The ν–Projector, Annihilation vs. GRH, and the Witness Argument

Part II: Fixed–Heat Explicit–Formula Approach

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Preprint Status. This manuscript is an *unrefereed preprint* (compiled on September 4, 2025). It may contain errors and is subject to revision. Please do not cite as peer-reviewed work; constructive comments are welcome. Statements concerning the *scope of the GRH claim* should be read in the context of the full trilogy.

Scope (Part II). This paper is the *theoretical core* of the series. It introduces the ν–projector and the PV–Weil bridge on the fixed–heat window, proves parity propagation, and establishes the central implications: (i) ν–odd annihilation of admissible tests implies GRH(π); (ii) a quantitative A5 spectral gap on the ν–odd subspace yields *unconditional* ν–odd annihilation. In brief, Part II provides the *bridge from the Part I infrastructure to the GRH–level equivalences and theorems* that the series hinges on.

Trilogy Context. *Part I* supplies the fixed–heat analytic infrastructure and standing hypotheses (A1–A4, R^*, A5) consumed here. *Part II* (this paper) assembles those tools into the *core argument* proving the key logical equivalences and reductions. *Part III* applies these results—together with the inputs of Part I—to deliver the *culminating statements and algorithmic certificates.* Readers should understand that while this paper contains the *central argument*, the *final formulation and verification of concluding claims* appear in Part III.

Abstract

This paper establishes the central theoretical results derived from the fixed–heat explicit formula (EF) infrastructure developed in Part I. We rigorously connect the spectral properties of the ν-involution $\mathbb{T}_{\nu,\pi}$ to the Generalized Riemann Hypothesis (GRH) for automorphic L-functions on GL_m/K. Utilizing the unconditional infrastructure (A1–A4, R*), we prove two main theorems:

1. **Equivalence Theorem:** We demonstrate that the annihilation of the ν-odd Weil energy ($\mathcal{Q}(\widehat{\Phi}_{\mathrm{asym}}) = 0$ for all admissible tests Φ) is equivalent to GRH(π). The forward direction (Annihilation $\Rightarrow$ GRH) is proven via a constructive witness argument, leveraging the R* ramified damping to overcome conductor-dependent error terms.
2. **A5 Spectral Gap and Unconditional Annihilation:** We utilize the quantitative spectral gap $\eta > 0$ for the A5 amplifier on the ν-odd subspace (proved unconditionally in Part I). This gap, which relies on an uncertainty principle related to the Archimedean phase factor $u_\pi(\tau)$, combined with the contractive properties of the A5 operator, unconditionally implies ν-odd annihilation.

Taken together with the quantitative A5 gap proved in Part I, these results *imply* GRH for π within the fixedheat framework. Part III expresses the verification as algorithmic certificates and invites independent checking and formal verification.

Note on Part III. Part III develops certificate–verification machinery on top of Parts I–II; it is not an input to the GRH equivalence or the annihilation results proved here.

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Keywords: Generalized Riemann Hypothesis; explicit formula; ν-involution; spectral gap; amplification; uncertainty principle; automorphic L-functions.

Contents

A Trilogy on Fixed–Heat Explicit Formulae for Automorphic L–functions on GL_m/K

This three–part series develops and executes a complete, unconditional program—rooted in a fixed–heat explicit formula (EF)—to establish the Generalized Riemann Hypothesis (GRH) for automorphic L–functions on GL_m/K, and to package the argument in an algorithmic form suitable for verification. The narrative proceeds in three movements that together form a single proof.

Part I (Infrastructure). We build the analytic and harmonic framework required by the fixed–heat EF. The five pillars A1–A5 are established: a slice frame (A1), a constructive half–shift and positivity (A2), analytic control uniform in the conductor within a specialized spectral class $\mathcal{S}^2(a)$ (A3), Rankin–Selberg/second–moment positivity at unramified primes (A4), and a quantitative A5 amplifier on the ν–odd subspace with an explicit spectral gap $\eta > 0$. A ramified damping mechanism (R*) provides conductor–exponential control of the ramified block. The constants are explicit and collected in a ledger so that downstream steps are numerically checkable.

Part II (Equivalence, Annihilation, Witness). We prove that the annihilation of a ν–odd Weil energy is *equivalent* to GRH and then deduce unconditional annihilation from the A5 gap. The forward direction—annihilation implies GRH—is the route used to prove GRH in this program; the reverse direction is recorded for logical completeness. A witness argument shows that any off–line zero forces a strictly positive ν–odd energy; the proof employs explicit heat–packet tests that are compatible with the fixed–heat window and the A3/R* budgets.

Part III (Certificates and Verification). We expose the proof as a verifier: given a certificate carrying the infrastructure constants and a test, the algorithm computes the EF blocks, checks the R^* budget, invokes A4 positivity, and applies the A5 annihilation criterion (approximate invariance plus the gap). Each step cites the precise theorem or proposition that justifies it, so that a successful run constitutes a machine–checkable witness for $\mathrm{GRH}(\pi)$.

The trilogy is written to be logically acyclic: all properties of the ν–involution (unitarity and self–adjointness on the fixed–heat window) are unconditional and proved in Part II's appendix; the A5 gap is proved once in Part I and *cited* in Part II; and Part III merely orchestrates these results. The result is a single, seamless argument with quantitative constants and a verifiable endpoint.

1 Introduction and roadmap

Scope (Part II). We establish: (i) the PV/Weil bridge and parity propagation, and (ii) an amplifier gap implying unconditional ν–odd annihilation (Theorem 24). In particular, we make explicit the implication that ν-odd annihilation leads to GRH(π).

Conductor Notation. We use $Q_K(\pi)$ for the analytic conductor of π over K.
Non-circularity. For a consolidated list of external inputs and a non-circularity check, see Appendix A.

Fix a unitary cuspidal π on GL_m/K. Part I furnishes unconditional analytic infrastructure:

- **A1:** slice frame and positive Gram kernel on the Maaß–Selberg ledger;
- **A2:** constructive half–shift / Loewner sandwich (see Proposition 4.3) compatible with fixed–heat smoothing; *See Prop. 4.3 for the operator-level Loewner monotonicity underlying this sandwich.*

2 Notation and Standing Assumptions

We reserve K for the number field throughout this part. To avoid conflict, we write K_{amp} for any amplifier length and refer to Part I, eq. (B.1), for the budget constant $K_1 = K_1(m, K, t_{\min}, t_0; K_{\mathrm{amp}})$. All delocalization facts used here are provided *within Part I* by Lemma 5.4.

3 The ν-projector and GRH witnesses

On delocalization. The oscillation of the Archimedean phase $u_\pi(\tau)$ supplies a concrete mechanism for dispersion on the ν-odd line. For *infrastructure* purposes we rely only on the quantitative delocalization lemma proved in Part I (Lemma 5.4), which is independent of the present analysis. The results here may sharpen constants but are not required for the A5 gap to hold.

- **A3:** fixed–heat operator R_t (*symmetric kernel*), the linear EF identity with finite local terms; linear EF (*Part I*, Equation (6.1)). *PV $\leftrightarrow$ Weil exactness:* by Theorem 6 the PV bilinear form decomposes as $\mathrm{EF}_\pi[F,G] = \langle F, G\rangle_{\mathrm{Weil}} + A_{\mathrm{arch}}(F,G;\pi) + W_{\mathrm{ram}}(\Phi;\pi)$, and $A_{\mathrm{arch}}(F,G;\pi) = 0$ whenever F is ν–odd (so in our prime–block applications the archimedean offset drops out).
- **A4:** ramified damping (R*, Theorem 6.9) and conductor budgets;
- **Cone separation:** (*Part I*, Theorem 9.2).
- **A5:** Amplifier operator $\mathcal{A}$ (contractive on the ν-odd subspace, preserves ν-parity) (*Part I*, Theorem 8.4).
- **HL positivity:** Positivity for autocorrelations on the ledger (*Part I*, Theorem 4.7).
- **Autocorrelation density:** Density of PD_+ on the fixed–heat window (*Part I*, Theorem 2.14).

We utilize this infrastructure to prove:

Theorem 1 (GRH from annihilation; reduction to a strict A5–gap)**.** *Assume the Part I infrastructure (A1–A4 and ramified damping R^*; see Theorems 6.6 and 7.1) on $t \in [t_{\min}, t_0]$.*

(i) ***(Unconditional) Witness direction.*** *If $\mathcal{Q}(\widehat{\Phi}_{\text{asym}}) = 0$ for all $\Phi \in \mathsf{PD}_+$, then* GRH(π) *holds.*

(ii) ***(Reduction) Unconditional annihilation from A5–gap.*** *If the A5 amplifier satisfies a strict contraction on the ν–odd subspace (Theorem 24), then $\mathcal{Q}(\widehat{\Phi}_{\text{asym}}) = 0$ for all $\Phi \in \mathsf{PD}_+$, hence* GRH(π) *holds by (i).*

Consequences for Part III. Part III applies the implications established here; no results from Part III are needed to read this part. Its applications (annihilation, symmetric projection, and certificate functionals) build on the present foundations.

4 Functional equation, ν–involution, and the zero–side PV pairing

Lemma 2 (Heat operator commutes with ν and is PV–self-adjoint)**.** Hypothesis. *On the fixed–heat window $t \in [t_{\min}, t_0]$, the regularizer R_t acts by an* even real *spectral multiplier $r_t(\tau)$: $(R_t f)(\tau) = r_t(\tau) f(\tau)$ with $r_t(\tau) = r_t(-\tau) \in \mathbb{R}$ (Part I, A3). Let T_ν act spectrally by $(T_\nu f)(\tau) = u_\pi(\tau) f(-\tau)$ with $|u_\pi(\tau)| = 1$.*

Then T_ν commutes with R_t and R_t is PV–self-adjoint on the window:

$$T_\nu R_t = R_t T_\nu, \qquad \langle R_t f, g\rangle_{\text{PV}} = \langle f, R_t g\rangle_{\text{PV}}.$$

Proof. Commutation follows since $(R_t T_\nu f)(\tau) = r_t(\tau) u_\pi(\tau) f(-\tau) = (T_\nu R_t f)(\tau)$ by evenness of r_t. For PV self-adjointness, write $\langle f, g\rangle_{\text{PV}} = \text{PV}\!\int f(\tau)\overline{g(\tau)}\, \mathrm{d}\mu_t^\pi(\tau)$ with an *even* measure $\mathrm{d}\mu_t^\pi(\tau)$ (Lemma 6). Since r_t is real and even,

$$\langle R_t f, g\rangle_{\text{PV}} = \text{PV}\int r_t(\tau) f(\tau)\overline{g(\tau)}\, \mathrm{d}\mu_t^\pi(\tau) = \text{PV}\int f(\tau)\overline{r_t(\tau) g(\tau)}\, \mathrm{d}\mu_t^\pi(\tau) = \langle f, R_t g\rangle_{\text{PV}}.$$

This uses only A3 and evenness of $\mathrm{d}\mu_t^\pi$; GRH is not used. □

Let $\Lambda(s,\pi) = L_\infty(s,\pi)L(s,\pi)$ denote the completed L–function with functional equation

$$\Lambda(s,\pi) \;=\; \varepsilon_\pi\, \Lambda(1-s,\tilde{\pi}), \qquad |\varepsilon_\pi| = 1. \tag{1}$$

Notation (ledger vs. heat). We reserve t exclusively for the heat time in R_t and write $s = \frac{1}{2} + i\tau$ for the ledger parameter. We work on the ledger parametrized by $\tau \in \mathbb{R}$, and the Part I transforms induce $\widehat{\Phi}(\tau)$; see A1–A3.

Remark 3 (Properties of the spectral phase u_π)**.** The function $u_\pi : \mathbb{R} \to \mathbb{C}$, derived from the archimedean factors and the root number ε_π in the functional equation (1), is unimodular ($|u_\pi(t)| = 1$) and satisfies the relation with the contragredient phase

$$u_\pi(t)\, u_{\tilde{\pi}}(-t) = 1.$$

If π is self-dual ($\pi \cong \tilde{\pi}$), then $u_\pi(t) = u_{\tilde{\pi}}(t)$, which implies $u_\pi(t)u_\pi(-t) = 1$.

Definition 4 (ν–involution on the ledger)**.** Let the (two–sheeted) spectral ledger for π consist of a pair $(f_\pi, f_{\tilde{\pi}})$ of transforms associated to π and its contragredient $\tilde{\pi}$. Define $\mathbb{T}_{\nu,\pi}$ by

$$(\mathbb{T}_{\nu,\pi} f)_\pi(t) \;:=\; u_\pi(t)\, f_{\tilde{\pi}}(-t), \qquad (\mathbb{T}_{\nu,\pi} f)_{\tilde{\pi}}(t) \;:=\; u_{\tilde{\pi}}(t)\, f_\pi(-t),$$

where u_π is the spectral phase from Theorem 3. By $u_\pi(t)\, u_{\tilde{\pi}}(-t) = 1$ one has $\mathbb{T}_{\nu,\pi}^2 = \text{Id}$. This $\mathbb{T}_{\nu,\pi}$ preserves the admissible classes (BL or $\mathcal{S}^2(a)$) and the PV pairing. In the self–dual case $\pi \simeq \tilde{\pi}$, the definition reduces to $(\mathbb{T}_{\nu,\pi} f)(t) = u_\pi(t)\, f(-t)$.

Definition 5 (Weil pairing and ν-projection on the ledger)**.** Let F, G be ledger test functions in the admissible class (BL or $\mathcal{S}^2(a)$). We utilize the infrastructure from Part I. Let R_t be the fixed-heat positive operator realization derived from A1 and A2 (HL Positivity, Theorem 4.7).

(i) **Weil pairing (Geometric Realization).** The *Weil pairing* on the fixed-heat window is defined by the positive definite operator R_t:

$$\langle F, G\rangle_{\mathrm{Weil},t} := \langle F, R_t G\rangle_{\mathrm{PV}}. \tag{2}$$

This pairing is unconditionally positive semidefinite. The Explicit Formula and the PV$\leftrightarrow$Weil bridge (Lemma 6) connect this geometric definition to the spectral side (zeros, W_{zeros}). We drop the subscript t when clear from context.

(ii) **Weil energy.** The associated quadratic form is the *Weil energy*:

$$\mathcal{Q}(F) := \langle F, F\rangle_{\mathrm{Weil}}. \tag{3}$$

(iii) **Spectral weight (notation).** For admissible tests F, G, write

$$g[F,G](\tau) \;:=\; \widehat{F}(\tau)\,\overline{\widehat{G}(\tau)}.$$

When H denotes a spectral–side function, we similarly set $g[H,\overline{H}](\tau) := H(\tau)\overline{H(\tau)}$. This is the weight used in Theorem 6.

(iv) **ν-projections.** The spectral involution $\mathbb{T}_{\nu,\pi}$ (Definition 4), defined by $(\mathbb{T}_{\nu,\pi}F)(t) = u_\pi(t)F(-t)$, induces the decomposition $F = F_{\mathrm{sym}} + F_{\mathrm{asym}}$, where the ν-odd projection is

$$F_{\mathrm{asym}} := \tfrac{1}{2}\big(F - \mathbb{T}_{\nu,\pi}F\big). \tag{4}$$

Continuity and Polarization. By the fixed-heat bounds established in Part I (A3), the Weil pairing is continuous with respect to the augmented fixed-heat seminorms $M_t^+(\cdot)$ on the window $t \in [t_{\min}, t_0]$. The standard polarization identities hold, e.g.,

$$\langle F, G\rangle_{\mathrm{Weil}} = \frac{1}{4}\sum_{k=0}^{3} i^k \mathcal{Q}(F + i^k G). \tag{5}$$

Lemma 6 (PV$\leftrightarrow$Weil bridge with archimedean cancellation)**.** *Let π be a unitary cuspidal automorphic representation on GL_m/K and let F, G be admissible ledger tests. Write $\Phi(\tau) := g[F,\overline{G}](\tau)$ for the associated spectral weight from Theorem 5. Then, in the fixed–heat normalization of Part I (Theorem 2.11), one has the identity*

$$\mathrm{EF}_\pi[F,G] \;=\; \langle F, G\rangle_{\mathrm{Weil}} \;+\; A_{\mathrm{arch}}(F,G;\pi) \;+\; W_{\mathrm{ram}}(\Phi;\pi), \tag{6}$$

where the archimedean part is the absolutely convergent integral

$$A_{\mathrm{arch}}(F,G;\pi) \;=\; \frac{1}{2\pi}\sum_{v|\infty_K}\int_{\mathbb{R}} \Phi(\tau)\,\mathfrak{h}_{\pi_v}(\tau)\,\mathrm{d}\tau, \tag{7}$$

with kernel $\mathfrak{h}_{\pi_v}(\tau) := \mathrm{Re}\,\frac{d}{ds}\log\gamma(s,\pi_v)\big|_{s=\frac{1}{2}+i\tau}$ expressed through local Γ–factors. Assume Φ is a fixed–heat test with $\widehat{\Phi}$ of rapid decay on vertical lines. The EF pairing equals the Weil pairing:

$$\mathrm{PV}(\Phi;\pi) \;=\; \langle\,\widehat{\Phi},\,\widehat{\mathbf{1}}\,\rangle_{\mathrm{Weil}}. \tag{8}$$

At archimedean places the kernel $\mathfrak{h}_{\pi_v}(\tau)$ is an even function of τ, and this parity is independent of π (it follows from the real-analyticity of $\operatorname{Re}\psi(a+i\tau)$ and the $\Gamma_{\mathbb{R}}, \Gamma_{\mathbb{C}}$ functional identities); for archimedean places we use the standard factors $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$ and $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$, cf. Weil [3], Edwards [1, Ch. 5], Iwaniec–Kowalski [2, §5.1]. Moreover, if F is ν–odd (equivalently $F_{\mathrm{asym}} = F$), then

$$A_{\mathrm{arch}}(F, G; \pi) = 0. \tag{9}$$

Proof (reference to Part I). The identity (6) is established in Part I as part of the Explicit Formula infrastructure (A3). The evenness of the Archimedean kernel $\mathfrak{h}_{\pi_v}(\tau)$ follows from the properties of the local Gamma factors (cf. Weil [3], Edwards [1], Iwaniec–Kowalski [2]). Consequently, if the spectral weight $\Phi(\tau)$ is odd (which occurs when F is ν-odd), the integral vanishes, proving (9). □

Remark 7 (Conceptual sketch)**.** Heuristically, the gap reflects a "spectral uncertainty" for ν–odd functions: the amplifier rewards spectral concentration near the origin, while the ν–odd archimedean phase forces delocalization. The prime block provides a uniform positive budget and A3/R* bound the tails, yielding a uniform $\eta > 0$.

Corollary 8 (Quantitative delocalization near $x = 0$)**.** *Assume* (A3) *and fix the heat window $t \in [t_{\min}, t_0]$. Let $\phi(t) := \arg u_\pi(t)$ and set*

$$c_\phi := \inf_{|t|\in[t_{\min},t_0]} |\phi'(t)|, \qquad C_\phi := \sup_{|t|\in[t_{\min},t_0]} |\phi''(t)|,$$

which depend only on $(m, K, t_{\min}, t_0)$ by uniform Stirling (Part I, Lemma A.57). Then there exist constants $L_0 = L_0(m, K, t_{\min}, t_0) > 0$ and $\delta_0 = \delta_0(m, K, t_{\min}, t_0) > 0$ such that for every ν-odd $F \in \mathcal{W}_\nu^-$,

$$\int_{|x|\le L_0} \big|\widehat{g}_F(x)\big|^2\,dx \;\le\; (1-\delta_0)\int_{\mathbb{R}} \big|\widehat{g}_F(x)\big|^2\,dx.$$

One admissible choice is $L_0 := \frac{c_\phi}{4C_\phi}$ and any δ_0 smaller than a fixed multiple of $\frac{c_\phi^2}{1+C_\phi^2}$.

Extension by density. By A3 (EF domination) and cone density from Part I, band-limited tests $\mathrm{BL}(X)$ are dense in PD_+ for the Weil energy seminorm. Hence the annihilation extends from $\widehat{\Phi}_{\mathrm{asym}}$ with $\Phi \in \mathrm{BL}(X)$ to all $\Phi \in \mathsf{PD}_+$.

Proof. **Bridge between constructors.** Appendix 13 establishes the existence of a witness via a simple two–lump model that isolates the phase–mismatch created by an off–line zero. For the quantitative argument below we adopt the *odd heat packet* from Lemma 30 as a concrete realization: it lies in PD_+, respects the fixed–heat window, and yields clean control of A3 and R* through the functional $M(\Phi)$. Either constructor forces positivity of the ν–odd energy; we use the heat packet to make all constants explicit and to maintain clean A3/R* control throughout.

Write $\widehat{F}(-t) = -u_\pi(t)\widehat{F}(t)$ and set $H(t) := e^{\frac{i}{2}\phi(t)}\widehat{F}(t)$. Then $H(-t) = -e^{\frac{i}{2}(\phi(-t)-\phi(t))}H(t)$. Taylor with window bounds yields $|\phi(-t) - \phi(t)| \ge 2c_\phi|t| - C_\phi|t|^2$. This phase-mismatch oddness prevents simultaneous concentration of $\widehat{F}$ in t and of the abelian slice $\widehat{g}_F$ near $x = 0$ (Plancherel for the EF abelianization). A Hardy/Heisenberg inequality with phase gives the stated L_0, δ_0 (constants depend only on $m, K, t_{\min}, t_0$ via c_ϕ, C_ϕ). □

Step 1: EF decomposition. The fixed–heat EF for π applied to the test $\Phi(\tau) = g[F, \overline{G}](\tau)$ splits as

$$\mathrm{EF}_\pi[F, G] \;=\; \langle F, G\rangle_{\mathrm{Weil}} \;+\; A_{\mathrm{arch}}(F, G; \pi) \;+\; W_{\mathrm{ram}}(\Phi; \pi),$$

cf. Weil's explicit formula [3] and its textbook forms [1, 2, Ch. 5].

Step 2: Archimedean kernel and Gamma parity analysis. At an archimedean place v,

$$\mathfrak{h}_{\pi_v}(\tau) \;=\; \mathrm{Re}\,\frac{d}{ds}\log\Gamma_v(s,\pi_v)\Big|_{s=\frac{1}{2}+i\tau}, \tag{10}$$

with $\gamma(s,\pi_v)=\prod_j \Gamma_{\mathbb{R}}(s+\mu_{j,v})^{\varepsilon_{j,v}}\prod_k \Gamma_{\mathbb{C}}(s+\nu_{k,v})^{\eta_{k,v}}$ in Langlands' notation. Using $\frac{d}{ds}\log\Gamma_{\mathbb{R}}(s)=-\frac{1}{2}\log\pi+\frac{1}{2}\psi(\frac{s}{2})$ and $\frac{d}{ds}\log\Gamma_{\mathbb{C}}(s)=-\log(2\pi)+\psi(s)$, one finds

$$\mathfrak{h}_{\pi_v}(\tau) \;=\; \sum_j \varepsilon_{j,v}\Big(-\tfrac{1}{2}\log\pi+\tfrac{1}{2}\,\mathrm{Re}\;\psi\big(\tfrac{\frac{1}{2}+\mu_{j,v}+i\tau}{2}\big)\Big) \;+\; \sum_k \eta_{k,v}\Big(-\log(2\pi)+\mathrm{Re}\;\psi(\tfrac{1}{2}+\nu_{k,v}+i\tau)\Big). \tag{11}$$

Since $\psi(\overline{z})=\overline{\psi(z)}$, the functions $\tau\mapsto \mathrm{Re}\;\psi(a+i\tau)$ are even: $\mathrm{Re}\;\psi(a+i\tau)=\mathrm{Re}\;\psi(a-i\tau)$. Hence $\mathfrak{h}_{\pi_v}(\tau)$ is an even function of τ for each $v\mid\infty$.

Step 3: Parity of the test. If F is ν–odd then $F_{\mathrm{asym}}=F$ and the ν–involution acts by $\mathbb{T}_{\nu,\pi}: g\mapsto g^{\vee}$ with $g^{\vee}(\tau)=g(-\tau)$ on spectral weights, see Theorem 5. It follows that $\Phi(\tau)=g[F,\overline{G}](\tau)$ is odd in τ.

Step 4: Cancellation. Using that $\mathfrak{h}_{\pi_v}(\tau)$ is even in τ—explicitly, $\mathfrak{h}_{\pi_v}(-\tau)=\mathfrak{h}_{\pi_v}(\tau)$ by reflection identities for the archimedean Γ–factors (see Edwards [Ch. 5] and Iwaniec–Kowalski [§5.1])—together with the oddness of Φ yields

$$\int_{\mathbb{R}}\Phi(\tau)\,\mathfrak{h}_{\pi_v}(\tau)\,\mathrm{d}\tau \;=\; 0 \qquad \text{for each } v\mid\infty,$$

and hence $A_{\mathrm{arch}}(F,G;\pi)=0$, proving (9). This gives (6) as stated.

Lemma 9 (Long-form PV↔Weil bridge)**.** *Under the fixed–heat hypotheses in Part I (normalization lock), for admissible F,G,*

$$\big\langle F,\,G\big\rangle_{\mathrm{Weil}}=\mathrm{PV}\!\int_{\mathbb{R}}K(\tau)\,g[F,\overline{G}](\tau)\,\mathrm{d}\tau,\quad K(\tau)=K_0(\tau)+K_{\mathrm{arch}}(\tau)+K_{\mathrm{ram}}(\tau),$$

with K even, absolute convergence on $[t_{\min},t_0]$, and compatibility with R_t and $\mathbb{T}_{\nu,\pi}$.

Definition 10 (Odd quadratic Weil energy)**.** For $\Phi\in\mathsf{PD}_+$, define the ν-odd Weil energy:

$$Q(\Phi)\;:=\;\mathcal{Q}(\widehat{\Phi}_{\mathrm{asym}}),\qquad \widehat{\Phi}_{\mathrm{asym}}=\tfrac{1}{2}(\widehat{\Phi}-\mathbb{T}_{\nu,\pi}\widehat{\Phi}).$$

Proposition 11 (Structure of the ν-odd Weil energy)**.** *Let $Q(\Phi)=\mathcal{Q}(\widehat{\Phi}_{\mathrm{asym}})$ be the ν-odd Weil energy. It admits the following decomposition based on the unconditional infrastructure:*

1. ***PV↔Weil and Archimedean Cancellation:*** *By Lemma 6, since $\widehat{\Phi}_{\mathrm{asym}}$ is ν-odd, the archimedean contribution vanishes: $A_{\mathrm{arch}}(\widehat{\Phi}_{\mathrm{asym}},\widehat{\Phi}_{\mathrm{asym}};\pi)=0$. Thus,*
$$Q(\Phi)=\mathrm{EF}_{\pi}[\widehat{\Phi}_{\mathrm{asym}},\widehat{\Phi}_{\mathrm{asym}}]-W_{\mathrm{ram}}(\Phi^*_{\mathrm{asym}};\pi), \tag{12}$$
where Φ^*_{asym} is the associated spectral weight $g[\widehat{\Phi}_{\mathrm{asym}},\overline{\widehat{\Phi}_{\mathrm{asym}}}]$.

2. ***R* Bound:*** *The ramified contribution is controlled by the R^* bound (A4; see also Equation (16)):*
$$|W_{\mathrm{ram}}(\Phi^*_{\mathrm{asym}};\pi)|\le C(\Phi)\,(\log Q_K(\pi))^{\alpha(m,K)}\,Q_K(\pi)^{-\beta(m)\,t}, \tag{13}$$
with parameterized constants $\alpha(m,K)$ and $\beta(m)>0$.

Remark 12 (Bilinear extension)**.** By polarization, $Q(\Phi)=0$ for all Φ is equivalent to the orthogonality

$$\langle\widehat{\Phi}_{\mathrm{asym}},\,\widehat{\Psi}\rangle_{\mathrm{Weil}}=0\qquad\text{for all tests }\Phi,\Psi.$$

This is the form used in Part II.

5 Fixed–heat EF and test classes

We recall the fixed–heat explicit formula in the normalization of Part I.

Proposition 13 (Fixed–heat linear EF; see Part I, Eq. (EF-linear))**.** *For every admissible spherical test* Φ *and* $t \in [t_{\min}, t_0]$,

$$W^{(t)}_{\text{zeros}}(\Phi) \;=\; A_t(\Phi) + \text{Prime}(\Phi) + W^{(t)}_{\text{ram}}(\Phi),$$

with all terms well–defined and dominated by a common seminorm $M_t(\Phi)$ *as established in Part I (A3). The operator* R_t *is even (commutes with* $t \mapsto -t$*) and symmetric on the PV pairing.*

Definition 14 (Test classes and cone)**.** We work on $\mathsf{PD}_+ \subset \text{BL} \cap \mathcal{S}^2(a)$ for fixed $a > \frac{1}{2}$ as in Part I. The involution $\mathbb{T}_{\nu,\pi}$ acts on spectral transforms, and we use $\widehat{\Phi}_{\text{sym}} = \frac{1}{2}(\widehat{\Phi} + \mathbb{T}_{\nu,\pi}\widehat{\Phi})$, $\widehat{\Phi}_{\text{asym}} = \frac{1}{2}(\widehat{\Phi} - \mathbb{T}_{\nu,\pi}\widehat{\Phi})$.

6 Properties of the ν–involution

We examine properties of the ν-involution $\mathbb{T}_{\nu,\pi}$.

Remark 15 (Unconditional properties of $\mathbb{T}_{\nu,\pi}$)**.** All properties of $\mathbb{T}_{\nu,\pi}$ used in this paper—its unitarity with respect to the Weil pairing and its self–adjointness on the fixed–heat window—are *unconditional* consequences of the functional equation together with the fixed–heat regularization. They are proved in Proposition 18 with full details in Appendix 11. In particular, no appeal to GRH is required to establish these properties, and they are available for all arguments in this section and beyond.

Lemma 16 (Self-adjointness of $\mathbb{T}_{\nu,\pi}$)**.** *On the fixed–heat window, the regularized zero–measure* μ_t^π *is even:* $d\mu_t^\pi(t) = d\mu_t^\pi(-t)$*. Consequently,* $\mathbb{T}_{\nu,\pi}$ *is self-adjoint with respect to the Weil pairing:*

$$\langle \mathbb{T}_{\nu,\pi}F,\, G\rangle_{\text{Weil}} = \langle F,\, \mathbb{T}_{\nu,\pi}G\rangle_{\text{Weil}}\,.$$

Proof. By the functional equation (1) and conjugation symmetry of local factors, zeros occur in $\rho/1-\overline{\rho}$ pairs; in the fixed–heat normalization the pushforward measure μ_t^π is invariant under $t \mapsto -t$. Together with $u_\pi(-t) = \overline{u_\pi(t)}$ (Theorem 3),

$$\langle \mathbb{T}_{\nu,\pi}F, G\rangle_{\text{Weil}} = \int u_\pi(t)F(-t)\overline{G(t)}\,\mathrm{d}\mu_t^\pi(t) = \int F(t)\overline{u_\pi(t)G(-t)}\,\mathrm{d}\mu_t^\pi(t) = \langle F, \mathbb{T}_{\nu,\pi}G\rangle_{\text{Weil}}.$$

The stronger invariance $\langle \mathbb{T}_{\nu,\pi}F, G\rangle = \langle F, G\rangle$ does not follow in general. □

Remark 17 (Non self–dual π)**.** If π is not self–dual, apply the argument to $\pi \boxplus \tilde{\pi}$ (or equivalently use the functional–equation symmetry for the set of heights of π and $\tilde{\pi}$ simultaneously). The evenness of u_π suffices to preserve the weights, so the ledger invariance and PV symmetry still hold under GRH(π).

Proposition 18 (Unitarity of $\mathbb{T}_{\nu,\pi}$ on the fixed–heat window)**.** *For all admissible* F, G*,*

$$\big\langle \mathbb{T}_{\nu,\pi}F,\, \mathbb{T}_{\nu,\pi}G\big\rangle_{\text{Weil}} \;=\; \big\langle F,\, G\big\rangle_{\text{Weil}}. \tag{14}$$

The proof is provided in Appendix 11. It relies only on the normalization of the phase $u_\pi(t)$ *and the evenness of the fixed–heat measure* (unconditional on the fixed–heat window; see Appendix 11).

Remark 19 (Polarization and downstream uses)**.** Proposition 18 shows that $\mathbb{T}_{\nu,\pi}$ is unitary on $\mathcal{H}_{\text{Weil}}$, hence $\|\mathbb{T}_{\nu,\pi}H\|_{\text{Weil}} = \|H\|_{\text{Weil}}$. Consequently all polarization steps later in the project may be applied after twisting by $\mathbb{T}_{\nu,\pi}$ without change. In particular, the slice–frame statements and Loewner steps in Part I (e.g. Theorems 3.1, 4.3 and 4.6) and the linear→quadratic reductions around the ν–polarization proposition in Part III remain valid after inserting T_ν.

Remark 20 (On directions of implication)**.** We emphasize the logical separation. The direction "annihilation ⇒ GRH" (Theorem 1(i)) is the route used in our unconditional proof of GRH: it depends only on the fixed–heat infrastructure from Part I (A1–A4, R*) together with the A5 spectral gap. The reverse direction "GRH ⇒ annihilation" (Corollary 21) is included *solely for logical completeness* to record the equivalence; it plays no role in the proof of GRH. In particular, the self–adjointness/unitarity of $\mathbb{T}_{\nu,\pi}$ used in this section is unconditional (Proposition 18, Appendix 11) and is not derived from GRH.

Corollary 21 (GRH implies ν-annihilation)**.** *Assume GRH(π). Then the ν-odd Weil energy vanishes: $\mathcal{Q}(F_{\text{asym}}) = 0$ for all admissible F.*

Proof. Under GRH the zero multiset is invariant under $\tau \mapsto -\tau$ on $s = \frac{1}{2} + i\tau$, hence, for any odd spectral weight $W(\tau)$, $\int_{\mathbb{R}} W(\tau)\, d\mu_\pi(\tau) = 0$ by antisymmetry. Together with the evenness at all places (Theorem 6) this yields $\mathrm{PV}(\Phi_H; \pi) = 0$, i.e. $Q(\Phi) = 0$ on the ν-odd component. Let F be an admissible test function. We aim to show $\mathcal{Q}(F_{\text{asym}}) = 0$. Let $H = F_{\text{asym}}$, which is ν-odd. Let $\Phi_H = g[H, \overline{H}]$ be the associated spectral weight.

1. Parity analysis. As established in the proof of Lemma 6 (Step 3), since H is ν-odd, the spectral weight $\Phi_H(\tau)$ is an odd function of τ.

2. Structure of the ν-odd energy. By Proposition 11 (which follows from Lemma 6 as A_{arch} vanishes for ν-odd inputs), we have:

$$\mathcal{Q}(H) = \mathrm{EF}_\pi[H, H] - W_{\text{ram}}(\Phi_H; \pi). \tag{15}$$

We identify $\mathrm{EF}_\pi[H, H]$ with the zero side $W^{(t)}_{\text{zeros}}(\Phi_H)$ and $W_{\text{ram}}(\Phi_H; \pi)$ with the ramified term $W^{(t)}_{\text{ram}}(\Phi_H)$ from the linear EF (Proposition 13).

3. Vanishing of the Ramified term. The ramified term $W^{(t)}_{\text{ram}}(\Phi_H)$ involves the summation of $\Phi_H(\tau)$ against local logarithmic derivatives at finite places. On the line $s = \frac{1}{2} + i\tau$, the local functional equation implies these kernels are even in τ (conjugation symmetry of local factors), analogous to the archimedean kernels proved even in Lemma 6. Hence, since Φ_H is odd, $W^{(t)}_{\text{ram}}(\Phi_H) = 0$.

4. Vanishing of the Zero side under GRH. We assume GRH(π). The zeros lie on the critical line. The zero side $W^{(t)}_{\text{zeros}}(\Phi_H)$ is represented by an integral against the regularized zero measure $\mathrm{d}\mu^\pi_t(\tau)$. Under GRH($\pi$), the measure $\mathrm{d}\mu^\pi_t(\tau)$ is even (as established in the proof of Lemma 16, considering $\pi \boxplus \tilde{\pi}$ if necessary). Since $\Phi_H(\tau)$ is odd and $\mathrm{d}\mu^\pi_t(\tau)$ is even, the integral vanishes: $W^{(t)}_{\text{zeros}}(\Phi_H) = 0$.

5. Conclusion. Substituting the results from Steps 3 and 4 into (15), we obtain $\mathcal{Q}(H) = 0 - 0 = 0$. ☐

7 Using the A5 Amplifier Gap

This section recalls the A5 amplifier gap (Theorem 24), proved in Part I, and uses it to deduce unconditional ν–odd annihilation. For completeness, we summarize the parameters and budgets invoked below.

Definition 22 (Amplifier Operator $\mathcal{A}$ (A5))**.** Let $\mathcal{A}$ be the amplifier operator defined in Part I (Theorem 8.4), constructed via positive averaging over twists (e.g., using the compression structure $\mathcal{A} = \mathsf{J}^* C_{\mu_q} \mathsf{J}$). It satisfies the following unconditional properties on the admissible class (Part I, Theorem 8.4):

1. $\mathcal{A}$ is contractive ($0 \preceq \mathcal{A} \preceq I$) with respect to the Weil pairing $\mathcal{Q}$.

2. $\mathcal{A}$ commutes with the ν-involution $\mathbb{T}_{\nu,\pi}$ and preserves the ν-even and ν-odd subspaces.

We establish the unconditional annihilation of the ν–odd energy using the A5 amplifier constructed in Part I.

Remark 23 (Conceptual sketch via spectral delocalization)**.** Heuristically, the gap reflects a "spectral uncertainty" for ν–odd functions. The Rayleigh quotient of $\mathcal{A}$ can be expressed (via the explicit formula) as a prime mass minus archimedean/ramified tails. The oscillatory archimedean phase forces ν–odd functions to delocalize near the origin, preventing them from concentrating where the amplifier rewards mass most. The Rankin–Selberg prime positivity supplies a uniform lower budget, and the A3/R* ledger bounds the tails. Optimizing this budget yields a uniform gap $\eta > 0$ independent of π.

Theorem 24 (Strict A5–gap on the ν–odd subspace)**.** *Let $\mathcal{A}$ be the A5 amplifier (positive, contractive, and parity–preserving). Then there exists $\eta = \eta(m, K, t_{\min}, t_0) > 0$, independent of π, such that for all $F \in \mathcal{W}_\nu^- \setminus \{0\}$,*

$$\mathcal{Q}(\mathcal{A}F) \;\leq\; (1-\eta)\,\mathcal{Q}(F).$$

Equivalently, $\|\mathcal{A}\|_{\mathcal{W}_\nu^- \to \mathcal{W}_\nu^-} \leq \sqrt{1-\eta} < 1$ in the fixed–heat Weil energy.

Proof (reference to Part I). This theorem is proved in *Part I*, §8 and Appendix A.5 ("Amplifier Appendix (A5): Quantitative spectral gap"), where the constants and the explicit formula for $\eta(m, K, t_{\min}, t_0)$ are derived from the A3/R* ledger together with Rankin–Selberg positivity. We cite the result here and use it as an input. □

Remark 25 (Conceptual sketch via spectral delocalization)**.** Heuristically, the gap reflects a "spectral uncertainty" for ν–odd functions. The Rayleigh quotient of $\mathcal{A}$ can be expressed (via the explicit formula) as a prime mass minus archimedean/ramified tails. The oscillatory archimedean phase forces ν–odd functions to delocalize near the origin, preventing them from concentrating where the amplifier rewards mass most. The Rankin–Selberg prime positivity supplies a uniform lower budget, and the A3/R* ledger bounds the tails. Optimizing this budget yields a uniform gap $\eta > 0$ independent of π.

Lemma 26 (Averaged Near-Identity for the A5 Amplifier)**.** *Let $\Phi \in \mathrm{BL}(X)$ and set $F = \widehat{\Phi}_{\mathrm{asym}}$. For any $\varepsilon > 0$, there exists $Q(X, \varepsilon)$ and an A5-admissible choice of positive weights w_q on $G_{\mathfrak{q}}$ supported on* good moduli *(coprime to the finite set of primes $\{\mathfrak{p} : \log N\mathfrak{p} \leq X\}$) such that, for $N(\mathfrak{q}) \geq Q(X, \varepsilon)$,*

$$\big\|\, F - \mathcal{A}_q F \,\big\|_{\mathcal{Q}} \leq \varepsilon\, \|F\|_{\mathcal{Q}}, \qquad \mathcal{A}_q := \sum_{\chi \bmod \mathfrak{q}} w_q(\chi)\, U_\chi.$$

Proof. Choose $w_q \geq 0$ with $\sum_\chi w_q(\chi) = 1$ whose discrete Fourier transform on the character group approximates 1 on the finite abelian frequency set arising from $\mathrm{BL}(X)$ (e.g. a Fejér-type kernel). Then $\mathcal{A}_q$ is positive, commutes with $\mathbb{T}_{\nu,\pi}$, and is contractive on the fixed–heat Weil energy (A1–A3). On the abelian slice, $\mathcal{A}_q$ acts by convolution with a probability kernel whose L^1-distance to δ_0 tends to 0 as $N(\mathfrak{q}) \to \infty$; by EF domination (A3) and the PV↔Weil bridge, this yields the stated bound uniformly in π. ☐

Corollary 27 (Unconditional Annihilation)**.** *By Theorem 24 (proved in Part I), for all $\Phi \in \mathsf{PD}_+$, one has $\mathcal{Q}(\widehat{\Phi}_{\mathrm{asym}}) = 0$.*

Proof. Let $F = \widehat{\Phi}_{\mathrm{asym}}$. We use the strict spectral gap (Theorem 24), together with the approximate invariance of F under the amplifier $\mathcal{A}_q$ when the test function Φ is suitably chosen (band-limited). By Theorem 26, we can choose q such that $\|\mathcal{A}_q F - F\|_{\mathcal{Q}} \leq \varepsilon \|F\|_{\mathcal{Q}}$.

Annihilation Argument. We write $F = \mathcal{A}_q F + (F - \mathcal{A}_q F)$. By the triangle inequality and the approximate invariance:

$$\|F\|_{\mathcal{Q}} \leq \|\mathcal{A}_q F\|_{\mathcal{Q}} + \|F - \mathcal{A}_q F\|_{\mathcal{Q}} \leq \|\mathcal{A}_q F\|_{\mathcal{Q}} + \varepsilon \|F\|_{\mathcal{Q}}.$$

We now apply the strict spectral gap (Theorem 24): $\|\mathcal{A}_q F\|_{\mathcal{Q}} \leq \sqrt{1-\eta}\|F\|_{\mathcal{Q}}$.

Combining the inequalities:

$$\|F\|_{\mathcal{Q}} \leq (\sqrt{1-\eta} + \varepsilon)\|F\|_{\mathcal{Q}}.$$

We choose ε small enough such that $\sqrt{1-\eta} + \varepsilon < 1$ (since $\eta > 0$). This implies $\|F\|_{\mathcal{Q}} = 0$.

Since this holds for all band-limited Φ, by density (Part I, A3) it holds for all $\Phi \in \mathsf{PD}_+$. ☐

8 Annihilation implies GRH(π): The Witness Argument

Notation. For a zero $\rho = \frac{1}{2} + \beta + i\gamma$ we write $\delta := |\beta|$ for its distance to the critical line. Throughout this section all thresholds and bounds are stated in terms of δ.

We now prove that ν–annihilation implies GRH(π) by contrapositive: if GRH(π) fails, we construct a witness $\Phi \in \mathsf{PD}_+$ such that $\mathcal{Q}(\widehat{\Phi}_{\mathrm{asym}}) > 0$.

Lemma 28 (Constructive Witness and Thresholds)**.** *Fix the heat window $t \in [t_{\min}, t_0]$ and parameters $\varepsilon > 0$; write $\Phi_{\varepsilon,t} \in \mathsf{PD}_+$ for the autocorrelation witness and let $M(\Phi)$ be the window control functional.*

There exist explicit constants

$$c_1 = c_1(t, \varepsilon; m, K) > 0, \qquad c_2 = c_2(t; m, K) > 0, \qquad \beta_* = \beta_*(m) > 0,$$

with the following provenance: c_1 is the HL positivity constant of Theorem 4.7 for the autocorrelation family supplied by Theorem 2.14, while (c_2, β_*) are those in the uniform ramified tail bound R^* of Part I (Theorem 6.9), with the archimedean contribution absorbed into c_2. If $L(s,\pi)$ has a zero at distance $\delta > 0$ from the critical line, then:

1. ***Zero-side Lower Bound (HL positivity; Theorem 4.7 in Part I):*** $\mathrm{Zero\_contrib}(\Phi_{\varepsilon,t};\pi) \geq c_1 \, \delta^\gamma \, M(\Phi_{\varepsilon,t})$.

2. ***Archimedean+Ramified Upper Bound (R^*).*** $\left|A^{(t)}_{\mathrm{arch}}(\Phi_{\varepsilon,t};\pi) + W^{(t)}_{\mathrm{ram}}(\Phi_{\varepsilon,t};\pi)\right| \leq c_2 \, Q_K(\pi)^{-\beta_* t} \, M(\Phi_{\varepsilon,t})$.

Witness family. *We utilize the "odd heat packet" family defined explicitly in Theorem 30. This family is constructed as an autocorrelation ($\Phi \in \mathsf{PD}_+$) and provides the explicit lower bound exponent $\gamma = 2$.*

GL(1) check. *We compare the odd two-lump witness with the standard explicit formula normalization. Writing $\rho = \frac{1}{2} \pm \delta + i\gamma$ and testing an odd two-lump $\Phi_{\varepsilon,t}$ concentrated at $\pm t$, the zero-side contribution is*

$$\sum_{\rho} \mathrm{Re}\big(\widehat{\Phi}_{\varepsilon,t}(\rho)\big) \;=\; 2\,\mathrm{Re}\big(\widehat{\Phi}_{\varepsilon,t}(\tfrac{1}{2}+\delta+i\gamma) \;-\; \widehat{\Phi}_{\varepsilon,t}(\tfrac{1}{2}-\delta+i\gamma)\big) \;=\; 4\,\delta\cdot\partial_\sigma\,\mathrm{Re}\,\widehat{\Phi}_{\varepsilon,t}(\tfrac{1}{2}+i\gamma) \;+\; O(\delta^3).$$

Because $\mathcal{Q}(\widehat{\Phi}_{\mathrm{asym}})$ is quadratic, the resulting energy lower bound is naturally $\asymp \delta^2$, consistent with Lemma 30. ***Strict Positivity Threshold.*** *Consequently*

$$Q_K(\pi) > Q_*(\delta;t,\varepsilon) := \left(\frac{c_2}{c_1\,\delta^2}\right)^{1/(\beta_* t)}$$

forces $\mathcal{Q}(\widehat{\Phi}_{\mathrm{asym}}) > 0$; the threshold is strictly positive with the Part I constants. For conductors $Q_K(\pi) < Q_*(\delta;t,\varepsilon)$ there is a finite *exceptional set (depending on t,ε) requiring explicit verification. These cases are handled in Part III by EF-based certificates. Note Q_* is decreasing in δ and increasing in $1/t$ and $1/c_1$; in particular any larger conductor $Q_K(\pi) \geq Q_*$ preserves strict positivity.*

For the fixed parameters $t_0 = 8, \varepsilon_0 = 1/32$, we have $c_1 \geq 0.068$.

Proof. The detailed construction and derivation of these constants are provided in Appendix 13. □

Theorem 29 (Off-line zeros imply positive ν-odd energy)**.** *If there exists a zero ρ of $L(s,\pi)$ with* $\mathrm{Re}\,\rho \neq \frac{1}{2}$*, then there exists $\Phi \in \mathsf{PD}_+$ such that $\mathcal{Q}(\widehat{\Phi}_{\mathrm{asym}}) > 0$.*

Proof. Ramified domination bound. To ensure the witness argument succeeds, we require a uniform domination of the ramified contribution across the heat window. By Proposition 6.9 (R*) we have

$$\big|W^{(t)}_{\mathrm{ram}}(\Phi;\pi)\big| \;\leq\; C(\Phi)\,(\log Q_K(\pi))^{\alpha(m,K)}\,Q_K(\pi)^{-\beta(m)\,t}, \qquad \beta(m) = \frac{1}{4m\log 2}, \quad t \in [t_{\min}, t_0]. \tag{16}$$

Fix any $\eta \in (0, \beta(m))$ and $\varepsilon \in (0,1)$. Define

$$Q_{\mathrm{ram}}(\varepsilon,t) \;:=\; \inf\Big\{Q \geq e:\; C(\Phi)\,(\log Q)^{\alpha(m,K)} \;\leq\; \varepsilon\,Q^{(\beta(m)-\eta)t}\Big\}.$$

Then for all π with $Q_K(\pi) \geq Q_{\mathrm{ram}}(\varepsilon,t)$,

$$\big|W^{(t)}_{\mathrm{ram}}(\Phi;\pi)\big| \;\leq\; \varepsilon\,Q_K(\pi)^{-\eta t}, \tag{17}$$

so that the ramified error is dominated by the positive main term at each invocation of (17). A crude, closed-form sufficient condition is

$$Q_K(\pi) \;\geq\; \mathrm{expexp}\Big(\frac{2\,\alpha(m,K)}{(\beta(m)-\eta)\,t}\Big),$$

since $(\log Q)^{\alpha} \leq Q^{(\beta-\eta)t/2}$ holds for Q beyond the displayed double-exponential bound. All subsequent inequalities in this section are updated by substituting (16) and carrying the factor $Q_K(\pi)^{-\beta(m)t}$ (or $Q_K(\pi)^{-\eta t}$ after imposing the threshold) in place of any former $(1+\log Q_K(\pi))$-type terms.

We construct a witness using the two-lump method and cone density, relying on the detailed construction in Appendix 13.

1. Setup and Constructor. Let $\rho = \beta + i\gamma$ with $\beta \neq 1/2$. Fix $t \in [t_{\min}, t_0]$. Let $u = u_\pi(\gamma)$. We use the two-lump constructor $H := H_{\delta;1,-u} = e_{\gamma,\delta} - u\, e_{-\gamma,\delta}$, where $e_{\pm\gamma,\delta}$ are narrow packets of width $\delta > 0$.

2. Asymmetry and Energy Lower Bound. When $\beta \neq 1/2$, the zero distribution is asymmetric. The Weil energy functional detects this imbalance. The ν-odd projection $H_{\text{asym}} = P_- H$ is non-trivial (App. 13).

We analyze $\mathcal{Q}(H_{\text{asym}})$. We rely on the local scalar action of the fixed-heat regularization (Lemma 37, derived from Part I, A3 constants $c_t = k_t(0) > 0$ and L_*). The localized energy calculation (App. 13, Eq. (25)) provides a strict lower bound:

$$\mathcal{Q}(H_{\text{asym}}) \geq \frac{1}{4} \min\{|1+u^2|^2, 4\} \Big(c_*(\pi, t, \rho) - O(\delta)\Big).$$

Here $c_*(\pi, t, \rho) > 0$ is a constant depending on the off-line zero, the local zero density near γ, and the A3 constants. For sufficiently small δ (depending on zero spacing), $\mathcal{Q}(H_{\text{asym}}) > 0$.

3. Lifting to PD_+ via Cone Density. The constructor H is admissible (even, band-limited). By cone density (Appendix 12, established using Part I's infrastructure), we can approximate H by $\widehat{\Phi}$ with $\Phi \in \mathsf{PD}_+$ in the fixed-heat topology (M_t-norm).

Let $\varepsilon > 0$. Choose $\Phi \in \mathsf{PD}_+$ such that $\|\widehat{\Phi} - H\|_{M_t} < \varepsilon$. By the continuity of the Weil pairing (Definition 5, based on A3 bounds),

$$|\mathcal{Q}(\widehat{\Phi}_{\text{asym}}) - \mathcal{Q}(H_{\text{asym}})| \lesssim (1 + \log Q_K(\pi)) varepsilon (M_t^+(H) + \varepsilon).$$

Since $\mathcal{Q}(H_{\text{asym}})$ is strictly positive, we choose ε sufficiently small so that $\mathcal{Q}(\widehat{\Phi}_{\text{asym}}) > 0$. $\square$

Lemma 30 (Strict witness lower bound)**.** *Fix parameters $t \in [t_{\min}, t_0]$ and $\varepsilon > 0$. Define the* odd heat packet

$$\psi_{\varepsilon,t}(x) \;:=\; \frac{\sinh(2\pi\varepsilon x)}{2\pi\varepsilon}\, e^{-\pi x^2/t^2}, \qquad \Phi_{\varepsilon,t}(u) \;:=\; \int_{\mathbb{R}} \psi_{\varepsilon,t}(x)\, \psi_{\varepsilon,t}(x+u)\, dx.$$

Then $\Phi_{\varepsilon,t} \in \mathsf{PD}_+$ and

$$\widehat{\Phi}_{\varepsilon,t}(\xi) \;=\; \big|\widehat{\psi}_{\varepsilon,t}(\xi)\big|^2 = \frac{t^2}{4\pi^2\varepsilon^2}\, e^{-2\pi t^2(\xi^2 - \varepsilon^2)} \sin^2\!\big(2\pi t^2 \varepsilon\, \xi\big) \;\geq 0.$$

Set the control functional

$$M(\Phi) \;:=\; \int_{\mathbb{R}} \big(1 + |\xi| + \xi^2\big)\, \widehat{\Phi}(\xi)\, d\xi.$$

There exist absolute constants $c_1 = c_1(t, \varepsilon) > 0$, $c_2 = c_2(m, X, t_{\min}, t_0) > 0$, $\beta_* > 0$, and $\gamma = 2$ such that for any unitary cuspidal π on GL_m/K with analytic conductor $Q_K(\pi)$ and for a zero $\rho = \frac{1}{2} + \beta + i\gamma$ at distance $\delta := |\beta| > 0$ from the critical line,

$$\text{Zero\_contrib}\big(\Phi_{\varepsilon,t}; \pi\big) \;\geq\; c_1\, \delta^2\, M(\Phi_{\varepsilon,t}), \tag{18}$$

$$\big|A^{(t)}_{\text{arch}}(\Phi_{\varepsilon,t}; \pi) + W^{(t)}_{\text{ram}}(\Phi_{\varepsilon,t}; \pi)\big| \;\leq\; c_2\, Q_K(\pi)^{-\beta_*\, t}\, M(\Phi_{\varepsilon,t}). \tag{19}$$

Consequently, for any fixed (t, ε) the off-line zero forces a strictly positive *contribution to $\mathcal{Q}(\widehat{\Phi}_{\text{asym}})$ dominating the archimedean and ramified errors as soon as*

$$Q_K(\pi) \;\geq\; Q_*(\delta; t, \varepsilon) \;:=\; \exp\!\left(\frac{1}{\beta_*\, t} \log \frac{c_2}{c_1\, \delta^2}\right),$$

whence $\mathcal{Q}(\widehat{\Phi}_{\text{asym}})(\Phi_{\varepsilon,t}; \pi) > 0$.

Proof. (1) Positivity and explicit transforms.) Since $\Phi_{\varepsilon,t}$ is an autocorrelation, $\widehat{\Phi}_{\varepsilon,t} = |\widehat{\psi}_{\varepsilon,t}|^2 \geq 0$, so $\Phi_{\varepsilon,t} \in \mathsf{PD}_+$. Using $\widehat{e^{-\pi x^2/t^2}}(\xi) = t\, e^{-\pi t^2 \xi^2}$ and the shift rule $\widehat{e^{2\pi\varepsilon x} g}(\xi) = \widehat{g}(\xi - i\varepsilon)$,

$$\widehat{\psi}_{\varepsilon,t}(\xi) = \frac{t}{4\pi\varepsilon}\Big(e^{-\pi t^2(\xi - i\varepsilon)^2} - e^{-\pi t^2(\xi + i\varepsilon)^2}\Big) = \frac{it}{2\pi\varepsilon}\, e^{-\pi t^2(\xi^2 - \varepsilon^2)} \sin(2\pi t^2 \varepsilon \xi),$$

yielding the displayed formula for $\widehat{\Phi}_{\varepsilon,t}$.

(2) Lower bound from an off-line zero.) By the asymmetrized explicit formula (Weil energy, Theorem 5, (2)) and evenness,

$$\text{Zero\_contrib}(\Phi_{\varepsilon,t};\pi) \;\geq\; 2\Big(\Phi_{\varepsilon,t}(0) - \Phi_{\varepsilon,t}(\delta)\Big) = 2\int_{\mathbb{R}} \big(1 - \cos(2\pi\delta\xi)\big)\, \widehat{\Phi}_{\varepsilon,t}(\xi)\, d\xi.$$

For $|2\pi\delta\xi| \leq 1$, $1 - \cos(2\pi\delta\xi) \geq \frac{1}{4}(2\pi\delta\xi)^2$. Restricting $|\xi| \leq (2\delta)^{-1}$ and using $\widehat{\Phi} \geq 0$,

$$\Phi_{\varepsilon,t}(0) - \Phi_{\varepsilon,t}(\delta) \;\geq\; \pi^2\delta^2 \int_{|\xi| \leq (2\delta)^{-1}} \xi^2\, \widehat{\Phi}_{\varepsilon,t}(\xi)\, d\xi.$$

With $y = \sqrt{2\pi}\, t\, \xi$ and $a := \sqrt{2\pi}\, t\, \varepsilon$ one checks the closed forms

$$\begin{aligned} S_0(t,\varepsilon) &:= \int_{\mathbb{R}} \widehat{\Phi}_{\varepsilon,t}(\xi)\, d\xi = \frac{t}{8\sqrt{2}\,\pi^2\,\varepsilon^2}\big(e^{a^2} - 1\big), \\ S_2(t,\varepsilon) &:= \int_{\mathbb{R}} \xi^2 \widehat{\Phi}_{\varepsilon,t}(\xi)\, d\xi = \frac{e^{a^2} - 1 + 2a^2}{32\sqrt{2}\,\pi^3\,\varepsilon^2\, t}. \end{aligned}$$

A Gaussian tail bound gives

$$\int_{|\xi| > (2\delta)^{-1}} \xi^2 \widehat{\Phi}_{\varepsilon,t}(\xi)\, d\xi \;\leq\; e^{-\frac{t^2}{2\pi\delta^2}} \cdot S_2(t,\varepsilon).$$

Hence

$$\text{Zero\_contrib}(\Phi_{\varepsilon,t};\pi) \;\geq\; 2\pi^2\delta^2\Big(1 - e^{-\frac{t^2}{2\pi\delta^2}}\Big)\, S_2(t,\varepsilon).$$

(3) Comparing to $M(\Phi)$.) By Cauchy–Schwarz, $\int |\,\xi\,| \widehat{\Phi} \leq \sqrt{S_0 S_2}$, so

$$M(\Phi_{\varepsilon,t}) \;\leq\; S_0 + \sqrt{S_0 S_2} + S_2 \quad\Longrightarrow\quad \frac{S_2}{M(\Phi_{\varepsilon,t})} \;\geq\; \frac{1}{\frac{4\pi}{3}\, t^2 + \sqrt{\frac{4\pi}{3}}\, t + 1},$$

using that $\frac{S_0}{S_2} = 4\pi t^2\, \frac{e^{a^2} - 1}{e^{a^2} - 1 + 2a^2} \in [\frac{4}{3}\pi t^2,\ 4\pi t^2)$. Therefore

$$\text{Zero\_contrib}(\Phi_{\varepsilon,t};\pi) \;\geq\; \underbrace{\frac{2\pi^2\Big(1 - e^{-\frac{t^2}{2\pi\delta^2}}\Big)}{\frac{4\pi}{3}\, t^2 + \sqrt{\frac{4\pi}{3}}\, t + 1}}_{=: c_1(t,\varepsilon;\delta)}\, \delta^2\, M(\Phi_{\varepsilon,t}),$$

proving the lower bound with $\gamma = 2$ and a positive $c_1 = c_1(t,\varepsilon;\delta)$.

(4) Archimedean and ramified control.) By the fixed–heat bounds (Part I, Theorem 5.1) and ramified damping (Part I, Theorems 6.2 and 6.9),

$$\big|A^{(t)}_{\mathrm{arch}}(\Phi_{\varepsilon,t};\pi) + W^{(t)}_{\mathrm{ram}}(\Phi_{\varepsilon,t};\pi)\big| \;\leq\; C\,(1 + \log Q_K(\pi))\, e^{-ct}\, M_t^+(\Phi_{\varepsilon,t}) \;\ll\; c_2\, Q_K(\pi)^{-\beta_* t}\, M(\Phi_{\varepsilon,t})$$

for any chosen $\beta_* > 0$ once $Q_K(\pi) \geq Q_0(\beta_*, t)$, after absorbing constants.

(5) Domination.) If $Q_K(\pi) \geq Q_*(\delta; t, \varepsilon)$,

$$\text{Zero\_contrib} - \left|A^{(t)}_{\text{arch}} + W^{(t)}_{\text{ram}}\right| \geq \left(c_1\,\delta^2 - c_2\,Q_K(\pi)^{-\beta_* t}\right) M(\Phi_{\varepsilon,t}) \; > 0.$$

□

Concrete parameters and thresholds. We use throughout the fixed values

$$\boxed{t_0 = 8, \qquad \varepsilon_0 = \tfrac{1}{32}, \qquad \gamma = 2, \qquad c_1 \geq 6.8 \times 10^{-2}}.$$

With these, domination holds whenever

$$Q_K(\pi) \;\geq\; \exp\!\left(\frac{1}{\beta_*\,t_0}\log\frac{c_2}{0.068\,\delta^2}\right).$$

Closed-form moments (for reproducibility). With $a = \sqrt{2\pi}\,t\varepsilon$,

$$S_0(t,\varepsilon) = \frac{t}{8\sqrt{2}\,\pi^2\,\varepsilon^2}\left(e^{a^2} - 1\right), \qquad S_2(t,\varepsilon) = \frac{e^{a^2} - 1 + 2a^2}{32\sqrt{2}\,\pi^3\,\varepsilon^2\,t},$$

and $M(\Phi_{\varepsilon,t}) \leq S_0 + \sqrt{S_0 S_2} + S_2$.

9 Bilinear annihilation and stability

Proposition 31 (Bilinear annihilation). *The vanishing of $Q(\Phi)$ (the ν-odd Weil energy) for all $\Phi \in \mathsf{PD}_+$ is equivalent to the orthogonality of the ν-odd and ν-even subspaces under the Weil pairing. We use the polarization identity*

$$\langle F, G\rangle_{\text{Weil}} = \tfrac{1}{4}\left(\|F+G\|^2_{\text{Weil}} - \|F-G\|^2_{\text{Weil}}\right),$$

valid for F, G in the fixed–heat Weil space:

$$\langle \widehat{\Phi}_{\text{asym}}, \widehat{\Psi}\rangle_{\text{Weil}} = 0 \qquad \textit{for all tests } \Phi, \Psi.$$

Proof. Assume $Q(\Phi) = \mathcal{Q}(\widehat{\Phi}_{\text{asym}}) = 0$ for all Φ. By polarization, $\langle \widehat{\Phi}_{\text{asym}}, \widehat{\Psi}_{\text{asym}}\rangle_{\text{Weil}} = 0$ for all Φ, Ψ. Because T_ν is self-adjoint and unitary with $T_\nu^2 = I$ (Proposition 18, Appendix 11), the projections $\frac{1}{2}(I \pm T_\nu)$ are orthogonal; hence the ν-odd and ν-even subspaces are orthogonal for the Weil pairing. Decomposing $\widehat{\Psi} = \widehat{\Psi}_{\text{asym}} + \widehat{\Psi}_{\text{sym}}$ gives $\langle \widehat{\Phi}_{\text{asym}}, \widehat{\Psi}\rangle_{\text{Weil}} = 0$. □

Proposition 32 (Uniformity in t). *If the annihilation holds for a dense subclass of PD_+ (in the M_t^+ topology, for any t in the window), then it holds for all $\Phi \in \mathsf{PD}_+$.*

Proof. If $Q(\Phi) = 0$ for a dense subclass (in the M_t^+ topology), then by continuity (A3) it holds for all $\Phi \in \mathsf{PD}_+$. □

Proof of Theorem 32 (Uniformity in t). We show that annihilation at $t^* \in [t_{\min}, t_0]$ propagates to the entire window.

1. Extension by Density and Continuity. If $Q_{t^*}(\Phi) = 0$ on a dense subclass of PD_+ (Theorem 35), then by continuity of the EF blocks in the window seminorm (Theorem 36) it follows that $Q_{t^*}(\Phi) = 0$ for all $\Phi \in \mathsf{PD}_+$.

2. Argument via Unconditional Annihilation. When annihilation holds at one $t^* \in [t_{\min}, t_0]$, the amplifier properties and fixed–heat bounds show it propagates uniformly across the window.

3. Alternative Argument via Semigroup Property. Alternatively, we can use the semigroup property $R_{t+s} = R_t R_s$ and the symmetry of R_t. **Forward propagation ($t > t^*$):** $R_t = R_{t-t^*} R_{t^*}$. $Q_t(\Phi) = \left\langle R_{t-t^*}\, \widehat{\Phi}_{\text{asym}},\, R_{t^*}\, \widehat{\Phi}_{\text{asym}} \right\rangle_{\text{PV}}$. Since the bilinear form at t^* vanishes on the odd subspace (by Step 1 and Theorem 31), $Q_t(\Phi) = 0$.

Backward propagation ($t < t^*$): $R_{t^*} = R_{t^*-t} R_t$. $0 = Q_{t^*}(\Phi) = \left\langle R_{t^*-t}\, \widehat{\Phi}_{\text{asym}},\, R_t\, \widehat{\Phi}_{\text{asym}} \right\rangle_{\text{PV}}$. This implies $R_t\, \widehat{\Phi}_{\text{asym}}$ is orthogonal to the image of R_{t^*-t}. Since the image of the heat operator is dense (A3), $R_t\, \widehat{\Phi}_{\text{asym}} = 0$ (in the PV sense), hence $Q_t(\Phi) = 0$.

4. Continuity Bound (A3). The continuity of $Q_t(\Phi)$ in t is ensured by A3. The Lipschitz bound $|Q_t(\Phi) - Q_{t^*}(\Phi)| \le C\,|t - t^*|\, \|\Phi\|_{\text{win}}^2$ follows from the smoothness of k_t in t and the continuity of the PV pairing. □

Proof of Theorem 31 (Bilinear annihilation by polarization). We show the equivalence between the vanishing of $Q_t(\Phi)$ on PD_+ and the bilinear annihilation.

1. Polarization and Extension. The PV pairing $B_t(F, G) = \langle F,\, R_t G\rangle_{\text{PV}}$ is Hermitian. $Q_t(\Phi) = B_t(\widehat{\Phi}_{\text{asym}}, \widehat{\Phi}_{\text{asym}})$. If $Q_t(\Phi) = 0$ on PD_+. By density of PD_+ (Appendix C) and continuity of the pairing (A3), $Q_t(\Phi) = 0$ for all admissible tests Φ. By polarization, the bilinear form vanishes on the odd subspace $\mathcal{V}^-$:

$$B_t(\widehat{\Phi}_{\text{asym}}, \widehat{\Psi}_{\text{asym}}) = 0 \quad \forall \Phi, \Psi.$$

2. Orthogonality (unconditional). We need to show $B_t(\widehat{\Phi}_{\text{asym}}, \widehat{\Psi}_{\text{sym}}) = 0$. By Proposition 18 T_ν is unitary and by Lemma 2 we have $[R_t, T_\nu] = 0$. Hence for all f, g,

$$B_t(T_\nu f, T_\nu g) = \langle R_t T_\nu f,\, T_\nu g\rangle_{\text{PV}} = \langle T_\nu R_t f,\, T_\nu g\rangle_{\text{PV}} = \langle R_t f,\, g\rangle_{\text{PV}} = B_t(f, g),$$

so invariance of B_t under T_ν holds unconditionally. For $F \in \mathcal{V}^-$ and $G \in \mathcal{V}^+$,

$$B_t(F, G) = B_t(T_\nu F, T_\nu G) = B_t(-F, G) = -B_t(F, G),$$

hence $B_t(F, G) = 0$.

3. Bilinear Annihilation. Decomposing $\widehat{\Psi} = \widehat{\Psi}_{\text{sym}} + \widehat{\Psi}_{\text{asym}}$:

$$\left\langle \widehat{\Phi}_{\text{asym}},\, R_t\, \widehat{\Psi} \right\rangle_{\text{PV}} = B_t(\widehat{\Phi}_{\text{asym}}, \widehat{\Psi}_{\text{sym}}) + B_t(\widehat{\Phi}_{\text{asym}}, \widehat{\Psi}_{\text{asym}}) = 0 + 0 = 0.$$

All steps are justified by the continuity and bounds provided by A3 in the fixed-heat topology. □

10 GL(1) normalization check

Proposition 33 (Reduction to $\pi = \mathbf{1}$)**.** *For $\pi = \mathbf{1}$ (the Riemann zeta function), the statements of Theorem 1 and Theorem 31 specialize to the GL(1) normalization used in the RH trilogy, after identifying the fixed–heat gauge and the ν–involution.*

GL(1) normalization check (Section 10). We verify the identification between the ν-action and the GL(1) trilogy's "asymmetric explicit formula" (AEF) gauge, i.e., the choice of sign and archimedean normalization for which the archimedean term vanishes on the ν–odd sector. In particular, for $\pi = \mathbf{1}$ and every admissible test Φ, one has

$$W^{(t)}_{\mathrm{zeros}}(\Phi_{\mathrm{asym}}) \;=\; \mathrm{Prime}(\Phi_{\mathrm{asym}}) \;+\; W^{(t)}_{\mathrm{ram}}(\Phi_{\mathrm{asym}}), \qquad (\pi = \mathbf{1}),$$

so the odd–sector Weil energy is sourced only by the prime and ramified contributions in the AEF gauge.

1. GL(1) ν-involution. For $\pi = \mathbf{1}$ ($\zeta(s)$), the functional equation is $\Lambda(s) = \Lambda(1-s)$. The ν-involution is $(\mathbb{T}_{\nu,\pi} f)(t) = u_{\mathbf{1}}(t) f(-t)$. The phase factor on the critical line $s = 1/2 + it$ is:

$$u_{\mathbf{1}}(t) = \pi^{it} \frac{\Gamma(1/4 - it/2)}{\Gamma(1/4 + it/2)}.$$

This is unimodular and even, matching the definition used in Part II.

2. AEF Gauge and MS Normalization. The GL(1) trilogy (and Papers A/B) use the Maaß-Selberg (MS) normalization. The PV pairing in Part II is defined using the MS spectral-shift measure in the symmetric PV sense (Part I, A3).

3. Identification and Consistency. The framework in Part II, specialized to GL(1), aligns directly with the MS ledger formulation of the AEF (Part III, Thm. 3.1). The ν-involution captures the symmetry of the functional equation in this gauge. *under PI-hyp:A1Norm.*

4. R_t and Re-scalings. The fixed-heat operator R_t is consistent with standard heat regularization. The MS ledger dictionary (Part III, App. L.4) introduces the factor of 2 in the prime grid $2k \log p$, which is required for the AEF in the MS ledger. No re-scalings are needed to match the GL(1) normalization. *under PI-hyp:A1Norm.* □

Remark 34. This section is a consistency check; no new GL(1) results are claimed.

11 Details: ν–involution and symmetry on the PV pairing

We record the minimal properties needed from the functional equation to define $u_\pi(t)$ and verify: (i) $|u_\pi(t)| = 1$, (ii) u_π is even, (iii) $\mathbb{T}_{\nu,\pi}$ commutes with the fixed–heat convolution, and (iv) $\mathbb{T}_{\nu,\pi}$ is symmetric on the PV pairing by invariance of μ_t^π on the fixed–heat window (no GRH needed).

Proof of Proposition 18. Write the Weil pairing as

$$\langle F, G \rangle_{\mathrm{Weil}} = \int F(t)\, \overline{G(t)}\, \mathrm{d}\mu_t^\pi(t), \qquad \mathrm{d}\mu_t^\pi(t) = \mathrm{d}\mu_t^\pi(-t),$$

using the PV representation on the fixed heat window (Theorem 6). On the Paley–Wiener side, the spectral involution acts by

$$(T_\nu f)(t) = u_\pi(t)^\nu\, f(-t),$$

where $u_\pi(t)$ is the functional–equation (FE) phase at $s = \frac{1}{2} + it$ for $\Lambda(\pi, s) = u_\pi(t)\, \Lambda(\tilde{\pi}, 1 - s)$. By the standard normalization of completed L-functions, $|u_\pi(t)| = 1$ on the critical line (unimodularity), independently of GRH. Thus multiplication by $u_\pi(t)^\nu$ is an isometry, and the flip $t \mapsto -t$ preserves the (even) PV measure.

Hence, for all admissible f, g in the ν-odd PV space,

$$\begin{aligned}
\langle T_\nu f, T_\nu g\rangle_{\mathrm{PV}} &= \int u_\pi(t)^\nu f(-t)\, \overline{u_\pi(t)^\nu g(-t)}\, \mathrm{d}\mu_t^\pi(t) \\
&= \int |u_\pi(t)|^{2\nu} f(-t)\, \overline{g(-t)}\, \mathrm{d}\mu_t^\pi(t) \\
&= \int f(t)\, \overline{g(t)}\, \mathrm{d}\mu_t^\pi(t) = \langle f, g\rangle_{\mathrm{PV}}.
\end{aligned}$$

By Theorem 6 (PV$\leftrightarrow$Weil exactness on the ν-odd sector), the transfer $\mathsf{W} : \mathrm{PV}_\nu^- \to \mathcal{W}_\nu^-$ intertwines T_ν and identifies the pairings:

$$\mathsf{W}(T_\nu f) = T_\nu(\mathsf{W} f), \qquad \langle \mathsf{W} f, \mathsf{W} g\rangle_{\mathrm{Weil}} = \langle f, g\rangle_{\mathrm{PV}}.$$

Transporting the PV isometry gives

$$\langle T_\nu F, T_\nu G\rangle_{\mathrm{Weil}} = \langle F, G\rangle_{\mathrm{Weil}} \quad (F = \mathsf{W} f,\ G = \mathsf{W} g).$$

No distribution of zeros and no GRH is used—only $|u_\pi(t)| = 1$, the evenness of $\mathrm{d}\mu_t^\pi$, and Theorem 6.

□

12 Details: cone density on the fixed–heat window

Weil/weighted Hilbert space and window norm. For $t \in [t_{\min}, t_0]$ set

$$\mathrm{d}\mu_t(x) := k_t(x)\, \mathrm{d}x, \qquad k_t(x) = \frac{1}{2\pi}\, e^{-x^2/(8t)}\, T_{1/2}(x)\ (> 0 \text{ on } [0, \infty)).$$

Write $L_t^2 := L^2\big((0, \infty), \mu_t\big)$ with norm $\|h\|_{2,t}^2 = \int_0^\infty |h(x)|^2\, \mathrm{d}\mu_t(x)$ and inner product $\langle h_1, h_2\rangle_{\mathrm{Weil},t} = \int_0^\infty h_1(x)\, \overline{h_2(x)}\, \mathrm{d}\mu_t(x)$ (the *Weil pairing* on the abelian line). On the test class we use the fixed–heat window seminorm

$$\|\Phi\|_{\mathrm{win}} \ := \sup_{t \in [t_{\min}, t_0]} M_t^+(\Phi), \qquad w_t(x) = e^{-x^2/(8t)}(1 + x)^{-2} \ \text{ as in Part I.}$$

By the same–weight factorization ((6.4)–(6.5)) one has a uniform domination $\widehat{k}_t(x) \ll_{m,[t_{\min}, t_0]} w_t(x)$; in particular $M_t^+(\cdot)$ is adapted simultaneously to zeros, primes and archimedean factors on the window.

Lemma 35 (PD$^+$–density in the Weil space)**.** *For each fixed $t \in [t_{\min}, t_0]$ the cone of abelian slices of positive–definite tests*

$$\mathcal{A}^+ \ := \ \big\{ \widehat{g}_\Phi :\ \Phi \in \mathsf{PD}_+ \big\} \ \subset \ L_t^2$$

is dense in the cone of nonnegative functions of L^2_t*. Consequently, the real linear span* $\mathrm{span}_{\mathbb{R}}(\mathcal{A}^+)$ *is dense in* L^2_t*. Equivalently: for every* $h \in L^2_t$ *and* $\varepsilon > 0$ *there exist* $\Phi_1, \Phi_2 \in \mathsf{PD}_+$ *with*

$$\big\| \, h - (\widehat{g}_{\Phi_1} - \widehat{g}_{\Phi_2}) \, \big\|_{2,t} < \varepsilon.$$

Moreover, the same statement holds uniformly on the window in the product topology induced by $\max_{t \in [t_{\min}, t_0]} \| \cdot \|_{2,t}$.

Proof. Fix t. Since k_t is smooth and strictly positive on $(0, \infty)$, nonnegative C_c^∞–functions are dense in the nonnegative cone of L^2_t. Let $b_\eta \geq 0$ be a fixed even C_c^∞ "bump" of mass 1 at scale $\eta > 0$, and approximate an arbitrary $h \geq 0$ by $h_\eta := \big(h \cdot \mathbf{1}_{[0,R]}\big) * b_\eta$ in $\| \cdot \|_{2,t}$ (choose $R \gg 1$ then $\eta \ll 1$). By the ν–slice frame (Theorem 3.1) there is a finite slice expansion $h_\eta(x) \approx \sum_{j=1}^J c_j \, \psi_j(x)$ with $c_j \geq 0$ and $\psi_j \geq 0$ supported near $x_j > 0$; the frame constants depend only on the structural data and are uniform in $t \in [t_{\min}, t_0]$. For each j choose a K–biinvariant bump $\varphi_{j,\eta}$ and set the positive–definite autocorrelation $\Phi_{j,\eta} := \varphi_{j,\eta} * \tilde{\varphi}_{j,\eta} \in \mathsf{PD}_+$. Then $\widehat{g}_{\Phi_{j,\eta}} \geq 0$ and, by the fixed–heat A3 bounds together with the same–weight factorization ((6.4)–(6.5)),

$$\big\| \, \psi_j - \widehat{g}_{\Phi_{j,\eta}} \, \big\|_{2,t} \ \ll_{m,[t_{\min},t_0]} \ \eta.$$

Summing with coefficients $c_j \geq 0$ gives a positive–definite $\Phi_\eta := \sum_{j=1}^J c_j \, \Phi_{j,\eta}$ with $\| \, h_\eta - \widehat{g}_{\Phi_\eta} \, \|_{2,t} \ll \eta$ and hence $\| \, h - \widehat{g}_{\Phi_\eta} \, \|_{2,t} < \varepsilon$ for η small. The window–uniform version follows by the compactness of $[t_{\min}, t_0]$ and the uniformity of the A3/weight constants. Density of the real span is obtained by Jordan decomposition $h = h_+ - h_-$ and applying the cone statement to $h_\pm$. □

Proposition 36 (Continuity of EF blocks in the window seminorm)**.** *Fix* $a > 1/2$ *and the heat window* $t \in [t_{\min}, t_0]$*. There exist explicit constants depending only on* a, m *and the window such that, for every admissible test* Φ *with abelian slice* $\widehat{g}_\Phi \in \mathcal{S}^2(a)$*, each block of the linear explicit formula is continuous with respect to* $\| \cdot \|_{\mathrm{win}}$*:*

1. ***Zero–side PV pairing.*** *Writing* $W^{(t)}_{\mathrm{zeros}}(\Phi)$ *for the zero–side PV functional (left side of (6.1)),*

$$\big| W^{(t)}_{\mathrm{zeros}}(\Phi) \big| \ \leq \ \Big(C_0(a,m) + C_1(m) \log Q_K(\pi) \Big) \, K_{\mathrm{A3}}(a, m, [t_{\min}, t_0]) \ \|\Phi\|_{\mathrm{win}},$$

   with $C_1(m) = \frac{3m+1}{2}$ *and an admissible* $C_0(a,m)$ *as in Theorem 5.1, and a computable* K_{A3} *coming from the weight domination (6.5) (A3).*

2. ***Archimedean integral.*** *With* $A_t(\Phi) = \int_{\mathbb{R}} \widehat{g}_\Phi(x) \, a_t(x) \, dx$ *as in (6.2) and* $a_t(x) \geq 0$ *(Theorem 5.8),*

$$|A_t(\Phi)| \ \leq \ \left(\sup_{t \in [t_{\min}, t_0]} \ \sup_{x \geq 0} \ \frac{a_t(x)}{w_t(x)} \right) \|\Phi\|_{\mathrm{win}} \ =: \ C_{\mathrm{Arch}} \ \|\Phi\|_{\mathrm{win}}.$$

3. ***Unramified prime block.*** *For the prime contribution* $\mathrm{Prime}(\Phi)$ *one has a uniform continuous budget bound*

$$\big| \mathrm{Prime}(\Phi) \big| \ \leq \ C_{\mathrm{P}}(a, m, [t_{\min}, t_0]) \ \|\Phi\|_{\mathrm{win}},$$

   where C_{P} *arises from the trace–formula bound on the continuous remainder (Theorem 5.9) together with (6.5); the discrete TF sum is nonnegative on* PD_+ *and harmless for continuity.*

4. ***Ramified block.*** *Uniformly in π and $t \in [t_{\min}, t_0]$,*

$$\left| W^{(t)}_{\mathrm{ram}}(\Phi; \pi) \right| \;\leq\; C_{m,t} \, (\log Q_K(\pi))^{\alpha} \, Q_K(\pi)^{-\beta t} \; \|\Phi\|_{\mathrm{win}},$$

with $\beta = \frac{1}{4m^2}$ and $C_{m,t}$ as in the Uniform Ramified Tail bound (R^*), Theorem 6.9.

In particular, every EF block is a bounded linear functional on $(\mathrm{BL} \cap \mathcal{S}^2(a), \| \cdot \|_{\mathrm{win}})$, with explicit norms $C_{\mathrm{Z}}, C_{\mathrm{Arch}}, C_{\mathrm{P}}, C_{\mathrm{R}}$ as above.

Proof. (1) is a direct consequence of Theorem 5.1 and the weight comparison (6.5), which transfers the $(\mathcal{S}^2(a))^*$–bound to $\| \cdot \|_{\mathrm{win}}$. (2) follows from (6.2), positivity of a_t (Theorem 5.8), and the definition of $\| \cdot \|_{\mathrm{win}}$. (3) is Theorem 5.9 combined with (6.5). (4) is precisely Theorem 6.9, after observing that $M_t^+(\Phi) \leq \|\Phi\|_{\mathrm{win}}$. $\square$

A (δ, ε) constructive approximant. Given $h \in L^2_t$, a tolerance $\varepsilon \in (0,1)$ and a packet scale $\delta \in (0,1)$, the following three–step scheme produces $\Phi_{\varepsilon,\delta} \in \mathsf{PD}_+$ with quantitative control.

1. **Truncate and smooth.** Choose $R = R(\varepsilon)$ so that $\|h - h \cdot \mathbf{1}_{[0,R]}\|_{2,t} < \varepsilon/3$ and set $h_\delta = (h \cdot \mathbf{1}_{[0,R]}) * b_\delta$ with b_δ a nonnegative bump at scale δ.
2. **Finite slice discretization (A1).** Use Theorem 3.1 to find $c_j \geq 0$ and centers $x_j \in (0, \infty)$ with $\left\| h_\delta - \sum_{j=1}^{J} c_j \, \psi_j \right\|_{2,t} < \varepsilon/3$.
3. **Lift to PD$^+$ heat–packets (A3).** For each j pick a spherical bump $\varphi_{j,\delta}$ and set $\Phi_{j,\delta} = \varphi_{j,\delta} * \tilde{\varphi}_{j,\delta}$, then $\Phi_{\varepsilon,\delta} := \sum_{j=1}^{J} c_j \, \Phi_{j,\delta}$. By (6.4)–(6.5), $\left\| \sum_j c_j \, \psi_j - \widehat{g}_{\Phi_{\varepsilon,\delta}} \right\|_{2,t} \ll \delta$ uniformly on $t \in [t_{\min}, t_0]$.

One–line error bound. With constants depending only on $(a, m, [t_{\min}, t_0])$,

$$\max_{t \in [t_{\min}, t_0]} \left\| h - \widehat{g}_{\Phi_{\varepsilon,\delta}} \right\|_{2,t} \;\leq\; \varepsilon + C_{\mathrm{A3}} \, \delta, \qquad \left| W^{(t)}_{\mathrm{ram}}(\Phi_{\varepsilon,\delta}; \pi) \right| \;\leq\; C_{m,t} \, (\log Q_K(\pi))^{\alpha} \, Q_K(\pi)^{-\beta t} \, \|\Phi_{\varepsilon,\delta}\|_{\mathrm{win}}$$

by Theorem 6.9.

13 A worked "test constructor" for an off–line zero

In this appendix we give an explicit, local constructor that produces a ledger packet whose ν–odd PV energy is strictly positive in the presence of an off–line zero. Throughout, let $\rho = \beta + i\gamma$ be a zero of $L(s, \pi)$ with $\beta \neq \frac{1}{2}$. Write $u := u_\pi(\gamma)$ for the unimodular, even factor from Theorem 4 and fix any $t \in [t_{\min}, t_0]$.

Local scalar action of R_t on narrow packets (A3 constants)

We isolate the "$R_t \approx$ scalar" estimate used in (20) with explicit constants from Part I (A3). Let k_t denote the even fixed–heat kernel on the ledger so that $R_t f = k_t * f$. By A3 there exist finite constants on the window $[t_{\min}, t_0]$:

$$K_\infty \;:=\; \sup_{t \in [t_{\min}, t_0]} \|k_t\|_{L^\infty}, \qquad L_* \;:=\; \sup_{t \in [t_{\min}, t_0]} \mathrm{Lip}(k_t), \qquad c_t \;:=\; k_t(0) \; (> 0 \text{ for each } t).$$

Lemma 37 (Local scalar action of R_t)**.** *Let $W \in C_c^\infty(\mathbb{R})$ be even, nonnegative, with $\int_{\mathbb{R}} W = 1$ and first moment $M_1(W) := \int_{\mathbb{R}} |u|\, W(u)\, du$. For $\delta \in (0, 1]$ and $x \in \mathbb{R}$ set $W_\delta(u) = \delta^{-1}W(u/\delta)$ and $e_{x,\delta}(t) = W_\delta(t - x)$. Then for every $t \in [t_{\min}, t_0]$,*

$$\|R_t e_{x,\delta} - c_t\, e_{x,\delta}\|_{L^2(\mathbb{R})} \;\le\; L_*\, M_1(W)\, \delta\; \|e_{x,\delta}\|_{L^2(\mathbb{R})}.$$

In particular, with $c_t > 0$ one has $R_t e_{x,\delta} = c_t e_{x,\delta} + E_{x,\delta}$ where $\|E_{x,\delta}\|_{L^2} \le C_t\, \delta \|e_{x,\delta}\|_{L^2}$ and $C_t \le L_* M_1(W)$ is uniform in x and $t \in [t_{\min}, t_0]$.

Proof. Write $(R_t e_{x,\delta})(t) = (k_t * W_\delta)(t - x)$ and define $f_t(z) := (k_t * W_\delta)(z)$. Then

$$f_t'(z) = \int_{\mathbb{R}} k_t'(z - u)\, W_\delta(u)\, du, \qquad |f_t'(z)| \le \mathrm{Lip}(k_t) \int_{\mathbb{R}} W_\delta(u)\, du \le L_*.$$

Hence $|f_t(z) - f_t(0)| \le L_*|z|$. Multiplying by $W_\delta(z)$ and using $c_t = f_t(0) = k_t(0)$, we obtain the pointwise bound

$$\big|(R_t e_{x,\delta})(x + z) - c_t\, e_{x,\delta}(x + z)\big| \;\le\; L_*\, |z|\, W_\delta(z).$$

Taking L^2 norm and changing variables $z = \delta u$ gives

$$\|R_t e_{x,\delta} - c_t e_{x,\delta}\|_{L^2} \;\le\; L_*\, \delta \left(\int_{\mathbb{R}} |u|^2 W(u)^2\, du\right)^{1/2} \;\le\; L_*\, M_1(W)\, \delta\; \|e_{x,\delta}\|_{L^2},$$

since $\|e_{x,\delta}\|_{L^2} = \delta^{-1/2}\|W\|_{L^2}$ and $M_1(W) \ge \|\, |u| W(u)\, \|_{L^2}/\|W\|_{L^2}$. This proves the claim. □

Derivation of the sharpened bound in Theorem 37 (Spectral interpretation). We interpret the localization on the spectral side (Fourier dual of the ledger). R_t acts as multiplication by $m_t(\omega) = \widehat{k_t}(\omega)$. The localized packet is $e_{x,\delta}(\omega) = W_\delta(\omega - x)$. The scalar c_t is the local value $m_t(x)$.

We aim to bound $\|D\|_{L^2} = \|(m_t(\omega) - m_t(x))W_\delta(\omega - x)\|_{L^2}$. We use Taylor expansion of $m_t(\omega)$ around x. Let $L_1 = \sup_{t,\omega} |m_t'(\omega)|$ and $L_2 = \sup_{t,\omega} |m_t''(\omega)|$. (These relate to the decay of k_t and its moments).

$$m_t(\omega) - m_t(x) = m_t'(x)(\omega - x) + R_1(\omega - x), \quad |R_1(u)| \le \tfrac{1}{2} L_2 u^2.$$

Let $u = \omega - x$. The difference is $D(u + x) = (m_t'(x)u + R_1(u))W_\delta(u)$. We bound the L^2 norm squared using $(a + b)^2 \le 2a^2 + 2b^2$:

$$\begin{aligned}\|D\|_{L^2}^2 &\le 2 \int (m_t'(x)u)^2 W_\delta(u)^2 du + 2 \int R_1(u)^2 W_\delta(u)^2 du \\ &\le 2L_1^2 \int u^2 W_\delta(u)^2 du + \tfrac{1}{2} L_2^2 \int u^4 W_\delta(u)^2 du.\end{aligned}$$

We change variables $u = \delta v$. $W_\delta(u) = \delta^{-1}W(v)$.

$$\int u^k W_\delta(u)^2 du = \delta^{k-1} \int v^k W(v)^2 dv.$$

$\|e_{x,\delta}\|_{L^2}^2 = \delta^{-1}\|W\|_{L^2}^2$. The relative error squared is:

$$\frac{\|D\|_{L^2}^2}{\|e_{x,\delta}\|_{L^2}^2} \le \delta^2 \left(2L_1^2 \frac{\int v^2 W(v)^2 dv}{\|W\|_{L^2}^2} + \tfrac{1}{2} L_2^2 \delta^2 \frac{\int v^4 W(v)^2 dv}{\|W\|_{L^2}^2}\right).$$

Let $\kappa_k(W) = \int v^k W(v)^2 dv / \|W\|_{L^2}^2$.

$$\frac{\|D\|_{L^2}}{\|e_{x,\delta}\|_{L^2}} \le \delta \sqrt{2L_1^2 \kappa_2(W) + \tfrac{1}{2} L_2^2 \delta^2 \kappa_4(W)}.$$

Using $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$:

$$\|R_t e_{x,\delta} - m_t(x) e_{x,\delta}\|_{L^2} \le \delta \left(\sqrt{2} L_1 \sqrt{\kappa_2(W)} + \tfrac{1}{\sqrt{2}} L_2 \sqrt{\kappa_4(W)}\, \delta \right) \|e_{x,\delta}\|_{L^2}.$$

This confirms the $O(\delta) + O(\delta^2)$ structure. The constants L_1, L_2 correspond to the smoothness properties (L_* and K_∞ constraints on the derivatives of m_t). $\square$

Step 0: a base window and its rescales

Fix an even, nonnegative $W \in C_c^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} W = 1$, and for $\delta > 0$ write

$$W_\delta(t) \;:=\; \delta^{-1} W(t/\delta), \qquad e_{\gamma,\delta}(t) := W_\delta(t-\gamma), \qquad e_{-\gamma,\delta}(t) := W_\delta(t+\gamma).$$

For $\delta \in (0, \delta_0)$ small enough (with δ_0 depending on the minimal spacing of zeros around γ), the two lumps $e_{\pm\gamma,\delta}$ have essentially disjoint support and

$$R_t e_{\pm\gamma,\delta} \;=\; c_t\, e_{\pm\gamma,\delta} \;+\; O(\delta) \quad \text{in the ledger } L^2 \text{ sense,} \tag{20}$$

with $c_t > 0$ depending only on the fixed–heat window; see Part I (A3) and the evenness of the heat kernel.

Lemma 38 (Localized action of $\mathbb{T}_{\nu,\pi}$)**.** *With $u := u_\pi(\gamma)$ as above, for $\delta \in (0,\delta_0)$ small,*

$$\mathbb{T}_{\nu,\pi} e_{\gamma,\delta} \;=\; u\, e_{-\gamma,\delta} \;+\; O(\delta), \qquad \mathbb{T}_{\nu,\pi} e_{-\gamma,\delta} \;=\; u\, e_{\gamma,\delta} \;+\; O(\delta),$$

where $O(\delta)$ is in the ledger L^2 norm and uniform for $t \in [t_{\min}, t_0]$.

Proof. We analyze the action of $\mathbb{T}_{\nu,\pi}$ on $e_{\gamma,\delta}$. By definition, $(\mathbb{T}_{\nu,\pi} f)(t) = u_\pi(t) f(-t)$.

$$(\mathbb{T}_{\nu,\pi} e_{\gamma,\delta})(t) = u_\pi(t) W_\delta(-t-\gamma) = u_\pi(t) W_\delta(t+\gamma),$$

since W_δ is even. We compare this to $u e_{-\gamma,\delta}(t) = u_\pi(\gamma) W_\delta(t+\gamma)$. The difference is $D(t) = (u_\pi(t) - u_\pi(\gamma)) W_\delta(t+\gamma)$.

Since $u_\pi(t)$ is smooth (derived from archimedean factors), it is locally Lipschitz. Let L_u be a uniform Lipschitz constant on the relevant spectral range. We use the fact that u_π is even (Remark 3), so $u_\pi(\gamma) = u_\pi(-\gamma) = u$.

The function $W_\delta(t+\gamma)$ is localized near $t = -\gamma$. For t in this neighborhood, we estimate the difference: $|u_\pi(t) - u| = |u_\pi(t) - u_\pi(-\gamma)| \le L_u |t - (-\gamma)| = L_u |t+\gamma|$.

Let C_W be the support radius of W. If $D(t) \ne 0$, then $|t+\gamma| \le C_W \delta$. Thus, $|D(t)| \le L_u (C_W \delta) W_\delta(t+\gamma)$.

We compute the L^2 norm of the difference:

$$\|\mathbb{T}_{\nu,\pi} e_{\gamma,\delta} - u e_{-\gamma,\delta}\|_{L^2}^2 = \int |D(t)|^2 dt \le (L_u C_W \delta)^2 \int W_\delta(t+\gamma)^2 dt.$$

Since $\int W_\delta(t+\gamma)^2 dt = \|e_{-\gamma,\delta}\|_{L^2}^2$, we have

$$\|\mathbb{T}_{\nu,\pi} e_{\gamma,\delta} - u e_{-\gamma,\delta}\|_{L^2} \le (L_u C_W) \delta \|e_{-\gamma,\delta}\|_{L^2} = O(\delta).$$

The argument for $\mathbb{T}_{\nu,\pi} e_{-\gamma,\delta}$ is analogous. $\square$

Step 1: a two–lump profile and its ν–odd projection

Let $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = |\beta| = 1$, and set the two–lump profile

$$H_{\delta;\alpha,\beta}(t) \;:=\; \alpha\, e_{\gamma,\delta}(t) + \beta\, e_{-\gamma,\delta}(t).$$

Define the odd projection $P_- := \frac{1}{2}(\mathrm{Id} - \mathbb{T}_{\nu,\pi})$, so that $\widehat{\Phi}_{\mathrm{asym}} = P_-\, \widehat{\Phi}$. By Theorem 38,

$$\begin{aligned} P_-\, H_{\delta;\alpha,\beta} &= \tfrac{1}{2}\Big(\alpha e_{\gamma,\delta} + \beta e_{-\gamma,\delta} - \alpha \mathbb{T}_{\nu,\pi} e_{\gamma,\delta} - \beta \mathbb{T}_{\nu,\pi} e_{-\gamma,\delta}\Big) \\ &= \tfrac{1}{2}\Big(\alpha e_{\gamma,\delta} + \beta e_{-\gamma,\delta} - \alpha u\, e_{-\gamma,\delta} - \beta u\, e_{\gamma,\delta}\Big) \;+\; O(\delta) \\ &= \tfrac{1}{2}\Big((\alpha - \beta u)\, e_{\gamma,\delta} + (\beta - \alpha u)\, e_{-\gamma,\delta}\Big) \;+\; O(\delta). \end{aligned} \tag{21}$$

Step 2: a phase choice that guarantees a nontrivial odd component

Choose the coefficients

$$\alpha \;:=\; 1, \qquad \beta \;:=\; -u.$$

With this choice, (21) gives

$$P_-\, H_{\delta;1,-u} \;=\; \tfrac{1}{2}\big((1+u^2)\, e_{\gamma,\delta} - 2u\, e_{-\gamma,\delta}\big) \;+\; O(\delta). \tag{22}$$

Since $u \in \mathbb{C}$ is unimodular, the pair of coefficients $1+u^2$ and u cannot vanish simultaneously; hence the odd projection has at least one nonzero lump uniformly in δ.

Step 3: positive PV energy from a single off–line zero

Define the PV energy functional

$$\mathcal{B}_t(F) \;:=\; \langle F,\, R_t F\rangle_{\mathrm{PV}}\,.$$

Using (20) and the disjoint support of $e_{\pm\gamma,\delta}$, we obtain

$$\mathcal{B}_t\big(P_- H_{\delta;1,-u}\big) = \tfrac{1}{4}\,\Big(|1+u^2|^2\; \langle e_{\gamma,\delta},\, R_t e_{\gamma,\delta}\rangle_{\mathrm{PV}} + 4|u|^2\; \langle e_{-\gamma,\delta},\, R_t e_{-\gamma,\delta}\rangle_{\mathrm{PV}}\Big) \;+\; O(\delta). \tag{23}$$

Now, because there is a zero at $\rho = \beta + i\gamma$ with $\beta \neq \frac{1}{2}$, the regularized zero–measure μ_t^π puts positive mass in any sufficiently small neighborhood of γ *that does not cancel under the ν–odd projection*; in particular, for δ small enough,

$$\langle e_{\gamma,\delta},\, R_t e_{\gamma,\delta}\rangle_{\mathrm{PV}} \;\geq\; c_t\, \mu_t^\pi\big([\gamma-\delta, \gamma+\delta]\big) \;-\; O(\delta) \;\geq\; c_*(\pi,t,\rho) \;-\; O(\delta), \tag{24}$$

with $c_t > 0$ from (20) and $c_*(\pi,t,\rho) > 0$ independent of δ. An analogous statement holds at $-\gamma$. Combining (23)–(24) and $|u| = 1$ yields

$$\mathcal{B}_t\big(P_- H_{\delta;1,-u}\big) \;\geq\; \frac{1}{4}\, \min\big\{|1+u^2|^2,\, 4\big\}\, \Big(c_*(\pi,t,\rho) - O(\delta)\Big). \tag{25}$$

Hence there exists $\delta_1 = \delta_1(\pi,t,\rho) > 0$ such that for all $0 < \delta < \delta_1$,

$$\mathcal{B}_t\big(P_- H_{\delta;1,-u}\big) \;>\; 0.$$

Step 4: lifting to PD_+ via cone density

By Theorem 2.14 there is $\Phi \in \mathsf{PD}_+$ with $\|\widehat{\Phi} - H_{\delta;1,-u}\|_{M_t} < \varepsilon$ (uniformly in $t \in [t_{\min}, t_0]$). Continuity of the PV pairing under M_t–control (Part I, A3) implies

$$Q_t(\Phi) \;=\; \Big\langle \widehat{\Phi}_{\text{asym}},\, R_t\, \widehat{\Phi}_{\text{asym}} \Big\rangle_{\text{PV}} \;=\; \mathcal{B}_t\big(P_- H_{\delta;1,-u}\big) \;+\; O(\varepsilon) + O(\delta).$$

Taking $\varepsilon \ll \delta \ll 1$ we obtain $Q_t(\Phi) > 0$, completing the construction.

Remark 39 (Deterministic recipe)**.** Given $\rho = \beta + i\gamma$ with $\beta \neq \frac{1}{2}$:

1. pick small $\delta > 0$, set $e_{\pm\gamma,\delta}$ from the base window W;
2. compute $u := u_\pi(\gamma)$ (unimodular);
3. form $H_{\delta;1,-u} = \; e_{\gamma,\delta} - u\, e_{-\gamma,\delta}$;
4. choose $\Phi \in \mathsf{PD}_+$ with $\widehat{\Phi} \approx H_{\delta;1,-u}$ by Theorem 2.14.

Then for some (equivalently, all) $t \in [t_{\min}, t_0]$ close enough to the fixed–heat scale used in Part I, one has $Q_t(\Phi) > 0$.

14 Normalization table

Plancherel factors, Mellin/Harish–Chandra conventions, ν–action, and heat normalization compatible with Eq. (EF-linear) in Part I.

Lemma 40 (Density of the PD_+ image)**.** *Let $(\mathcal{H}_a, \|\cdot\|_{\mathcal{S}^2(a)})$ be the weighted Weil/Archimedean–Paley–Wiener Hilbert space used throughout, with norm*

$$\|h\|^2_{\mathcal{S}^2(a)} = \int_{\mathbb{R}} |h(\xi)|^2 \, \mathrm{d}\mu_{\infty,a}(\xi) \;+\; \sum_p \sum_{m\ge1} w_a(p^m)\, \Big(|h(m\log p)|^2 + |h(-m\log p)|^2\Big),$$

where the measure $\mu_{\infty,a}$ and weights $w_a(p^m) > 0$ satisfy (A3) *(Part I). For $\Phi \in \mathcal{S}(\mathbb{R})$ denote by g_Φ the EF test function produced by the standard synthesis (A1–A2), and by $\widehat{g}_\Phi$ its spectral profile. If $\Phi \in \mathsf{PD}_+$ (even, positive–definite), then $\widehat{g}_\Phi \in \mathcal{H}_a$ and $\widehat{g}_\Phi \ge 0$.*

Then:

(i) For every even $h \in C_c^\infty(\mathbb{R})$ with $h \ge 0$ and every $\varepsilon > 0$ there exists $\Phi \in \mathsf{PD}_+$ such that

$$\|\widehat{g}_\Phi - h\|_{\mathcal{S}^2(a)} \le \varepsilon.$$

In particular, the cone $\{\widehat{g}_\Phi : \Phi \in \mathsf{PD}_+\}$ is dense in the positive cone of $\mathcal{H}_a$.

(ii) The closed linear *span of $\{\widehat{g}_\Phi : \Phi \in \mathsf{PD}_+\}$ equals $\mathcal{H}_a$. Equivalently, for any even $h \in C_c^\infty(\mathbb{R})$ there exist $\Phi^+, \Phi^- \in \mathsf{PD}_+$ with*

$$\big\|((\widehat{g}_{\Phi^+} - \widehat{g}_{\Phi^-}) - h\big\|_{\mathcal{S}^2(a)} \le \varepsilon.$$

Proof. Step 1 (Heat packets and archimedean multiplier). For $\eta \in (0,1]$ and $\xi_0 \in \mathbb{R}$ define the even, positive–definite "heat packet"

$$\Phi_{\eta,\xi_0}(x) := e^{-\pi\eta x^2} \cos(2\pi\xi_0 x).$$

By Bochner's theorem, $\Phi_{\eta,\xi_0} \in \mathsf{PD}_+$. The EF synthesis (A1–A2) together with archimedean regularity (A.R) / prop. Rast yields a smooth even multiplier $m_a(\xi)$ such that

$$\widehat{g}_{\Phi_{\eta,\xi_0}}(\xi) = m_a(\xi)\, e^{-\pi(\xi-\xi_0)^2/\eta}. \tag{26}$$

On each compact window $[-R,R]$ there are constants $0 < c_{a,R} \le m_a(\xi) \le C_{a,R} < \infty$.

Step 2 (Approximate identity in $\mathcal{S}^2(a)$). Let $\phi_\eta(\xi) := e^{-\pi\eta^{-1}\xi^2}$, so $\phi_\eta = \widehat{\Phi_{\eta,0}}$. Assumption (A3) implies the Gaussian multipliers form a contractive approximate identity on $(\mathcal{H}_a, \|\cdot\|_{\mathcal{S}^2(a)})$:

$$\|h - (h * \phi_\eta)\|_{\mathcal{S}^2(a)} \xrightarrow[\eta\downarrow 0]{} 0 \qquad (h \in \mathcal{H}_a). \tag{27}$$

Fix $\eta > 0$ so that $\|h - h_\eta\|_{\mathcal{S}^2(a)} \le \varepsilon/3$, where $h_\eta := h * \phi_\eta$.

Step 3 (Positive synthesis on a compact spectral window). Choose $R > 0$ with $\|h_\eta \cdot \mathbf{1}_{|\xi|>R}\|_{\mathcal{S}^2(a)} \le \varepsilon/3$ (possible by (A3)). On the grid $\Lambda_{R,\delta} := \{-R, -R+\delta, \dots, R\}$ set nonnegative coefficients

$$c_j := \frac{\delta}{m_a(\xi_j)}\, h_\eta(\xi_j)\ \ge 0, \qquad \xi_j \in \Lambda_{R,\delta},$$

and define the finite positive combination

$$H_{\eta,\delta,R}(\xi) := \sum_{\xi_j \in \Lambda_{R,\delta}} c_j\, \widehat{g}_{\Phi_{\eta,\xi_j}}(\xi) = \sum_{\xi_j \in \Lambda_{R,\delta}} \delta\, h_\eta(\xi_j)\, e^{-\pi(\xi-\xi_j)^2/\eta}.$$

Standard Gaussian quadrature shows $H_{\eta,\delta,R} \to h_\eta$ in $L^2([-R,R], \mathrm{d}\mu_{\infty,a})$ as $\delta \downarrow 0$; the bounds on m_a transfer this to the same window for $H_{\eta,\delta,R} - h_\eta$. For the discrete part of $\|\cdot\|_{\mathcal{S}^2(a)}$, (A3) makes the tail beyond $|\xi| > R$ arbitrarily small and there are only finitely many interior samples $\xi = \pm m \log p$ to control, which are uniformly approximated by the grid as $\delta \downarrow 0$. Hence

$$\|H_{\eta,\delta,R} - h_\eta\|_{\mathcal{S}^2(a)} \le \varepsilon/3 \qquad (\delta \text{ small}). \tag{28}$$

Step 4 (Conclusion and linear span). Let $\Phi := \sum_{\xi_j \in \Lambda_{R,\delta}} c_j\, \Phi_{\eta,\xi_j} \in \mathsf{PD}_+$. Then $\widehat{g}_\Phi = H_{\eta,\delta,R}$, and by (27) and (28),

$$\|\widehat{g}_\Phi - h\|_{\mathcal{S}^2(a)} \le \|H_{\eta,\delta,R} - h_\eta\|_{\mathcal{S}^2(a)} + \|h_\eta - h\|_{\mathcal{S}^2(a)} \le \varepsilon.$$

This proves (i). For (ii) write $h = h_+ - h_-$ with $h_\pm = \frac{1}{2}(|h| \pm h) \in C_c^\infty$, $h_\pm \ge 0$, and approximate each $h_\pm$ by $\widehat{g}_{\Phi^\pm}$ as in (i). □

Proposition 41 (Continuity of EF linear functionals)**.** *On $(\mathcal{H}_a, \|\cdot\|_{\mathcal{S}^2(a)})$ the linear functionals used in the explicit formula extend continuously with the following bounds (constants depend only on a and the data in* (A3) *and* (A.R) */ prop. Rast):*

(a) Zero-side PV pairing. *With the symmetric zero-counting measure* $\mathrm{d}N_a$*,*

$$\mathcal{L}_{\mathrm{zero}}(h) := \mathrm{PV}\int_{\mathbb{R}} h(\xi)\, \mathrm{d}N_a(\xi), \qquad |\mathcal{L}_{\mathrm{zero}}(h)| \le C_{\mathrm{zero}}(a)^{1/2}\, \|h\|_{\mathcal{S}^2(a)},$$

since (A3) gives $\int_{\mathbb{R}}(1+\xi^2)^{-1}\, \mathrm{d}N_a(\xi) \le C_{\mathrm{zero}}(a)$.

(b) Archimedean integral. *With kernel* $K_\infty(\xi)$,

$$\mathcal{L}_\infty(h) := \int_{\mathbb{R}} h(\xi)\, K_\infty(\xi)\, \mathrm{d}\xi, \qquad |\mathcal{L}_\infty(h)| \le C_\infty(a)\, \|h\|_{\mathcal{S}^2(a)},$$

because $K_\infty/m_a \in L^2(\mathrm{d}\mu_{\infty,a})$ *by (A.R) / prop.* ***Rast.***

(c) Prime block (unramified).

$$\mathcal{L}_{\mathrm{prime}}(h) := 2\ \mathrm{Re} \sum_{p \notin S} \sum_{m \ge 1} \frac{A(p^m)}{m\, p^{m/2}}\, e^{-ma\log p}\, h(m \log p),$$

and (A3) implies

$$|\mathcal{L}_{\mathrm{prime}}(h)| \le \left(\sum_{p \notin S} \sum_{m \ge 1} \frac{|A(p^m)|^2}{m^2 p^m}\, e^{-2ma\log p}\, w_a(p^m)^{-1} \right)^{1/2} \|h\|_{\mathcal{S}^2(a)}.$$

(d) Ramified block. *The finite sum over* $p \in S$ *satisfies*

$$|\mathcal{L}_{\mathrm{ram}}(h)| \le \left(\sum_{p \in S} \sum_{m \ge 1} \frac{|A(p^m)|^2}{m^2 p^m}\, e^{-2ma\log p}\, w_a(p^m)^{-1} \right)^{1/2} \|h\|_{\mathcal{S}^2(a)}.$$

Consequently, all cone–separation steps applied to these functionals are valid in the $\mathcal{S}^2(a)$ *topology.*

Constructive approximation recipe (two steps). Given even $h \in C_c^\infty(\mathbb{R})$ with $h \ge 0$ and a tolerance $\varepsilon > 0$:

1. *Heat regularization.* Pick $\eta \in (0,1]$ so that $\|h - (h * \phi_\eta)\|_{\mathcal{S}^2(a)} \le \varepsilon/2$, with $\phi_\eta(\xi) = e^{-\pi\eta^{-1}\xi^2}$.
2. *Positive Gaussian synthesis.* Choose $R > 0$ so that $\|(h * \phi_\eta)\, \mathbf{1}_{|\xi|>R}\|_{\mathcal{S}^2(a)} \le \varepsilon/4$. On the grid $\Lambda_{R,\delta}$ set

$$c_j := \frac{\delta}{m_a(\xi_j)}\, (h * \phi_\eta)(\xi_j) \ge 0, \qquad \Phi := \sum_{\xi_j \in \Lambda_{R,\delta}} c_j\, \Phi_{\eta,\xi_j} \in \mathsf{PD}_+.$$

   For δ sufficiently small (Gaussian quadrature on $[-R,R]$), $\left\|\widehat{g}_\Phi - (h * \phi_\eta)\right\|_{\mathcal{S}^2(a)} \le \varepsilon/4$, hence $\|\widehat{g}_\Phi - h\|_{\mathcal{S}^2(a)} \le \varepsilon$.

Practical stopping criteria. It suffices to enforce

$$\|h - h * \phi_\eta\|_{\mathcal{S}^2(a)} \le \varepsilon/2, \quad \|(h * \phi_\eta)\mathbf{1}_{|\xi|>R}\|_{\mathcal{S}^2(a)} \le \varepsilon/4, \quad \delta \le \frac{\varepsilon}{4\, L_{a,R}\, \sqrt{\mu_{\infty,a}([-R-1, R+1])}},$$

where $L_{a,R}$ is a Lipschitz constant of $h * \phi_\eta$ on $[-R-1, R+1]$ and $\mu_{\infty,a}([-R-1,R+1])$ is the window mass.

A Non-circularity audit

We list external inputs used in this paper; each is unconditional and independent of RH/GRH.

- Harish–Chandra Plancherel theorem (real reductive groups): spectral decomposition for $L^2(G)$; unconditional; independent of RH/GRH.
- Casselman newvectors (nonarchimedean GL_n): existence/uniqueness of newvectors and explicit models; unconditional; independent of RH/GRH.
- Bushnell–Henniart local bounds: conductor and ϵ-factor size/normalization controls; unconditional; independent of RH/GRH.
- Moy–Prasad depth theory: group/representation filtrations and depth; unconditional; independent of RH/GRH.
- Stirling's formulae for Γ and ψ: archimedean asymptotics for gamma and digamma factors; unconditional; independent of RH/GRH.
- Rankin–Selberg local zeta integrals: convergence/meromorphic continuation and local functional equations; unconditional; independent of RH/GRH.
- Tate's thesis (local functional equation): normalization of local L-, ϵ- and γ-factors; unconditional; independent of RH/GRH.

References

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Corollary 42 (Annihilation Threshold)**.** *Let $\eta \in (0,1)$ be the A5 spectral gap (Theorem 24). If $\varepsilon \in (0,1)$ satisfies*

$$\sqrt{1-\eta} + \varepsilon \ < \ 1,$$

then the ν–odd energy annihilates for the corresponding amplifier choice; i.e., $\mathcal{Q}(\widehat{\Phi}_{\mathrm{asym}}) = 0$ for all Φ covered by the approximate invariance bound.

Lemma 43 (Witness Lower Bound)**.** *Let $\Phi_{\varepsilon,t} \in \overline{\mathsf{PD}_+}$ be the odd heat packet used in the witness construction with parameter $\varepsilon > 0$ and time $t \in [t_{\min}, t_0]$. If a zero of $L(s,\pi)$ lies in the forbidden box at distance $\delta > 0$ from the central line, then*

$$\mathrm{Zero\_contrib}(\Phi_{\varepsilon,t}) \ \geq \ c_1\, \delta^2,$$

with an explicit constant $c_1 = c_1(m, K, \varepsilon, t) > 0$ given in the construction. (For other admissible test families, replace 2 by the corresponding exponent γ.)

Theorem 44 (Main Unconditional Implication)**.** *If the ν–odd Weil energy functional annihilates for all $\Phi \in \overline{\mathsf{PD}_+}$, then $\mathrm{GRH}(\pi)$ holds. Moreover, it suffices to verify annihilation on a finite ε–net of $\overline{\mathsf{PD}_+}$ with a certified modulus of continuity bound.*