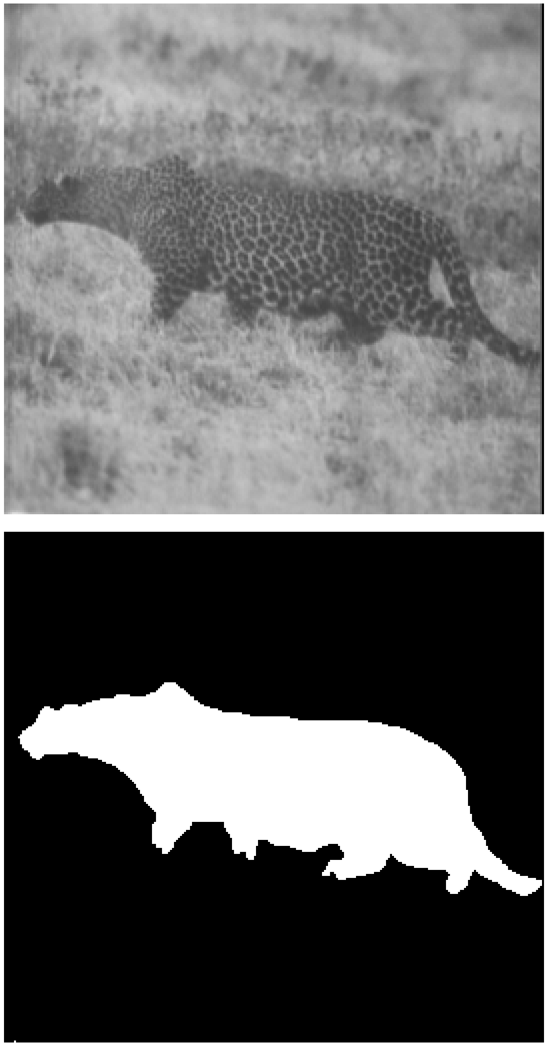
Expectation-Maximization

Nuno Vasconcelos

ECE Department, UCSD

Recall

- last class, we will have “Cheetah Day”
- what:
  - 4 teams, average of 6 people
  - each team will write a report on the 4 cheetah problems
  - each team will give a presentation on one of the problems
- I am waiting to hear on the teams



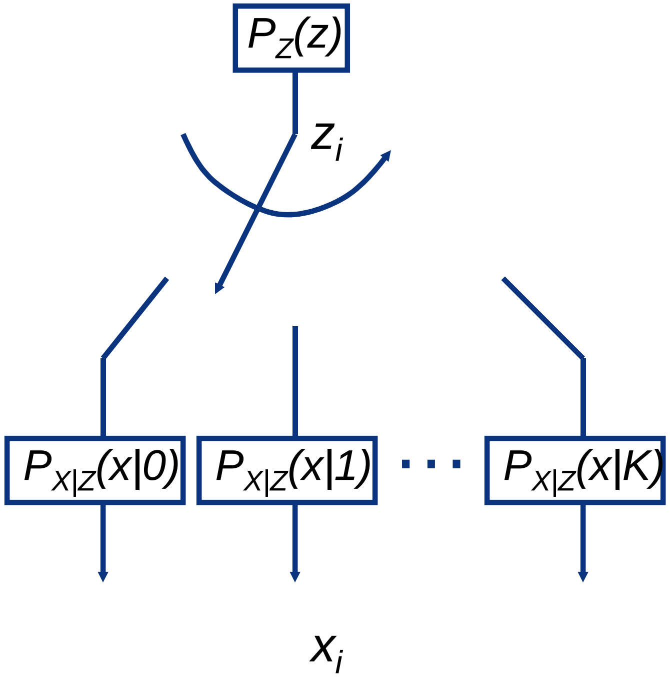
Plan for today

- we have been talking about mixture models
- last time we introduced the basics of EM
- today we study the application of EM for ML estimation of mixture parameters

- next class:
  - proof that EM maximizes likelihood of incomplete data

mixture model

- two types of random variables
  - Z – hidden state variable
  - X – observed variable
- observations sampled with a two-step procedure
  - a state (class) is sampled from the distribution of the hidden variable



$$P_Z(z) \rightarrow z_i$$

  - an observation is drawn from the class conditional density for the selected state

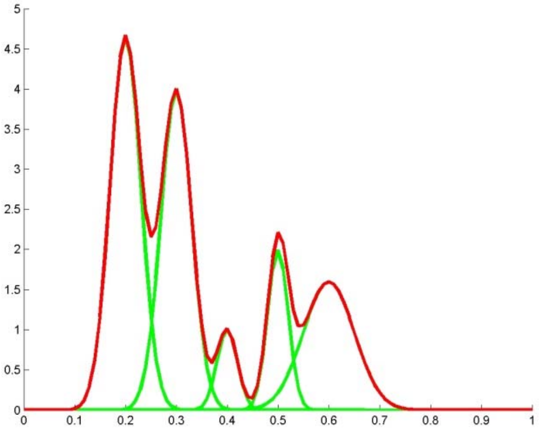
$$P_{X|Z}(x|z_i) \rightarrow x_i$$

mixture model

- the sample consists of pairs (x_i,z_i)

  $$D = \{(x_1,z_1), \ldots, (x_n,z_n)\}$$

  but we never get to see the z_i



- the pdf of the observed data is

$$\begin{aligned} P_{\mathbf{X}}(\mathbf{x}) &= \sum_{c=1}^{C} P_{\mathbf{X}|Z}(\mathbf{x}|c) P_Z(c) \\ &= \sum_{c=1}^{C} P_{\mathbf{X}|Z}(\mathbf{x}|c)\pi_c \end{aligned}$$

# of mixture components

component “weight”

c^th “mixture component”

The basics of EM

- as usual, we start from an iid sample $D = \{x_1,\ldots,x_N\}$
- goal is to find parameters Ψ^* that maximize likelihood with respect to D

$$
\begin{aligned}
\Psi^\star &= \arg\max_{\Psi} P_{\mathbf{X}}(\mathcal{D};\Psi) \\
&= \arg\max_{\Psi} \int P_{\mathbf{X}|Z}(\mathcal{D}|z;\Psi) P_Z(z;\Psi) dz
\end{aligned}
$$

- the set

$$D_c = \{(x_1,z_1), \ldots, (x_N,z_N)\}$$

  is called the complete data

- the set

$$D = \{x_1, \ldots, x_N\}$$

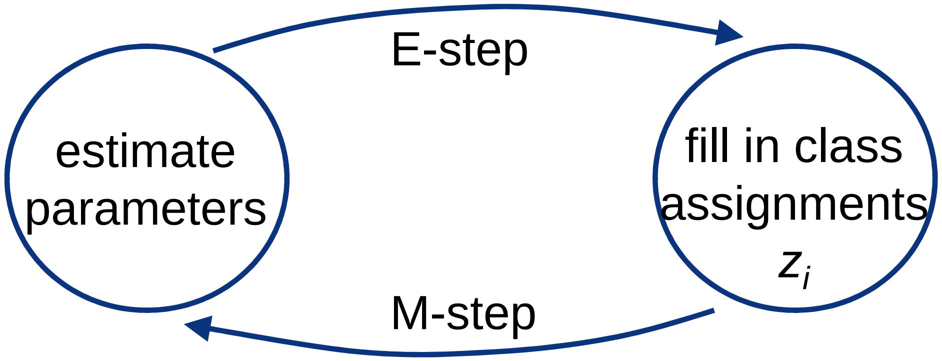
  is called the incomplete data

Learning with incomplete data (EM)

- the basic idea is quite simple

1. start with an initial parameter estimate $\Psi^{(0)}$
2. **E-step:** given current parameters $\Psi^{(i)}$ and observations in D, "guess" what the values of the z_i are
3. **M-step:** with the new z_i, we have a complete data problem, solve this problem for the parameters, i.e. compute $\Psi^{(i+1)}$
4. go to 2.

- this can be summarized as



Classification-maximization

- C-step:
  - given estimates $\Psi^{(i)} = \{\Psi^{(i)}_1, \ldots, \Psi^{(i)}_C\}$
  - determine z_i by the BDR

$$z_l = \arg\max_c P_{\mathbf{X}|Z}\left(\mathbf{x}_l|c; \Psi_c^{(i)}\right)\pi_c^{(i)}, l \in \{1, \ldots, n\}$$

  - split the training set according to the labels z_i

$$D^1 = \{x_i|z_i{=}1\}, \quad D^2 = \{x_i|z_i{=}2\}, \quad \ldots \quad , D^C = \{x_i|z_i{=}C\}$$

- M-step:
  - as before, determine the parameters of each class independently

$$\Psi_c^{(i+1)} = \arg\max_{\Psi,\pi} P_{\mathbf{X}|Z}(\mathcal{D}^c|c, \Psi)\pi$$

For Gaussian mixtures

- C-step:
  - $$z_l = \arg\max_c \left\{ -\frac{1}{2}\left(\mathbf{x}_l - \mu_c^{(i)}\right)^T \left(\Sigma_c^{(i)}\right)^{-1} \left(\mathbf{x}_l - \mu_c^{(i)}\right) - \frac{1}{2} log \left|\Sigma_c^{(i)}\right| + \log \pi_c^{(i)} \right\}, l \in \{1, \ldots, n\}$$
  - split the training set according to the labels z_i

    $D^1 = \{x_i|z_i=1\}$, $D^2 = \{x_i|z_i=2\}$, … , $D^C = \{x_i|z_i=C\}$

- M-step:
  - $$\pi_c^{(i+1)} = \frac{|\{\mathbf{x}_i \in \mathcal{D}^c\}|}{n} \qquad \mu_c^{(i+1)} = \frac{1}{|\{\mathbf{x}_i \in \mathcal{D}^c\}|} \sum_{i|\mathbf{x}_i \in \mathcal{D}^c} \mathbf{x}_i$$

    $$\Sigma_c^{(i+1)} = \frac{1}{|\{\mathbf{x}_i \in \mathcal{D}^c\}|} \sum_{i|\mathbf{x}_i \in \mathcal{D}^c} \left(\mathbf{x}_i - \mu_c^{(i+1)}\right)\left(\mathbf{x}_i - \mu_c^{(i+1)}\right)^T$$

K-means

- when covariances are identity and priors uniform
- C-step:
  - $z_l = \arg\min_c ||\mathbf{x}_l - \mu_c^{(i)}||^2, \qquad l \in \{1, \ldots, n\}$
  - split the training set according to the labels z_i

    $D^1 = \{x_i|z_i=1\}$, $D^2 = \{x_i|z_i=2\}$, … , $D^C = \{x_i|z_i=C\}$
- M-step:
  - $\mu_c^{(i+1)} = \frac{1}{|\{\mathbf{x}_i \in \mathcal{D}^c\}|} \sum_{i|\mathbf{x}_i \in \mathcal{D}^c} \mathbf{x}_i$
- this is the K-means algorithm, aka generalized Loyd algorithm, aka LBG algorithm in the vector quantization literature:
  - “assign points to the closest mean; recompute the means”

The Q function

- is defined as

$$Q(\Psi;\Psi^{(n)}) = E_{Z|\mathbf{X};\Psi^{(n)}}\left[\log P_{\mathbf{X},Z}(\mathcal{D},\{z_1,\ldots,z_N\};\Psi)|\mathcal{D}\right]$$

- and is a bit tricky:
  - it is the expected value of likelihood with respect to complete data (joint X and Z)
  - given that we observed incomplete data (X=$\mathcal{D}$)
  - note that the likelihood is a function of Ψ (the parameters that we want to determine)
  - but to compute the expected value we need to use the parameter values from the previous iteration (because we need a distribution for Z|X)
- the EM algorithm is, therefore, as follows

Expectation-maximization

- E-step:
  - given estimates $\Psi^{(n)} = \{\Psi^{(n)}{}_1, \ldots, \Psi^{(n)}{}_C\}$
  - compute expected log-likelihood of complete data

$$Q(\Psi; \Psi^{(n)}) = E_{Z|\mathbf{X};\Psi^{(n)}}\left[\log P_{\mathbf{X},Z}(\mathcal{D}, \{z_1, \ldots, z_N\}; \Psi)|\mathcal{D}\right]$$

- M-step:
  - find parameter set that maximizes this expected log-likelihood

$$\Psi^{(n+1)} = \arg\max_{\Psi} Q(\Psi; \Psi^{(n)})$$

- let's make this more concrete by looking at the mixture case

Expectation-maximization

- to derive an EM algorithm you need to do the following

→ 1. write down the likelihood of the COMPLETE data
2. E-step: write down the Q function, i.e. its expectation given the observed data
3. M-step: solve the maximization, deriving a closed-form solution if there is one

EM for mixtures (step 1)

- the first thing we always do in a EM problem is
  - compute the likelihood of the COMPLETE data
- very neat trick to use when z is discrete (classes)
  - instead of using *z in {1, 2, ..., C}*
  - use a binary vector of size equal to the # of classes

$$\mathbf{z} \in \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

  - where *z = j* in the *z in {1, 2, ..., C}* notation, now becomes

$$\mathbf{z} = \mathbf{e}_j = \begin{bmatrix} 0 & \\ \vdots & \\ 1 & (j^{th} position) \\ \vdots & \\ 0 & \end{bmatrix}$$

EM for mixtures (step 1)

- we can now write the complete data likelihood as

$$\begin{aligned} P_{\mathbf{X},\mathbf{Z}}(\mathbf{x},\mathbf{z};\Psi) &= P_{\mathbf{X}|\mathbf{Z}}(\mathbf{x}|\mathbf{z};\Psi)P_{\mathbf{Z}}(\mathbf{z};\Psi) \\ &= \prod_{j=1}^{C}\left[P_{\mathbf{X}|\mathbf{Z}}(\mathbf{x}|\mathbf{e}_j,\Psi)\pi_j\right]^{z_j} \end{aligned}$$

- for example, if $z = k$ in the *z in {1, 2, ..., C}* notation,

$$\begin{aligned} P_{\mathbf{X},Z}(\mathbf{x},k;\Psi) &= P_{\mathbf{X},\mathbf{Z}}(\mathbf{x},\mathbf{e}_k;\Psi) \\ &= \left[P_{\mathbf{X}|\mathbf{Z}}(\mathbf{x}|\mathbf{e}_k,\Psi)\pi_k\right]^{1}\prod_{j\neq k}\left[P_{\mathbf{X}|\mathbf{Z}}(\mathbf{x}|\mathbf{e}_j,\Psi)\pi_j\right]^{0} \end{aligned}$$

- the advantage is that

$$\log P_{\mathbf{X},\mathbf{Z}}(\mathbf{x},\mathbf{z};\Psi) = \sum_{j=1}^{C} z_j \log\left[P_{\mathbf{X}|\mathbf{Z}}(\mathbf{x}|\mathbf{e}_j,\Psi)\pi_j\right]$$

- becomes LINEAR in the components z_j !!!

The assignment vector trick

- this is similar to something that we used already
- Bernoulli random variable

$$P_Z(z) = \begin{cases} p & z = 1 \\ 1-p & z = 0 \end{cases}$$

- can be written as

$$P_Z(z) = p^z(1-p)^{1-z}$$

- or, using $z \in \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ instead of $z \in \{0,1\}$, as

$$P_Z(z) = p^{z_1}(1-p)^{z_2}$$

EM for mixtures (step 1)

- for the complete iid dataset $D_c = \{(x_1,z_1), \ldots, (x_N,z_N)\}$

$$\begin{aligned} P_{\mathbf{X},Z}(\mathcal{D},\{\mathbf{z}_1,\ldots,\mathbf{z}_N\};\Psi) &= \prod_{i=1}^{N} P_{\mathbf{X},Z}(\mathbf{x}_i,\mathbf{z}_i;\Psi) \\ &= \prod_{i=1}^{N}\prod_{j=1}^{C}\left[P_{\mathbf{X}|\mathbf{Z}}(\mathbf{x}_i|\mathbf{e}_j,\Psi)\pi_j\right]^{z_{ij}} \end{aligned}$$

- and the complete data log-likelihood is

$$\log P_{\mathbf{X},Z}(\mathcal{D},\{\mathbf{z}_1,\ldots,\mathbf{z}_N\};\Psi) = \sum_{i,j} z_{ij} \log\left[P_{\mathbf{X}|\mathbf{Z}}(\mathbf{x}_i|\mathbf{e}_j,\Psi)\pi_j\right]$$

- this does not depend on z and simply becomes a constant for the expectation that we have to compute in the E-step

Expectation-maximization

- to derive an EM algorithm you need to do the following
  1. write down the likelihood of the COMPLETE data
  2. E-step: write down the Q function, i.e. its expectation given the observed data
  3. M-step: solve the maximization, deriving a closed-form solution if there is one
- important E-step advice:
  - do not compute terms that you do not need
  - at the end of the day we only care about the parameters
  - terms of Q that do not depend on the parameters are useless, e.g. in

    $$Q = f(z, \Psi) + \log(\sin z)$$

    the expected value of *log(sin z)* appears to be difficult and is completely unnecessary, since it is dropped in the M-step

EM for mixtures (step 2)

- once we have the complete data likelihood

$$
\begin{aligned}
Q(\Psi;\Psi^{(n)}) &= E_{Z|\mathbf{X};\Psi^{(n)}}\left[\log P_{\mathbf{X},Z}(\mathcal{D},\{z_1,\ldots,z_N\};\Psi)|\mathcal{D}\right] \\
&= \sum_{i,j} E_{Z|\mathbf{X};\Psi^{(n)}}[z_{ij}|\mathcal{D}] \log\left[P_{\mathbf{X}|\mathbf{Z}}(\mathbf{x}_i|\mathbf{e}_j,\Psi)\pi_j\right]
\end{aligned}
$$

- i.e. to compute the Q function we only need to compute

$$
E_{Z|\mathbf{X};\Psi^{(n)}}[z_{ij}|\mathcal{D}],\ \forall i,j
$$

- note that this expectation can only be computed because we use $\Psi^{(n)}$

- note that the Q function will be a function of both Ψ and $\Psi^{(n)}$

EM for mixtures (step 2)

- since z_{ij} is binary and only depends on x_i

$$E_{\mathbf{Z}|\mathbf{X};\Psi^{(n)}}[z_{ij}|\mathcal{D}] \quad = \quad P_{\mathbf{Z}|\mathbf{X}}(z_{ij} = 1|\mathbf{x}_i; \Psi^{(n)}) = P_{\mathbf{Z}|\mathbf{X}}(\mathbf{e}_j|\mathbf{x}_i; \Psi^{(n)})$$

- the E-step reduces to computing the posterior probability of each point under each class!
- defining

$$h_{ij} = P_{\mathbf{Z}|\mathbf{X}}(\mathbf{e}_j|\mathbf{x}_i; \Psi^{(n)})$$

- the Q function is

$$Q(\Psi; \Psi^{(n)}) \quad = \quad \sum_{i,j} h_{ij} \log \left[P_{\mathbf{X}|\mathbf{Z}}(\mathbf{x}_i|\mathbf{e}_j, \Psi)\pi_j \right]$$

Expectation-maximization

- to derive an EM algorithm you need to do the following

1. write down the likelihood of the COMPLETE data

$$\log P_{\mathbf{X},Z}(\mathcal{D},\{\mathbf{z}_1,\ldots,\mathbf{z}_N\};\Psi) \;=\; \sum_{i,j} z_{ij}\log\left[P_{\mathbf{X}|\mathbf{Z}}(\mathbf{x}_i|\mathbf{e}_j,\Psi)\pi_j\right]$$

2. E-step: write down the Q function, i.e. its expectation given the observed data

$$h_{ij} = P_{\mathbf{Z}|\mathbf{X}}(\mathbf{e}_j|\mathbf{x}_i;\Psi^{(n)})$$

$$Q(\Psi;\Psi^{(n)}) \;=\; \sum_{i,j} h_{ij}\log\left[P_{\mathbf{X}|\mathbf{Z}}(\mathbf{x}_i|\mathbf{e}_j,\Psi)\pi_j\right]$$

3. M-step: solve the maximization, deriving a closed-form solution if there is one

$$\Psi^{(n+1)} \;=\; \arg\max_{\Psi}\sum_{ij} h_{ij}\log\left[P_{\mathbf{X}|\mathbf{Z}}(\mathbf{x}_i|\mathbf{e}_j,\Psi)\pi_j\right]$$

EM vs CM

- let’s compare this with the CM algorithm
  - the C-step

$$\mathbf{z}_i = \arg\max_j P_{\mathbf{Z}|\mathbf{X}}\left(\mathbf{e}_j|\mathbf{x}_i;\Psi^{(n)}\right)$$

assigns each point to the class of largest posterior

  - the E-step

$$h_{ij} = P_{\mathbf{Z}|\mathbf{X}}(\mathbf{e}_j|\mathbf{x}_i)$$

assigns the point to all classes with weight given by the posterior

- for this, EM is said to make “soft-assignments”
  - it does not commit to any of the classes (unless the posterior is one for that class), i.e. it is less greedy
  - no longer partition space into rigid cells, but now the boundaries are soft

EM vs CM

- what about the M-steps?
  - for CM

$$\begin{aligned}\Psi_j^{(n+1)} &= \arg\max_{\Psi} P_{\mathbf{X}|\mathbf{Z}}(\mathcal{D}^j|\mathbf{e}_j,\Psi)\pi \\ &= \arg\max_{\Psi} \sum_{i|z_i=j} \log[P_{\mathbf{X}|\mathbf{Z}}(\mathbf{x}_i|\mathbf{e}_j,\Psi)\pi] \\ &= \arg\max_{\Psi} \sum_i \delta_{z_i=j} \log[P_{\mathbf{X}|\mathbf{Z}}(\mathbf{x}_i|\mathbf{e}_j,\Psi)\pi]\end{aligned}$$

  - for EM

$$\Psi^{(n+1)} = \arg\max_{\Psi} \sum_{ij} h_{ij} \log\left[P_{\mathbf{X}|\mathbf{Z}}(\mathbf{x}_i|\mathbf{e}_j,\Psi)\pi_j\right]$$

- these are the same if we threshold the h_{ij} to make, for each i, $max_j\ h_{ij} = 1$ and all other $h_{ij} = 0$
- M-steps the same up to the difference of assignments

EM for Gaussian mixtures

- in summary:
  - CM = EM + hard assignments
  - CM special case, cannot be better
- let's look at the special case of Gaussian mixtures
- **E-step:**

$$
\begin{aligned}
h_{ij} &= P_{\mathbf{Z}|\mathbf{X}}(\mathbf{e}_j|\mathbf{x}_i;\Psi^{(n)}) \\
&= \frac{\mathcal{G}\left(\mathbf{x}_i,\mu_j^{(n)},\sigma_j^{(n)}\right)\pi_j^{(n)}}{\sum_{k=1}^{C}\mathcal{G}\left(\mathbf{x}_i,\mu_k^{(n)},\sigma_k^{(n)}\right)\pi_k^{(n)}}
\end{aligned}
$$

M-step for Gaussian mixtures

- **M-step:**

$$\begin{aligned}\Psi^{(n+1)} &= \arg\max_{\Psi} \sum_{ij} h_{ij} \log \left[\mathcal{G}\left(\mathbf{x}_i, \mu_j, \sigma_j\right) \pi_j\right] \\ &= \arg\min_{\Psi} \sum_{ij} \frac{h_{ij}(\mathbf{x}_i - \mu_j)^2}{2\sigma_j^2} + \frac{h_{ij}}{2} \log \sigma_j^2 - h_{ij} \log \pi_j\end{aligned}$$

- important note:
  - in the M-step, the optimization must be subject to whatever constraint may hold
  - in particular, we always have the constraint $\sum_j \pi_j = 1$
  - as usual we introduce a Lagrangian

$$L = \sum_{ij} \left[\frac{h_{ij}(\mathbf{x}_i - \mu_j)^2}{2\sigma_j^2} + \frac{h_{ij}}{2} \log \sigma_j^2 - h_{ij} \log \pi_j\right] + \lambda \left(\sum_j \pi_j - 1\right)$$

M-step for Gaussian mixtures

- Lagrangian

$$L = \sum_{ij}\left[\frac{h_{ij}(\mathbf{x}_i-\mu_j)^2}{2\sigma_j^2}+\frac{h_{ij}}{2}\log\sigma_j^2-h_{ij}\log\pi_j\right]+\lambda\left(\sum_j\pi_j-1\right)$$

- setting derivatives to zero

$$\frac{\partial L}{\partial \mu_j} = -\sum_i\frac{h_{ij}(\mathbf{x}_i-\mu_j)}{\sigma_j^2}=0$$

$$\frac{\partial L}{\partial \sigma_j^2} = -\sum_i\left[\frac{h_{ij}(\mathbf{x}_i-\mu_j)^2}{\sigma_j^4}-\frac{h_{ij}}{\sigma_j^2}\right]=0$$

$$\frac{\partial L}{\partial \pi_j} = -\sum_i\frac{h_i}{\pi_j}+\lambda=0$$

$$\frac{\partial L}{\partial \lambda} = \sum_j\pi_j-1=0$$

M-step for Gaussian mixtures

- leads to the update equations

$$\mu_j^{(n+1)} = \frac{\sum_i h_{ij}\mathbf{x}_i}{\sum_i h_{ij}} \qquad \pi_j^{(n+1)} = \frac{1}{n}\sum_i h_{ij}$$

$$\sigma_j^{2(n+1)} = \frac{\sum_i h_{ij}(\mathbf{x}_i - \mu_j)^2}{\sum_i h_{ij}}$$

- comparing to those of CM

$$\pi_c^{(n+1)} = \frac{|\{\mathbf{x}_i \in \mathcal{D}^c\}|}{N} \qquad \mu_c^{(n+1)} = \frac{1}{|\{\mathbf{x}_i \in \mathcal{D}^c\}|}\sum_{i|\mathbf{x}_i \in \mathcal{D}^c} \mathbf{x}_i$$

$$\Sigma_c^{(n+1)} = \frac{1}{|\{\mathbf{x}_i \in \mathcal{D}^c\}|}\sum_{i|\mathbf{x}_i \in \mathcal{D}^c} \left(\mathbf{x}_i - \mu_c^{(n+1)}\right)\left(\mathbf{x}_i - \mu_c^{(n+1)}\right)^T$$

- they are the same up to hard vs soft assignments.

Expectation-maximization

- note that the procedure is the same for all mixtures

1. write down the likelihood of the COMPLETE data

$$\log P_{\mathbf{X},Z}(\mathcal{D}, \{\mathbf{z}_1, \ldots, \mathbf{z}_N\}; \Psi) \quad = \quad \sum_{i,j} z_{ij} \log \left[P_{\mathbf{X}|\mathbf{Z}}(\mathbf{x}_i|\mathbf{e}_j, \Psi)\pi_j \right]$$

2. E-step: write down the Q function, i.e. its expectation given the observed data

$$h_{ij} = P_{\mathbf{Z}|\mathbf{X}}(\mathbf{e}_j|\mathbf{x}_i; \Psi^{(n)})$$

$$Q(\Psi; \Psi^{(n)}) \quad = \quad \sum_{i,j} h_{ij} \log \left[P_{\mathbf{X}|\mathbf{Z}}(\mathbf{x}_i|\mathbf{e}_j, \Psi)\pi_j \right]$$

3. M-step: solve the maximization, deriving a closed-form solution if there is one

$$\Psi^{(n+1)} \quad = \quad \arg\max_{\Psi} \sum_{ij} h_{ij} \log \left[P_{\mathbf{X}|\mathbf{Z}}(\mathbf{x}_i|\mathbf{e}_j, \Psi)\pi_j \right]$$

Expectation-maximization

- E.g. for a mixture of exponential distributions

$$P_X(x) = \sum_{i=1}^{C} \pi_i \lambda_i e^{-\lambda_i x}$$

1. E-step: write down the Q function, i.e. its expectation given the observed data

$$h_{ij} = P_{Z|X}(j \,|\, x_i) = \frac{\pi_j \lambda_j e^{-\lambda_j x_i}}{\sum_{c=1}^{C} \pi_c \lambda_c e^{-\lambda_c x_i}}$$

2. M-step: solve the maximization, deriving a closed-form solution if there is one

$$Q(\Psi;\Psi^{(n)}) \;=\; \sum_{i,j} h_{ij} \log\left[P_{\mathbf{X}|\mathbf{Z}}(\mathbf{x}_i|\mathbf{e}_j,\Psi)\pi_j\right]$$

M-step for exponential mixtures

- **M-step:**

$$\Psi^{(n+1)} = \arg\max_{\Psi} \sum_{ij} h_{ij} \log\left[\pi_j \lambda_j e^{-\lambda_j x_i}\right]$$

$$= \arg\min_{\Psi} \sum_{ij} h_{ij} \left(\lambda_j x_i - \log\left[\pi_j \lambda_j\right]\right)$$

- the Lagragian is

$$L = \sum_{ij} h_{ij} \left(\lambda_j x_i - \log \lambda_j - \log \pi_j\right) + \kappa \left(\sum_j \pi_j - 1\right)$$

M-step for exponential mixtures

- $L = \sum_{ij} h_{ij}\left(\lambda_j x_i - \log\lambda_j - \log\pi_j\right) + \kappa\left(\sum_j \pi_j - 1\right)$

and has minimum at

$$\frac{\partial L}{\partial \lambda_k} = \sum_i h_{ik}\left(x_i - \frac{1}{\lambda_k}\right) = 0$$

$$\frac{\partial L}{\partial \pi_k} = -\sum_i \frac{h_{ik}}{\pi_k} + \kappa = 0$$

$$\frac{\partial L}{\partial \kappa} = \sum_j \pi_j - 1 = 0$$

$$\Longleftrightarrow$$

$$\boxed{\begin{aligned} \frac{1}{\lambda_k} &= \frac{\sum_i h_{ik} x_i}{\sum_i h_{ik}} \\ \kappa &= \sum_{ij} h_{ik} \\ \pi_k &= \frac{\sum_i h_{ik}}{\sum_{ij} h_{ik}} \end{aligned}}$$

EM algorithm

- note, however, that EM is much more general than this recipe for mixtures
- it can be applied for any problem where we have observed and hidden random variables
- here is a very simple example
  - X observer Gaussian variable, $X \sim N(\mu,1)$,
  - Z hidden exponential variable
  - It is known that Z is independent of X
  - sample $D = \{x_1, \ldots, x_n\}$ of iid observations from X
- note that the assumption of independence does not really make sense (why?)
- how does this affect EM?

Example

- toy model: *X* iid, *Z* iid, $X_i \sim N(\mu,1)$, $Z_i \sim \lambda e^{-\lambda z}$, *X* independent of *Z*
- $Q(\Psi; \Psi^{(n)}) = E_{Z|\mathbf{X};\Psi^{(n)}}\left[\log P_{\mathbf{X},Z}(\mathcal{D}, \{z_1, \ldots, z_N\}; \Psi)|\mathcal{D}\right]$

$$\begin{aligned}
&= E_{Z|\mathbf{X};\Psi^{(n)}}\left[-\sum_k \frac{(x_k-\mu)^2}{2} - \frac{N}{2}\log 2\pi - \lambda\sum_k z_k + N\log\lambda|\mathcal{D}\right] \\
&= -\sum_k \frac{(x_k-\mu)^2}{2} - \frac{N}{2}\log 2\pi - \lambda\sum_k E_{Z|\mathbf{X};\Psi^{(n)}}[z_k|x_k] + N\log\lambda \\
&= -\sum_k \frac{(x_k-\mu)^2}{2} - \frac{N}{2}\log 2\pi - \lambda\sum_k E_{Z_k;\Psi^{(n)}}[z_k] + N\log\lambda \\
&= -\sum_k \frac{(x_k-\mu)^2}{2} - \frac{N}{2}\log 2\pi - N\lambda E_{Z;\Psi^{(n)}}[z] + N\log\lambda \\
&= -\sum_k \frac{(x_k-\mu)^2}{2} - \frac{N}{2}\log 2\pi - N\frac{\lambda}{\lambda^{(n)}} + N\log\lambda
\end{aligned}$$

Example

- $$\Psi^{(n+1)} = \arg\max_{\Psi} Q(\Psi; \Psi^{(n)})$$

- $$Q(\Psi; \Psi^{(n)}) = -\sum_k \frac{(x_k - \mu)^2}{2} - \frac{N}{2}\log 2\pi - N\frac{\lambda}{\lambda^{(n)}} + N\log\lambda$$

$$\frac{\partial Q}{\partial \mu} = 0 \Leftrightarrow \boxed{\mu^{(n+1)} = \frac{1}{n}\sum_k x_k} \qquad \frac{\partial Q}{\partial \lambda} = 0 \Leftrightarrow \boxed{\lambda^{(n+1)} = \lambda^{(n)}}$$

- this makes sense:
  - since hidden variables *Z* are independent of observed *X*
  - ML estimate of μ is always the same: the sample mean, no dependence on z_i
  - ML estimate of λ is always the initial estimate $\lambda^{(0)}$: since the observations are independent of the z_i we have no information on what λ should be, other than initial guess.
- note that model does not make sense, not EM solution

Any Questions?