX, credit, weather, etc.).

6. Examples of Underlyings and Risk Transfers

Exhibit 4: Common Underlyings and Risk Use Cases

Underlying	**Example and Risk Type**
Equity (Stock)	Forward on Acme shares → risk: stock price movements.
Bond	Forward on 30-year U.S. Treasury bond → risk: interest rate or bond price.
Index	S&P 500 forward → hedge or gain exposure to equity index performance.
Currency	Forward on GBP/USD → hedge FX risk from future payments/receipts.
Interest Rate	Forward rate agreement (FRA) → hedge future borrowing or lending rates.
Commodity	Wheat or oil forward → hedge producer or consumer price risk.
Credit	Credit default swap (CDS) → transfer risk of borrower default.
Other	Weather, carbon, longevity, or crypto derivatives → hedge non-financial risks.

7. Hedging Rule of Thumb

"Do in the futures market what you must do in the future."

- Need to buy in future → go **long** futures/forwards.
- Need to sell in future → go **short** futures/forwards.

Exhibit 5: Core Derivative Instruments

Instrument	**Definition and Key Characteristics**
Forwards	Customized OTC agreement to buy/sell at a future date for fixed price.
Futures	Standardized version of a forward, traded on exchanges.
Options	Right (not obligation) to buy/sell underlying at strike price. Call = buy; Put = sell.
Swaps	Exchange of future cash flows (e.g., fixed vs. floating interest payments).
Credit Derivatives	Transfer default or credit spread risk (e.g., CDS).

8. Major Derivative Types Overview

LOS 68.b: Derivative Market Structures – Exchange-Traded vs. OTC

1. Overview

- Two primary venues for derivative trading:

1. **Exchange-Traded Derivatives (ETDs)**
2. **Over-the-Counter (OTC) Derivatives**

- Differ mainly by standardization, counterparty risk, liquidity, and regulation.

(a) Exchange-Traded Derivatives (ETDs)

- Traded on centralized exchanges (e.g., CME Group, B3 Brazil, NSE India).
- **Standardized contracts** with uniform terms (size, maturity, underlying, settlement).
- Backed by a **central clearinghouse (CCH)** that performs novation — takes opposite side of every trade.
- **Margin requirements:** Both sides post initial margin + variation margin (mark-to-market daily).
- High **liquidity, transparency**, and low default risk.

Exhibit 6: Exchange-Traded Derivative Features

Feature	**Description**
Trading Venue	Centralized exchange with electronic/physical trading.
Standardization	Contract terms predefined by the exchange.
Clearing Mechanism	Central clearinghouse guarantees settlement (novation).
Margining	Initial + maintenance margin required; daily marking-to-market.
Liquidity and Transparency	High — public price discovery and regulatory oversight.
Counterparty Risk	Minimal — guaranteed by clearinghouse.
Exit Mechanism	Easy offset by taking opposite position (liquid secondary market).

(b) Over-the-Counter (OTC) Derivatives

- Custom bilateral contracts negotiated privately between counterparties.
- Examples: forwards, swaps, bespoke options.
- **Highly customizable:** contract size, settlement, underlying, dates, etc.
- **Counterparty risk:** each party exposed to the other's default.
- Less regulated, less transparent, and less liquid.

Exhibit 7: OTC Derivative Features

Feature	**Description**
Trading Venue	Decentralized (phone, Bloomberg, interdealer platforms).
Standardization	Customizable — tailored to user needs.
Clearing	Bilateral settlement or via central counterparty (CCP) for mandated products.
Regulation	Light regulation (though stricter post-2008 reforms).
Liquidity / Transparency	Limited; trades not publicly reported (except SEFs).
Counterparty Risk	High (reduced for cleared swaps).
Collateralization	Often no mandatory margining for uncleared trades.

Example – Central Clearing Mandate (Post-2008 Reform):

- Many swaps must now clear through a **Central Counterparty (CCP)**.
- CCP replaces bilateral trade with two offsetting trades (reducing bilateral risk).
- Trades reported to **Swap Execution Facility (SEF)** for transparency.
- **Advantage:** lower counterparty risk; **Disadvantage:** risk concentration in CCP.

Exhibit 8: Exchange-Traded vs. OTC Derivatives

Characteristic	**Exchange-Traded**	**OTC (Dealer Market)**
Trading Venue	Centralized exchange	Decentralized bilateral network
Contract Form	Standardized	Customized
Clearing	Central clearinghouse (CCH)	Bilateral or CCP (for cleared swaps)
Counterparty Risk	Minimal (CCH guarantees)	Significant (unless cleared)
Regulation	Highly regulated	Light (some post-crisis reforms)
Liquidity	High	Variable / low
Transparency	High (public data)	Low (private negotiation)
Transaction Costs	Low (standardized)	Higher (customized)
Collateral / Margin	Mandatory margining	Not always required
Best Use Case	Speculative or standardized hedging needs	Bespoke risk management

2. Comparison Summary

Key Takeaways Summary

- **Derivatives:** Instruments whose value derives from an underlying asset, rate, or index.
- **Key Features:** Underlying, price/strike, contract size, settlement date, zero-value initiation.
- **Positions:**
  - **Long:** Gains when underlying ↑.
  - **Short:** Gains when underlying ↓.
- **Use:** Hedging, speculation, arbitrage.
- **Exchange-Traded:** Standardized, liquid, cleared, low risk.
- **OTC:** Customized, less transparent, higher counterparty risk.
- **Post-2008 Reform:** CCPs introduced for many swaps ⇒ centralized risk management.

Exhibit 9: Formula and Concept Recap

Concept	**Formula / Rule**
Forward Contract Payoff (Long)	$V_T = S_T - F_0$
Forward Contract Payoff (Short)	$V_T = F_0 - S_T$
Notional Exposure	Contract Size × Underlying Price
Leverage Effect	Small change in underlying → large change in derivative value
Hedging Rule	"Do in the futures market what you must do in the future."
Clearinghouse (CCH) Function	Novation + margining + default guarantee

Module 69.1–69.2: Forwards, Futures, Swaps, Options, and Credit Derivatives

LOS 69.a: Define and Compare Basic Derivative Instruments

1. Forward Contracts

- **Definition:** A bilateral agreement where the **buyer (long)** agrees to purchase, and the **seller (short)** agrees to deliver, an asset at a specified price and date in the future.
- **Payoff at Settlement:**

$$V_T^{\text{Long}} = S_T - F_0 \quad \text{and} \quad V_T^{\text{Short}} = F_0 - S_T$$

- **Characteristics:**
  - OTC contract — customized, privately negotiated.
  - No cash exchanged at initiation (value = 0).
  - Subject to counterparty credit risk.
  - Typically settled by delivery or cash.
- **Use:** Hedging or speculation on future price changes.

Example: Forward Contract on Stock Buy 100 shares at \$30 in 3 months.

$$\text{Gain/Loss}_{\text{Long}} = 100 \times (S_T - 30)$$

If $S_T = 40$ => Gain = +\$1,000; if $S_T = 25$ => Loss = –\$500.

2. Futures Contracts

- **Definition:** Standardized, exchange-traded version of a forward contract.
- **Differences vs. Forwards:**
  - Traded on organized exchanges.
  - Backed by a **clearinghouse** (no counterparty risk).
  - **Daily mark-to-market:** Profits/losses settled daily.
  - **Margin requirements:** Initial + maintenance margin deposits required.
  - High liquidity and price transparency.

Exhibit 1: Mark-to-Market Process for 100 oz Gold Futures

Day	Settlement Price	Change (\$/oz)	Buyer's Balance (\$)	Seller's Balance (\$)
0	1,950	—	5,000 (Initial)	5,000 (Initial)
1	1,947.5	–2.5	4,750	5,250
2	1,945	–2.5	4,500 ⇒ margin call +500 → 5,000	5,500

Example: Gold Futures Margining

Futures Margin Formulas

$$\text{Daily Gain/Loss} = (\text{Price}_t - \text{Price}_{t-1}) \times \text{Contract Size}$$

Price Limits and Circuit Breakers:

- Exchanges impose daily maximum price moves to control volatility.
- If exceeded, trading halts temporarily ("limit up" / "limit down").

Exhibit 2: Key Differences between Forwards and Futures

Feature	**Forward**	**Futures**
Trading Venue	OTC (private)	Exchange-traded
Standardization	Customized	Standardized
Liquidity	Low	High
Counterparty Risk	High	Minimal (clearinghouse)
Settlement	Single (at maturity)	Daily mark-to-market
Margins	None required	Initial + maintenance margin
Price Limits	None	Imposed by exchange
Transparency	Low	High

3. Comparison: Forwards vs. Futures

LOS 69.a (continued): Swaps, Credit Derivatives, and Options

4. Swaps

- **Definition:** Agreement to exchange a series of cash flows on periodic settlement dates over time.
- **Most common: Interest rate swap** — one party pays fixed, the other pays floating.
- **Equivalent to:** A series of forward contracts.
- **Exposure:** Counterparty credit risk unless cleared by a central counterparty (CCP).

Example: Fixed-for-Floating Interest Rate Swap

$$\text{Notional Principal} = \$10,000,000$$

$$\text{Fixed Rate} = 2\% \quad \Rightarrow \text{Quarterly Fixed Payment} = 10,000,000 \times \frac{0.02}{4} = 50,000$$

$$\text{Floating Rate (SOFR)} = 1.6\% \quad \Rightarrow \text{Floating Payment} = 10,000,000 \times \frac{0.016}{4} = 40,000$$

Net Payment: Fixed payer pays 10,000 to the floating payer.

- **Hedging Example:** A firm with floating-rate debt enters a fixed-pay swap to convert exposure into fixed-rate liability.

5. Credit Default Swaps (CDS)

- **Definition:** Protection buyer pays periodic premiums; protection seller compensates buyer if a credit event occurs.
- **Credit event:** Default, bankruptcy, or restructuring of the reference entity.

- **Payoff:** Protection seller pays loss amount = (1 − recovery rate) × notional.
- **Risk Profiles:**
  - Protection buyer = long credit risk protection (short credit exposure).
  - Protection seller = earns premium but assumes credit risk.

CDS Analogy

$$\text{CDS Spread} \approx \text{Default Probability} \times \text{Loss Given Default (LGD)}$$

6. Options (Calls and Puts)

- **Call Option:** Right (not obligation) to **buy** underlying at strike price X.
- **Put Option:** Right (not obligation) to **sell** underlying at strike price X.
- **Option Buyer (Holder):** Pays **premium**; has limited loss (premium) and unlimited profit potential (for calls).
- **Option Seller (Writer):** Receives premium; faces opposite payoff.

Option Value at Expiration

$$\text{Call Payoff} = \max(0, S_T - X) \qquad \text{Put Payoff} = \max(0, X - S_T)$$

Profit Formulas

$$\text{Call Buyer Profit} = \max(0, S_T - X) - \text{Premium}$$

$$\text{Put Buyer Profit} = \max(0, X - S_T) - \text{Premium}$$
$$\text{Writer Profit} = \text{Premium} - \text{Option Payoff}$$

LOS 69.b: Option Payoffs and Profit Diagrams

1. Call Option Example

$$\text{Exercise Price} = \$50, \quad \text{Premium} = \$5$$

- **Breakeven:** $X + \text{Premium} = 55$
- **Max Loss (Buyer):** \$5 (premium)
- **Max Loss (Writer):** Unlimited
- **Max Profit (Writer):** \$5

Exhibit 3: Call Option Profits at Expiration

Stock Price (S)	**Call Value**	**Buyer Profit**	**Writer Profit**
40	0	–5	+5
50	0	–5	+5
55	5	0 (breakeven)	0
60	10	+5	–5

Exhibit 4: Put Option Profits at Expiration

Stock Price (S)	**Put Value**	**Buyer Profit**	**Writer Profit**
60	0	–5	+5
50	0	–5	+5
45	5	0 (breakeven)	0
40	10	+5	–5

2. Put Option Example

$$\text{Exercise Price} = \$50, \quad \text{Premium} = \$5$$

- **Breakeven:** $X - \text{Premium} = 45$
- **Max Gain (Buyer):** $X - \text{Premium} = 45$
- **Max Loss (Buyer):** Premium = \$5
- **Max Loss (Writer):** \$45

Exhibit 5: Option Position Payoff Summary

Position	**Exposure to Underlying**	**Profit When...**
Call Buyer (Long Call)	Long	Price rises ($S_T \uparrow$)
Call Writer (Short Call)	Short	Price falls ($S_T \downarrow$)
Put Buyer (Long Put)	Short	Price falls ($S_T \downarrow$)
Put Writer (Short Put)	Long	Price rises ($S_T \uparrow$)

3. Summary of Option Exposures

4. Example: Combined Option Profit Given: $X = 40$, Call Premium = 3, Put Premium = 0.75.

- Call profitable when $S > 43$.
- Put profitable when $S < 39.25$.

Exhibit 6: Option Profits at Expiration

Stock Price (S)	Long Call	Short Call	Long Put	Short Put
35	–3	+3	+4.25	–4.25
43	0	0	–0.75	+0.75

LOS 69.c: Forward Commitments vs. Contingent Claims

Exhibit 7: Comparison of Derivative Claim Types

Feature	Forward Commitment	Contingent Claim
Definition	Obligation to transact in the future.	Payoff depends on occurrence of specific event.
Examples	Forwards, futures, swaps.	Options, CDS.
Obligation	Both parties must perform.	Only exercised if event favorable.
Initial Value	Typically zero at inception.	Buyer pays upfront premium.
Payoff Trigger	Always executed at maturity.	Only if condition met (e.g., $S_T > X$ or credit default).

Interpretation:

- Forward commitments = linear payoffs (obligation).
- Contingent claims = nonlinear payoffs (optionality).

Key Takeaways Summary

- **Forwards:** Customized OTC; one-time settlement; counterparty risk.
- **Futures:** Standardized, exchange-traded, daily mark-to-market, low counterparty risk.
- **Swaps:** Series of forward commitments; most common = interest rate swap.
- **Credit Derivatives (CDS):** Payoff contingent on credit event; used for credit risk hedging.
- **Options:** Nonlinear contingent claims; provide asymmetric risk exposures.
- **Forward Commitments vs. Contingent Claims:** Obligation vs. optional payoff.

Exhibit 8: Formula Recap

Derivative Type	**Key Formula**
Forward Payoff (Long)	$S_T - F_0$
Futures Daily Gain	$(P_t - P_{t-1}) \times \text{Contract Size}$
Swap Fixed Payment	$\text{Notional} \times \frac{\text{Fixed Rate}}{m}$
Call Payoff	$\max(0, S_T - X)$
Put Payoff	$\max(0, X - S_T)$
Call Profit (Buyer)	$\max(0, S_T - X) - C_0$
Put Profit (Buyer)	$\max(0, X - S_T) - P_0$
CDS Payout	$(1 - \text{Recovery Rate}) \times \text{Notional}$

Module 70.1: Uses, Benefits, and Risks of Derivatives

LOS 70.a: Benefits and Risks of Derivative Instruments

1. Advantages of Derivatives

- Derivatives allow investors and issuers to **alter, transfer, or manage risk exposures** without transacting directly in the underlying cash markets.
- They also enhance market efficiency, improve price discovery, and reduce operational costs.

2. Risks of Derivatives

- Despite their advantages, derivatives entail several types of risks.

3. Implicit Leverage Formula

$$\text{Leverage Ratio} = \frac{1}{\text{Margin Requirement}}$$

Example: Margin = 4% => leverage = 25:1. A 1% adverse price move causes:

$$\text{Loss on Margin} = 1\% \times 25 = 25\% \text{ of equity.}$$

4. Summary Diagram: Derivative Benefits vs. Risks

Benefits:	Risk transfer, information discovery, efficiency, leverage, liquidity.
Risks:	Leverage, basis mismatch, liquidity stress, counterparty risk, systemic contagion.

Exhibit 1: Summary of Derivative Advantages

Advantage Category	**Explanation and Examples**
(1) Risk Transfer and Management	• **Hedging:** Reduce exposure to undesirable price movements. • Example: A manufacturer hedges FX risk of future USD receipts by selling USD forward. • Example: A floating-rate note issuer enters a swap to pay fixed and receive floating. • Example: An equity investor buys a put option to limit downside risk. • **Speculation:** Gain exposure to desired risk (e.g., buying calls for upside participation).
(2) Information Discovery	• Derivative prices reveal market expectations: – **Option prices** => imply expected volatility (*implied volatility*). – **Futures/forwards** => imply expected future prices of underlying. – **Interest rate futures** => imply expected future short-term rates or policy changes.
(3) Operational Advantages	• **Ease of short sales:** No need to borrow the underlying asset (e.g., selling futures). • **Lower transaction costs:** Avoid costs of transportation, insurance, and settlement. • **Greater leverage:** Small margin requirement allows large notional exposure. • **Greater liquidity:** Low capital requirement enables rapid large-scale adjustments.
(4) Improved Market Efficiency	• Lower trading costs and ease of arbitrage $\rightarrow$ fewer mispricings. • Example: Arbitrage through index futures keeps spot and futures prices aligned.

LOS 70.b: Derivative Use by Issuers vs. Investors

1. Corporate Issuers (Hedgers / Risk Managers)

- Firms use derivatives to manage earnings and balance sheet volatility due to market, interest rate, FX, or commodity price movements.
- **Issuers of derivatives** include corporations creating swap or forward exposures.

Exhibit 2: Major Risks Associated with Derivatives

Risk Type	Explanation and Example
(1) Implicit Leverage Risk	• Small initial margin = high leverage. • Example: Futures margin = 5% => leverage = 20:1. A 1% price drop => 20% loss on margin. • Amplifies both gains and losses; may lead to margin calls or forced liquidation.
(2) Basis Risk	• Occurs when the hedge and the exposure are imperfectly correlated. • Example: A fund hedges portfolio of 50 large-cap stocks with S&P 500 futures → imperfect hedge. • Example: Farmer sells October corn futures for a September harvest → mismatch in timing.
(3) Liquidity Risk	• Hedge cash flows and underlying cash flows do not align. • Example: Farmer short wheat futures → margin calls when price rises even if harvest value offsets losses. • Lack of liquidity to meet interim margin calls can destroy hedge effectiveness.
(4) Counterparty Credit Risk	• Risk that the counterparty fails to perform. • Forwards: Both parties face risk (bilateral). • Options: Only buyer faces risk (seller has received premium). • Futures: Counterparty risk minimized via central clearinghouse and daily margining.
(5) Systemic Risk	• Excessive speculation and interconnected exposures may threaten entire markets. • Post-2008 reforms: central clearing mandates for swaps aim to reduce systemic counterparty risk.

2. Hedge Accounting Classifications

- **Cash Flow Hedge:** Reduces variability in future cash flows. Example: FX forward for future foreign currency receipts.
- **Fair Value Hedge:** Reduces volatility in balance sheet values. Example: Interest rate swap to stabilize fixed-rate debt valuation.

Exhibit 3: Examples of Derivative Use by Issuers

Exposure / Problem	Derivative Solution and Result
Foreign currency receipts	Use **forward contracts** to lock in domestic-currency value (cash flow hedge).
Fixed-rate debt valued at fair value	Enter **interest rate swap** as floating-rate payer → reduces fair value volatility (fair value hedge).
Commodity inventory at fair value	Sell **commodity forwards** to offset inventory value changes.
Foreign subsidiary value volatility	Use **FX forwards/futures** as a **net investment hedge** to stabilize reported equity value.

- **Net Investment Hedge:** Protects against FX changes in foreign subsidiaries. Example: FX forward on subsidiary's equity value.

3. Benefits to Issuers

- Reduce earnings volatility.
- Stabilize reported financials.
- Align economic and accounting exposures via hedge accounting.

4. Investors (Buy-side Users)

- Use derivatives to hedge, modify, or amplify exposure to underlying risks (equities, rates, FX, commodities).

Exhibit 4: Examples of Derivative Use by Investors

Objective	Derivative Example and Effect
Hedging Risk Exposure	Buy index put to limit downside risk while maintaining upside potential.
Speculative Exposure (Leverage)	Buy silver forwards to gain exposure with low capital outlay.
Duration Management	Enter interest rate swap (pay floating / receive fixed) to increase portfolio duration.
Market Timing / Tactical Allocation	Buy S&P 500 futures to temporarily increase equity exposure, or sell to reduce it.

Exhibit 5: Comparison of Derivative Use

Aspect	Issuers (Corporates)	Investors (Buy-side)
Primary Goal	Hedge financial reporting and cash flow volatility.	Adjust portfolio risk/return profile.
Typical Instruments	Swaps, forwards, futures (hedges).	Options, futures, swaps (speculative or hedge).
Accounting Impact	Subject to hedge accounting (IFRS/GAAP rules).	Reflected at fair value in portfolio returns.
Time Horizon	Medium-to-long term balance sheet exposures.	Short-term tactical or strategic positions.
Example Use	FX forward for export receipts.	Index puts to protect equity portfolio.

5. Key Comparison: Issuers vs. Investors

6. Conceptual Summary Diagram

Issuers:	Manage risk on balance sheet items (FX, interest rates, commodities).
Investors:	Manage portfolio exposures (hedge, leverage, or speculate).

Key Formulas and Concepts Recap

Exhibit 6: Quantitative and Conceptual Recap

Concept	**Formula / Key Idea**
Leverage Ratio	$\text{Leverage} = \frac{1}{\text{Margin Requirement}}$
Basis Risk	Basis = Spot Price − Futures Price
Counterparty Risk	Higher for OTC forwards/swaps; minimal for exchange-traded futures.
Hedge Effectiveness	Correlation between underlying exposure and derivative hedge.
Cash Flow Hedge	Stabilizes future cash flows (e.g., FX forward).
Fair Value Hedge	Stabilizes asset/liability valuations (e.g., interest rate swap).
Net Investment Hedge	Offsets FX changes on foreign subsidiaries.
Liquidity Risk	Mismatch between hedge cash flow timing and underlying exposure.

Final Summary Box

Advantages:	Risk transfer, low cost, liquidity, leverage, price discovery, efficiency.
Risks:	Leverage, basis mismatch, liquidity stress, counterparty, systemic risk.
Issuers:	Use derivatives to hedge accounting and operational exposures.
Investors:	Use derivatives to hedge, speculate, or modify portfolio exposures.

Module 71.1: Arbitrage, Replication, and Carrying Costs

LOS 71.a: Arbitrage and Replication in Derivative Pricing

1. No-Arbitrage Principle

- **Arbitrage:** A simultaneous purchase and sale of identical (or equivalent) assets to earn a **risk-free profit with zero net investment**.
- **No-Arbitrage Condition:** If two portfolios generate the same future payoff under all states, they must have the same value today.

$$\text{If Payoff}_A = \text{Payoff}_B \Rightarrow \text{Value}_A = \text{Value}_B$$

- Any deviation creates an arbitrage opportunity that will quickly be eliminated by market forces.

2. Example: No-Arbitrage Forward Price Derivation Given:

$$S_0 = \$30, \quad R_f = 5\%, \quad T = 1 \text{ year}, \quad \text{No dividends.}$$

Portfolio 1 (Synthetic Stock):

- Buy bond paying $F_0(1)$ at $t = 1 \rightarrow$ Cost $= F_0(1)/(1.05)$.
- Enter **long forward** (zero cost).
- Payoff at $t = 1$: S_1.

Portfolio 2 (Actual Stock):

- Buy Acme share today $\rightarrow$ Cost $= 30$.
- Payoff at $t = 1$: S_1.

By Law of One Price:

$$F_0(1)/1.05 = 30 \quad \Rightarrow \quad \boxed{F_0(1) = 30(1.05) = 31.50}$$

No-Arbitrage Forward Price:

$$\boxed{F_0(T) = S_0(1 + R_f)^T}$$

3. Arbitrage Scenarios

$$\boxed{\text{Arbitrage condition: } F_0(T) = S_0(1 + R_f)^T}$$

4. Replication Concept Replication: Constructing a portfolio of cash-market instruments that produces the same payoff as the derivative under all states of the world.

Therefore: When forward is correctly priced,

$$F_0(T) = S_0(1 + R_f)^T$$

Intuition:

- A long forward = long spot + short bond.
- A short forward = short spot + long bond.

LOS 71.b: Spot Price, Expected Future Price, and Cost of Carry

1. Forward Pricing with Costs and Benefits Base Formula (Discrete Compounding):

$$\boxed{F_0(T) = [S_0 + PV_0(\text{costs}) - PV_0(\text{benefits})](1 + R_f)^T}$$

- **Costs of carry:** Storage, insurance, financing, etc. $\rightarrow$ increase forward price.
- **Benefits of holding:** Dividends, coupon income, convenience yield $\rightarrow$ decrease forward price.

Exhibit 1: Forward Mispricing and Arbitrage Actions

Case	**Condition**	**Arbitrage Strategy**
Forward price too high	$F_0(1) > 31.50$	• Sell (short) the forward. • Buy the underlying (Acme share). • Borrow to finance purchase if necessary. • Deliver share at $t = 1$, receive F_0, repay loan $\Rightarrow$ profit $= F_0 - S_0(1 + R_f)$.
Forward price too low	$F_0(1) < 31.50$	• Buy the forward. • Short sell the underlying, invest proceeds at R_f. • Receive $S_0(1 + R_f)$ at $t = 1$, use F_0 to close short $\Rightarrow$ profit $= S_0(1+R_f) - F_0$.

Exhibit 2: Replicating Forward Positions

Position	**Replicating Strategy (Same Payoff at T)**
Long Forward	Borrow S_0 at R_f, buy one share. At T: Payoff $= S_T - S_0(1+R_f)^T = S_T - F_0(T)$.
Short Forward	Short one share, invest proceeds S_0 at R_f. At T: Payoff $= S_0(1 + R_f)^T - S_T = F_0(T) - S_T$.

Example 1: With Storage Cost Only

$$S_0 = 100, \quad PV_0(\text{storage}) = 2, \quad R_f = 5\%, \quad T = 1$$

$$F_0(1) = (100 + 2)(1.05) = 107.10$$

Example 2: With Dividend Benefit

$$S_0 = 30, \quad \text{Dividend} = 1, \quad R_f = 5\%$$

$$F_0(1) = [30 - PV_0(1)](1.05) = 30(1.05) - 1 = 31.50 - 1 = 30.50$$

Interpretation:

- Benefits (↑) => Forward price (↓)
- Costs (↑) => Forward price (↑)

2. Cost of Carry with Continuous Compounding Without Costs/Benefits:

$$F_0(T) = S_0 e^{rT}$$

With Costs (rate = c):

$$F_0(T) = S_0 e^{(r+c)T}$$

With Benefits (rate = b):

$$F_0(T) = S_0 e^{(r+c-b)T}$$

Interpretation:

- r = risk-free rate (financing cost)
- c = storage cost rate
- b = benefit yield (dividend or convenience yield)

3. Example: Continuous Compounding

$$S_0 = 1{,}550, \quad r = 3\%, \quad b = 1.3\%, \quad T = 0.5$$

$$F_0 = 1{,}550 \times e^{(0.03-0.013)(0.5)} = 1{,}550 \times e^{0.0085} = \boxed{1{,}563.23}$$

Interpretation: Forward price reflects interest cost (carry) net of benefits.

4. Cost of Carry Definition

$$\text{Cost of Carry} = \text{Financing Cost (r + c)} - \text{Benefits (b)}$$

Effect on Forward Price:

- If $b > (r+c) \to$ Forward price < Spot (backwardation).
- If $b < (r+c) \to$ Forward price > Spot (contango).

$$\boxed{F_0(T) = S_0 e^{(\text{Cost of Carry})T}}$$

Forward Contracts on Currencies

1. Covered Interest Rate Parity (CIRP)

$$\boxed{F_0 = S_0 \times \frac{(1+R_{\text{price}})}{(1+R_{\text{base}})}}$$

Continuous Compounding Form:

$$F_0 = S_0 e^{(r_{\text{price}} - r_{\text{base}})T}$$

2. Example: USD/EUR Forward Arbitrage Given:

$$S_0 = 1.10\,(\text{USD/EUR}), \quad R_{\text{USD}} = 2\%, \quad R_{\text{EUR}} = 3\%, \quad T = 1$$

$$F_0 = 1.10 \times \frac{1.02}{1.03} = \boxed{1.0893}$$

or equivalently,

$$F_0 = 1.10e^{(0.0198-0.0296)} = 1.0893$$

Interpretation:

- Forward EUR price falls (USD expected to appreciate).
- Higher foreign (EUR) rate => base currency (USD) expected to strengthen.

Exhibit 3: Currency Arbitrage Transactions (USD-based Investor)

Step	**Transaction and Cash Flows**
1. Borrow USD	Borrow \$100 at 2% interest.
2. Convert to EUR	\$100 / 1.10 = €90.91.
3. Invest in EUR asset	€90.91 × 1.03 = €93.64 at year-end.
4. Use Forward to convert back	Sell €93.64 at forward rate 1.0893 USD/EUR → \$102.00.
5. Repay USD loan	\$100 × 1.02 = \$102.00 → No arbitrage profit.

3. Arbitrage Illustration Interpretation: If forward rate deviates from parity, arbitrage exists:

$$F_{\text{too high}} \Rightarrow \text{Sell forward (USD overpriced)} F_{\text{too low}} \Rightarrow \text{Buy forward (USD underpriced)}$$

Key Takeaways Summary

- **Arbitrage:** Exploiting price differences to earn risk-free profits.
- **Replication:** Reconstructing derivative payoffs using spot and risk-free assets.
- **No-Arbitrage Forward Price (discrete):** $F_0(T) = S_0(1 + R_f)^T$
- **Continuous compounding:** $F_0(T) = S_0e^{(r+c-b)T}$
- **Cost of Carry:** $(r + c - b)$ — determines contango or backwardation.
- **Currency forwards:** $F_0 = S_0e^{(r_{\text{price}}-r_{\text{base}})T}$

Summary Rule: $\begin{cases} F_0 > S_0(1 + R_f)^T & \Rightarrow \text{Sell forward, buy underlying} \\ F_0 < S_0(1 + R_f)^T & \Rightarrow \text{Buy forward, short underlying} \end{cases}$

Exhibit 4: Formula Recap

Concept	**Formula / Relationship**
No-arbitrage forward (discrete)	$F_0(T) = S_0(1 + R_f)^T$
No-arbitrage forward (continuous)	$F_0(T) = S_0 e^{rT}$
Forward with costs/benefits	$F_0(T) = [S_0 + PV(\text{cost}) - PV(\text{benefit})](1 + R_f)^T$
Continuous compounding (general)	$F_0(T) = S_0 e^{(r+c-b)T}$
Currency forward (discrete)	$F_0 = S_0 \times \frac{(1+R_{\text{price}})}{(1+R_{\text{base}})}$
Currency forward (continuous)	$F_0 = S_0 e^{(r_{\text{price}} - r_{\text{base}})T}$
Cost of Carry	Cost of Carry $= (r + c - b)$
Replication (Long Forward)	Buy spot, borrow at R_f
Replication (Short Forward)	Short spot, lend at R_f

Module 72.1: Forward Contract Valuation

LOS 72.a: Value and Price of a Forward Contract

1. Key Distinction: Price vs. Value

- **Forward Price** ($F_0(T)$): the agreed-upon delivery price set *at initiation.*
- **Forward Value** ($V_t(T)$): the *current market value* of an existing forward to either party.

Exhibit 1: Difference Between Forward Price and Forward Value

Term	**Meaning and Notes**
Forward Price ($F_0(T)$)	The delivery price that makes the contract's initial value zero. Set by no-arbitrage.
Forward Value ($V_t(T)$)	The market value of the contract at time t (may be positive or negative as spot price changes).

2. Valuation at Initiation

$$F_0(T) = S_0(1 + R_f)^T$$

$$V_0(T) = S_0 - F_0(T)(1 + R_f)^{-T} = 0$$

Interpretation: At initiation, the forward price is chosen such that the contract has zero value to both parties (no arbitrage).

3. Valuation During Contract Life At time $t < T$, the **value to the long (buyer)** is:

$$\boxed{V_t(T) = S_t - F_0(T)(1 + R_f)^{-(T-t)}}$$

Logic:

- S_t: current spot value of the underlying.
- $F_0(T)(1+R_f)^{-(T-t)}$: present value of forward delivery price.
- If $S_t > PV(F_0) \rightarrow$ gain to long; if $S_t < PV(F_0) \rightarrow$ loss to long.

Replication: Sell the asset short at S_t and invest $F_0(T)(1+R_f)^{-(T-t)}$ at the risk-free rate $\rightarrow$ locks in the current forward value.

4. Valuation at Expiration

$$\boxed{V_T(T) = S_T - F_0(T)}$$

- **Long forward payoff:** $S_T - F_0(T)$
- **Short forward payoff:** $F_0(T) - S_T$

Interpretation:

- If $S_T > F_0(T) \rightarrow$ gain for the long.
- If $S_T < F_0(T) \rightarrow$ gain for the short.

5. General Case with Costs and Benefits

$$\boxed{V_t(T) = [S_t + PV_t(\text{costs}) - PV_t(\text{benefits})] - F_0(T)(1+R_f)^{-(T-t)}}$$

- **Costs:** storage, insurance, financing.
- **Benefits:** dividends, coupons, convenience yield.
- **Net effect:** costs increase, benefits decrease forward value.

6. Example: Forward Value During Life Given:

$$S_0 = 100,\ R_f = 4\%,\ T = 1,\ \text{so}\ F_0 = 104.$$

After 6 months ($t = 0.5$ yr):

$$S_t = 103$$

Value to Long:

$$V_t = 103 - 104(1.04)^{-0.5} = 103 - 101.96 = \boxed{+1.04}$$

Interpretation: The long could close the contract now and lock in a profit of \$1.04 per unit.

7. Summary Table of Forward Valuation

Exhibit 2: Forward Price and Value Summary

Timing	**Forward Price**	**Value to Long**
Initiation	$F_0(T) = S_0(1 + R_f)^T$	0
During Life	Fixed at $F_0(T)$	$S_t - F_0(T)(1 + R_f)^{-(T-t)}$
At Expiration	$F_0(T)$	$S_T - F_0(T)$
With Costs/Benefits	$[S_0 + PV_0(c) - PV_0(b)](1 + R_f)^T$	$[S_t + PV_t(c) - PV_t(b)] - PV(F_0)$

LOS 72.b: Forward Rates and Interest Rate Forward Contracts

1. Definition: Forward Rate

- A **forward rate** is the implied interest rate for a loan or investment *that begins in the future.*
- Denoted as:

$$F_{m,n} = \text{Forward rate for an } n\text{-year loan starting in } m \text{ years.}$$

- Examples:

$$F_{1,1} \text{ (1y1y)} = \text{1-year loan starting 1 year from now.}$$

$$F_{2,1} \text{ (2y1y)} = \text{1-year loan starting 2 years from now.}$$

2. Forward Rate Derivation

$$(1 + Z_2)^2 = (1 + Z_1)(1 + F_{1,1})$$

$$\boxed{F_{1,1} = \frac{(1 + Z_2)^2}{(1 + Z_1)} - 1}$$

General Formula:

$$(1 + Z_{m+n})^{m+n} = (1 + Z_m)^m (1 + F_{m,n})^n$$

$$\boxed{F_{m,n} = \left(\frac{(1 + Z_{m+n})^{m+n}}{(1 + Z_m)^m} \right)^{1/n} - 1}$$

3. Example: Implied Forward Rate Given:

$$Z_2 = 2\%, \quad Z_3 = 3\%$$

Find $F_{2,1}$.

$$(1 + 0.03)^3 = (1 + 0.02)^2 (1 + F_{2,1})$$

$$1.0927 = 1.0404(1 + F_{2,1}) \Rightarrow F_{2,1} = 5.03\%$$

Interpretation: The implied 1-year rate starting 2 years from now is 5.03%.

4. Forward Rate Agreement (FRA)

- A forward contract on an interest rate.
- **Parties:**
  - **Fixed-rate payer (long):** pays fixed forward rate, receives floating.
  - **Floating-rate payer (short):** pays realized reference rate, receives fixed.
- Used by financial institutions to hedge or speculate on future interest rates.

5. FRA Payoff Formula (to Fixed-Rate Payer)

$$\boxed{\text{Payoff} = \frac{(\text{Reference Rate} - \text{Forward Rate}) \times \text{Notional} \times (d/360)}{1 + \text{Reference Rate} \times (d/360)}}$$

where d = days in interest period.

6. Example: FRA Valuation Given:

- 3m6m FRA (loan starts 3 months from now, lasts 6 months)
- Notional = \$1 million
- Forward rate = 1.3%, realized 6-month rate = 1.5%

$$\text{Payoff} = \frac{(0.015 - 0.013) \times 1{,}000{,}000 \times 0.5}{1 + 0.015 \times 0.5} = \frac{1{,}000}{1.0075} = \boxed{\$993.}$$

Interpretation: Since the floating rate > forward rate, the fixed-rate payer receives a gain.

7. FRA Uses

- **Banks / Issuers:** Hedge future funding costs or interest income.
- **Investors:** Speculate on rate movements or hedge bond duration.
- **Building Blocks:** Multi-period FRAs = Interest rate swaps.

8. Summary of Forward Rate Notation

Exhibit 3: Forward Rate Interpretation

Notation	Meaning
1y1y or $F_{1,1}$	1-year loan beginning 1 year from today
2y1y or $F_{2,1}$	1-year loan beginning 2 years from today
3y2y or $F_{3,2}$	2-year loan beginning 3 years from today
3m6m	6-month rate beginning 3 months from today

Exhibit 4: Formula and Relationship Summary

Concept	Formula / Interpretation
Forward Price (no cost)	$F_0(T) = S_0(1 + R_f)^T$
Forward Value at t	$V_t(T) = S_t - F_0(T)(1 + R_f)^{-(T-t)}$
Forward Value with costs/benefits	$[S_t + PV_t(c) - PV_t(b)] - PV(F_0)$
Expiration Payoff (Long)	$S_T - F_0(T)$
Implied Forward Rate (discrete)	$(1 + Z_{m+n})^{m+n} = (1 + Z_m)^m(1 + F_{m,n})^n$
Forward Rate (continuous)	$F_{m,n} = \frac{Z_{m+n}(m+n) - Z_m m}{n}$
FRA Payoff (fixed-rate payer)	$\frac{(R_{\text{ref}} - R_{\text{fwd}})N(d/360)}{1 + R_{\text{ref}}(d/360)}$

Key Formula Recap

Conceptual Summary Box

Forward Contract: At initiation $V_0 = 0,$ Price fixed at $F_0(T)$.
During Life: Value changes with spot price: $V_t = S_t - PV(F_0)$.
At Expiration:$V_T = S_T - F_0$.
Forward Rates: Implied from zero rates: $(1 + Z_{m+n})^{m+n} = (1 + Z_m)^m(1 + F_{m,n})^n$.
FRAs: Single-period interest rate swaps for hedging future rates.

Module 73.1: Futures Valuation

LOS 73.a: Comparing the Value and Price of Forwards and Futures

1. Key Concept: Price vs. Value

- **Forward Contract:**
  - The **forward price** (F_0) is fixed at initiation.
  - The **value** (V_t) fluctuates with the spot price of the underlying but is *not settled* until maturity.
- **Futures Contract:**
  - Both the **price and value** change daily.

– Gains/losses are realized and settled daily through the **mark-to-market (MTM)** process.

2. Daily Mark-to-Market (MTM) Example: Example: Futures on 100 oz gold at $1,870

Exhibit 1: Daily Settlement Illustration (Simplified)

Day	Settlement Price ($/oz)	Change ($/oz)	Daily Gain/Loss ($)
0	1,870	–	–
1	1,872	+2	+200
2	1,868	–4	–400
3	1,869	+1	+100

Explanation:

- The contract's **price changes daily**, and each day's gain/loss is settled.
- After settlement, the **contract value resets to zero**.
- Forward contracts, by contrast, do not have daily settlements; their values simply accumulate until expiration.

3. Futures Price Definition (Interest Rate Futures)

$$\boxed{\text{Futures Price} = 100 - (100 \times \text{MRR}_{A,B-A})}$$

Example:

$$\text{If futures price} = 97, \quad \Rightarrow \text{MRR}_{6m,6m} = 3\%.$$

Interpretation:

- Futures prices move inversely to market reference rates (MRR).
- As interest rates rise → futures price decreases.

4. Basis Point Value (BPV)

$$\boxed{\text{BPV} = \text{Notional Principal} \times \text{Period} \times 0.01\%}$$

Example:

$$\text{BPV} = 1{,}000{,}000 \times \frac{0.0001}{2} = 50.$$

Interpretation: A one-basis-point change in the MRR changes the contract value by €50.

5. Forward vs. Futures Comparison

Exhibit 2: Comparison of Forwards and Futures

Feature	Forward Contract	Futures Contract
Trading venue	OTC (customized)	Exchange-traded (standardized)
Settlement	Single payment at expiration	Daily mark-to-market (MTM) settlements
Credit risk	Counterparty risk (no clearinghouse)	Minimal counterparty risk (clearinghouse)
Liquidity	Less liquid (custom contracts)	Highly liquid
Value behavior	Value changes continuously; no interim cash flows	Value resets to zero daily through mark-to-market (MTM)
Price behavior	Essentially fixed over contract life (set at F_0)	Fluctuates daily with market expectations
Correlation effect	Less affected by interest-rate correlation	Correlation between interest rates and futures prices affects valuation

LOS 73.b: Why Forward and Futures Prices Differ

1. Daily Settlement Impact

- **Forwards:** No daily cash flow until maturity → gains/losses accrue.
- **Futures:** Daily gains/losses settled → immediate reinvestment or withdrawal affects future interest income.

Exhibit 3: Effect of Correlation on Futures vs. Forwards

Correlation	Outcome	Interpretation
Positive ($r \uparrow$, Futures Price $\uparrow$)	Futures more valuable than forwards.	Long gains (positive MTM) coincide with higher rates, allowing higher reinvestment returns.
Negative ($r \uparrow$, Futures Price $\downarrow$)	Futures less valuable than forwards.	Gains occur when rates are low → less reinvestment income.
Zero / Constant Rates	No difference.	Futures and forwards priced equally.

2. Correlation Between Interest Rates and Futures Prices

$$\boxed{\text{If Corr}(r, \text{Futures Price}) > 0 \Rightarrow F_{\text{fut}} > F_{\text{fwd}}}$$

3. Example: Interest Rate Futures Convexity Bias Given:

- \$1,000,000 6-month interest rate future priced at 97.50 (i.e. MRR = 2.5%).
- Settlement in six months.

- Each 1-bp change = \$50.

Scenario A: MRR rises to 2.51%

$$\text{Gain to long (forward)} = \frac{50}{1 + 0.0251/2} = 49.3803$$

Scenario B: MRR falls to 2.49%

$$\text{Loss to long (forward)} = -\frac{50}{1 + 0.0249/2} = -49.3852$$

Interpretation:

- The value change is asymmetric — a decrease in rates increases value slightly more than an equivalent rate increase decreases it.
- This curvature is called **convexity bias**.

4. Convexity Bias Summary

$$\boxed{\begin{array}{r}\text{Futures exhibit greater linear sensitivity (daily MTM),}\\ \text{Forwards exhibit convexity bias (nonlinear response).}\end{array}}$$

Analogy: Like bond convexity — value gain from rate decreases > value loss from rate increases of equal magnitude.

Exhibit 4: Interest Rate Futures vs. FRAs

Feature	**Interest Rate Futures** vs. **FRA**
Market Type	Exchange-traded (standardized) vs. OTC (customized).
Quotation	Price basis: $100-(100\times\text{MRR})$ vs. Rate basis (direct interest rate).
Settlement	Daily MTM vs. Single cash settlement at start of forward period.
Valuation	Value resets to 0 daily vs. Value changes continuously until maturity.
Convexity Effect	Linear (no convexity) vs. Convexity bias present (asymmetric sensitivity).

5. Interest Rate Futures vs. Forward Rate Agreements (FRA)

6. Summary Example: Correlation and Futures Superiority Scenario: Positive Correlation between Interest Rates and Futures Price

- When rates ↑, futures prices ↑, gains settled daily → funds reinvested at high rates.
- When rates ↓, losses settled daily → funds paid at low opportunity cost.

Conclusion:

$$\boxed{F_{\text{fut}} > F_{\text{fwd}} \text{ when correlation } \rho(r, F_{\text{price}}) > 0.}$$

Conceptual Flow Summary

Forwards:	One-time settlement at maturity; fixed price F_0.
Futures:	Daily settlement (MTM); price changes each day.
Correlation Effect:	$\rho(r, F_{\text{price}}) > 0 \Rightarrow F_{\text{fut}} > F_{\text{fwd}}$.
Convexity Bias:	Futures less convex $\rightarrow$ differ for long-term interest contracts.

Formula Summary Table

Exhibit 5: Key Formulas Recap

Concept	**Formula / Relationship**
Futures Price (interest rate futures)	$P_{\text{fut}} = 100 - (100 \times \text{MRR})$
BPV (Basis Point Value)	$\text{BPV} = \text{Notional} \times \text{Period} \times 0.0001$
Forward Value	$V_t = S_t - F_0(1 + R_f)^{-(T-t)}$
Forward Expiration Value	$V_T = S_T - F_0$
Futures vs. Forwards (Correlation Rule)	$\rho(r, F_{\text{price}}) > 0 \Rightarrow F_{\text{fut}} > F_{\text{fwd}}$
Futures Convexity Bias	Forward Value Change ($\downarrow$r) > |Forward Value Change ($\uparrow$r)|
Interest Rate Futures Increment	$\Delta V = \text{BPV} \times \Delta\text{MRR}$ (bps)

Key Takeaways

1. **Forwards:**Fixed forward price, single settlement, value fluctuates.
2. **Futures:**Daily MTM; both price and value change daily.
3. **Correlation:**If $\rho(r, F_{\text{price}}) > 0 \Rightarrow F_{\text{fut}} > F_{\text{fwd}}$.
4. **Convexity Bias:**Forwards show non-linear (convex) rate sensitivity.
5. **Practical Impact:**In short maturities, $F_{\text{fut}} \approx F_{\text{fwd}}$.

Module 74.1: Swap Valuation

LOS 74.a: Swaps vs. Series of Forward Rate Agreements (FRAs)

1. Basic Definition

- An **interest rate swap (IRS)** is a derivative contract in which:
  - One party pays a **fixed rate** on a notional principal.
  - The other pays a **floating rate** (typically a market reference rate, MRR).
- Payments are netted — only the *difference* between fixed and floating is exchanged.

$$\boxed{\text{Net Payment at period } n = (\text{MRR}_n - F) \times \text{Notional} \times \frac{\text{Days}}{360}}$$

2. Structure Example: 1-Year Quarterly Swap

$$\text{Fixed Rate} = F, \quad \text{Floating Rate} = \text{90-day MRR}_n$$

Exhibit 1: Quarterly Cash Flows (Fixed-for-Floating Swap)

Quarter (n)	Floating Rate (MRR_n)	Fixed Rate (F)	Net Payment to Fixed Payer
1	Known at initiation (MRR_1)	F	MRR_1 - F
2	Unknown (MRR_2)	F	MRR_2 - F
3	Unknown (MRR_3)	F	MRR_3 - F
4	Unknown (MRR_4)	F	MRR_4 - F

Interpretation:

- The first floating rate payment (based on current MRR_1) is *known* at time 0.
- Future floating payments ($\text{MRR}_{2\text{-}4}$) are *unknown* and depend on future rates.

3. Swap = Series of Forward Rate Agreements (FRAs)

A fixed-for-floating swap is equivalent to a portfolio of long FRAs for the fixed payer.

Structure:

- FRA 1 → Settlement in 90 days: Pays $(\text{MRR}_2 - F)$
- FRA 2 → Settlement in 180 days: Pays $(\text{MRR}_3 - F)$
- FRA 3 → Settlement in 270 days: Pays $(\text{MRR}_4 - F)$

However:

- Each FRA has a contract rate equal to F, but FRAs individually may not be zero-value at initiation.
- The **sum of all FRA values** at initiation equals zero — defining the **par swap rate**.

4. No-Arbitrage Interpretation

- The fixed payer's position (pay fixed, receive floating) can be **replicated** by:

$$\boxed{\text{Borrowing at the fixed rate and lending at the floating rate.}}$$

- The par swap rate F_{par} is the rate that makes the *value of the swap at initiation* equal to zero.

$$\boxed{\text{PV(Fixed Leg)} = \text{PV(Floating Leg)}}$$

Result:

$$V_0^{\text{swap}} = 0$$

Exhibit 2: Swap vs. Series of FRAs

Characteristic	**Swap**	**Series of FRAs**
Underlying	Floating reference rate (e.g., 90-day MRR)	Same underlying rate (each FRA references the same index)
Structure	Multiple fixed-versus-floating exchanges across reset dates	Each FRA covers one fixed-versus-floating period
Payment timing	Net interest differential settled at each reset (often netted)	Payments typically occur at the start of each FRA period (PV-adjusted)
Valuation at initiation	Par swap rate chosen so total PV of payments = 0	Individual FRAs may have nonzero initial values; the sum can replicate the swap PV profile
Cash-flow equivalence	Sum of all FRAs can replicate swap cash flows if notional and conventions match	Equivalent when notionals, accrual conventions and payment dates align

5. Relationship Summary

6. Example: Swap as Series of FRAs Given: 1-year quarterly swap (4 periods), fixed rate = 4%, notional = \$1,000,000.

Swap as Series of FRAs — Net Flows per Quarter

Quarter	**Forward 90-day MRR**	**Net Flow (MRR − 4%)** $\times$ $\frac{1}{4}\times$ **\$1,000,000**
1	4.00% (known)	\$0
2	4.25%	+\$625
3	3.75%	-\$625
4	4.10%	+\$250

LOS 74.b: Swap Value and Swap Price

1. Distinction Between Price and Value

- **Price:** The fixed rate specified in the swap contract — the *par swap rate.*
- **Value:** The difference between the PV of floating and fixed legs — can change over time.

$$\boxed{\text{Swap Price} = F_{\text{par}} \quad ; \quad \text{Swap Value at time } t = PV(\text{Float Leg}) - PV(\text{Fixed Leg})}$$

2. Swap Valuation Logic

At initiation:

$$V_0^{\text{swap}} = PV(\text{Float}) - PV(\text{Fixed}) = 0$$

During the swap life:

$$V_t^{\text{swap}} = PV_t(\text{Expected Floating Payments}) - PV_t(\text{Fixed Payments})$$

At any given time, changes in expected future short-term rates will shift the swap's value:

- If expected future MRRs $\uparrow$ $\rightarrow$ value of fixed payer's position $\uparrow$
- If expected future MRRs $\downarrow$ $\rightarrow$ value of fixed payer's position $\downarrow$

3. Determining the Par Swap Rate

Let:

- S_i = Spot rate for period i (e.g., 90-day, 180-day, 270-day, 360-day)
- MRR_i = Implied forward rate for period i
- F = Fixed swap rate (unknown)

$$\boxed{PV(\text{Fixed Leg}) = PV(\text{Floating Leg})}$$

$$F = \frac{1 - (1 + S_4)^{-4}}{\frac{1}{4}\left[(1 + S_1)^{-1} + (1 + S_2)^{-2} + (1 + S_3)^{-3} + (1 + S_4)^{-4}\right]}$$

Interpretation: The par swap rate is the fixed rate F that makes the swap's net PV = 0.

4. Swap Value After Initiation

If market rates change:

$$V_t^{\text{swap}} = PV_t(\text{Float Leg at new MRR}) - PV_t(\text{Fixed Leg at F})$$

$$> 0 \Rightarrow \text{Value to fixed payer if rates rise.}$$

5. **Example: Swap Valuation Concept Given:**
   - Notional = \$1,000,000
   - Quarterly swap, 1 year remaining
   - Fixed rate = 3%
   - Current implied forward 90-day MRRs = 2.8%, 3.2%, 3.4%, 3.6%

   Step 1: Compute PV(Fixed Leg):

   $$PV(\text{Fixed}) = 0.0075(1 + 0.028/4)^{-1} + 0.0075(1 + 0.032/4)^{-2} + \dots$$

   Step 2: Compute PV(Floating Leg):

   $$PV(\text{Float}) = \sum \frac{\text{Expected MRR}_i}{4(1 + S_i)^i}$$

   Step 3: Swap Value:

   $$V = PV(\text{Float}) - PV(\text{Fixed})$$

   If V ¿ 0 → gain to fixed payer (rates have risen).

Exhibit 3: Swap Price vs. Swap Value

Concept	**Definition / Behavior**
Swap Price	Fixed rate in the contract (F_{par}). Constant throughout.
Swap Value	Changes with market expectations of future floating rates.
At Initiation	PV(Fixed) = PV(Floating) → Value = 0.
If Expected Rates Rise	Fixed payer gains; floating payer loses.
If Expected Rates Fall	Fixed payer loses; floating payer gains.

6. **Relationship Summary**

7. **Conceptual Illustration**

$$\text{Swap} = \text{Series of FRAs (long fixed, short floating)} \Rightarrow \begin{cases} \text{Price} = F_{\text{par}} \\ \text{Value} = PV_{\text{Float}} - PV_{\text{Fixed}} \end{cases}$$

Summary of Key Formulas

Key Takeaways

1. **Swap = Series of FRAs:** Each period behaves like a forward on MRR.
2. **Price (Par Swap Rate):** Fixed rate ensuring PV(Fixed) = PV(Floating).
3. **Value:** Difference in PVs — changes with expected rates.
4. **Direction:** ↑ Expected MRR ⇒↑ Value to fixed payer.
5. **Initiation:** Always zero value under no-arbitrage.

Exhibit 4: Core Swap Formulas

Concept	**Formula**
Swap Net Payment	$(\text{MRR}_n - F) \times \text{Notional} \times \frac{d}{360}$
Par Swap Rate	$F_{\text{par}} = \dfrac{1-(1+S_N)^{-N}}{\sum_{i=1}^{N}(1+S_i)^{-i} \times \Delta t_i}$
Swap Value	$V_t = PV_t(\text{Float}) - PV_t(\text{Fixed})$
Value to Fixed Payer	Increases when expected short-term rates rise.
Value to Floating Payer	Increases when expected short-term rates fall.

Module 75.1: Option Valuation

LOS 75.a: Exercise Value, Moneyness, and Time Value of an Option

1. Moneyness Definitions

- **Moneyness** indicates whether immediate exercise yields a gain.
- Let S = current price of the underlying, and X = exercise (strike) price.

Exhibit 1: Moneyness Conditions

Option Type	**In the Money (ITM)**	**At the Money (ATM)**	**Out of the Money (OTM)**
Call Option	$S - X > 0$	$S = X$	$S - X < 0$
Put Option	$X - S > 0$	$S = X$	$X - S < 0$

Example: July 40 Call & Put, stock price = \$37

$$\text{Call: } S - X = 37 - 40 = -3 \Rightarrow 3 \text{ out of the money.}$$
$$\text{Put: } X - S = 40 - 37 = +3 \Rightarrow 3 \text{ in the money.}$$

2. Exercise (Intrinsic) Value

$$\boxed{\text{Exercise Value (Intrinsic Value)} = \max(0, \text{Amount In the Money})}$$

$$\text{Call: } \max(0, S - X)$$
$$\text{Put: } \max(0, X - S)$$

Interpretation: Intrinsic value = immediate exercise payoff (if exercised now).

3. Time Value

$$\boxed{\text{Option Premium} = \text{Exercise Value} + \text{Time Value}}$$

Time Value (Speculative Value):

$$\text{Time Value} = \text{Market Price of Option} - \text{Intrinsic Value}$$

- Reflects potential for future favorable price moves.
- Always positive before expiration.
- Decreases as expiration approaches (*time decay*).

$$\boxed{\text{At expiration: Time Value} = 0}$$

Exhibit 2: Option Premium Decomposition

Option	Market Price	Intrinsic Value	Time Value
Call @ \$5, $S - X = 3$	\$5	\$3	\$2
Put @ \$4, $X - S = 0$	\$4	\$0	\$4

4. Illustration: Option Value Composition

LOS 75.b: Arbitrage and Replication in Option Pricing

1. Difference from Forward Commitments

- **Forward commitments:** Initial value = 0 for both parties.
- **Options (contingent claims):** Initial value > 0 (buyer pays premium).
- Forward payoffs are **symmetric**; option payoffs are **one-sided**.

$$\text{Call payoff: } \max(0, S_T - X), \quad \text{Put payoff: } \max(0, X - S_T)$$

Replication: Option values derived by replicating option payoff through combinations of:

- The underlying asset (long/short),
- A pure discount bond (borrowing/lending at R_f).

2. Upper Bounds (No Arbitrage)

- **Call option:**

$$\boxed{c_t \leq S_t}$$

  No one pays more than the current asset price for a call.

- **Put option:**

$$\boxed{p_t \leq X(1 + R_f)^{-(T-t)}}$$

  Max payoff discounted to present (since puts are European).

3. Lower Bounds (No Arbitrage) **For Call Options:**

$$\boxed{c_0 \geq \max[0, S_0 - X(1 + R_f)^{-T}]}$$

Derivation: Construct a zero-risk portfolio:

- Long call ($+c_0$)
- Long risk-free bond paying X at T
- Short one share of stock ($-S_0$)

$$c_0 - S_0 + X(1 + R_f)^{-T} \geq 0 \Rightarrow c_0 \geq S_0 - X(1 + R_f)^{-T}$$

For Put Options:

$$\boxed{p_0 \geq \max[0, X(1 + R_f)^{-T} - S_0]}$$

Derivation: Construct:

- Long put ($+p_0$)
- Long stock ($+S_0$)
- Short risk-free bond (borrow $X(1 + R_f)^{-T}$)

$$p_0 + S_0 - X(1 + R_f)^{-T} \geq 0 \Rightarrow p_0 \geq X(1 + R_f)^{-T} - S_0$$

Exhibit 3: Price Bounds for European Options

Option Type	**Lower Bound**	**Upper Bound**
Call Option	$\max[0, S_0 - X(1 + R_f)^{-T}]$	S_0
Put Option	$\max[0, X(1 + R_f)^{-T} - S_0]$	$X(1 + R_f)^{-T}$

4. Summary of Option Boundaries **Interpretation:**

- Lower bound ensures no negative value (no arbitrage).
- Upper bound ensures no overpricing (cheaper to transact directly in underlying).

LOS 75.c: Factors Affecting Option Value

1. Six Key Determinants

Exhibit 4: Factors Affecting Option Values

Factor	Effect on Call Value	Effect on Put Value
1. Price of Underlying ($S \uparrow$)	$\uparrow$ Call Value	$\downarrow$ Put Value
2. Exercise Price ($X \uparrow$)	$\downarrow$ Call Value	$\uparrow$ Put Value
3. Risk-Free Rate ($R_f \uparrow$)	$\uparrow$ Call Value	$\downarrow$ Put Value
4. Volatility ($\sigma \uparrow$)	$\uparrow$ Call Value	$\uparrow$ Put Value
5. Time to Expiration ($T \uparrow$)	$\uparrow$ Call Value	$\uparrow$ (usually) but may $\downarrow$ for deep ITM puts
6. Costs / Benefits of Holding Asset	Dividends $\downarrow$ Call Value Storage costs $\uparrow$ Call Value	Dividends $\uparrow$ Put Value Storage costs $\downarrow$ Put Value

2. Conceptual Intuition for Risk-Free Rate Impact

- **Call holder:** Pays strike X in future $\rightarrow$ higher R_f reduces PV(X) $\rightarrow$ call becomes more valuable.
- **Put holder:** Receives X in future $\rightarrow$ higher R_f reduces PV(X) $\rightarrow$ put becomes less valuable.

$$\boxed{R_f \uparrow \Rightarrow C \uparrow, \quad P \downarrow}$$

3. Volatility Effect

- Volatility increases potential payoff range.
- Options have *limited downside (premium)* but *unlimited upside (for calls)*.
- Hence, higher volatility $\Rightarrow$ higher value for both calls and puts.

4. Time to Expiration

- Longer time $\Rightarrow$ more uncertainty $\Rightarrow$ higher value.
- At expiration, time value = 0.
- Exception: Deep in-the-money European puts (delayed receipt reduces PV).

5. Example: Combined Effects

$$\text{Given: } S_0 = 50, X = 55, R_f = 5\%, \sigma = 20\%, T = 1.$$

- If S_0 increases to 60 $\rightarrow$ call value rises, put value falls.
- If σ increases to 30% $\rightarrow$ both call and put values rise.
- If R_f decreases to 2% $\rightarrow$ call value falls, put value rises.

Exhibit 5: Core Relationships and Bounds

Concept	Formula / Relationship
Intrinsic (Exercise) Value	Call: $\max(0, S - X)$; Put: $\max(0, X - S)$
Option Premium	Premium = Intrinsic + Time Value
Call Lower Bound	$c_0 \geq \max[0, S_0 - X(1 + R_f)^{-T}]$
Put Lower Bound	$p_0 \geq \max[0, X(1 + R_f)^{-T} - S_0]$
Call Upper Bound	$c_t \leq S_t$
Put Upper Bound	$p_t \leq X(1 + R_f)^{-(T-t)}$
Risk-Free Rate Impact	$R_f \uparrow \Rightarrow C \uparrow,\ P \downarrow$
Volatility Impact	$\sigma \uparrow \Rightarrow C \uparrow,\ P \uparrow$
Time Decay	As $T \to 0$, Time Value $\to 0$

Key Formula Summary

Summary Box: Conceptual Overview

Moneyness: Determines ITM, OTM, ATM status based on S vs. X.

Intrinsic Value: $\max(0, S - X)$ (call) or $\max(0, X - S)$ (put).

Time Value: Premium − Intrinsic; reflects future potential.

Boundaries: Ensure no-arbitrage range for option prices.

Determinants: $S, X, R_f, \sigma, T,$ and holding costs/benefits.

Key Dynamics: $\begin{cases} S \uparrow \Rightarrow C \uparrow, P \downarrow \\ X \uparrow \Rightarrow C \downarrow, P \uparrow \\ R_f \uparrow \Rightarrow C \uparrow, P \downarrow \\ \sigma \uparrow \Rightarrow C, P \uparrow \end{cases}$

Module 76.1: Put–Call Parity

LOS 76.a: Put–Call Parity for European Options

1. Overview and Key Concept

- **Put–Call Parity (PCP)** defines the equilibrium relationship between European call and put options that share:
  - The same underlying asset.
  - The same strike price (X).
  - The same expiration date (T).
- Based on the principle of **no arbitrage**: portfolios with identical payoffs must have identical prices.

Exhibit 1: Key Portfolios in Put–Call Parity

Portfolio	**Composition**	**Payoff at Expiration ($t = T$)**
Fiduciary Call	Long call ($+c$) and long risk-free bond paying X at T	• If $S_T > X$: $X + (S_T - X) = S_T$ • If $S_T \leq X$: X
Protective Put	Long stock ($+S$) and long put ($+p$)	• If $S_T > X$: S_T • If $S_T \leq X$: $S_T + (X - S_T) = X$

2. Core Portfolio Structures **Result:** At expiration, both fiduciary call and protective put pay *identical outcomes*:

$$\text{Payoff} = \max(S_T, X)$$

3. Put–Call Parity Equation By the **no-arbitrage condition**:

$$\boxed{c + X(1 + R_f)^{-T} = S + p}$$

- LHS: Value of **Fiduciary Call**
- RHS: Value of **Protective Put**

4. Rearranged Equivalent Forms

$$\boxed{\begin{aligned} c &= S + p - X(1 + R_f)^{-T} \\ p &= c - S + X(1 + R_f)^{-T} \\ S &= c - p + X(1 + R_f)^{-T} \\ X(1 + R_f)^{-T} &= S + p - c \end{aligned}}$$

Interpretation: Each of the four securities can be *synthetically replicated* using the other three.

Exhibit 2: Synthetic Positions from Put–Call Parity

Security to Replicate	**Synthetic Combination (Long = +, Short = −)**
Stock (S)	$+c - p + X(1 + R_f)^{-T}$
Call (c)	$+S + p - X(1 + R_f)^{-T}$
Put (p)	$+c - S + X(1 + R_f)^{-T}$
Risk-Free Bond (PV of X)	$+S + p - c$

5. Synthetic Equivalent Constructions Example: Synthetic Stock

$$S = c - p + X(1 + R_f)^{-T}$$

To create a synthetic stock: Long call, short put, and long risk-free bond.

6. Example: Call Option Valuation Using PCP Given:

$$S_0 = 52, \quad X = 50, \quad R_f = 5\%, \quad T = 0.25, \quad p_0 = 1.50$$

Find: c_0.

$$\begin{aligned} c_0 &= S_0 + p_0 - X(1 + R_f)^{-T} \\ &= 52 + 1.50 - 50(1.05)^{-0.25} \\ &= 52 + 1.50 - 49.39 = \boxed{4.11} \end{aligned}$$

Interpretation: A 3-month, $50 call should be priced at approximately $4.11 to prevent arbitrage.

7. Intuition of Parity

- Both portfolios guarantee $\max(S_T, X)$ at expiration.
- Arbitrage ensures that current values are equal.
- If violated, traders can lock in riskless profits by buying the underpriced portfolio and selling the overpriced one.

8. Arbitrage Example (If Parity Violated) Suppose:

$$c + X(1 + R_f)^{-T} > S + p$$

Strategy:

- Sell fiduciary call (LHS), receive cash.
- Buy protective put (RHS).
- Lock in arbitrage gain with zero risk.

LOS 76.b: Put–Call Forward Parity for European Options

1. Conceptual Difference

- Instead of using the current spot price (S_0), we substitute a synthetic forward price.
- A **forward contract** is equivalent to borrowing or lending at R_f and agreeing to buy the asset later at $F_0(T)$.

$$\boxed{F_0(T) = S_0(1 + R_f)^T}$$

Synthetic Asset via Forward:

Long Forward + Risk-Free Bond (PV of $F_0(T)$) $\Rightarrow$ Equivalent to owning asset at T.

2. Derivation of Put–Call Forward Parity

$$\text{Start with: } c + X(1+R_f)^{-T} = S + p$$

Replace the spot price S with its synthetic equivalent $F_0(T)(1+R_f)^{-T}$:

$$\boxed{F_0(T)(1+R_f)^{-T} + p_0 = c_0 + X(1+R_f)^{-T}}$$

$$\boxed{p_0 - c_0 = [X - F_0(T)](1+R_f)^{-T}}$$

3. Interpretation and Insights

- This relationship links option prices directly to the forward price.
- When the forward is at fair value (no arbitrage), PCP ensures consistency between forward-based and spot-based valuation.

If $X = F_0(T)$:

$$p_0 = c_0$$

(both options are equally valuable).

4. Example: Put–Call Forward Parity Application Given:

$$F_0(T) = 105, \quad X = 100, \quad R_f = 4\%, \quad T = 1, \quad c_0 = 8$$

Find: p_0.

$$p_0 = c_0 + [X - F_0(T)](1+R_f)^{-T} = 8 + (100-105)(1.04)^{-1} = 8 - 4.81 = \boxed{3.19}$$

5. Corporate Finance Interpretation (Options on Firm Value) Equity as a Call Option:

$$\text{Equity Value} = \max(0, V_T - D)$$

- V_T: Firm value at time T.
- D: Debt due at maturity (exercise price).

Debt as Risk-Free Bond – Short Put:

$$\text{Debt Value} = D(1+R_f)^{-T} - p$$

or equivalently:

$$\text{Debt payoff} = \min(V_T, D)$$

Interpretation:

- Equity = Call on firm value.
- Debt = Risk-free bond minus put on firm value.

Exhibit 3: Core Formulas – Put–Call Parity Relationships

Concept	**Formula / Expression**
Spot-based PCP	$c + X(1 + R_f)^{-T} = S + p$
Call (explicit form)	$c = S + p - X(1 + R_f)^{-T}$
Put (explicit form)	$p = c - S + X(1 + R_f)^{-T}$
Synthetic Stock	$S = c - p + X(1 + R_f)^{-T}$
Forward-based PCP	$F_0(T)(1 + R_f)^{-T} + p_0 = c_0 + X(1 + R_f)^{-T}$
Put–Call Spread Relation	$p_0 - c_0 = [X - F_0(T)](1 + R_f)^{-T}$
Equity as Option	$E = \max(0, V_T - D)$
Debt as Option	$D_{\text{holders}} = D(1 + R_f)^{-T} - p$

Summary of Key Equations

Conceptual Summary Box

1. Put–Call Parity (Spot): $c + PV(X) = S + p$

2. Put–Call Forward Parity: $F_0(T)PV + p = c + PV(X)$

3. Synthetic Equivalents: $\begin{cases} S = c - p + PV(X) \\ c = S + p - PV(X) \\ p = c - S + PV(X) \end{cases}$

4. Corporate Analogy: $\begin{cases} \text{Equity} = \text{Call on firm value, strike} = \text{debt due.} \\ \text{Debt} = \text{Risk-free bond - Put on firm value.} \end{cases}$

Key Takeaways

- Put–call parity ensures **pricing consistency** between puts, calls, and the underlying asset.
- Applies only to **European options** with identical X and T.
- Synthetic creation allows replication and arbitrage-free valuation.
- Put–call-forward parity extends the concept to forward contracts.
- Firm valuation can be modeled via option theory:
  - Equity = Call option on assets.
  - Debt = Bond – Put option on assets.

Module 77.1: Binomial Model for Option Values

LOS 77.a: Valuing a Derivative Using a One-Period Binomial Model

1. Concept Overview

- The **binomial model** assumes that over one period, the underlying asset price can take one of two possible values:

$$S_u = S_0 \times u \quad \text{(up-move)}, \qquad S_d = S_0 \times d \quad \text{(down-move)}$$

- Inputs required:
  - Current asset price S_0.
  - Exercise price X.
  - Up and down factors: u and d.
  - Risk-free rate R_f.
- Used to derive the **no-arbitrage fair value** of the option at time 0.

2. Example: One-Period Call Option

$$S_0 = 50, \quad X = 55, \quad S_u = 60, \quad S_d = 42, \quad R_f = 3\%$$

$$u = \frac{60}{50} = 1.20, \quad d = \frac{42}{50} = 0.84$$

Call payoffs at expiration:

$$C_u = \max(0, 60 - 55) = 5, \quad C_d = \max(0, 42 - 55) = 0$$

3. Hedge Portfolio Construction (Replication Approach)

- Construct a portfolio of h shares of stock and one short call (–1 call):

$$V_0 = hS_0 - c_0$$

- Portfolio payoffs:

$$V_u = hS_u - C_u$$
$$V_d = hS_d - C_d$$

- Set $V_u = V_d$ (riskless portfolio condition):

$$hS_u - C_u = hS_d - C_d$$

Solve for h:

$$h = \frac{C_u - C_d}{S_u - S_d} = \frac{5 - 0}{60 - 42} = 0.278$$

4. Portfolio Value and Option Price Value of the hedged portfolio:

$$V_u = 0.278(60) - 5 = 11.68, \quad V_d = 0.278(42) - 0 = 11.68$$

Since $V_u = V_d$, this is a risk-free portfolio.
Present value of risk-free portfolio:

$$V_0 = \frac{11.68}{1.03} = 11.34$$

Solve for the option price c_0:

$$V_0 = hS_0 - c_0 \Rightarrow c_0 = hS_0 - V_0 = 0.278(50) - 11.34 = \boxed{2.56}$$

5. Interpretation of Hedge Ratio (h)

- $h = 0.278$ means: to replicate one call option, hold 0.278 shares of stock.
- This ratio (also called **Delta,** Δ) indicates option sensitivity to underlying price movement.

$$\boxed{\Delta = \frac{C_u - C_d}{S_u - S_d}}$$

Exhibit 1: Summary of One-Period Binomial Call Valuation

Parameter	**Value / Formula**
Current Stock Price	$S_0 = 50$
Up Price / Down Price	$S_u = 60,\ S_d = 42$
Call Payoffs	$C_u = 5,\ C_d = 0$
Hedge Ratio	$h = \frac{5-0}{60-42} = 0.278$
Risk-Free Rate	$R_f = 3\%$
Riskless Portfolio Value	$V_u = V_d = 11.68$
Present Value of Portfolio	$V_0 = \frac{11.68}{1.03} = 11.34$
Option Value	$c_0 = 0.278(50) - 11.34 = 2.56$

6. Summary Equation (Replication Method)

$$\boxed{c_0 = hS_0 - \frac{hS_d - C_d}{(1 + R_f)}}$$

LOS 77.b: Risk Neutral Valuation

1. Concept of Risk Neutrality

- Risk-neutral pricing assumes investors are indifferent to risk.
- Under risk neutrality, expected returns on all assets = risk-free rate.
- Therefore, we can price options by discounting the **expected payoff at the risk-free rate**.

2. Risk-Neutral Probability Derivation

$$p^* = \frac{(1 + R_f) - d}{u - d}$$

- p^* = risk-neutral probability of an up-move.
- $1 - p^*$ = probability of a down-move.

3. Example: Risk-Neutral Valuation of a Call Option Given:

$$S_0 = 30, \quad X = 30, \quad u = 1.15, \quad d = 0.87, \quad R_f = 7\%$$

$$S_u = 30(1.15) = 34.50, \quad S_d = 30(0.87) = 26.10$$

$$C_u = \max(0, 34.5 - 30) = 4.5, \quad C_d = \max(0, 26.1 - 30) = 0$$

Step 1: Compute Risk-Neutral Probabilities

$$p^* = \frac{1.07 - 0.87}{1.15 - 0.87} = \frac{0.20}{0.28} = 0.715$$

$$1 - p^* = 0.285$$

Step 2: Expected Option Payoff (Risk-Neutral World)

$$E(C_T) = (p^* \times C_u) + [(1 - p^*) \times C_d] = (0.715)(4.5) + (0.285)(0) = 3.22$$

Step 3: Discount Expected Payoff to Present

$$c_0 = \frac{E(C_T)}{1.07} = \frac{3.22}{1.07} = \boxed{3.01}$$

4. Example: Risk-Neutral Valuation of a Put Option Given same data: $S_0 = 30,\ X = 30,\ u = 1.15,\ d = 0.87,\ R_f = 7\%$

Put Payoffs:

$$P_u = \max(0, 30 - 34.5) = 0, \quad P_d = \max(0, 30 - 26.1) = 3.9$$

Expected Put Payoff:

$$E(P_T) = (p^* \times P_u) + [(1 - p^*) \times P_d] = (0.715)(0) + (0.285)(3.9) = 1.11$$

Discount to Present:

$$p_0 = \frac{E(P_T)}{1.07} = \frac{1.11}{1.07} = \boxed{1.04}$$

Exhibit 2: Risk-Neutral Pricing Summary

Parameter	**Formula / Result**
Risk-Neutral Probability	$p^* = \dfrac{(1+R_f)-d}{u-d}$
Expected Option Payoff	$E(V_T) = p^*V_u + (1-p^*)V_d$
Option Present Value	$V_0 = \dfrac{E(V_T)}{(1+R_f)}$
Call Example Result	$c_0 = 3.01$
Put Example Result	$p_0 = 1.04$

5. Key Insights and Interpretation

- Risk-neutral probabilities are **not real probabilities**; they are derived from arbitrage-free pricing.
- They allow us to compute expected option values without needing investor risk preferences.
- Option value = discounted expected payoff under the risk-neutral measure.

$$\boxed{V_0 = \frac{p^*V_u + (1-p^*)V_d}{1+R_f}}$$

Summary Comparison: Replication vs. Risk-Neutral Methods

Exhibit 3: Comparison of Binomial Approaches

Method	**Replication Portfolio Approach**	**Risk-Neutral Valuation**
Logic	Construct hedge with no risk (Δ shares, short option)	Use arbitrage-free probabilities p^*
Key Step	Solve $V_u = V_d$ for hedge ratio h	Compute $p^* = \frac{(1+R_f)-d}{u-d}$
Option Value Formula	$c_0 = hS_0 - \frac{V_u}{1+R_f}$	$c_0 = \frac{p^*C_u+(1-p^*)C_d}{1+R_f}$
Uses Actual Probabilities?	No	No (uses risk-neutral)
Underlying Assumption	No-arbitrage equilibrium	Risk neutrality
Output	Identical fair value for option under both methods	Same as replication method

Exhibit 4: Formula Summary

Concept	**Formula**
Up / Down Factors	$u = \frac{S_u}{S_0}, \quad d = \frac{S_d}{S_0}$
Call Payoffs	$C_u = \max(0, S_u - X),\ C_d = \max(0, S_d - X)$
Hedge Ratio (Delta)	$h = \frac{C_u - C_d}{S_u - S_d}$
Portfolio Value (Riskless)	$V_0 = \frac{hS_d - C_d}{1+R_f}$
Call Price (Replication)	$c_0 = hS_0 - V_0$
Risk-Neutral Probability	$p^* = \frac{(1+R_f)-d}{u-d}$
Option Value (Risk-Neutral)	$V_0 = \frac{p^* V_u + (1-p^*)V_d}{1+R_f}$
Put–Call Parity Check	$c_0 - p_0 = S_0 - X(1+R_f)^{-T}$

Summary of Key Formulas

Key Takeaways Box

1. **One-Period Binomial Model:** Price changes → Up/Down moves.
2. **Replication Approach:** Construct riskless hedge, use $V_u = V_d$.
3. **Risk-Neutral Approach:** Use $p^* = \dfrac{(1+R_f)-d}{u-d}$, discount expected payoff.
4. **Delta (h):** Shares per option to hedge.
5. **Equivalence:** Both methods yield identical option values under no-arbitrage.