HOLONOMY UNIVERSALITY BEYOND THE DISK: RIEMANN MAPS, DOMAINS, AND HARDY/SMIRNOV CLASSES ON A SINGLE HEISENBERG SLICE

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Abstract. We demonstrate the universality of the Heisenberg slice $H_1 \cong S^1 \times \mathbb{R}$ as the native space for one-variable complex analysis, extending the holonomy realization from the unit circle to arbitrary Jordan domains and annuli. The geometry of any domain $\Omega \subset \mathbb{C}$ (itself a Heisenberg shadow) is encoded entirely within weights on this universal space. Let $\Phi : \mathbb{D} \to \Omega$ be a Riemann map. Pulling back contour integrals on $\partial\Omega$ converts all Cauchy, Laurent, and residue functionals into holonomy integrals on H_1 against the fixed kernel $\alpha_H = d\theta$ with conformal weight

$$\boxed{\mathcal{W}_{\Omega,z_0,n}(w) \;=\; \frac{w\,\Phi'(w)}{\big(\Phi(w)-z_0\big)^{n+1}}}.$$

Clarification. The identities below are classical consequences of the conformal change of variables $z = \Phi(w)$; the role of the Heisenberg slice is to provide a uniform representation on the fixed base H_1, with all domain dependence encoded in the conformal weight $\mathcal{W}_{\Omega,z_0,n}$. Thus for f holomorphic near $\overline{\Omega}$,

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi}\int_0^{2\pi} f\big(\Phi(\mathrm{e}^{J\theta})\big)\,\frac{\mathrm{e}^{J\theta}\,\Phi'(\mathrm{e}^{J\theta})}{\big(\Phi(\mathrm{e}^{J\theta})-z_0\big)^{n+1}}\;\alpha_H,$$

and for meromorphic g with no poles on $\partial\Omega$,

$$\frac{1}{2\pi J}\oint_{\partial\Omega} g(z)\,dz = \frac{1}{2\pi}\int_0^{2\pi}\big(\mathrm{e}^{J\theta}\,\Phi'(\mathrm{e}^{J\theta})\big)\,g\big(\Phi(\mathrm{e}^{J\theta})\big)\;\alpha_H = \sum_{a\in\Omega}\operatorname*{Res}_{z=a} g.$$

We treat annuli and multiply connected domains via oriented sums of boundary components (each pulled back to S^1), discuss Hardy/Smirnov classes on Ω by pullback from $H^p(\mathbb{D})$, and formulate residues as *vertical holonomy* of a weighted connection. All domain geometry is absorbed into explicit weights; the kernel and slice are unchanged.

Contents

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1. Introduction

Context from Paper I. Paper I established the kernel identity on the unit circle

$$\alpha_C := \frac{dz}{z} = J\,\alpha_H \quad \text{on } |z| = 1, \tag{1}$$

with $\alpha_H = d\theta$ along $H_1 \cong S^1 \times \mathbb{R}$. This revealed that complex analysis on the disk is natively the geometry of holonomy on H_1, and that the complex plane $\mathbb{C}$ is a projection, or Heisenberg shadow, of this structure.

Aim of this paper. We show that the *entire one-variable analytic functional calculus*, regardless of domain complexity, lives universally on the same H_1 slice. The act of changing domains (e.g., via a Riemann map) is merely the management of *conformal weights* on this fixed space.

Main claims. For a bounded simply connected $C^{1,\alpha}$ Jordan domain $(0 < \alpha \le 1)$ Ω and a boundary-extendable Riemann map $\Phi : \mathbb{D} \to \Omega$:

- Cauchy evaluation/derivatives at $z_0 \in \Omega$ become holonomy integrals on H_1 with weight $\mathcal{W}_{\Omega,z_0,n}$.
- Residues on Ω correspond to vertical holonomy of the pulled-back connection $dt + \big(w\,\Phi'(w)\,g(\Phi(w))\big)\,\alpha_H$.
- Hardy and Smirnov classes on Ω are the pullback of $H^p(\mathbb{D})$ and $E^p(\mathbb{D})$ via Φ, with norms realized on S^1 against α_H.
- Annuli and multiply connected domains reduce to sums over boundary components, each handled by the same S^1-holonomy with the appropriate conformal weight and orientation.

2. Preliminaries

2.1. **Heisenberg slice and circle parameter.**

We retain $H_1 \cong S^1 \times \mathbb{R}$, $\alpha_H = d\theta$, and $w = \mathrm{e}^{J\theta}$ from Paper I. Along $|w| = 1$,

$$dw = J\,w\,d\theta = J\,w\,\alpha_H.$$

2.2. **Domains and Riemann maps.**

Let $\Omega \subset \mathbb{C}$ be a bounded simply connected Jordan domain with $C^{1,\alpha}$ boundary $(0 < \alpha \le 1)$. By Carathéodory's theorem, any conformal map $\Phi : \mathbb{D} \to \Omega$ extends to a homeomorphism $\Phi : \overline{\mathbb{D}} \to \overline{\Omega}$; Φ' extends continuously and is nonvanishing on S^1.

We parametrize $\partial\Omega$ by

$$z(\theta) = \Phi(w(\theta)), \quad w(\theta) = \mathrm{e}^{J\theta}, \quad 0 \le \theta < 2\pi.$$

Then

$$dz = \Phi'(w)\,dw = \Phi'(w)\,J\,w\,\alpha_H.$$

Remark 2.1 (Boundary regularity)**.** For a bounded simply connected domain Ω with Jordan boundary:

- If $\partial\Omega$ is C^1, then Φ' extends continuously and is nonvanishing on S^1; all identities here are classical line integrals.
- If $\partial\Omega$ is rectifiable (finite length), then Φ extends continuously to S^1, non-tangential boundary values of Φ' exist a.e., and the weights $\mathcal{W}_{\Omega,z_0,n} \in L^1(S^1)$; the resulting H_1 integrals are interpreted as a.e. boundary limits/principal values and agree with arc-length integrals on $\partial\Omega$.
- If $\partial\Omega$ has corners (piecewise C^1), then near a corner of interior angle $\beta\pi$ one has $\Phi'(e^{J\theta}) \asymp |\theta - \theta_0|^{\beta-1}$; the weights may develop power singularities but remain integrable for $\beta > 0$, with principal-value interpretations when needed.

See, e.g., Pommerenke, *Boundary Behaviour of Conformal Maps*; Duren, *Theory of H^p Spaces*; and Garnett–Marshall, *Harmonic Measure*, for precise statements.

Remark 2.2 (Orientation)**.** The parameter θ increases counterclockwise on S^1; under Φ this induces the positive orientation on $\partial\Omega$.

3. Conformal pullback of Cauchy kernels

Lemma 3.1 (Pulled-back Cauchy kernel)**.** *Fix $z_0 \in \Omega$. For $n \geq 0$ and f holomorphic on a neighborhood of $\overline{\Omega}$,*

$$\frac{1}{2\pi J}\oint_{\partial\Omega} \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{1}{2\pi}\int_0^{2\pi} f\big(\Phi(\mathrm{e}^{J\theta})\big)\,\underbrace{\frac{\mathrm{e}^{J\theta}\,\Phi'(\mathrm{e}^{J\theta})}{\big(\Phi(\mathrm{e}^{J\theta})-z_0\big)^{n+1}}}_{\mathcal{W}_{\Omega,z_0,n}(\mathrm{e}^{J\theta})}\ \alpha_H.$$

Proof. Substitute $z = \Phi(w)$ and $dz = \Phi'(w)\,dw = \Phi'(w)\,J\,w\,\alpha_H$; divide by $2\pi J$ to cancel J. □

Theorem 3.2 (Evaluation and derivatives on Ω via H_1)**.** *Under the hypotheses of Lemma 3.1,*

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi}\int_0^{2\pi} f\big(\Phi(\mathrm{e}^{J\theta})\big)\,\mathcal{W}_{\Omega,z_0,n}(\mathrm{e}^{J\theta})\ \alpha_H.$$

Remark 3.3 (Choice of basepoint)**.** If $\Phi(0) = z_0$, the weight simplifies to $\mathcal{W}_{\Omega,\Phi(0),n}(w) = w\,\Phi'(w)/(\Phi(w)-\Phi(0))^{n+1}$. Different choices of z_0 correspond to precomposing Φ with a disk automorphism sending 0 to $w_0 = \Phi^{-1}(z_0)$.

4. Residues as vertical holonomy on $\partial\Omega$

Theorem 4.1 (Residue/holonomy on general domains)**.** *Let g be meromorphic on an open set containing $\overline{\Omega}$ with no pole on $\partial\Omega$. Then*

$$\frac{1}{2\pi J}\oint_{\partial\Omega} g(z)\,dz = \frac{1}{2\pi}\int_0^{2\pi} \big(\mathrm{e}^{J\theta}\,\Phi'(\mathrm{e}^{J\theta})\big)\,g\big(\Phi(\mathrm{e}^{J\theta})\big)\ \alpha_H = \sum_{a\in\Omega}\operatorname*{Res}_{z=a} g.$$

Equivalently, the connection

$$A_{\Omega,g} \;=\; dt + \big(\,w\,\Phi'(w)\,g(\Phi(w))\,\big)\,\alpha_H$$

Using $dz = \Phi'(w)\,J\,w\,\alpha_H$ on $|w| = 1$, this connection can equivalently be written intrinsically *along $\partial\Omega$ as*

$$A_g \;=\; dt + \frac{1}{J}\,g(z)\,dz,$$

which is independent of the choice of Riemann map Φ. has vertical holonomy

$$\Delta t \;=\; -\int_0^{2\pi} \big(\,w\,\Phi'(w)\,g(\Phi(w))\,\big)\,\alpha_H \;=\; -\,2\pi\sum_{a\in\Omega}\operatorname*{Res}_{z=a} g.$$

Proof. Pull back dz as above and apply the residue theorem. □

Remark 4.2 (The $w\,\Phi'(w)$ factor)**.** As in Paper I where $dz = J\,z\,\alpha_H$ produced the z-factor, here $dz = \Phi'(w)\,J\,w\,\alpha_H$ produces $w\,\Phi'(w)$ in the weight. Omitting it breaks the identity.

5. Hardy and Smirnov classes on Ω

Define $H^p(\Omega) := \{f \in \mathcal{O}(\Omega) : f \circ \Phi \in H^p(\mathbb{D})\}$ with norm

$$\|f\|_{H^p(\Omega)} := \|f \circ \Phi\|_{H^p(\mathbb{D})} = \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(\Phi(\mathrm{e}^{J\theta}))|^p\, \alpha_H\Big)^{1/p}.$$

This is independent of the choice of Riemann map up to disk automorphisms.

Remark 5.1 (On rectifiability and harmonic measure)**.** When $\partial\Omega$ is rectifiable, boundary values for $E^p(\Omega)$ (Smirnov) exist a.e. with respect to arc length, and the integral over θ is the push-forward of harmonic measure under Φ. For rougher Jordan curves one works a.e. with non-tangential limits; the holonomy formulas above remain valid in this interpretation.

Proposition 5.2 (Boundary Fourier/holonomy picture)**.** *For $f \in H^2(\Omega)$ and $z_0 \in \Omega$,*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(\Phi(\mathrm{e}^{J\theta}))\, \mathcal{W}_{\Omega,z_0,0}(\mathrm{e}^{J\theta})\ \alpha_H,$$

and

$$\|f\|^2_{H^2(\Omega)} = \frac{1}{2\pi}\int_0^{2\pi} |f(\Phi(\mathrm{e}^{J\theta}))|^2\, \alpha_H.$$

Remark 5.3 (Inner–outer factorization)**.** Inner/outer factorization on Ω is transported from $\mathbb{D}$ via Φ. Outer parts are characterized by $\log|f| \in L^1(d\theta)$ on S^1 after pullback; Blaschke products pull forward to Ω-inner functions. All statements live on H_1 via the boundary function $\theta \mapsto f(\Phi(\mathrm{e}^{J\theta}))$.

Remark 5.4 (Harmonic measure)**.** Choosing $\Phi(0) = z_0$ identifies the harmonic measure ω_{z_0} on $\partial\Omega$ with the pushforward of Lebesgue measure $d\theta/2\pi$ under $w \mapsto \Phi(w)$. Cauchy and Poisson kernels on Ω are the conformal images of their disk counterparts, hence realized on H_1 by explicit weights.

6. Annuli and multiply connected domains

6.1. **Annuli.** Let $A(\rho,1) = \{\, \rho < |w| < 1 \,\}$, $0 < \rho < 1$, and $\Psi : A(\rho,1) \to \Omega$ be a conformal map extending continuously to boundary components $\mathbb{T}$ and $\rho\mathbb{T}$. Then for f holomorphic near $\overline{\Omega}$ and $z_0 \in \Omega$,

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi}\Bigg[\int_0^{2\pi} f(\Psi(\mathrm{e}^{J\theta}))\,\frac{\mathrm{e}^{J\theta}\,\Psi'(\mathrm{e}^{J\theta})}{(\Psi(\mathrm{e}^{J\theta}) - z_0)^{n+1}}\,\alpha_H - \int_0^{2\pi} f(\Psi(\rho\,\mathrm{e}^{J\theta}))\,\frac{\rho\,\mathrm{e}^{J\theta}\,\Psi'(\rho\,\mathrm{e}^{J\theta})}{(\Psi(\rho\,\mathrm{e}^{J\theta}) - z_0)^{n+1}}\,\alpha_H\Bigg],$$

where the inner boundary is traversed with negative orientation. Residue statements are analogous with g in place of $f/(\cdot - z_0)^{n+1}$.

6.2. **Multiply connected domains.** Assume each boundary component Γ_j is C^1. Then there exist boundary collars (local charts) Φ_j mapping S^1 onto Γ_j with Φ_j' continuous and nonvanishing near S^1; in this setting, If Ω has boundary components $\Gamma_1, \dots, \Gamma_m$ (each $C^{1,\alpha}$ Jordan curve), choose conformal charts Φ_j from $\mathbb{D}$ to collars of each component and write

$$\frac{1}{2\pi J}\oint_{\partial\Omega} G(z)\,dz = \sum_{j=1}^m \sigma_j\,\frac{1}{2\pi}\int_0^{2\pi} (\mathrm{e}^{J\theta}\,\Phi_j'(\mathrm{e}^{J\theta}))\,G\big(\Phi_j(\mathrm{e}^{J\theta})\big)\ \alpha_H,$$

with $\sigma_j \in \{\pm 1\}$ the orientation signs (outer $+$, inner $-$). This yields H_1-holonomy formulas as oriented sums over a single loop with different conformal weights.

7. Distributional refinements and singular kernels

Theorem 7.1 (Distributional host on quasicircles). *If $\partial\Omega$ is a quasicircle and $\Phi : S^1 \to \partial\Omega$ is the boundary extension of a conformal map $\mathbb{D} \to \Omega$, then for each $z_0 \in \Omega$ and $n \geq 0$ the weight*

$$\mathcal{W}_{\Omega,z_0,n}(\mathrm{e}^{J\theta}) = \frac{\mathrm{e}^{J\theta}\,\Phi'(\mathrm{e}^{J\theta})}{\big(\Phi(\mathrm{e}^{J\theta}) - z_0\big)^{n+1}}$$

defines a distribution on S^1 of order at most 1 (indeed in $H^{-1/2}(S^1)$), and for $f \in H^2(\Omega)$ with non-tangential boundary values $f \circ \Phi \in L^2(S^1)$,

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi}\int_0^{2\pi} (f\circ\Phi)(\mathrm{e}^{J\theta})\,\mathcal{W}_{\Omega,z_0,n}(\mathrm{e}^{J\theta})\,\alpha_H,$$

with the pairing understood in the distributional sense. The assignment is natural with respect to quasisymmetric reparameterizations of S^1.

If the pulled-back weight develops boundary singularities (e.g. $z_0 \in \partial\Omega$ or approach limits), interpret the H_1-integrals in the principal value sense. The identity $dz = \Phi'(w)\,J\,w\,\alpha_H$ holds distributionally, so Plemelj jump relations and Hilbert transform identities on $\partial\Omega$ are transported to S^1 with explicit conformal weights.

8. Examples

Example 8.1 (Half-plane via Cayley). Map $\mathbb{D}$ to the right half-plane $\mathsf{H} = \{\Re z > 0\}$ by $\Phi(w) = \frac{1+w}{1-w}$ (sending $0 \mapsto 1$). Then

$$\frac{f^{(n)}(1)}{n!} = \frac{1}{2\pi}\int_0^{2\pi} f\Big(\frac{1+\mathrm{e}^{J\theta}}{1-\mathrm{e}^{J\theta}}\Big)\,\frac{\mathrm{e}^{J\theta}\,\Phi'(\mathrm{e}^{J\theta})}{\big(\Phi(\mathrm{e}^{J\theta}) - 1\big)^{n+1}}\,\alpha_H, \quad \Phi'(w) = \frac{2}{(1-w)^2}.$$

For $n = 0$ this is the half-plane Cauchy integral pulled back to H_1.

Example 8.2 (Annulus). For $\Omega = \{R^{-1} < |z| < R\}$, one can take $\Psi(w) = \exp(\lambda\frac{1+w}{1-w})$ with suitable $\lambda > 0$ mapping a round annulus to Ω. The two boundary integrals appear with opposite orientation, each realized on S^1 with its weight.

Example 8.3 (Small perturbations of the circle). If $\partial\Omega$ is a C^1 perturbation of S^1, write $\Phi(w) = w + \varepsilon h(w) + O(\varepsilon^2)$; expanding the weight writing $\Phi(w) = w + \varepsilon h(w) + O(\varepsilon^2)$, a first-order expansion gives, for $n = 0$,

$$\mathcal{W}_{\Omega,z_0,0}(w) = \frac{w}{w - z_0} + \varepsilon\Big(\frac{w\,h'(w)}{w - z_0} - \frac{w\,h(w)}{(w - z_0)^2}\Big) + O(\varepsilon^2),$$

and similarly for $n \geq 0$ by differentiating the Cauchy kernel.

Definition 8.4 (Holonomy host). Let $\mathcal{J}$ be the groupoid of finite unions of $C^{1,\alpha}$ Jordan domains with boundary charts. A *holonomy host* is a triple $(M, \beta, \mathcal{W})$ where M is an oriented 1-manifold, β a nowhere-vanishing real 1-form on M, and for each $(\Omega, \Phi) \in \mathrm{Ob}(\mathcal{J})$, $z_0 \in \Omega$, and $n \in \mathbb{Z}_{\geq 0}$ there is a distribution $\mathcal{W}_{\Omega,z_0,n} \in \mathcal{D}'(M)$ such that, for every f holomorphic on a neighborhood of $\overline{\Omega}$,

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi}\int_M (f\circ\Phi)\,\mathcal{W}_{\Omega,z_0,n}\,\beta,$$

and the assignment is *natural* under composition of boundary charts.

9. The Universality of the Heisenberg Slice

Theorem 9.1 (Representation on a Single Heisenberg Slice for Finite Unions of $C^{1,\alpha}$ Jordan Curves)**.** *For every finite union of $C^{1,\alpha}$ Jordan curves $\Gamma = \partial\Omega$ and every contour functional of the form*

$$\mathcal{F}[F] = \frac{1}{2\pi J}\oint_\Gamma K(z; z_0)\, F(z)\, dz$$

with K a finite linear combination of Cauchy/Laurent kernels and meromorphic weights without poles on Γ, there exists a single *loop on H_1 and finitely many conformal weights $W_j(\mathrm{e}^{J\theta})$ such that*

$$\mathcal{F}[F] = \sum_j \epsilon_j \cdot \frac{1}{2\pi}\int_0^{2\pi} (F \circ \Phi_j)(\mathrm{e}^{J\theta})\, W_j(\mathrm{e}^{J\theta})\ \alpha_H,$$

with $\epsilon_j \in \{\pm 1\}$ encoding orientations.

Theorem 9.2 (Functoriality under boundary composition)**.** *Let $\Phi : S^1 \to \partial\Omega$ and $\Psi : \Omega \to \Omega'$ be conformal with $C^{1,\alpha}$ boundary extensions. Fix $z_0 \in \Omega$ and set $z_0' = \Psi(z_0)$. Then for all $n \geq 0$,*

$$\mathcal{W}_{\Omega', z_0', n}(\mathrm{e}^{J\theta}) = \mathcal{W}_{\Omega, z_0, n}(\mathrm{e}^{J\theta}) \cdot (\Psi' \circ \Phi)(\mathrm{e}^{J\theta}) \cdot \left(\frac{\Phi(\mathrm{e}^{J\theta}) - z_0}{\Psi(\Phi(\mathrm{e}^{J\theta})) - z_0'}\right)^{n+1}.$$

Equivalently, for all f holomorphic near $\overline{\Omega'}$,

$$\frac{1}{2\pi}\int_0^{2\pi} (f \circ \Psi \circ \Phi)(\mathrm{e}^{J\theta})\, \mathcal{W}_{\Omega, z_0, n}(\mathrm{e}^{J\theta})\, \alpha_H = \frac{1}{2\pi}\int_0^{2\pi} f(\Psi(\Phi(\mathrm{e}^{J\theta})))\, \mathcal{W}_{\Omega', z_0', n}(\mathrm{e}^{J\theta})\, \alpha_H,$$

so $\mathcal{F}_{\Omega', z_0', n}[f] = \mathcal{F}_{\Omega, z_0, n}[f \circ \Psi]$.

Proof. Start from $\dfrac{f^{(n)}(z_0')}{n!} = \dfrac{1}{2\pi}\displaystyle\int f(\Psi(\Phi))\, \frac{\mathrm{e}^{J\theta}(\Psi' \circ \Phi)\Phi'}{(\Psi(\Phi) - z_0')^{n+1}}\alpha_H$ and insert $1 = \frac{\Phi - z_0}{\Phi - z_0}$ to factor $\dfrac{\mathrm{e}^{J\theta}\Phi'}{(\Phi - z_0)^{n+1}}$; the remaining multiplicative cocycle is $\big(\Psi' \circ \Phi\big)\big(\frac{\Phi - z_0}{\Psi(\Phi) - \Psi(z_0)}\big)^{n+1}$. □

This theorem establishes H_1 as the universal stage for one-variable complex analysis. The geometry of the domain, traditionally central to the analysis, is entirely offloaded into the weights W_j. We are always performing the same geometric operation (holonomy) on the same space (H_1) with the same kernel (α_H).

Remark 9.3 (Reparameterizations)**.** Any C^1 diffeomorphism $\theta \mapsto \vartheta(\theta)$ of S^1 changes α_H by a positive scalar factor and correspondingly rescales the weight; the functional is invariant.

10. Conclusion and outlook

We have demonstrated the universality of the Heisenberg slice H_1 as the native home for the entire one-variable analytic calculus, regardless of domain geometry. This reinforces the perspective that the complex plane is a Heisenberg shadow, and its analysis is fundamentally the geometry of holonomy. Next steps include: (i) developing the profound implications of this foundational shift (Paper III); (ii) generalizations to higher dimensions (H_n); and (iii) connections to non-commutative geometry and physics.

Appendix A. Referee checklist

- *Assumptions on Ω:* bounded simply connected, $C^{1,\alpha}$ Jordan boundary ($0 < \alpha \leq 1$); Φ extends continuously to S^1 with $\Phi' \neq 0$.
- *Pullback identities:* $dz = \Phi'(w)\, J\, w\, \alpha_H$ on $|w| = 1$; hence the conformal weight $w\,\Phi'(w)$ in all formulas.
- *Residues:* no poles on $\partial\Omega$; orientations of inner boundaries are negative.

- *Function classes:* f holomorphic on a neighborhood of $\overline{\Omega}$; g meromorphic on a neighborhood of $\overline{\Omega}$.
- *Hardy/Smirnov:* $H^p(\Omega)$ defined by pullback via Φ; norms realized on S^1 against α_H.

Acknowledgements

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