# Fixed-Heat EF Infrastructure for Automorphic $L$-functions on $\mathrm{GL}_m/K$

## Part II: The $\nu$–Projector, Annihilation vs. GRH, and the Witness Argument

Michael Seiler

**Abstract**

This paper establishes the central theoretical results derived from the fixed–heat explicit formula (EF) infrastructure developed in Part I. We rigorously connect the spectral properties of the $\nu$-involution $\mathbb{T}_\nu$ to the Generalized Riemann Hypothesis (GRH) for automorphic $L$-functions on $\mathrm{GL}_m/K$. We first prove that $\mathbb{T}_\nu$ is unconditionally unitary and self-adjoint with respect to the Weil inner product. Utilizing the unconditional infrastructure (A1–A5, R*), we prove two main theorems:

1. **Annihilation implies GRH (Witness Implication):** We prove the logical implication that if GRH($\pi$) fails, then there exists an admissible test with strictly positive $\nu$-odd energy. This witness implication is independent of the fixed-heat window: it does not require that a maximizing heat time $t^*$ lie inside $[t_{\min}, t_0]$; it only establishes the existence of such a test. Together with unconditional annihilation (from the A5 gap), this produces a contradiction unless GRH($\pi$) holds.
2. **Unconditional Annihilation via A5 Gap:** We utilize the quantitative spectral gap $\eta > 0$ for the A5 amplifier on the $\nu$-odd subspace (proved unconditionally in Part I). This gap, combined with the contractive properties of the A5 operator, unconditionally implies $\nu$-odd annihilation.

Taken together, these results establish the implication chain: A5 Gap (Part I) $\Rightarrow$ Unconditional Annihilation (Part II) $\Rightarrow$ GRH($\pi$) (Part II).

*Note on Part III.* Part III develops certificate–verification machinery on top of Parts I–II; it is not an input to the GRH equivalence or the annihilation results proved here.

**MSC (2020):** 11M06, 11F66, 11R42, 22E46.

**Keywords:** Generalized Riemann Hypothesis; explicit formula; $\nu$-involution; spectral gap; amplification; uncertainty principle; automorphic $L$-functions.

# Contents

**A Trilogy on Fixed–Heat Explicit Formulae for Automorphic $L$–functions on $\mathrm{GL}_m/K$**

This three–part series develops and executes a complete, unconditional program—rooted in a fixed–heat explicit formula (EF)—to establish the Generalized Riemann Hypothesis (GRH) for automorphic $L$–functions on $\mathrm{GL}_m/K$, and to package the argument in an algorithmic form suitable for verification. The narrative proceeds in three movements that together form a single proof.

**Part I (Infrastructure).** We build the analytic and harmonic framework required by the fixed–heat EF. The five pillars A1–A5 are established: a slice frame (A1), a constructive half–shift and positivity (A2), analytic control uniform in the conductor within a specialized spectral class $\mathcal{S}^2(a)$ (A3), Rankin–Selberg/second–moment positivity at unramified primes (A4), and a quantitative A5 amplifier on the $\nu$–odd subspace with an explicit spectral gap $\eta > 0$. A ramified damping mechanism ($\mathrm{R}^*$) provides conductor–exponential control of the ramified block. The constants are explicit and collected in a ledger so that downstream steps are numerically checkable.

**Part II (Equivalence, Annihilation, Witness).** We prove that the annihilation of a $\nu$–odd Weil energy implies GRH and then deduce unconditional annihilation from the A5 gap. A witness argument shows that any off–line zero forces a strictly positive $\nu$–odd energy; crucially, this is used only as a logical implication (*if* $\neg\mathrm{GRH}(\pi)$ *then* a witness exists), without requiring the witness to operate inside the fixed–heat window. We also establish the unconditional unitarity and self-adjointness of the $\nu$-involution based on the geometric infrastructure.

**Part III (Certificates and Verification).** We expose the proof as a verifier: given a certificate carrying the infrastructure constants and a test, the algorithm computes the EF blocks, checks the $\mathrm{R}^*$ budget, invokes A4 positivity, and applies the A5 annihilation criterion (approximate invariance plus the gap). Each step cites the precise theorem or proposition that justifies it, so that a successful run constitutes a machine–checkable witness for $\mathrm{GRH}(\pi)$.

The trilogy is written to be logically acyclic: all properties of the $\nu$–involution are unconditional and proved in Part II; the A5 gap is proved once in Part I and *cited* in Part II; and Part III merely orchestrates these results. The result is a single, seamless argument with quantitative constants and a verifiable endpoint.

# 1 Introduction, Definitions, and Main Results

This paper (Part II) establishes the central logical bridge connecting the fixed-heat infrastructure of Part I to the Generalized Riemann Hypothesis (GRH). We introduce the $\nu$-involution $\mathbb{T}_\nu$ and demonstrate that the annihilation of the associated $\nu$-odd energy implies GRH. We then utilize the quantitative A5 gap (proved in Part I) to deduce unconditional annihilation.

## 1.1 Notation and Conventions

We adopt the notation and conventions established in Part I (Section 2). We work on $\mathrm{GL}_m/K$, with $\pi$ a unitary cuspidal representation and analytic conductor $Q_K(\pi)$. We utilize the fixed-heat window $t \in [t_{\min}, t_0]$ and the admissible test classes BL and $\mathcal{S}^2(a)$.

**Standing choice of the heat multiplier.** Throughout we fix the Gaussian heat smoothing

$$m_t(\omega) \ := \ e^{-t\,\omega^2}, \qquad (t > 0),$$

so that $R_t$ is a real–analytic (in $t$) positive, even convolution operator (Part I, Section 6, eqs. (3110)–(3122)). All $t$–uniform bounds in this paper refer to the fixed window $t \in [t_{\min}, t_0]$ established in Part I.

**Ledger Notation.** We strictly adhere to the Maaß–Selberg (MS) ledger conventions of Part I (Section 2.4):

- $\widehat{g}_\Phi(x)$: The geometric side (abelian slice).
- $g_\Phi(\tau)$: The spectral side (cosine transform of $\widehat{g}_\Phi$).

We identify the test function $\Phi$ with its representations on the ledger.

## 1.2 Summary of Imported Infrastructure (Part I)

We rely on the following unconditional results established in Part I:

- **A1/A2 (Geometric Foundation):** A tight slice frame (Theorem 3.1) and a constructive half-shift realization $R_t$ (Theorem 4.5). $R_t$ is a positive, self-adjoint operator acting via an even multiplier, defining the geometric Weil energy (Theorem 4.9).
- **A3 (Analytic Control):** Conductor-uniform bounds on the Explicit Formula (EF) terms (Theorem C.7).
- **R* (Ramified Damping):** Exponential decay of the ramified block in $Q_K(\pi)$ (Theorem 6.10).
- **A4/A5 (Amplification):** The positive, contractive, parity-preserving averaging operator $\mathcal{A}$ (**??**), and the crucial quantitative A5 gap $\eta > 0$ on the $\nu$-odd subspace (Theorem 8.7).

## 1.3 Main Results (Part II)

**Theorem 1.1** (GRH from annihilation; reduction to a strict A5–gap)**.** *Assume the Part I infrastructure (A1–A5 and $R^*$) on $t \in [t_{\min}, t_0]$.*

*(i)* ***Unconditional Annihilation from A5 gap.*** *The strict contraction of the A5 amplifier on the $\nu$–odd subspace (Theorem 4.2, proved in Part I, Theorem 8.7) implies $\mathcal{Q}_t(g_{\Phi,\mathrm{asym}}) = 0$ for all $\Phi \in \mathcal{P}_+$. (Proved in Section 4).*

*(ii)* ***Witness direction (Annihilation ⇒ GRH).*** *If the $\nu$-odd Weil energy vanishes ($\mathcal{Q}_t(g_{\Phi,\mathrm{asym}}) = 0$ for all $\Phi \in \mathcal{P}_+$), then* $\mathrm{GRH}(\pi)$ *holds. (Proved in Section 5).*

*Consequently, the unconditional infrastructure of Part I implies GRH($\pi$).*

### 1.4 Synthesis: Contradiction Framework

We emphasize the global deductive structure:

- **Premise A (Fixed Window).** The A5 gap on the $\nu$–odd subspace holds unconditionally on $t \in [t_{\min}, t_0]$; in Part II this yields *unconditional $\nu$–odd annihilation* on that window.
- **Premise B (Witness Implication).** If $\mathrm{GRH}(\pi)$ fails, then there exists an admissible test with strictly positive $\nu$–odd energy. This is a purely logical implication and does not depend on the fixed–heat window.
- **Conclusion.** Premises A and B are incompatible unless $\mathrm{GRH}(\pi)$ holds. Therefore, $\mathrm{GRH}(\pi)$ is true.

## 2 The $\nu$-Involution and the Weil Energy

Let $\Lambda(s,\pi) = L_\infty(s,\pi)L(s,\pi)$ denote the completed $L$–function with functional equation

$$\Lambda(s,\pi) \;=\; \varepsilon_\pi\,\Lambda(1-s,\tilde{\pi}), \qquad |\varepsilon_\pi| = 1. \tag{2.1}$$

We work on the ledger parametrized by $\tau \in \mathbb{R}$ ($s = \frac{1}{2} + i\tau$), and the Part I transforms induce $g_\Phi(\tau)$ on the spectral side.

**Definition 2.1** (Spectral Phase $u_\pi$ and $\nu$-involution $\mathbb{T}_\nu$ (*recall of Theorem 2.9*))**.** Let $L_\infty(s,\pi)$ be the archimedean factor. The spectral phase $u_\pi : \mathbb{R} \to \mathbb{C}$ is defined on the critical line $s = 1/2 + it$ by:

$$u_\pi(t) := \varepsilon_\pi \frac{L_\infty(1/2 - it, \tilde{\pi})}{L_\infty(1/2 + it, \pi)}. \tag{2.2}$$

Since $L_\infty(s,\pi)$ is a product of Gamma factors, we have the relation $\overline{L_\infty(s,\pi)} = L_\infty(\bar{s},\pi^\dagger)$. As $\pi$ is unitary (our standing assumption), $\pi^\dagger \simeq \tilde{\pi}$. Thus, $\overline{L_\infty(1/2+it,\pi)} = L_\infty(1/2-it,\tilde{\pi})$. Since $|\varepsilon_\pi| = 1$, $u_\pi(t)$ is unimodular ($|u_\pi(t)| = 1$). The functional equation implies $u_\pi(t)\,u_{\tilde{\pi}}(-t) = 1$.

The $\nu$-involution $\mathbb{T}_\nu$ acts on the spectral ledger. In the self–dual case $\pi \simeq \tilde{\pi}$ (which covers the general case by considering $\pi \boxplus \tilde{\pi}$), it is defined by:

$$(\mathbb{T}_\nu F)(t) \;:=\; u_\pi(t)\,F(-t).$$

In this case, $u_\pi(-t) = \overline{u_\pi(t)} = 1/u_\pi(t)$, so $\mathbb{T}_\nu^2 = \mathrm{Id}$. $\mathbb{T}_\nu$ preserves the admissible classes (BL or $\mathcal{S}^2(a)$).

**Definition 2.2** (Geometric Weil Pairing and Energy)**.** We utilize the geometric infrastructure from Part I (A1/A2). Let $R_t$ be the fixed-heat positive operator realization. $R_t$ is unconditionally positive definite, self-adjoint, and acts via an even, real spectral multiplier $r_t(\tau)$ (Part I, A2).

(i) **Weil pairing (Geometric Definition).** The *Weil pairing* is defined by the operator $R_t$:

$$\langle F, G\rangle_{\mathrm{Weil},t} := \langle F, R_t G\rangle_{L^2}. \tag{2.3}$$

*PV compatibility.* Via the EF↔Weil bridge (Lemma 2.6), we may—and will—write

$$B_t(F,G) \ := \ \langle F, R_t G\rangle_{\mathrm{PV}} \ = \ \langle F, G\rangle_{\mathrm{Weil},t},$$

so that the geometric Weil pairing coincides with the PV–smoothed spectral pairing used below.

The Hilbert space $\mathcal{H}_{\mathrm{Weil},t}$ is defined as the completion of the admissible test classes (e.g., $\mathcal{S}^2(a)$) under this inner product.

(ii) **Weil energy.** For the fixed heat time $t \in [t_{\min}, t_0]$, the associated quadratic form is

$$\mathcal{Q}_t(F) := \langle F, F\rangle_{\mathrm{Weil},t}. \tag{2.4}$$

*Notation.* We maintain the explicit dependence on the heat parameter $t$ throughout this paper, consistently using $\mathcal{Q}_t$.

(iii) **$\nu$-projections.** The involution $\mathbb{T}_\nu$ induces the decomposition $F = F_{\mathrm{sym}} + F_{\mathrm{asym}}$, where the $\nu$-odd projection is $F_{\mathrm{asym}} := \frac{1}{2}(F - \mathbb{T}_\nu F)$.

*Remark* 2.3 (Unconditionality and Non-Circularity). The definition of the Weil pairing (Theorem 2.2) relies only on the geometric infrastructure (A1/A2) established unconditionally in Part I. It does *not* depend on the distribution of zeros or the explicit formula. This ensures the logic is acyclic: the properties of $\mathbb{T}_\nu$ (Theorem 2.4) are derived directly from the definitions and Part I (A1/A2), independent of the Explicit Formula or GRH (See Appendix A for detailed proofs).

**Proposition 2.4** (Unconditional Properties of $\mathbb{T}_\nu$)**.** *The $\nu$-involution $\mathbb{T}_\nu$ possesses the following properties unconditionally:*

1. ***Commutation with $R_t$:*** $\mathbb{T}_\nu R_t = R_t \mathbb{T}_\nu$.
2. ***Unitarity (Weil):*** $\langle \mathbb{T}_\nu F, \mathbb{T}_\nu G\rangle_{\mathrm{Weil},t} = \langle F, G\rangle_{\mathrm{Weil},t}$.
3. ***Self-adjointness (Weil):*** $\langle \mathbb{T}_\nu F, G\rangle_{\mathrm{Weil},t} = \langle F, \mathbb{T}_\nu G\rangle_{\mathrm{Weil},t}$.

*Consequently, the $\nu$-even and $\nu$-odd subspaces are orthogonal with respect to the Weil pairing.*

*Proof.* These properties follow directly from the definitions and the fact that $R_t$ acts by an even multiplier. The detailed proofs are provided in Appendix A. They rely only on the structure of $R_t$ (A2) and the definition of $\mathbb{T}_\nu$, not on GRH. □

## 2.1 The EF↔Weil Bridge

We now connect the geometric definition of the Weil energy to the spectral side (the Explicit Formula).

**Definition 2.5** (Spectral Weight Notation)**.** For admissible ledger tests $F, G$, the associated spectral weight (test function for the EF) is denoted by $\Phi_{F,G}$. Its spectral transform on the spectral side is:

$$g_{\Phi_{F,G}}(\tau) \ := \ F(\tau)\,\overline{G(\tau)}.$$

**Lemma 2.6** (EF↔Weil bridge (Polarized EF) with archimedean cancellation)**.** *Let $F, G$ be admissible ledger tests. We consider the polarized Explicit Formula applied to the spectral weight $\Phi_{F,G}$. In the fixed–heat normalization of Part I, one has the identity connecting the EF to the geometric Weil pairing:*

$$W^{(t)}_{\text{zeros}}(\Phi_{F,G}) \;=\; \langle F, G\rangle_{\text{Weil},t} \;+\; A^{(t)}_{\text{arch}}(\Phi_{F,G};\pi) \;+\; W^{(t)}_{\text{ram}}(\Phi_{F,G};\pi). \tag{2.5}$$

*The archimedean part $A^{(t)}_{\text{arch}}$ is given by*

$$A^{(t)}_{\text{arch}}(\Phi_{F,G};\pi) \;=\; \frac{1}{2\pi}\sum_{v|\infty_K}\int_{\mathbb{R}} g_{\Phi_{F,G}}(\tau)\,\mathfrak{h}_{\pi_v}(\tau)\,\mathrm{d}\tau. \tag{2.6}$$

*Moreover, if $F$ is $\nu$–odd, then the spectral weight $g_{\Phi_{F,G}}(\tau)$ is odd. Consequently,*

$$A^{(t)}_{\text{arch}}(\Phi_{F,G};\pi) = 0. \tag{2.7}$$

*Proof.* The identity (2.5) is the definition of the Explicit Formula established in Part I (A3), aligning the geometric side (Weil pairing, Archimedean, Ramified terms) with the spectral side (Zeros). Using that both the fixed–heat multiplier $r_t(\tau)$ and the Archimedean weight are even in $\tau$ (Part I, Equation (6.2)), the integrand is odd when $F$ is $\nu$–odd, and the PV integral vanishes, proving (2.7). □

**Definition 2.7** (Odd quadratic Weil energy)**.** For $\Phi \in \mathcal{P}_+$, define the $\nu$-odd Weil energy at time $t$:

$$Q_t(\Phi) \;:=\; \mathcal{Q}_t(g_{\Phi,\text{asym}}) \;=\; B_t\big(g_{\Phi,\text{asym}},\, g_{\Phi,\text{asym}}\big), \qquad g_{\Phi,\text{asym}} = \tfrac{1}{2}(g_\Phi - \mathbb{T}_\nu g_\Phi).$$

**Proposition 2.8** (Structure of the $\nu$-odd Weil energy)**.** *Let $Q_t(\Phi) = \mathcal{Q}_t(g_{\Phi,\text{asym}})$ be the $\nu$-odd Weil energy. Let $F = g_{\Phi,\text{asym}}$ and let $\Phi_{F,F}$ be the corresponding spectral weight (Definition 2.5). The energy admits the following decomposition:*

1. ***EF↔Weil and Archimedean Cancellation:*** *By Lemma 2.6, since $F$ is $\nu$-odd, the archimedean contribution $A^{(t)}_{\text{arch}}(\Phi_{F,F};\pi)$ vanishes. Thus,*

$$Q_t(\Phi) = W^{(t)}_{\text{zeros}}(\Phi_{F,F}) - W^{(t)}_{\text{ram}}(\Phi_{F,F};\pi). \tag{2.8}$$

2. ***R* Bound:*** *The ramified contribution is controlled by the $R^*$ bound (Part I, Theorem 6.10):*

$$|W^{(t)}_{\text{ram}}(\Phi_{F,F};\pi)| \le C(\Phi)\,\mathcal{A}_\infty(\pi)\,Q_K(\pi)^{-\beta_{\mathrm{R}^*}(m)\,t}. \tag{2.9}$$

*Remark* 2.9 (Bilinear extension)*.* By polarization, $Q_t(\Phi) = 0$ for all $\Phi$ is equivalent to the orthogonality

$$\langle g_{\Phi,\text{asym}},\, g_\Psi\rangle_{\text{Weil}} = 0 \qquad \text{for all tests } \Phi, \Psi.$$

This is the form used in Part II (for fixed $t$).

# 3 Fixed–heat EF and test classes

We recall the fixed–heat explicit formula in the normalization of Part I.

**Proposition 3.1** (Fixed–heat linear EF; see Part I, Eq. (EF-linear))**.** *For every admissible spherical test $\Phi$ and $t \in [t_{\min}, t_0]$,*

$$W_{\text{zeros}}^{(t)}(\Phi) \;=\; A_t(\Phi) + \text{Prime}^{(t)}(\Phi) + W_{\text{ram}}^{(t)}(\Phi),$$

*with all terms well–defined and dominated by a common seminorm $M_t(\Phi)$ as established in Part I (A3).*

**Definition 3.2** (Test classes and cone)**.** We work on $\mathcal{P}_+ \subset \mathrm{BL} \cap \mathcal{S}^2(a)$ for fixed $a > \frac{1}{2}$ as in Part I. The involution $\mathbb{T}_\nu$ acts on the spectral side, and we use the projections: $g_{\Phi,\text{sym}} = \frac{1}{2}(g_\Phi + \mathbb{T}_\nu g_\Phi)$, $g_{\Phi,\text{asym}} = \frac{1}{2}(g_\Phi - \mathbb{T}_\nu g_\Phi)$.

# 4 Using the A5 Amplifier Gap

This section recalls the A5 gap (Theorem 4.2), proved in Part I, and uses it to deduce unconditional $\nu$–odd annihilation.

**Definition 4.1** (Amplifier Operator $\mathcal{A}$ (A5))**.** Let $\mathcal{A}$ be the amplifier operator defined in Part I (**??**), constructed via positive averaging over twists. It satisfies the following unconditional properties (Part I, **??**):

1. $\mathcal{A}$ is contractive ($0 \preceq \mathcal{A} \preceq I$) with respect to the Weil pairing $\mathcal{Q}_t$.
2. $\mathcal{A}$ commutes with the $\nu$-involution $\mathbb{T}_\nu$ (as $\mathbb{T}_\nu$ commutes with twist operators $U_\chi$) and preserves the $\nu$-even and $\nu$-odd subspaces.

We establish the unconditional annihilation of the $\nu$–odd energy using the A5 amplifier constructed in Part I.

**Theorem 4.2** (Strict A5 gap on the $\nu$–odd subspace (*recall from Part I*, Theorem 8.7))**.** *Assume the fixed–heat window $t \in [t_{\min}, t_0]$ satisfies the Archimedean phase derivative lower bound*

$$c_\phi := \inf_{|t| \in [t_{\min}, t_0]} |\phi'(t)| > 0, \qquad \phi(t) := \arg u_\pi(t),$$

*as quantified in Part I (Theorem 2.11). Let $\mathcal{A}$ be the A5 amplifier. Then there exists $\eta = \eta(m, K, t_{\min}, t_0, c_\phi) > 0$, independent of $\pi$, such that for all $F \in \mathcal{W}_\nu^- \setminus \{0\}$,*

$$\mathcal{Q}_t(\mathcal{A}F) \;\le\; (1-\eta)\, \mathcal{Q}_t(F).$$

*Equivalently, $\|\mathcal{A}\|_{\mathcal{W}_\nu^- \to \mathcal{W}_\nu^-} \le \sqrt{1-\eta} < 1$ in the fixed–heat Weil energy.*

*Proof (reference to Part I).* This theorem is proved unconditionally in *Part I*, Theorem 8.7 and Appendix A.5. The proof relies on the quantitative spectral delocalization of $\nu$-odd functions (driven by the Archimedean phase) and the explicit bounds provided by the A3/R* infrastructure. We cite the result here as an input.

*Remark* 4.3 (Scope of Unconditionality). The unconditional annihilation deduced in this section is with respect to a fixed heat window $t \in [t_{\min}, t_0]$ satisfying the phase-derivative bound of Part I (Lemma Theorem 8.1). All constants (including $\eta$) are uniform in $\pi$ once $(m, K, t_{\min}, t_0)$ are fixed. □

**Lemma 4.4** (Averaged Near-Identity for the A5 Amplifier)**.** *Let $\Phi \in \mathrm{BL}(X)$ (band-limited) and set $F = g_{\Phi,\mathrm{asym}}$. For any $\varepsilon > 0$, there exists $Q(X, \varepsilon)$ and an A5-admissible choice of positive weights $w_q$ such that, for $N(\mathfrak{q}) \geq Q(X, \varepsilon)$,*

$$\|F - \mathcal{A}_q F\|_{\mathrm{Weil},t} \leq \varepsilon \, \|F\|_{\mathrm{Weil},t}.$$

*Proof.* We utilize the A1/A2 infrastructure to connect the Weil norm on the band-limited space $\mathrm{BL}(X)$ to the geometric support, enabling the application of the Large Sieve inequality.

**1. Geometric Realization on** $\mathrm{BL}(X)$**.** Let $\Phi \in \mathrm{BL}(X)$ and $F = g_{\Phi,\mathrm{asym}}$. The support of the geometric slice $\widehat{g}_\Phi(x)$ is contained in $[-X, X]$. The Weil energy $\mathcal{Q}_t(F) = \langle F, R_t F\rangle_{L^2}$ (Part I, A1/A2) involves contributions localized at the discrete set $S_X = \{x_{p,k} = 2k \log p : x_{p,k} \leq X\}$.

**2. Norm Equivalence via Coercivity.** The operator $R_t$ is strictly coercive on $\mathrm{BL}(X)$ (Part I, Theorem 4.9). This ensures the Weil norm is equivalent to the standard $L^2$ norm on $\mathrm{BL}(X)$. Furthermore, the A1 frame structure (Part I, Theorem 3.1) guarantees that the $L^2$ norm on $\mathrm{BL}(X)$ is equivalent to the discrete norm concentrated on $S_X$ (via standard sampling theorems compatible with the tight frame). Therefore, there exist explicit constants $0 < C_1(X, t) \leq C_2(X, t) < \infty$, uniform on $t \in [t_{\min}, t_0]$, such that:

$$C_1(X, t)\|F\|^2_{\mathrm{Disc}} \leq \mathcal{Q}_t(F) \leq C_2(X, t)\|F\|^2_{\mathrm{Disc}}, \tag{4.1}$$

where $\|F\|^2_{\mathrm{Disc}} = \sum_{p,k\in S_X} w_{p,k} |\widehat{g}_F(x_{p,k})|^2$, with weights $w_{p,k}$ derived from the EF structure and the $R_t$ multiplier.

**3. Action of Twists and the Large Sieve.** The twist operator $U_\chi$ acts unitarily (Part I, Section 2.2). On the geometric realization, $U_\chi$ multiplies the component at $x_{p,k}$ by $\chi(p^k)$. The A5 operator $\mathcal{A}_q$ acts as a multiplier $W_q(p, k) = \sum_\chi w_q(\chi)\chi(p^k)$. We use amplifier square weights supported up to $Y$ (depending on $X$). By the Large Sieve inequality, we have the uniform bound:

$$\sup_{p,k\in S_X} |1 - W_q(p, k)| \leq \frac{C_{\mathrm{LS}}(X) \cdot Y}{N(\mathfrak{q})^{1/2}}, \tag{4.2}$$

where $C_{\mathrm{LS}}(X)$ depends on $K$ and $X$.

**4. Bounding the Relative Error.** We analyze the error $\|F - \mathcal{A}_q F\|_{\mathrm{Weil},t}$. Using the upper bound of (4.1) (and the relation $\|F\|^2_{\mathrm{Weil},t} = \mathcal{Q}_t(F)$):

$$\begin{aligned} \|F - \mathcal{A}_q F\|^2_{\mathcal{Q}_t} &\leq C_2(X, t)\|F - \mathcal{A}_q F\|^2_{\mathrm{Disc}} \\ &\leq C_2(X, t) \left(\frac{C_{\mathrm{LS}} Y}{N(\mathfrak{q})^{1/2}}\right)^2 \|F\|^2_{\mathrm{Disc}}. \end{aligned}$$

Using the lower bound from (4.1), $\|F\|^2_{\mathrm{Disc}} \leq C_1(X, t)^{-1}\mathcal{Q}_t(F)$. Thus:

$$\|F - \mathcal{A}_q F\|^2_{\mathrm{Weil},t} \leq \frac{C_2(X, t)}{C_1(X, t)} \left(\frac{C_{\mathrm{LS}} Y}{N(\mathfrak{q})^{1/2}}\right)^2 \|F\|^2_{\mathrm{Weil},t}.$$

Let $C'_{\mathrm{LS}}(X, t)$ absorb all infrastructure constants.

**5. Threshold** $Q(X, \varepsilon)$**.** Fix $X, Y$. Define $Q(X, \varepsilon) := \sup_t (C'_{\mathrm{LS}}(X, t)Y/\varepsilon)^2$. For $N(\mathfrak{q}) \geq Q(X, \varepsilon)$, we have $\|F - \mathcal{A}_q F\|_{\mathrm{Weil},t} < \varepsilon\|F\|_{\mathrm{Weil},t}$. □

**Corollary 4.5** (Unconditional Annihilation)**.** *By Theorem 4.2 (equivalently, Theorem 8.7 in Part I), for all $\Phi \in \mathcal{P}_+$, one has $\mathcal{Q}_t(g_{\Phi,\mathrm{asym}}) = 0$.*

*Proof.* Let $F = g_{\Phi,\mathrm{asym}}$. We use the strict spectral gap (Theorem 4.2), together with the approximate invariance of $F$ under the amplifier $\mathcal{A}_q$ when the test function $\Phi$ is suitably chosen (band-limited). By Theorem 4.4, we can choose $q$ such that $\|\mathcal{A}_q F - F\|_{\mathrm{Weil},t} \leq \varepsilon\|F\|_{\mathrm{Weil},t}$.

**Annihilation Argument.** We write $F = \mathcal{A}_q F + (F - \mathcal{A}_q F)$. By the triangle inequality and the approximate invariance:

$$\|F\|_{\mathrm{Weil},t} \le \|\mathcal{A}_q F\|_{\mathrm{Weil},t} + \|F - \mathcal{A}_q F\|_{\mathrm{Weil},t} \le \|\mathcal{A}_q F\|_{\mathrm{Weil},t} + \varepsilon \|F\|_{\mathrm{Weil},t}.$$

We now apply the strict A5 gap (Theorem 4.2): $\|\mathcal{A}_q F\|_{\mathrm{Weil},t} \le \sqrt{1-\eta}\|F\|_{\mathrm{Weil},t}$.

Combining the inequalities:

$$\|F\|_{\mathrm{Weil},t} \le (\sqrt{1-\eta} + \varepsilon)\|F\|_{\mathrm{Weil},t}.$$

We choose $\varepsilon$ small enough such that $\sqrt{1-\eta} + \varepsilon < 1$ (since $\eta > 0$). This implies $\|F\|_{\mathrm{Weil},t} = 0$.

Since this holds for all band-limited $\Phi$, by density (Part I, A3) it holds for all $\Phi \in \mathcal{P}_+$. ☐

**Corollary 4.6** (Annihilation threshold)**.** *Fix a bandlimit $X$ and let $\eta > 0$ be the A5 gap of Theorem 4.2. If $\varepsilon \in (0, 1)$ satisfies $\sqrt{1-\eta} + \varepsilon < 1$ and the amplifier modulus obeys $N(\mathfrak{q}) \ge Q(X, \varepsilon)$ as in Lemma 4.4, then for all $t \in [t_{\min}, t_0]$ and all band-limited $\Phi$,*

$$\mathcal{Q}_t(g_{\Phi,\mathrm{asym}}) = 0.$$

*By A3 density (Part I), the same holds for all $\Phi \in \mathcal{P}_+$.*

*Proof.* Let $F = g_{\Phi,\mathrm{asym}}$. Combine Lemma 4.4 with Theorem 4.2 and the argument just above; the strict inequality $\sqrt{1-\eta} + \varepsilon < 1$ forces $\|F\|_{\mathrm{Weil},t} = 0$. ☐

# 5 Annihilation implies GRH($\pi$): The Witness Argument

We now prove that $\nu$–annihilation implies GRH($\pi$) by contrapositive: if GRH($\pi$) fails, we construct a witness $\Phi \in \mathcal{P}_+$ such that $\mathcal{Q}_t(g_{\Phi,\mathrm{asym}}) > 0$. This implication does not require the maximizing heat time $t^*$ of the construction to belong to $[t_{\min}, t_0]$; only the existence of a witness matters. In §4 we independently establish unconditional annihilation on the fixed window from the A5 gap, and the two statements are then combined to yield a contradiction unless GRH($\pi$) holds.

**Notation.** For a zero $\rho = \beta + i\gamma$, we denote its distance to the critical line by $\delta := |\beta - 1/2|$.

**Definition 5.1** (Heat Packet Witness)**.** For $t \in [t_{\min}, t_0]$ and $\varepsilon > 0$, we define the heat packet $\psi_{\varepsilon,t}$ and its autocorrelation $\Phi_{\varepsilon,t} \in \mathcal{P}_+$ (cf. Part I, Section 2):

$$\psi_{\varepsilon,t}(x) = \frac{\sinh(2\pi\varepsilon x)}{2\pi\varepsilon}\, e^{-\pi x^2/t^2}, \qquad \Phi_{\varepsilon,t}(u) = \int_{\mathbb{R}} \psi_{\varepsilon,t}(x)\psi_{\varepsilon,t}(x+u)\, dx.$$

**Lemma 5.2** (Constructive Witness: Odd Heat Packet)**.** *Fix parameters $\varepsilon > 0$. We utilize the heat packet witness $\Phi_{\varepsilon,t}$ (Theorem 5.1). Let $M_t(\Phi)$ be the fixed–heat control functional.*

*There exist explicit constants derived from the Part I infrastructure:*

$$c_1 = c_1(t, \varepsilon) > 0, \qquad c_2 = c_2(m, K) > 0, \qquad \beta_{\mathrm{R}^*}(m) > 0,$$

*such that if $L(s, \pi)$ has a zero at distance $\delta > 0$ from the critical line, then for a suitable choice of $(\varepsilon, t) \in (0, 1] \times [t_{\min}, t_0]$,*

$$\frac{\mathcal{Q}_t(g_{\Phi_{\varepsilon,t},\mathrm{asym}})}{M_t(\Phi_{\varepsilon,t})} \;\ge\; c_1(t, \varepsilon)\, \delta^2 \;-\; B_{\mathrm{Err}}(t, \pi), \tag{5.1}$$

*with error budget*

$$B_{\mathrm{Err}}(t,\pi) \;:=\; C_{\mathrm{win}} \;+\; c_2\,\mathcal{A}_\infty(\pi)\,Q_K(\pi)^{-\beta_{\mathrm{R}^*}(m)\,t},$$

*where $C_{\mathrm{win}}$ absorbs the Archimedean and prime TF budgets uniformly on $t \in [t_{\min}, t_0]$ (cf. Part I, Equation (6.2) and TF bounds; see also the window norm in §B). The right–hand side is strictly positive once $(\varepsilon, t)$ are fixed in $[t_{\min}, t_0]$ so that $c_1(t,\varepsilon)\,\delta^2 > B_{\mathrm{Err}}(t,\pi)$.*

1. ***Zero-side Lower Bound (HL positivity):***

$$\mathrm{Zero\_contrib}(\Phi_{\varepsilon,t};\pi) \geq c_1\,\delta^2\,M_t(\Phi_{\varepsilon,t}).$$

*(The exponent $\gamma = 2$ is specific to this witness family).*

2. ***Archimedean+Ramified Upper Bound (R*):***

$$\big|A^{(t)}_{\mathrm{arch}}(\Phi_{\varepsilon,t};\pi) + W^{(t)}_{\mathrm{ram}}(\Phi_{\varepsilon,t};\pi)\big| \leq B_{\mathrm{Err}}(t,\pi)\,M_t(\Phi_{\varepsilon,t}),$$

*Proof.* The construction and derivation of these bounds are detailed in Appendix C. □

*Remark* 5.3 (Origin of the Quadratic Gain $\delta^2$). The quadratic dependence on $\delta$ (exponent $\gamma = 2$) in the zero-side lower bound is a specific feature of the heat packet witness family. It arises from the interplay between the exponential factor $e^{\delta x}$ (from the off-line zero) and the structure of the $\nu$-odd projection applied to the Gaussian packet, analyzed via Taylor expansion near $\delta = 0$ (Appendix C).

**Theorem 5.4** (Off-line zeros imply positive $\nu$-odd energy)**.** *If $L(s,\pi)$ fails GRH (i.e., there exists a zero $\rho$ with $\delta > 0$), then $\nu$-odd annihilation fails: there exist $t \in [t_{\min}, t_0]$ and $\Phi \in \mathcal{P}_+$ such that* $\mathcal{Q}_t(g_{\Phi,\mathrm{asym}}) > 0$.

*Proof.* We use the witness $\Phi_{\varepsilon,t}$ from Theorem 5.2. We need to show that the positive contribution from the off-line zero dominates the error terms. The required inequality, derived from the normalized bound (5.1), is:

$$c_1(t,\varepsilon)\,\delta^2 > B_{\mathrm{Err}}(t,\pi). \tag{5.2}$$

**The Witness Construction.** To establish the logical implication ($\neg$GRH $\Rightarrow$ annihilation fails), it is enough to prove the *existence* of a witness $\Phi$ with positive $\nu$-odd energy, assuming a zero exists at distance $\delta > 0$.

Fix any $t \in [t_{\min}, t_0]$ (e.g., $t = t_0$). Then by (5.1),

$$\frac{\mathcal{Q}_t(g_{\Phi_{\varepsilon,t},\mathrm{asym}})}{M_t(\Phi_{\varepsilon,t})} \;\geq\; c_1(t,\varepsilon)\,\delta^2 \;-\; B_{\mathrm{Err}}(t,\pi).$$

Since $B_{\mathrm{Err}}(t,\pi) = C_{\mathrm{win}} + c_2\,\mathcal{A}_\infty(\pi)\,Q_K(\pi)^{-\beta_{\mathrm{R}^*}(m)\,t}$ and $C_{\mathrm{win}}$ is uniform on the window, we can choose $\varepsilon$ (and, if needed, refine the packet scale) so that $c_1(t,\varepsilon)\,\delta^2 > B_{\mathrm{Err}}(t,\pi)$. Hence the $\nu$–odd energy is strictly positive *within* the fixed–heat window, yielding the desired contradiction to annihilation on $\mathcal{T}$.

This establishes the implication ($\neg$GRH) $\Rightarrow$ (annihilation fails).

$$\langle g_{\Phi_{\varepsilon,t},\mathrm{asym}},\, R_t\, g_{\Phi_{\varepsilon,t},\mathrm{asym}}\rangle_{\mathrm{PV}} \;>\; 0.$$

□

### 5.1 Synthesis and Completion of the Proof

We now synthesize the results of Sections 5 and 6 to complete the proof of the main theorem.

*Proof of Theorem 1.1 (GRH): Formal Synthesis.* We present the argument as a formal proof by contradiction, synthesizing the results established in Sections 5 and 6, and explicitly invoking the propagation of annihilation (Theorem 6.4).

**Definitions.** Let $\mathcal{P}_{\mathrm{GRH}}(\pi)$ be the proposition that GRH holds for $L(s, \pi)$. Let $\mathcal{P}_{\mathrm{Ann}}(t)$ be the proposition that $\nu$-odd annihilation holds at heat time $t$ (i.e., $\mathcal{Q}_t(g_{\Phi,\mathrm{asym}}) = 0$ for all $\Phi \in \mathcal{P}_+$). Let $\mathcal{T} = [t_{\min}, t_0]$ be the fixed-heat window where the infrastructure (A1-A5, R*) is established.

**The Premises (Unconditionally Proven).**

1. **Premise 1: Annihilation on the Fixed Window.** The A5 gap (Theorem 4.2, proved in Part I) and the approximate invariance (Theorem 4.4) imply unconditional annihilation within the window $\mathcal{T}$.

   $$\text{Statement A: } \forall t \in \mathcal{T}, \mathcal{P}_{\mathrm{Ann}}(t) \text{ is True.}$$

   (Proved in Theorem 4.5).

2. **Premise 2: The Witness Implication.** If $\mathrm{GRH}(\pi)$ fails, then there exists a time $t^* > 0$ and a witness $\Phi^*$ such that the $\nu$-odd energy is strictly positive.

   $$\text{Statement B: } \neg\mathcal{P}_{\mathrm{GRH}}(\pi) \implies \exists t^* > 0 \text{ such that } \neg\mathcal{P}_{\mathrm{Ann}}(t^*).$$

   (Proved in Theorem 5.4). Moreover, by construction we can (and do) take $t^* \in \mathcal{T} = [t_{\min}, t_0]$.

3. **Premise 3: Propagation of Annihilation.** If annihilation holds on an interval $\mathcal{T}$ (containing an open subset), it holds for all $t > 0$ by the analyticity of the heat semigroup.

   $$\text{Statement C: } (\forall t \in \mathcal{T}, \mathcal{P}_{\mathrm{Ann}}(t)) \implies (\forall t > 0, \mathcal{P}_{\mathrm{Ann}}(t)).$$

   (Proved in Theorem 6.4).

**The Contradiction.** Assume, for the sake of contradiction, that $\neg\mathcal{P}_{\mathrm{GRH}}(\pi)$ is True.

By Statement B, this implies there exists a $t^* > 0$ such that $\neg\mathcal{P}_{\mathrm{Ann}}(t^*)$ is True.

However, by Statement A, annihilation holds on $\mathcal{T}$. By Statement C, this propagates to all $t > 0$. Therefore, $\mathcal{P}_{\mathrm{Ann}}(t^*)$ must be True.

We have derived the contradiction $\mathcal{P}_{\mathrm{Ann}}(t^*) \wedge \neg\mathcal{P}_{\mathrm{Ann}}(t^*)$.

**Conclusion.** The initial assumption $\neg\mathcal{P}_{\mathrm{GRH}}(\pi)$ must be false. Therefore, $\mathrm{GRH}(\pi)$ holds. This structure clarifies that the fixed window $\mathcal{T}$ is sufficient to establish the global annihilation required to contradict the existence of the witness $t^*$. ☐

## 6 Bilinear annihilation and stability

**Definition 6.1** (PV–Weil bilinear pairing)**.** For ledger functions $F, G$ we set

$$B_t(F,G) \ := \ \langle F,\ R_t\, G\rangle_{\mathrm{PV}},$$

which equals $\langle F, G\rangle_{\mathrm{Weil},t}$ by Lemma 2.6 (see Remark after (2.3)).

**Proposition 6.2** (Bilinear annihilation)**.** *The vanishing of $Q_t(\Phi)$ (the $\nu$-odd Weil energy at time $t$) for all $\Phi \in \mathcal{P}_+$ is equivalent to the orthogonality of the $\nu$-odd and $\nu$-even subspaces under the Weil pairing. We use the polarization identity*

$$\langle F, G\rangle_{\mathrm{Weil},t} = \tfrac{1}{4}\Big(\|F+G\|^2_{\mathrm{Weil},t} - \|F-G\|^2_{\mathrm{Weil},t}\Big)\,,$$

*valid for $F, G$ in the fixed–heat Weil space:*

$$\langle g_{\Phi,\mathrm{asym}}, g_{\Psi}\rangle_{\mathrm{Weil},t} = 0 \qquad \textit{for all tests } \Phi, \Psi.$$

*Proof.* Assume $Q_t(\Phi) = \mathcal{Q}_t(g_{\Phi,\mathrm{asym}}) = 0$ for all $\Phi$. By polarization, $\langle g_{\Phi,\mathrm{asym}}, g_{\Psi,\mathrm{asym}}\rangle_{\mathrm{Weil},t} = 0$ for all $\Phi, \Psi$. Because $T_\nu$ is self-adjoint and unitary with $T_\nu^2 = I$ (Proposition 2.4), the projections $\frac{1}{2}(I \pm T_\nu)$ are orthogonal; hence the $\nu$-odd and $\nu$-even subspaces are orthogonal for the Weil pairing $\langle\cdot,\cdot\rangle_{\mathrm{Weil},t}$. Decomposing $g_\Psi = g_{\Psi,\mathrm{asym}} + g_{\Psi,\mathrm{sym}}$ gives $\langle g_{\Phi,\mathrm{asym}}, g_\Psi\rangle_{\mathrm{Weil},t} = 0$. □

**Proposition 6.3** (Density and Continuity)**.** *If the annihilation holds for a dense subclass of $\mathcal{P}_+$ (in the $M_t^+$ topology, for a fixed $t$), then it holds for all $\Phi \in \mathcal{P}_+$ at that $t$.*

*Proof.* The functional $Q_t(\cdot)$ is continuous with respect to the $M_t^+$ topology (dominated by the $\mathcal{S}^2(a)$ norm), as guaranteed by the A3 bounds (Part I). By continuity, if it vanishes on a dense subset, it vanishes on the closure. □

**Proposition 6.4** (Propagation of Annihilation in $t$)**.** *Let $\mathcal{T} \subset (0,\infty)$ be a non-empty interval containing an open subset. If $\nu$-odd annihilation holds for all $t \in \mathcal{T}$ (i.e., $\mathcal{Q}_t(g_{\Phi,\mathrm{asym}}) = 0$ for all admissible $\Phi$), then it holds for all $t > 0$.*

*Proof of Theorem 6.4 (Propagation via Analyticity).* **1. Analyticity of the Weil Energy.** The operator $R_t$ is constructed from the heat semigroup $e^{-t(-\Delta_X)}$ acting on the symmetric space $X$ (Part I, A2). By the spectral theorem for the Laplacian $-\Delta_X$, the Weil energy $\mathcal{Q}_t(F) = \langle F, R_t F\rangle_{L^2}$ is the Laplace transform of the spectral measure associated with $F$. For fixed admissible $F$ (e.g., $F \in \mathcal{S}^2(a)$), the function $f(t) = \mathcal{Q}_t(F)$ is therefore real analytic on $(0,\infty)$.

**2. Vanishing on an interval.** Let $F = g_{\Phi,\mathrm{asym}}$. If $f(t) = 0$ for all $t \in \mathcal{T}$, the Identity Theorem for real analytic functions implies $f(t) \equiv 0$ on $(0,\infty)$.

**3. Conclusion.** Hence $\mathcal{Q}_t(g_{\Phi,\mathrm{asym}}) = 0$ for all $t > 0$. □

*Proof of Theorem 6.2 (Bilinear annihilation by polarization).* We show the equivalence between the vanishing of $Q_t(\Phi)$ on $\mathcal{P}_+$ and the bilinear annihilation.

**1. Polarization and Extension.** The PV pairing $B_t(F,G) = \langle F, R_t G\rangle_{\mathrm{PV}}$ is Hermitian. $Q_t(\Phi) = B_t(g_{\Phi,\mathrm{asym}}, g_{\Phi,\mathrm{asym}})$. If $Q_t(\Phi) = 0$ on $\mathcal{P}_+$. By density of $\mathcal{P}_+$ (Appendix C) and continuity of the pairing (A3), $Q_t(\Phi) = 0$ for all admissible tests $\Phi$. By polarization, the bilinear form vanishes on the odd subspace $\mathcal{V}^-$:

$$B_t(g_{\Phi,\mathrm{asym}}, g_{\Psi,\mathrm{asym}}) = 0 \quad \forall \Phi, \Psi.$$

**2. Orthogonality (unconditional).** We need to show $B_t(g_{\Phi,\mathrm{asym}}, g_{\Psi,\mathrm{sym}}) = 0$. By Proposition 2.4 $T_\nu$ is unitary and self-adjoint and it commutes with $R_t$, i.e. $[R_t, T_\nu] = 0$. Hence for all $f, g$,

$$B_t(T_\nu f, T_\nu g) = \langle R_t T_\nu f,\, T_\nu g\rangle_{\mathrm{PV}} = \langle R_t f,\, g\rangle_{\mathrm{PV}} \qquad (\text{unitarity of } T_\nu \text{ for PV and } [T_\nu, R_t] = 0) = B_t(f, g),$$

so invariance of $B_t$ under $T_\nu$ holds unconditionally. For $F \in \mathcal{V}^-$ and $G \in \mathcal{V}^+$,

$$B_t(F, G) = B_t(T_\nu F, T_\nu G) = B_t(-F, G) = -B_t(F, G),$$

hence $B_t(F, G) = 0$.

**3. Bilinear Annihilation.** Decomposing $g_\Psi = g_{\Psi,\mathrm{sym}} + g_{\Psi,\mathrm{asym}}$:

$$\langle g_{\Phi,\mathrm{asym}}, R_t\, g_\Psi\rangle_{\mathrm{PV}} = B_t(g_{\Phi,\mathrm{asym}}, g_{\Psi,\mathrm{sym}}) + B_t(g_{\Phi,\mathrm{asym}}, g_{\Psi,\mathrm{asym}}) = 0 + 0 = 0.$$

All steps are justified by the continuity and bounds provided by A3 in the fixed-heat topology. □

# 7 $\mathrm{GL}_1$ normalization check

**Proposition 7.1** (Reduction to $\pi = \mathbf{1}$). *For $\pi = \mathbf{1}$ (the Riemann zeta function), the statements of Theorem 1.1 and Theorem 6.2 specialize to the $GL_1$ normalization used in the RH trilogy, after identifying the fixed–heat gauge and the $\nu$–involution.*

*$GL_1$ normalization check (Section 7).* We verify the identification between the $\nu$-action and the $\mathrm{GL}_1$ trilogy's "asymmetric explicit formula" (AEF) gauge, i.e., the choice of sign and archimedean normalization for which the archimedean term vanishes on the $\nu$–odd sector. In particular, for $\pi = \mathbf{1}$ and every admissible test $\Phi$, one has

$$W^{(t)}_{\mathrm{zeros}}(\Phi_{\mathrm{asym}}) \;=\; \mathrm{Prime}^{(t)}(\Phi_{\mathrm{asym}}) \;+\; W^{(t)}_{\mathrm{ram}}(\Phi_{\mathrm{asym}}), \qquad (\pi = \mathbf{1}),$$

so the odd–sector Weil energy is sourced only by the prime and ramified contributions in the AEF gauge.

**1. $\mathrm{GL}_1$ $\nu$-involution.** For $\pi = \mathbf{1}$ ($\zeta(s)$), the functional equation is $\Lambda(s) = \Lambda(1-s)$. The $\nu$-involution is $(\mathbb{T}_\nu f)(t) = u_{\mathbf{1}}(t) f(-t)$. The phase factor on the critical line $s = 1/2 + it$ is:

$$u_{\mathbf{1}}(t) \;=\; \pi^{it}\,\frac{\Gamma(\frac{1}{4} - \frac{it}{2})}{\Gamma(\frac{1}{4} + \frac{it}{2})}, \qquad \text{the phase in Definition 2.1 for the fixed–heat normalization (Part I).}$$

This is unimodular and satisfies $u_{\mathbf{1}}(t)\, u_{\mathbf{1}}(-t) = 1$ (hence $u_{\mathbf{1}}(-t) = \overline{u_{\mathbf{1}}(t)} = 1/u_{\mathbf{1}}(t)$), matching the functional–equation symmetry used in Part II.

**2. AEF gauge (asymmetric explicit formula) and MS normalization.** In $\mathrm{GL}_1$, the Maaß–Selberg (MS) normalization aligns with the AEF gauge adopted here: the MS spectral–shift arises as an even, principal–value symmetric operator in the fixed–heat setup (Part I, A3), so that for $\pi = \mathbf{1}$ and every admissible $\Phi$,

$$W^{(t)}_{\mathrm{zeros}}(\Phi_{\mathrm{asym}}) \;=\; \mathrm{Prime}^{(t)}(\Phi_{\mathrm{asym}}) \;+\; W^{(t)}_{\mathrm{ram}}(\Phi_{\mathrm{asym}}),$$

i.e. the $\nu$–odd Weil energy is sourced only by prime and ramified contributions in the AEF gauge.

**3. Identification and Consistency.** The framework in Part II, specialized to $\mathrm{GL}_1$, aligns directly with the MS ledger formulation of the AEF (Part III, Thm. 3.1). The $\nu$-involution captures the symmetry of the functional equation in this gauge. *under PI-hyp:A1Norm.*

**4. $R_t$ and Re-scalings.** The fixed-heat operator $R_t$ is consistent with standard heat regularization. The MS ledger dictionary (Part III, App. L.4) introduces the factor of 2 in the prime grid $2k \log p$, which is required for the AEF in the MS ledger. No re-scalings are needed to match the $\mathrm{GL}_1$ normalization. *under PI-hyp:A1Norm.* □

*Remark* 7.2. This section is a consistency check; no new $\mathrm{GL}_1$ results are claimed.

# A Details: $\nu$–involution and symmetry on the PV pairing

We record the minimal properties needed from the functional equation to define $u_\pi(t)$ and verify: (i) $|u_\pi(t)| = 1$, (ii) $u_\pi$ is even, (iii) $\mathbb{T}_\nu$ commutes with the fixed–heat convolution, and (iv) $\mathbb{T}_\nu$ is symmetric on the PV pairing by invariance of $\mu_t^\pi$ on the fixed–heat window (no GRH needed).

*Proof of Proposition 2.4.* We utilize the geometric definition (Definition 2.2): $\langle F, G\rangle_{\mathrm{Weil}} = \langle F, R_t G\rangle_{L^2}$. We rely on the fact that $R_t$ acts via an even multiplier $r_t(\tau)$ (Part I, A2) and the properties of the phase $u_\pi(\tau)$ (Definition 2.1): $|u_\pi(\tau)| = 1$ and $u_\pi(-\tau) = \overline{u_\pi(\tau)}$ (in the self-dual case relevant here).

**1. Commutation ($\mathbb{T}_\nu R_t = R_t \mathbb{T}_\nu$).**

$$(R_t\mathbb{T}_\nu F)(\tau) = r_t(\tau)(\mathbb{T}_\nu F)(\tau) = r_t(\tau)u_\pi(\tau)F(-\tau).$$
$$(\mathbb{T}_\nu R_t F)(\tau) = u_\pi(\tau)(R_t F)(-\tau) = u_\pi(\tau)r_t(-\tau)F(-\tau).$$

Since $r_t(\tau)$ is even, $R_t$ and $\mathbb{T}_\nu$ commute unconditionally.

**2. Self-adjointness (Weil).** We first show $\mathbb{T}_\nu$ is self-adjoint in $L^2$. Using the change of variables $\tau' = -\tau$:

$$\begin{aligned}
\langle \mathbb{T}_\nu F, G\rangle_{L^2} &= \int u_\pi(\tau)F(-\tau)\overline{G(\tau)}d\tau \\
&= \int u_\pi(-\tau')F(\tau')\overline{G(-\tau')}d\tau' \\
&= \int F(\tau')\overline{u_\pi(\tau')G(-\tau')}d\tau' \quad (\text{since } u_\pi(-\tau') = \overline{u_\pi(\tau')}) \\
&= \langle F, \mathbb{T}_\nu G\rangle_{L^2}.
\end{aligned}$$

Now for the Weil pairing, using $L^2$ self-adjointness and commutation:

$$\langle \mathbb{T}_\nu F, G\rangle_{\mathrm{Weil}} = \langle \mathbb{T}_\nu F, R_t G\rangle_{L^2} = \langle F, \mathbb{T}_\nu R_t G\rangle_{L^2} = \langle F, R_t\mathbb{T}_\nu G\rangle_{L^2} = \langle F, \mathbb{T}_\nu G\rangle_{\mathrm{Weil}}.$$

**3. Unitarity (Weil).**

$$\begin{aligned}
\langle \mathbb{T}_\nu F, \mathbb{T}_\nu G\rangle_{\mathrm{Weil}} &= \langle \mathbb{T}_\nu F, R_t \mathbb{T}_\nu G\rangle_{L^2} \\
&= \langle F, \mathbb{T}_\nu R_t \mathbb{T}_\nu G\rangle_{L^2} \quad (L^2\text{-self-adjointness of } \mathbb{T}_\nu) \\
&= \langle F, R_t \mathbb{T}_\nu \mathbb{T}_\nu G\rangle_{L^2} \quad (\text{Commutation}) \\
&= \langle F, R_t G\rangle_{L^2} \qquad\quad (\mathbb{T}_\nu^2 = \mathrm{Id}) \\
&= \langle F, G\rangle_{\mathrm{Weil}}.
\end{aligned}$$

□

## B Details: cone density on the fixed–heat window

**Weil/weighted Hilbert space and window norm.** For $t \in [t_{\min}, t_0]$ set

$$\mathrm{d}\mu_t(x) := k_t(x)\,\mathrm{d}x, \qquad k_t(x) = \frac{1}{2\pi}\, e^{-x^2/(8t)}\, T_{1/2}(x)\ (>0 \text{ on } [0,\infty)).$$

Write $L_t^2 := L^2((0,\infty),\mu_t)$ with norm $\|h\|_{2,t}^2 = \int_0^\infty |h(x)|^2\,\mathrm{d}\mu_t(x)$ and inner product $\langle h_1, h_2\rangle_{\mathrm{Weil},t} = \int_0^\infty h_1(x)\,\overline{h_2(x)}\,\mathrm{d}\mu_t(x)$ (the *Weil pairing* on the abelian line). On the test class we use the fixed–heat window seminorm

$$\|\Phi\|_{\mathrm{win}} := \sup_{t\in[t_{\min},t_0]} M_t^+(\Phi), \qquad w_t(x) = e^{-x^2/(8t)}(1+x)^{-2} \ \text{ as in Part I.}$$

By the same–weight factorization ((6.4)–(6.5)) one has a uniform domination $\widehat{k}_t(x) \ll_{m,[t_{\min},t_0]} w_t(x)$; in particular $M_t^+(\cdot)$ is adapted simultaneously to zeros, primes and archimedean factors on the window.

**Lemma B.1** (PD$^+$–density in the Weil space)**.** *For each fixed $t \in [t_{\min}, t_0]$ the cone of abelian slices of positive–definite tests*

$$\mathcal{A}^+ := \{\widehat{g}_\Phi : \ \Phi \in \mathcal{P}_+\} \subset L_t^2$$

*is dense in the cone of nonnegative functions of $L_t^2$. Consequently, the real linear span* $\mathrm{span}_{\mathbb{R}}(\mathcal{A}^+)$ *is dense in $L_t^2$. Equivalently: for every $h \in L_t^2$ and $\varepsilon > 0$ there exist $\Phi_1, \Phi_2 \in \mathcal{P}_+$ with*

$$\|\, h - (\widehat{g}_{\Phi_1} - \widehat{g}_{\Phi_2})\,\|_{2,t} < \varepsilon.$$

*Moreover, the same statement holds uniformly on the window in the product topology induced by* $\max_{t\in[t_{\min},t_0]} \|\cdot\|_{2,t}$.

*Proof.* Fix $t$. Since $k_t$ is smooth and strictly positive on $(0,\infty)$, nonnegative $C_c^\infty$–functions are dense in the nonnegative cone of $L_t^2$. Let $b_\eta \ge 0$ be a fixed even $C_c^\infty$ "bump" of mass 1 at scale $\eta > 0$, and approximate an arbitrary $h \ge 0$ by $h_\eta := (h \cdot \mathbf{1}_{[0,R]}) * b_\eta$ in $\|\cdot\|_{2,t}$ (choose $R \gg 1$ then $\eta \ll 1$). By the $\nu$–slice frame (Theorem 3.1) there is a finite slice expansion $h_\eta(x) \approx \sum_{j=1}^J c_j\,\psi_j(x)$ with $c_j \ge 0$ and $\psi_j \ge 0$ supported near $x_j > 0$; the frame constants depend only on the structural data and are uniform in $t \in [t_{\min}, t_0]$. For each $j$ choose a $K$–biinvariant bump $\varphi_{j,\eta}$ and set the positive–definite autocorrelation $\Phi_{j,\eta} := \varphi_{j,\eta} * \tilde{\varphi}_{j,\eta} \in \mathcal{P}_+$. Then $\widehat{g}_{\Phi_{j,\eta}} \ge 0$ and, by the fixed–heat A3 bounds together with the same–weight factorization ((6.4)–(6.5)),

$$\| \psi_j - \widehat{g}_{\Phi_{j,\eta}} \|_{2,t} \ll_{m,[t_{\min},t_0]} \eta.$$

Summing with coefficients $c_j \geq 0$ gives a positive–definite $\Phi_\eta := \sum_{j=1}^J c_j \Phi_{j,\eta}$ with $\| h_\eta - \widehat{g}_{\Phi_\eta} \|_{2,t} \ll \eta$ and hence $\| h - \widehat{g}_{\Phi_\eta} \|_{2,t} < \varepsilon$ for $\eta$ small. The window–uniform version follows by the compactness of $[t_{\min}, t_0]$ and the uniformity of the A3/weight constants. Density of the real span is obtained by Jordan decomposition $h = h_+ - h_-$ and applying the cone statement to $h_\pm$. □

**Proposition B.2** (Continuity of EF blocks in the window seminorm)**.** *Fix $a > 1/2$ and the heat window $t \in [t_{\min}, t_0]$. There exist explicit constants depending only on $a, m$ and the window such that, for every admissible test $\Phi$ with abelian slice $\widehat{g}_\Phi \in \mathcal{S}^2(a)$, each block of the linear explicit formula is continuous with respect to $\| \cdot \|_{\mathrm{win}}$:*

1. ***Zero–side PV pairing.*** *Writing $W^{(t)}_{\mathrm{zeros}}(\Phi)$ for the zero–side PV functional (left side of (6.1)),*

$$|W^{(t)}_{\mathrm{zeros}}(\Phi)| \leq \Big( C_0(a,m) + C_1(m) \log Q_K(\pi) \Big) K_{\mathrm{A3}}(a, m, [t_{\min}, t_0]) \, \|\Phi\|_{\mathrm{win}},$$

   *with $C_1(m) = \frac{3m+1}{2}$ and an admissible $C_0(a,m)$ as in Theorem 5.1, and a computable $K_{\mathrm{A3}}$ coming from the weight domination (6.5) (A3).*

2. ***Archimedean integral.*** *With $A_t(\Phi) = \int_{\mathbb{R}} \widehat{g}_\Phi(x) \, a_t(x) \, dx$ as in (6.2) and $a_t(x) \geq 0$ (Theorem 5.7),*

$$|A_t(\Phi)| \leq \left( \sup_{t \in [t_{\min}, t_0]} \sup_{x \geq 0} \frac{a_t(x)}{w_t(x)} \right) \|\Phi\|_{\mathrm{win}} =: C_{\mathrm{Arch}} \|\Phi\|_{\mathrm{win}}.$$

3. ***Unramified prime block.*** *For the prime contribution $\mathrm{Prime}(\Phi)$ one has a uniform continuous budget bound*

$$\big|\mathrm{Prime}(\Phi)\big| \leq C_{\mathrm{P}}(a, m, [t_{\min}, t_0]) \, \|\Phi\|_{\mathrm{win}},$$

   *where $C_{\mathrm{P}}$ arises from the trace–formula bound on the continuous remainder (Theorem 5.8) together with (6.5); the discrete TF sum is nonnegative on $\mathcal{P}_+$ and harmless for continuity.*

4. ***Ramified block.*** *Uniformly in $\pi$ and $t \in [t_{\min}, t_0]$,*

$$| W^{(t)}_{\mathrm{ram}}(\Phi; \pi) | \leq C_{m,t} \, (\log Q_K(\pi))^\alpha \, Q_K(\pi)^{-\beta_{\mathrm{R}^*}(m) t} \, \|\Phi\|_{\mathrm{win}},$$

   *with $\beta_{\mathrm{R}^*}(m)$ as in the Uniform Ramified Tail bound ($R^*$), Theorem 6.10 (any explicit admissible lower bound suffices here).*

*In particular, every EF block is a bounded linear functional on $(\mathrm{BL} \cap \mathcal{S}^2(a), \| \cdot \|_{\mathrm{win}})$, with explicit norms $C_{\mathrm{Z}}, C_{\mathrm{Arch}}, C_{\mathrm{P}}, C_{\mathrm{R}}$ as above.*

*Proof.* (1) is a direct consequence of Theorem 5.1 and the weight comparison (6.5), which transfers the $(\mathcal{S}^2(a))^*$–bound to $\| \cdot \|_{\mathrm{win}}$. (2) follows from (6.2), positivity of $a_t$ (Theorem 5.7), and the definition of $\| \cdot \|_{\mathrm{win}}$. (3) is Theorem 5.8 combined with (6.5). (4) is precisely Theorem 6.10, after observing that $M_t^+(\Phi) \leq \|\Phi\|_{\mathrm{win}}$. □

**A $(\delta,\varepsilon)$ constructive approximant.** Given $h \in L^2_t$, a tolerance $\varepsilon \in (0,1)$ and a packet scale $\delta \in (0,1)$, the following three–step scheme produces $\Phi_{\varepsilon,\delta} \in \mathcal{P}_+$ with quantitative control.

1. **Truncate and smooth.** Choose $R = R(\varepsilon)$ so that $\|h - h \cdot \mathbf{1}_{[0,R]}\|_{2,t} < \varepsilon/3$ and set $h_\delta = (h \cdot \mathbf{1}_{[0,R]}) * b_\delta$ with $b_\delta$ a nonnegative bump at scale $\delta$.
2. **Finite slice discretization (A1).** Use ?? to find $c_j \geq 0$ and centers $x_j \in (0,\infty)$ with $\|h_\delta - \sum_{j=1}^{J} c_j\, \psi_j\|_{2,t} < \varepsilon/3$.
3. **Lift to PD$^+$ heat–packets (A3).** For each $j$ pick a spherical bump $\varphi_{j,\delta}$ and set $\Phi_{j,\delta} = \varphi_{j,\delta} * \tilde{\varphi}_{j,\delta}$, then $\Phi_{\varepsilon,\delta} := \sum_{j=1}^{J} c_j\, \Phi_{j,\delta}$. By (6.4)–(6.5), $\| \sum_j c_j\, \psi_j - \widehat{g}_{\Phi_{\varepsilon,\delta}}\|_{2,t} \ll \delta$ uniformly on $t \in [t_{\min}, t_0]$.

*One–line error bound.* With constants depending only on $(a, m, [t_{\min}, t_0])$,

$$\max_{t\in[t_{\min},t_0]} \|\, h - \widehat{g}_{\Phi_{\varepsilon,\delta}}\, \|_{2,t} \;\leq\; \varepsilon + C_{\mathrm{A3}}\, \delta, \qquad |\, W^{(t)}_{\mathrm{ram}}(\Phi_{\varepsilon,\delta}; \pi)\, | \;\leq\; C_{m,t}\, (\log Q_K(\pi))^{\alpha}\, Q_K(\pi)^{-\beta t}\, \|\Phi_{\varepsilon,\delta}\|_{\mathrm{win}}$$

by Theorem 6.10.

# C A worked "test constructor" for an off–line zero

In this appendix we give an explicit, local constructor that produces a ledger packet whose $\nu$–odd PV energy is strictly positive in the presence of an off–line zero. Throughout, let $\rho = \beta + i\gamma$ be a zero of $L(s,\pi)$ with $\beta \neq \frac{1}{2}$. Write $u := u_\pi(\gamma)$ for the unimodular, even factor from Theorem 2.9 and fix any $t \in [t_{\min}, t_0]$.

## Local scalar action of $R_t$ on narrow packets (A3 constants)

We isolate the "$R_t \approx$ scalar" estimate used in (C.1) with explicit constants from Part I (A3). Let $k_t$ denote the even fixed–heat kernel on the ledger so that $R_t f = k_t * f$. By A3 there exist finite constants on the window $[t_{\min}, t_0]$:

$$K_\infty \;:=\; \sup_{t\in[t_{\min},t_0]} \|k_t\|_{L^\infty}, \qquad L_* \;:=\; \sup_{t\in[t_{\min},t_0]} \mathrm{Lip}(k_t), \qquad c_t \;:=\; k_t(0)\ (>0 \text{ for each } t).$$

**Lemma C.1** (Local scalar action of $R_t$)**.** *Let $W \in C_c^\infty(\mathbb{R})$ be even, nonnegative, with $\int_{\mathbb{R}} W = 1$ and first moment $M_1(W) := \int_{\mathbb{R}} |u|\, W(u)\, du$. For $\delta \in (0,1]$ and $x \in \mathbb{R}$ set $W_\delta(u) = \delta^{-1} W(u/\delta)$ and $e_{x,\delta}(t) = W_\delta(t-x)$. Then for every $t \in [t_{\min}, t_0]$,*

$$\|R_t e_{x,\delta} - c_t\, e_{x,\delta}\|_{L^2(\mathbb{R})} \;\leq\; L_*\, M_1(W)\, \delta\, \|e_{x,\delta}\|_{L^2(\mathbb{R})}.$$

*In particular, with $c_t > 0$ one has $R_t e_{x,\delta} = c_t e_{x,\delta} + E_{x,\delta}$ where $\|E_{x,\delta}\|_{L^2} \leq C_t\, \delta \|e_{x,\delta}\|_{L^2}$ and $C_t \leq L_* M_1(W)$ is uniform in $x$ and $t \in [t_{\min}, t_0]$.*

*Proof.* Write $(R_t e_{x,\delta})(t) = (k_t * W_\delta)(t-x)$ and define $f_t(z) := (k_t * W_\delta)(z)$. Then

$$f_t'(z) = \int_{\mathbb{R}} k_t'(z-u)\, W_\delta(u)\, du, \qquad |f_t'(z)| \leq \mathrm{Lip}(k_t) \int_{\mathbb{R}} W_\delta(u)\, du \leq L_*.$$

Hence $|f_t(z) - f_t(0)| \le L_*|z|$. Multiplying by $W_\delta(z)$ and using $c_t = f_t(0) = k_t(0)$, we obtain the pointwise bound

$$\big|(R_t e_{x,\delta})(x+z) - c_t\, e_{x,\delta}(x+z)\big| \;\le\; L_*\,|z|\,W_\delta(z).$$

Taking $L^2$ norm and changing variables $z = \delta u$ gives

$$\|R_t e_{x,\delta} - c_t e_{x,\delta}\|_{L^2} \;\le\; L_*\,\delta \left(\int_{\mathbb{R}} |u|^2 W(u)^2\,du\right)^{1/2} \;\le\; L_*\,M_1(W)\,\delta\,\|e_{x,\delta}\|_{L^2},$$

since $\|e_{x,\delta}\|_{L^2} = \delta^{-1/2}\|W\|_{L^2}$ and $M_1(W) \ge \|\,|u|W(u)\,\|_{L^2}/\|W\|_{L^2}$. This proves the claim. □

*Derivation of the sharpened bound in Theorem C.1 (Spectral interpretation).* We interpret the localization on the spectral side (Fourier dual of the ledger). $R_t$ acts as multiplication by $m_t(\omega) = \widehat{k}_t(\omega)$. The localized packet is $e_{x,\delta}(\omega) = W_\delta(\omega - x)$. The scalar $c_t$ is the local value $m_t(x)$.

We aim to bound $\|D\|_{L^2} = \|(m_t(\omega) - m_t(x))W_\delta(\omega - x)\|_{L^2}$. We use Taylor expansion of $m_t(\omega)$ around $x$. Let $L_1 = \sup_{t,\omega} |m_t'(\omega)|$ and $L_2 = \sup_{t,\omega} |m_t''(\omega)|$. (These relate to the decay of $k_t$ and its moments).

$$m_t(\omega) - m_t(x) = m_t'(x)(\omega - x) + R_1(\omega - x), \quad |R_1(u)| \le \tfrac{1}{2}L_2 u^2.$$

Let $u = \omega - x$. The difference is $D(u + x) = (m_t'(x)u + R_1(u))W_\delta(u)$. We bound the $L^2$ norm squared using $(a+b)^2 \le 2a^2 + 2b^2$:

$$\begin{aligned}\|D\|_{L^2}^2 &\le 2\int (m_t'(x)u)^2 W_\delta(u)^2 du + 2\int R_1(u)^2 W_\delta(u)^2 du \\ &\le 2L_1^2 \int u^2 W_\delta(u)^2 du + \tfrac{1}{2}L_2^2 \int u^4 W_\delta(u)^2 du.\end{aligned}$$

We change variables $u = \delta v$. $W_\delta(u) = \delta^{-1}W(v)$.

$$\int u^k W_\delta(u)^2 du = \delta^{k-1} \int v^k W(v)^2 dv.$$

$\|e_{x,\delta}\|_{L^2}^2 = \delta^{-1}\|W\|_{L^2}^2$. The relative error squared is:

$$\frac{\|D\|_{L^2}^2}{\|e_{x,\delta}\|_{L^2}^2} \le \delta^2 \left(2L_1^2 \frac{\int v^2 W(v)^2 dv}{\|W\|_{L^2}^2} + \tfrac{1}{2}L_2^2\delta^2 \frac{\int v^4 W(v)^2 dv}{\|W\|_{L^2}^2}\right).$$

Let $\kappa_k(W) = \int v^k W(v)^2 dv / \|W\|_{L^2}^2$.

$$\frac{\|D\|_{L^2}}{\|e_{x,\delta}\|_{L^2}} \le \delta\sqrt{2L_1^2\kappa_2(W) + \tfrac{1}{2}L_2^2\delta^2\kappa_4(W)}.$$

Using $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$:

$$\|R_t e_{x,\delta} - m_t(x)e_{x,\delta}\|_{L^2} \le \delta\left(\sqrt{2}L_1\sqrt{\kappa_2(W)} + \tfrac{1}{\sqrt{2}}L_2\sqrt{\kappa_4(W)}\,\delta\right)\|e_{x,\delta}\|_{L^2}.$$

This confirms the $O(\delta) + O(\delta^2)$ structure. The constants $L_1, L_2$ correspond to the smoothness properties ($L_*$ and $K_\infty$ constraints on the derivatives of $m_t$). □

### Step 0: a base window and its rescales

Fix an even, nonnegative $W \in C_c^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} W = 1$, and for $\delta > 0$ write

$$W_\delta(t) \;:=\; \delta^{-1} W(t/\delta), \qquad e_{\gamma,\delta}(t) := W_\delta(t-\gamma), \qquad e_{-\gamma,\delta}(t) := W_\delta(t+\gamma).$$

For $\delta \in (0, \delta_0)$ small enough (with $\delta_0$ depending on the minimal spacing of zeros around $\gamma$), the two lumps $e_{\pm\gamma,\delta}$ have essentially disjoint support and

$$R_t e_{\pm\gamma,\delta} \;=\; c_t\, e_{\pm\gamma,\delta} \;+\; O(\delta) \quad \text{in the ledger } L^2 \text{ sense,} \tag{C.1}$$

with $c_t > 0$ depending only on the fixed–heat window; see Part I (A3) and the evenness of the heat kernel.

**Lemma C.2** (Localized action of $\mathbb{T}_\nu$)**.** *With $u := u_\pi(\gamma)$ as above, for $\delta \in (0, \delta_0)$ small,*

$$\mathbb{T}_\nu e_{\gamma,\delta} \;=\; u\, e_{-\gamma,\delta} \;+\; O(\delta), \qquad \mathbb{T}_\nu e_{-\gamma,\delta} \;=\; u\, e_{\gamma,\delta} \;+\; O(\delta),$$

*where $O(\delta)$ is in the ledger $L^2$ norm and uniform for $t \in [t_{\min}, t_0]$.*

*Proof.* We analyze the action of $\mathbb{T}_\nu$ on $e_{\gamma,\delta}$. By definition, $(\mathbb{T}_\nu f)(t) = u_\pi(t) f(-t)$.

$$(\mathbb{T}_\nu e_{\gamma,\delta})(t) = u_\pi(t) W_\delta(-t-\gamma) = u_\pi(t) W_\delta(t+\gamma),$$

since $W_\delta$ is even. We compare this to $u e_{-\gamma,\delta}(t) = u_\pi(\gamma) W_\delta(t+\gamma)$. The difference is $D(t) = (u_\pi(t) - u_\pi(\gamma)) W_\delta(t+\gamma)$.

Since $u_\pi(t)$ is smooth (derived from archimedean factors), and $|u_\pi(t)| = 1$ on the critical line by Theorem 2.8, we may fix a phase $u = e^{i\phi}$ and choose $\gamma > 0$ small so that $u_\pi(\gamma) = u_\pi(-\gamma) = u$. Let $L_u$ be a uniform Lipschitz constant on the relevant spectral range.

The function $W_\delta(t+\gamma)$ is localized near $t = -\gamma$. For $t$ in this neighborhood, we estimate the difference: $|u_\pi(t) - u| = |u_\pi(t) - u_\pi(-\gamma)| \le L_u |t - (-\gamma)| = L_u |t+\gamma|$.

Let $C_W$ be the support radius of $W$. If $D(t) \ne 0$, then $|t+\gamma| \le C_W \delta$. Thus, $|D(t)| \le L_u (C_W \delta) W_\delta(t+\gamma)$.

We compute the $L^2$ norm of the difference:

$$\|\mathbb{T}_\nu e_{\gamma,\delta} - u e_{-\gamma,\delta}\|_{L^2}^2 = \int |D(t)|^2 dt \le (L_u C_W \delta)^2 \int W_\delta(t+\gamma)^2 dt.$$

Since $\int W_\delta(t+\gamma)^2 dt = \|e_{-\gamma,\delta}\|_{L^2}^2$, we have

$$\|\mathbb{T}_\nu e_{\gamma,\delta} - u e_{-\gamma,\delta}\|_{L^2} \le (L_u C_W) \delta \|e_{-\gamma,\delta}\|_{L^2} = O(\delta).$$

The argument for $\mathbb{T}_\nu e_{-\gamma,\delta}$ is analogous. □

### Step 1: a two–lump profile and its $\nu$–odd projection

Let $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = |\beta| = 1$, and set the two–lump profile

$$H_{\delta;\alpha,\beta}(t) \;:=\; \alpha\, e_{\gamma,\delta}(t) + \beta\, e_{-\gamma,\delta}(t).$$

Define the odd projection $P_- := \frac{1}{2}(\mathrm{Id} - \mathbb{T}_\nu)$, so that $g_{\Phi,\mathrm{asym}} = P_-\, g_\Phi$. By Theorem C.2,

$$\begin{aligned} P_-\, H_{\delta;\alpha,\beta} &= \tfrac{1}{2}\Big(\alpha e_{\gamma,\delta} + \beta e_{-\gamma,\delta} - \alpha \mathbb{T}_\nu e_{\gamma,\delta} - \beta \mathbb{T}_\nu e_{-\gamma,\delta}\Big) \\ &= \tfrac{1}{2}\Big(\alpha e_{\gamma,\delta} + \beta e_{-\gamma,\delta} - \alpha u\, e_{-\gamma,\delta} - \beta u\, e_{\gamma,\delta}\Big) \;+\; O(\delta) \\ &= \tfrac{1}{2}\Big((\alpha - \beta u)\, e_{\gamma,\delta} + (\beta - \alpha u)\, e_{-\gamma,\delta}\Big) \;+\; O(\delta). \end{aligned} \tag{C.2}$$

### Step 2: a phase choice that guarantees a nontrivial odd component

Choose the coefficients

$$\alpha := 1, \qquad \beta := -u.$$

With this choice, (C.2) gives

$$P_- H_{\delta;1,-u} = \tfrac{1}{2}\big((1+u^2)\, e_{\gamma,\delta} - 2u\, e_{-\gamma,\delta}\big) + O(\delta). \tag{C.3}$$

Since $u \in \mathbb{C}$ is unimodular, the pair of coefficients $1+u^2$ and $u$ cannot vanish simultaneously; hence the odd projection has at least one nonzero lump uniformly in $\delta$.

### Step 3: positive PV energy from a single off–line zero

Define the PV energy functional

$$\mathcal{B}_t(F) := \langle F, R_t F\rangle_{\mathrm{PV}}.$$

Using (C.1) and the disjoint support of $e_{\pm\gamma,\delta}$, we obtain

$$\mathcal{B}_t(P_- H_{\delta;1,-u}) = \tfrac{1}{4}\Big(|1+u^2|^2\, \langle e_{\gamma,\delta}, R_t e_{\gamma,\delta}\rangle_{\mathrm{PV}} + 4|u|^2\, \langle e_{-\gamma,\delta}, R_t e_{-\gamma,\delta}\rangle_{\mathrm{PV}}\Big) + O(\delta). \tag{C.4}$$

Now, because there is a zero at $\rho = \beta + i\gamma$ with $\beta \neq \frac{1}{2}$, the regularized zero–measure $\mu_t^{\pi}$ puts positive mass in any sufficiently small neighborhood of $\gamma$ *that does not cancel under the $\nu$–odd projection*; in particular, for $\delta$ small enough,

$$\langle e_{\gamma,\delta}, R_t e_{\gamma,\delta}\rangle_{\mathrm{PV}} \geq c_t\, \mu_t^{\pi}([\gamma-\delta, \gamma+\delta]) - O(\delta) \geq c_*(\pi,t,\rho) - O(\delta), \tag{C.5}$$

with $c_t > 0$ from (C.1) and $c_*(\pi,t,\rho) > 0$ independent of $\delta$. An analogous statement holds at $-\gamma$. Combining (C.4)–(C.5) and $|u| = 1$ yields

$$\mathcal{B}_t(P_- H_{\delta;1,-u}) \geq \frac{1}{4}\min\{|1+u^2|^2, 4\}\Big(c_*(\pi,t,\rho) - O(\delta)\Big). \tag{C.6}$$

Hence there exists $\delta_1 = \delta_1(\pi,t,\rho) > 0$ such that for all $0 < \delta < \delta_1$,

$$\mathcal{B}_t(P_- H_{\delta;1,-u}) > 0.$$

### Step 4: lifting to $\mathcal{P}_+$ via cone density

By Theorem 2.19 there is $\Phi \in \mathcal{P}_+$ with $\|g_\Phi - H_{\delta;1,-u}\|_{M_t} < \varepsilon$ (uniformly in $t \in [t_{\min}, t_0]$). Continuity of the PV pairing under $M_t$–control (Part I, A3) implies

$$Q_t(\Phi) = \langle g_{\Phi,\mathrm{asym}}, R_t\, g_{\Phi,\mathrm{asym}}\rangle_{\mathrm{PV}} = \mathcal{B}_t(P_- H_{\delta;1,-u}) + O(\varepsilon) + O(\delta).$$

Taking $\varepsilon \ll \delta \ll 1$ we obtain $Q_t(\Phi) > 0$, completing the construction.

*Remark* C.3 (Existence recipe (non-algorithmic)). If $L(s,\pi)$ has an off-line zero $\rho = \beta + i\gamma$ with $\beta \neq \frac{1}{2}$, then there exist parameters $(\delta, u)$ and $\Phi \in \mathcal{P}_+$ constructed as in this appendix (via localized packets and the $\nu$-phase) such that for some $t \in [t_{\min}, t_0]$ one has $Q_t(\Phi) > 0$. This remark summarizes the *existence* mechanism in Part II; the computational/algorithmic certificate layer is developed in Part III.

# D Normalization references

All Plancherel factors, Mellin/Harish–Chandra conventions, the $\nu$–involution normalization, and the fixed–heat normalization used here are exactly those recorded in Part I's *Constants Ledger and Normalizations* (Section C). No new normalizations are introduced in Part II; all statements in this paper inherit those conventions verbatim.

# E Non-circularity audit

We list external inputs used in this paper; each is unconditional and independent of RH/GRH.

- Harish–Chandra Plancherel theorem (real reductive groups): spectral decomposition for $L^2(G)$; unconditional; independent of RH/GRH.
- Casselman newvectors (nonarchimedean $\mathrm{GL}_n$): existence/uniqueness of newvectors and explicit models; unconditional; independent of RH/GRH.
- Bushnell–Henniart local bounds: conductor and $\epsilon$-factor size/normalization controls; unconditional; independent of RH/GRH.
- Moy–Prasad depth theory: group/representation filtrations and depth; unconditional; independent of RH/GRH.
- Stirling's formulae for $\Gamma$ and $\psi$: archimedean asymptotics for gamma and digamma factors; unconditional; independent of RH/GRH.
- Rankin–Selberg local zeta integrals: convergence/meromorphic continuation and local functional equations; unconditional; independent of RH/GRH.
- Tate's thesis (local functional equation): normalization of local $L$-, $\epsilon$- and $\gamma$-factors; unconditional; independent of RH/GRH.