# An Adelic Distributional Framework for the Symmetric Explicit Formula on a Band-Limited Class*

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**Preprint status (unrefereed).** This is an *unrefereed* preprint; community verification is invited.
**Scope (Paper II).** Fixes the Maaß – Selberg normalization and constructs the adelic geometric functional $Q_{\mathrm{MS}}(g) = P(g) - \mathcal{A}(g)$ on band-limited and exponentially-damped classes. *No Riemann Hypothesis (RH) claim is asserted in this paper.*
**Companion papers (Zenodo DOIs).** Paper I (10.5281/zenodo.16930060): automorphic derivation of the asymmetric explicit formula; *no RH claim.* Paper III (10.5281/zenodo.16930094): *states a claimed proof of RH as a preprint, pending peer review*, within the same ledger.

**Abstract**

We construct the geometric side of the GL(1) symmetric explicit formula, $Q_{\mathrm{MS}}(g) = P(g) - \mathcal{A}(g)$ (the Maaß – Selberg base functional), within an adelic distributional framework, focusing on band-limited and exponentially-damped test classes. We rigorously fix the constants according to the Maaß – Selberg normalization, ensuring compatibility with the automorphic derivation presented in the companion paper (Paper I, [4]). The Archimedean term $\mathcal{A}(g)$ is evaluated through a careful regularization procedure, connecting the distributional pairing on the modulus line to the spectral definition involving the digamma function. The finite-place contributions yield the prime block $P(g)$, a weighted sum over prime powers. The local distributional contributions $D_v$ are explicitly calculated and summed to the global functional $Q_{\mathrm{MS}}(g)$. This construction reveals the underlying positivity structure of the geometric side, which is governed by a positive kernel (the half-shift kernel $T_{1/2}$). A comprehensive normalization ledger synchronizes conventions across this series of papers. All constructions are carried out without assuming the Riemann Hypothesis.

**Contributions.** This paper provides: (i) a self-contained adelic distributional construction of the geometric side $Q_{\mathrm{MS}}(g)$ on these classes; (ii) a rigorous identification, enforced by the normalization, of the Archimedean boundary kernel as the half-shift kernel $T_{1/2}$; and (iii) the necessary framework and constant synchronization for the geometric side of the explicit formula (derived fully in Paper I), with all analytic interchanges rigorously justified and without assuming RH.

**Keywords:** Explicit formula, adelic distributions, Riemann zeta function, Weil distribution, GL(1), Heisenberg/CR boundary, Maaß – Selberg normalization, half-shift kernel.
**MSC 2020:** Primary 11M06; Secondary 11F72, 11R42.

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*Reader's map.* §2–§4 fix the pairing spaces and the Maaß – Selberg ledger; §5 evaluates the Archimedean block; §6–§7 assemble the global identity $Q_{\mathrm{MS}} = P - A$. RH appears only as *classical* cross-references (Weil positivity), not as a claim in this paper.

## Contents



# 1 Introduction

**Claim boundaries.** Throughout this paper we *prove identities and normalizations* for the geometric side $Q_{\mathrm{MS}}(g) = P(g) - \mathcal{A}(g)$ in the Maaß – Selberg ledger and *do not* assert a proof of RH. Any RH-related statement here is strictly *classical equivalence or implication phrasing* (e.g., Weil's criterion) and is used for context only. The *preprint claim* regarding RH appears solely in Paper III ([5]) and is offered for community review.

The classical explicit formulas, originating with Riemann and developed by Guinand and Weil, establish a profound duality between the spectral properties of the Riemann zeta function (encoded in the distribution of its nontrivial zeros) and its arithmetic properties (encoded in the distribution of prime numbers). This paper provides a rigorous, self-contained derivation of the geometric side of the symmetric explicit formula within an adelic framework for GL(1). Our primary objective is to establish the fundamental identity

$$Q_{\mathrm{MS}}(g) = P(g) - \mathcal{A}(g), \tag{1.1}$$

for a specific band-limited test class (Definition 2.9). Here, $Q_{\mathrm{MS}}(g)$ denotes the Maaß–Selberg base geometric functional, $P(g)$ the arithmetic (prime) block, and $\mathcal{A}(g)$ the Archimedean block.

*Remark* 1.1 (Terminology: Symmetric vs. Asymmetric). The title refers to the "Symmetric Explicit Formula" because the underlying framework is the symmetric adelic structure of GL(1), which naturally leads to the construction of the geometric side using even test functions $g$. When this geometric side is equated to the spectral side (the zeros), the resulting identity (e.g., (1.3) and Section A) utilizes one-sided Laplace weights $\mathcal{H}(g;\rho)$. This specific form, where the geometric input is symmetric (even $g$) but the spectral weights are asymmetric (one-sided Laplace transforms), is termed the "Asymmetric Explicit Formula" (as derived in Paper I [4]). The two forms are mathematically equivalent due to the functional equation and the symmetry of the zeros ($\rho \leftrightarrow 1-\rho$).

**Analytic Setup and Adelic Framework.** *Notation.* We operate within the adelic framework established by Tate and Weil. The global *base* functional $Q_{\mathrm{MS}}(g)$ is defined as the sum of local distributional pairings over all places $v$ of $\mathbb{Q}$: *Normalization pointer.* Haar- and cosine-transform conventions (and the Maaß–Selberg ledger) are fixed in Section 2 and Section A.

$$Q_{\mathrm{MS}}(g) \coloneqq \sum_v \langle D_v, W\rangle. \tag{1.2}$$

We systematically evaluate the local distributions $D_v$ and their pairings with the associated Weil test function $W$. At finite places $p$, the distribution $D_p$ corresponds to the logarithmic derivative of the local zeta factor, yielding the prime block $P(g)$. At the infinite place $\infty$, the distribution $D_\infty$ yields the Archimedean contribution $-\mathcal{A}(g)$.

A crucial aspect of this work is the rigorous justification of the Archimedean pairing. This requires a careful regularization procedure (Section 5) to handle the interplay between the kernel singularities and the test functions. This procedure ensures precise alignment with the Maaß–Selberg normalization utilized in the automorphic derivation of the full explicit formula presented in the companion paper [4].

**Geometric Positivity and the Riemann Hypothesis.** A key outcome of this construction is the explicit revelation of the positivity structure inherent in the geometric side. The *Full Geometric Functional*, $Q_{\mathrm{geom}}(g)$, combines the base functional $Q_{\mathrm{MS}}(g)$ with boundary terms arising from regularization ($\frac{1}{2}g(0)$) and the Poisson block ($2J(g)$; see Definition 2.12):

$$Q_{\mathrm{geom}}(g) \coloneqq Q_{\mathrm{MS}}(g) + \frac{1}{2}g(0) + 2J(g) = P(g) - \mathcal{A}(g) + \frac{1}{2}g(0) + 2J(g).$$

We demonstrate that $Q_{\mathrm{geom}}(g)$ is governed by a manifestly positive kernel arising from a mechanism we term the "torsion filter" $C_{1/2}$ (Section 2.4).

This structure is central to the connection with the Riemann Hypothesis (RH) via the full explicit formula established in [4], which equates the spectral side to the full geometric functional:

$$2\sum_\rho \mathcal{H}(g;\rho) \;=\; Q_{\mathrm{geom}}(g). \tag{1.3}$$

(See Appendix A for the definition of the spectral weights $\mathcal{H}$). The results herein demonstrate that the entire geometric side (the RHS of (1.3)) is represented by positive coefficients (for $P(g)$) and a strictly positive density (for the continuous part), as summarized in Proposition 6.2.

**Proposition 1.2** (Positivity and the Weil Criterion in this ledger)**.** *Within the Maaß – Selberg normalization fixed here, let $g_\Phi = \Phi * \Phi^\vee$ be an autocorrelation with $\Phi \in \mathcal{S}(\mathbb{A})$.*

*(i)* ***(Arithmetic Positivity)*** *The full geometric functional $Q_{\mathrm{geom}}(g_\Phi)$ is nonnegative. Consequently, the spectral side $2\sum_\rho \mathcal{H}(g_\Phi;\rho)$ is also nonnegative. This holds unconditionally (independent of RH).*

*(ii)* ***(Classical Weil Criterion)*** *The Riemann Hypothesis is* equivalent *to the nonnegativity of the Maaß – Selberg base functional:*

$$Q_{\mathrm{MS}}(g_\Phi) \;=\; P(g_\Phi) - \mathcal{A}(g_\Phi) \;\geq 0 \qquad \textit{for all autocorrelations } g_\Phi.$$

*Remark* 1.3 (Sign conventions). The minus sign in $\langle D_\infty, W\rangle = -\mathcal{A}(g)$ reflects the Maaß – Selberg choice of spectral phase derivative and boundary pairing; it is consistent with $Q_{\mathrm{MS}}(g) = P(g) - \mathcal{A}(g)$ in (1.1).

*Proof sketch (Classical arguments rephrased in the Maaß – Selberg ledger).* (i) (Arithmetic Positivity) For an autocorrelation $g_\Phi$, $g_\Phi$ is positive definite. By Bochner's theorem, its cosine profile $g_{\mathrm{b}\Phi}$ is non-negative. The full geometric functional is given by $Q_{\mathrm{geom}}(g_\Phi) = \int g_{\mathrm{b}\Phi}\, d\nu$ (see Lemma 1.5). Since $d\nu$ is the positive Weil measure (defined later in Definition 2.15), $Q_{\mathrm{geom}}(g_\Phi) \geq 0$. By the explicit formula (1.3), the spectral side $2\sum_\rho \mathcal{H}(g_\Phi;\rho)$ must also be non-negative.

(ii) (Weil Criterion) This is the classical statement [6]. RH is equivalent to the positive definiteness of the Weil distribution, which corresponds here to $Q_{\mathrm{MS}}(g_\Phi) = \sum_v \langle D_v, W_\Phi\rangle \geq 0$ for all $g_\Phi$. □

*Remark* 1.4 (Adelic-Spectral Dictionary). The spectral function $g_\Phi$ associated with an adelic autocorrelation $\Phi * \Phi^\vee$ (where $\Phi \in \mathcal{S}(\mathbb{A})$) arises from the adelic Poisson summation framework. For GL(1), the structure ensures that $g_\Phi$ is even. Crucially, the cosine profile $g_{\mathrm{b}\Phi}$ inherits rapid decay from the Schwartz conditions on $\Phi$. Specifically, $g_\Phi \in \mathcal{S}(\mathbb{R})$ and $g_{\mathrm{b}\Phi}$ exhibits rapid decay (faster than any required exponential $e^{-\sigma|x|}$), thus $g_\Phi \in \mathcal{A}_{\mathrm{Exp}}$. This guarantees $g_{\mathrm{b}\Phi} \in L^1(d\nu)$ and justifies the application of the derived identities to autocorrelations, as well as the continuity arguments in Lemma 1.5.

**Lemma 1.5** (Bridge to autocorrelations and closure in the Maaß – Selberg ledger)**.** *Let $\Phi \in \mathcal{S}(\mathbb{A})$ and $g_\Phi := \Phi * \Phi^\vee$ be its autocorrelation on the spectral line. Let $d\nu$ be the prime–continuous Weil measure (Definition 2.15). As noted in Remark 1.4, $g_\Phi \in \mathcal{A}_{\mathrm{Exp}}$, which implies $g_{\mathrm{b}\Phi} \in L^1(d\nu)$. Then:*

*(i) (Adelic–spectral agreement) In the Maaß – Selberg normalization fixed here,*

$$Q_{\mathrm{MS}}(g_\Phi) \;=\; \sum_v \langle D_v, W_\Phi\rangle, \qquad Q_{\mathrm{geom}}(g_\Phi) \;=\; \int_{(0,\infty)} g_{\mathrm{b}\Phi}(x)\,\mathrm{d}\nu(x),$$

*and, equivalently,*

$$Q_{\mathrm{geom}}(g_\Phi) \;=\; Q_{\mathrm{MS}}(g_\Phi) + \tfrac{1}{2} g_\Phi(0) + 2J(g_\Phi).$$

*Here $W_\Phi$ is the Weil modulus-line test attached to $g_\Phi$ by* (2.6)*.*

*(ii) (Closure/continuity) If $g_{\mathrm{b},n} \to g_{\mathrm{b}\Phi}$ in $L^1(d\nu)$ with each $g_n \in \mathcal{A}_{\mathrm{BL}}$, then*

$$\lim_{n\to\infty} Q_{\mathrm{geom}}(g_n) \;=\; Q_{\mathrm{geom}}(g_\Phi), \qquad \lim_{n\to\infty} \int g_{\mathrm{b},n}\,\mathrm{d}\nu \;=\; \int g_{\mathrm{b}\Phi}\,\mathrm{d}\nu.$$

*In particular, since $d\nu$ is a positive measure, $Q_{\mathrm{geom}}(g_\Phi) \geq 0$ whenever $g_{\mathrm{b}\Phi} \geq 0$ a.e.*

*Proof.* For (i), the identity $Q_{\mathrm{MS}}(g_\Phi) = \sum_v \langle D_v, W_\Phi\rangle$ is Definition 3.1, evaluated as $P(g_\Phi) - \mathcal{A}(g_\Phi)$ in Sections 4 and 5. The identity $Q_{\mathrm{geom}}(g_\Phi) = \int g_{\mathrm{b}\Phi}\,\mathrm{d}\nu$ follows from Definition 2.15 (the Weil measure), which combines the prime block $P(g_\Phi)$ and the continuous block (Remark 2.13). The equivalence $Q_{\mathrm{geom}}(g_\Phi) = Q_{\mathrm{MS}}(g_\Phi) + \frac{1}{2}g_\Phi(0) + 2J(g_\Phi)$ follows from the regrouping identity (Remark 5.6), where the continuous block equals $-\mathcal{A}(g_\Phi) + \frac{1}{2}g_\Phi(0) + 2J(g_\Phi)$. The Maaß–Selberg normalization ensures consistency (Proposition 5.3).

For (ii), use Lemma 2.18 and the continuity of the linear functional $g_\mathrm{b} \mapsto \int g_\mathrm{b}\,\mathrm{d}\nu$ in the $L^1(d\nu)$ topology. □

*Remark* 1.6 (Scope and Context). *No new RH criterion is proposed here.* Proposition 1.2 simply restates the classical Weil positivity equivalence in our fixed Maaß–Selberg normalization and on the test classes used here. The contribution of this paper is to construct $Q_{\mathrm{MS}}(g) = P(g) - \mathcal{A}(g)$ rigorously, fix constants, and expose the continuous positivity structure (via the torsion filter) as input to Paper III [5]. Any *claimed* RH result is confined to Paper III ([5]) and is *pending peer review.*

*Remark* 1.7 (Test Class Properties). *Crosswalk.* The band-limited class $\mathcal{A}_{\mathrm{BL}}$ is used as the primary class for derivations, as it simplifies convergence arguments (e.g., finite prime sums). The results are then extended to the exponentially damped class $\mathcal{A}_{\mathrm{Exp}}$ (Theorem 2.9) and autocorrelations (Theorem 1.2, Theorem 1.5) via continuity arguments. (We use $\mathcal{A}_{\mathrm{BL}}$ consistently here; it is denoted $\mathcal{S}_{\mathrm{BL}}$ in some companion contexts.)

The relevant topology for this extension is the $L^1(d\nu)$ topology (Theorem 2.15). Since the profiles of $\mathcal{A}_{\mathrm{BL}}$ are dense in $L^1(d\nu)$ (Theorem 2.18), and the functional $Q_{\mathrm{geom}}(g) = \int g_\mathrm{b}\,d\nu$ is continuous in this topology, the identities established on $\mathcal{A}_{\mathrm{BL}}$ hold throughout $L^1(d\nu)$. This topology naturally controls both the continuous density ($\rho_W$) and the atomic prime lattice.

# 2 Conventions, Test Functions, and Analytical Prerequisites

## 2.1 Analytic Objects and Normalizations

We denote the completed Riemann zeta function and the associated real gamma factor as:

$$\Lambda(s) := \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s), \qquad \Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big). \tag{2.1}$$

The logarithmic derivative involves the digamma function $\psi(s) = \Gamma'(s)/\Gamma(s)$.

*Remark* 2.1 (Phase measure vs. Plancherel). *Normalization lock.* We pair throughout against the Maaß–Selberg (Birman–Kreĭn) spectral–shift measure $\mathrm{d}\xi(\tau) = \frac{1}{2\pi}\,\varphi'(\tau)\,\mathrm{d}\tau$ in the *symmetric* principal–value sense, and there is no $\delta_0$ mass at $\tau = 0$. We never use the Selberg Plancherel density $\mathrm{d}\tau/(4\pi)$ in pairings. *Scattering note.* For reference, the Eisenstein scattering coefficient is $\widetilde{S}(s) = \Lambda(2s-1)/\Lambda(2s)$, and $\varphi'(\tau) = \mathrm{Im}\,\partial_\tau \log \widetilde{S}(\frac{1}{2} + i\tau)$ under this ledger (see also Lemma 2.6).

*Corollary* 2.2 (Ledger lock). *All pairings in this paper use the measure established formally in Paper III (the* Maaß–Selberg ledger lock; no $\delta_0$ atom *theorem) [5]: $d\xi(\tau) = \frac{1}{2\pi}\varphi'(\tau)\,d\tau$, with symmetric PV and no $\delta_0$ at $\tau = 0$. The proof relies on the explicit cancellation of Poisson kernels (Paper III, proof of Thm. 1.8; see also Lemma 2.7 herein).*

*Ledger crosswalk.* This lock matches the density $\rho_W$ in the positivity paper [5] and the phase–measure ledger in the automorphic paper [4]; see Paper III (continuous density) and Paper I (App. A).

*Normalization pointer.* The PV/no-$\delta_0$ convention is proved locally in Lemma 2.7. For a single canonical ledger of constants and pairings used across the trilogy, see Section A.

**Definition 2.3** (The Weil continuous density $\rho_W$)**.** The continuous geometric density on the modulus line $(0,\infty)$ is

$$\rho_W(x) := \frac{1}{2\pi}\left(\frac{1}{1+e^{-x/2}} + 2e^{-x/2}\right) \mathbf{1}_{(0,\infty)}(x). \tag{2.2}$$

This density encodes the Archimedean, boundary, and Poisson contributions in the Maaß – Selberg ledger; it is derived rigorously in Section 5 (see Remark 5.6).

**Definition 2.4** (Spectral functional in Maaß – Selberg normalization)**.** We pair throughout against the Birman–Krein spectral–shift measure in the *symmetric* principal–value sense with no atom at $\tau = 0$:

$$\mathcal{I}(g) := \frac{1}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}} g(\tau)\,\varphi'(\tau)\,\mathrm{d}\tau \quad (\textit{symmetric PV}).$$

*Symmetric PV convention:* $\mathrm{PV}\int_{-\infty}^{\infty} f(\tau)\,\mathrm{d}\tau := \lim_{T\to\infty}\int_{-T}^{T} f(\tau)\,\mathrm{d}\tau$. In the Maaß – Selberg ledger there is no $\delta_0$ atom (Lemma 2.7).

**Lemma 2.5** (Pointwise bounds and $L^1$-equivalence for $\rho_W$)**.** *The Weil density $\rho_W(x)$ (Definition 2.3) satisfies the following bounds for all $x > 0$:*

$$\frac{1}{2\pi} < \rho_W(x) < \frac{5}{4\pi}.$$

*Consequently, the $L^1(\rho_W\,\mathrm{d}x)$ norm is equivalent to the standard $L^1(\mathrm{d}x)$ norm on $(0,\infty)$.*

*Proof.* Put $y = e^{-x/2}$. Since $x > 0$, we have $y \in (0,1)$. Then $\rho_W(x) = \frac{1}{2\pi} f(y)$, where $f(y) = \frac{1}{1+y} + 2y$. We analyze $f(y)$ on $(0,1)$. The derivative is $f'(y) = -\frac{1}{(1+y)^2} + 2$. For $y \in (0,1)$, we have $1 < 1+y < 2$, so $1 < (1+y)^2 < 4$. This implies $\frac{1}{4} < \frac{1}{(1+y)^2} < 1$. Thus, $f'(y) = 2 - \frac{1}{(1+y)^2} > 2 - 1 = 1$. Since $f'(y) > 0$, $f(y)$ is strictly increasing on $(0,1)$. The infimum is $\lim_{y\to 0^+} f(y) = 1$ and the supremum is $\lim_{y\to 1^-} f(y) = \frac{1}{2} + 2 = \frac{5}{2}$. This yields the strict bounds $\frac{1}{2\pi} < \rho_W(x) < \frac{5/2}{2\pi} = \frac{5}{4\pi}$. ☐

*Consequence.* In particular, $\rho_W \in L^1((0,\infty)) \cap L^\infty((0,\infty))$, so the pairing $g_{\mathrm{b}} \mapsto \int_0^\infty g_{\mathrm{b}}(x)\,\rho_W(x)\,\mathrm{d}x$ is well defined for $g_{\mathrm{b}} \in L^1$ and bounded by $\|\rho_W\|_\infty\,\|g_{\mathrm{b}}\|_{L^1}$. Moreover, by the bounds above, $\|\cdot\|_{L^1(\rho_W\,\mathrm{d}x)}$ is equivalent to $\|\cdot\|_{L^1(\mathrm{d}x)}$ on $(0,\infty)$.

**Lemma 2.6** (Boundary phase derivative)**.** *In $\mathcal{S}'(\mathbb{R})$, taken in the* symmetric *principal–value sense, one has*

$$\varphi'(\tau) = \mathrm{PV}\ 4\,\mathrm{Re}\Big(\frac{\Lambda'}{\Lambda}(2i\tau)\Big), \qquad \Lambda(s) = \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s).$$

*Moreover, $\varphi'(\tau) = O(\log(2+|\tau|))$ as $|\tau| \to \infty$ (cf. Paper I, Lemma 1.11), hence $\varphi'$ defines a tempered distribution. The absence of a $\delta_0$ atom is proven below.*

**Lemma 2.7** (No $\delta_0$ atom: Rigorous proof via regularization)**.** *The measure $d\xi(\tau) = \frac{1}{2\pi}\varphi'(\tau)\,\mathrm{d}\tau$ has no $\delta_0$ atom in the Maaß – Selberg normalization.*

*Proof.* We analyze the limit $\varepsilon \downarrow 0$ of the regularized pairing $\langle \varphi'_\varepsilon, g\rangle$ for $g \in \mathcal{S}(\mathbb{R})$ (or $\mathcal{A}_{\mathrm{Exp}}$), justifying the application of DCT via an $\varepsilon$-independent $L^1$ majorant. Let $s = \frac{1}{2} + \varepsilon + i\tau$ and $P_\varepsilon(\tau) = \frac{\varepsilon}{\varepsilon^2+\tau^2}$.

*1. Decomposition and Cancellation.* We utilize the standard decompositions near the singularities:

- $\psi(z) = -1/z + \psi_{\mathrm{reg}}(z)$; $\psi_{\mathrm{reg}}(z)$ is holomorphic for $\mathrm{Re}(z) > -1$.
- $\frac{\zeta'}{\zeta}(s) = -1/(s-1) + Z_{\mathrm{reg}}(s)$; $Z_{\mathrm{reg}}(s)$ is holomorphic near $s = 1$.

Analyzing the components of $\varphi'_\varepsilon(\tau)$ near $\tau = 0$:

$$\operatorname{Re}\psi(\varepsilon + i\tau) = \operatorname{Re}\left(\frac{-1}{\varepsilon + i\tau}\right) + \operatorname{Re}\psi_{\mathrm{reg}}(\varepsilon + i\tau) = -P_\varepsilon(\tau) + R_1(\tau, \varepsilon).$$

$$\begin{aligned}-2\operatorname{Re}\frac{\zeta'}{\zeta}(1 + 2\varepsilon + 2i\tau) &= -2\operatorname{Re}\left(\frac{-1}{2\varepsilon + 2i\tau}\right) - 2\operatorname{Re} Z_{\mathrm{reg}}(1 + 2\varepsilon + 2i\tau)\\ &= +P_\varepsilon(\tau) + R_2(\tau, \varepsilon).\end{aligned}$$

The Poisson kernels cancel exactly. The full kernel $\varphi'_\varepsilon(\tau)$ consists entirely of the remainder terms.

*2. Uniform Domimation and DCT Justification.* We construct an $\varepsilon$-independent $L^1$ majorant for $g(\tau)\varphi'_\varepsilon(\tau)$ for $\varepsilon \in (0, 1]$.

*Local Bounds ($|\tau| \le 1$):* We rigorously establish uniform bounds for the remainders.

- $R_1(\tau, \varepsilon) = \operatorname{Re}\psi_{\mathrm{reg}}(\varepsilon + i\tau)$. We define the compact domain $K_1 = \{\varepsilon + i\tau : \varepsilon \in [0, 1], |\tau| \le 1\}$. Since $\psi_{\mathrm{reg}}$ is continuous on the compact set $K_1$, it is bounded (Extreme Value Theorem). Let $C_1 = \sup_{z\in K_1} |\psi_{\mathrm{reg}}(z)|$. This constant is finite and independent of $\varepsilon$. Thus $|R_1(\tau, \varepsilon)| \le C_1$ uniformly.
- $R_2(\tau, \varepsilon) = -2\operatorname{Re} Z_{\mathrm{reg}}(1 + 2\varepsilon + 2i\tau)$. We define the compact domain $K_2 = [1, 3] + i[-2, 2]$. $Z_{\mathrm{reg}}$ is holomorphic (and thus continuous) on the compact set $K_2$. Let $C_2 = 2\sup_{s\in K_2} |Z_{\mathrm{reg}}(s)|$. This constant is finite and independent of $\varepsilon$. Thus $|R_2(\tau, \varepsilon)| \le C_2$ uniformly.

The other components (e.g., $-\operatorname{Re}\psi(\frac{1}{2} + \varepsilon + i\tau)$) are similarly bounded. Therefore, for $|\tau| \le 1$ and $0 < \varepsilon \le 1$, $|\varphi'_\varepsilon(\tau)| \le C_{\mathrm{local}}$.

*Global Bounds ($|\tau| > 1$):* By standard analysis (Stirling's formula and bounds for $\zeta'/\zeta$ uniform on vertical strips away from poles; see Paper I, Lemma 2.3), we have the uniform global bound $|\varphi'_\varepsilon(\tau)| \le C_{\mathrm{global}} \log(2 + |\tau|)$, independent of $\varepsilon \in (0, 1]$.

*Majorant Construction:* Define the majorant:

$$M(\tau) := |g(\tau)| \cdot \max(C_{\mathrm{local}}, C_{\mathrm{global}} \log(2 + |\tau|)).$$

Since $g \in \mathcal{A}_{\mathrm{Exp}}$ (or $\mathcal{S}(\mathbb{R})$), $|g(\tau)| = O((1 + |\tau|)^{-2})$. Thus $M(\tau) \in L^1(\mathbb{R})$ and is independent of $\varepsilon$.

*Conclusion:* Since $|g(\tau)\varphi'_\varepsilon(\tau)| \le M(\tau) \in L^1(\mathbb{R})$, the Dominated Convergence Theorem (DCT) applies. The limit $\varepsilon \downarrow 0$ passes inside the integral. As the convergence is dominated (not merely weak-*), no atomic mass is generated at $\tau = 0$. ☐

*Remark* 2.8 (Measure Convention). We conduct integration on the modulus line $\mathbb{R}^\times_{>0}$ using the additive coordinate $u = \log q$. We employ the additive Haar measure $\mathrm{d}u$ on $\mathbb{R}$, which under $q = e^u$ corresponds to the multiplicative Haar measure $\mathrm{d}^\times q = \mathrm{d}q/q$ on $\mathbb{R}^\times_{>0}$.
*Dictionary Jacobian.* Passing to the symmetric profile via $x = 2u$ gives $\mathrm{d}x = 2\,\mathrm{d}u$; this explains the consistent factors of 2 in $G_{\mathrm{b}}$ and $W$, removes half–factor ambiguity in pairings, and fixes the *prime lattice* at $x = 2k \log p$.

## 2.2 Test Functions and Transforms

We lock the cosine and profile dictionary *before* any local pairings, so that later Tonelli/DCT interchanges cite a single source of domination. The class and all labels in this subsection are canonical across the trilogy.

**Definition 2.9** (Band-limited class and exponentially-damped extension $\mathcal{A}_{\mathrm{Exp}}$)**.** A real, even $g : \mathbb{R} \to \mathbb{R}$ belongs to the *band-limited class* ($\mathcal{A}_{\mathrm{BL}}$, also denoted $\mathcal{S}_{\mathrm{BL}}$ in companion papers) if its cosine transform

$$g_{\mathrm{b}}(x) \;:=\; \int_{\mathbb{R}} g(\tau) \cos(x\tau)\,\mathrm{d}\tau$$

lies in $C_c^\infty(\mathbb{R})$, and the even inversion holds:

$$g(\tau) \;=\; \frac{1}{\pi}\int_0^\infty g_{\mathrm{b}}(x)\,\cos(x\tau)\,\mathrm{d}x \qquad (\tau\in\mathbb{R}). \tag{2.3}$$

*Normalization.* We use the unnormalized cosine transform with Lebesgue measure; factors of $\pi$ appear explicitly in inversion and in the Archimedean block. Since $g$ is even, $g_{\mathrm{b}}$ is also even. As $g_{\mathrm{b}} \in C_c^\infty(\mathbb{R})$, there exists $X > 0$ with $\operatorname{supp}(g_{\mathrm{b}}) \subseteq [-X, X]$. The inversion in (2.3) (integrating over $[0,\infty)$) holds in the usual $L^1$ (and distributional) sense.

The *exponentially-damped class* $\mathcal{A}_{\mathrm{Exp}}$ consists of even $g \in C^\infty(\mathbb{R})$ satisfying $|g(\tau)| \le C_g(1+|\tau|)^{-2}$ (sufficient for pairing against tempered distributions) such that its cosine transform $g_{\mathrm{b}}$ satisfies

$$|g_{\mathrm{b}}(x)| \;\le\; C\,e^{-\sigma|x|} \quad \text{for some } C>0 \text{ and } \textit{strict } \sigma > 1/2.$$

*Note on decay:* Since $\varphi'(\tau) = O(\log|\tau|)$ (Lemma 2.6), the decay condition $|g(\tau)| = O((1+|\tau|)^{-2})$ ensures $|g(\tau)\varphi'(\tau)| \in L^1(\mathbb{R})$, thus the spectral pairing $\mathcal{I}(g)$ (Definition 2.4) converges absolutely.

**Assumption 2.10** (Admissible test class)**.** Throughout, $g$ denotes a real, even test function that is either band-limited ($g \in \mathcal{A}_{\mathrm{BL}}$) or exponentially damped ($g \in \mathcal{A}_{\mathrm{Exp}}$).

*Why strict?* The same strict $\sigma > 1/2$ supplies: (i) absolute convergence of $P(g)$ via $\sum_{p,k}(\log p)\,|g_{\mathrm{b}}(2k\log p)| < \infty$, and (ii) an $\alpha$-independent $L^1$ majorant on $K_\alpha \cdot g_{\mathrm{b}}$ in the Archimedean limits (DCT/Tonelli in §5).

*Remark* 2.11 (Fourier pair and cosine specialization).

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(\tau)\,e^{-i\tau\xi}\,\mathrm{d}\tau, \qquad f(\tau) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\xi)\,e^{i\tau\xi}\,\mathrm{d}\xi.$$

For even $g$ we use the cosine specialization:

$$\widehat{g}(x) = \int_{\mathbb{R}} g(\tau)\cos(x\tau)\,\mathrm{d}\tau, \quad g(\tau) = \frac{1}{\pi}\int_0^\infty \widehat{g}(x)\cos(x\tau)\,\mathrm{d}x.$$

For a *real even* $g$ we denote its Fourier transform by $g_{\mathrm{b}}(x)$ (i.e. $g_{\mathrm{b}}(x) = \widehat{g}(x)$), and we reserve $g_{\mathrm{b}}$ (never a bare hat) in the prime/continuous blocks to prevent normalization drift. We will informally refer to

$$\mathrm{d}\mu_g(x) = \frac{1}{2\pi}\,g_{\mathrm{b}}(|x|)\,\mathrm{d}x \qquad (\text{here } g_{\mathrm{b}} = \widehat{g} \text{ in the cosine dictionary}) \tag{2.4}$$

as the *Bochner profile measure.* For a general even $g$ this is a (possibly signed) measure; when $g = \Phi * \Phi^\vee$ (autocorrelation) it is positive. All phase-measure pairings use the *symmetric* PV in the Maaß–Selberg ledger; see Thm. 2.1.

**Definition 2.12** (Derived profiles and the Poisson block)**.** For $g \in \mathcal{A}_{\mathrm{Exp}}$ (hence for $\mathcal{A}_{\mathrm{BL}}$) set:

(i) **Symmetric profile** (even on $\mathbb{R}$):

$$G_{\mathrm{b}}(x) \;:=\; 2\sinh\!\Big(\frac{|x|}{2}\Big)\,g_{\mathrm{b}}(2x). \tag{2.5}$$

If $\operatorname{supp}(g_{\mathrm{b}}) \subseteq [-X, X]$, then $\operatorname{supp}(G_{\mathrm{b}}) \subseteq [-X/2, X/2]$, so $2k\log p \le X \iff k\log p \le X/2$.

(ii) **Weil test on the modulus half–line** ($u \ge 0$):

$$W(u) \;:=\; \frac{1}{\pi}\,(1-e^{-u})\,g_{\mathrm{b}}(2u). \tag{2.6}$$

(iii) **Poisson block** (pole of $\zeta$ at $s = 1$):

$$J(g) \;\coloneqq\; \frac{1}{2\pi}\int_0^\infty e^{-x/2}\, g_{\mathrm{b}}(x)\,\mathrm{d}x \;=\; \frac{1}{\pi}\int_0^\infty e^{-y}\, g_{\mathrm{b}}(2y)\,\mathrm{d}y \text{ via the substitution } y = x/2. \tag{2.7}$$

*Remark* 2.13 (Dictionary and continuous density locks)*.* For $u > 0$ we have the basic dictionary

$$W(u) \;=\; \frac{1}{\pi}\, e^{-u/2}\, G_{\mathrm{b}}(u) \quad (u > 0), \qquad (1 - p^{-k})\, g_{\mathrm{b}}(2k\log p) \;=\; p^{-k/2}\, G_{\mathrm{b}}(k\log p). \tag{2.8}$$

Combining the Archimedean block with the boundary and Poisson terms yields the continuous block:

$$W_{\mathrm{cont}}(g) \;=\; \int_0^\infty g_{\mathrm{b}}(x)\, \rho_W(x)\,\mathrm{d}x, \tag{2.9}$$

where $\rho_W(x)$ is the Weil continuous density (Definition 2.3). These identities are in the same Maaß−Selberg normalization as the automorphic paper. *All* later uses of Tonelli/DCT on $W_{\mathrm{cont}}$ and $P(g)$ rely on the strict $\sigma > 1/2$ tilt recorded in Thm. 2.9.

### 2.3 Analytical Justifications and Test Spaces

We establish the prerequisite lemmas justifying the distributional pairings and the analytical maneuvers required for the evaluation of the Archimedean block.

**Definition 2.14** (Distributional Test Spaces)**.** We define the required regularity properties for test functions on the modulus line:

- *Boundary-safe (half-line)* $\mathcal{D}_0$: Functions $\varphi \in C^\infty((0,\infty))$ such that $\varphi$ and all its derivatives extend continuously to $u = 0$ (denoted $\varphi \in C^\infty([0,\infty))$), satisfying the boundary condition $\varphi(0) = 0$.
- *Adapted even (full line)* $\mathcal{D}^\flat_{\mathrm{ev}}$: $\Phi \in C^\infty(\mathbb{R})$ such that $\Phi$ is even and $\Phi(0) = 0$.

We utilize the following subspaces characterized by support/decay, corresponding to the image of the spectral test classes under the mapping $g \mapsto W$:

(i) **Compact support ($\mathcal{D}_{0,c}$)**: The subspace of $\mathcal{D}_0$ consisting of functions with compact support. If $g \in \mathcal{A}_{\mathrm{BL}}$, then $W \in \mathcal{D}_{0,c}$.

(ii) **Exponential decay ($\mathcal{D}_0^{(\sigma)}$)**: The subspace of $\mathcal{D}_0$ consisting of functions with decay $O(e^{-2\sigma u})$. If $g \in \mathcal{A}_{\mathrm{Exp}}$ (with parameter $\sigma > 1/2$), then $W \in \mathcal{D}_0^{(\sigma)}$.

These ensure $W$ and all kernels pair absolutely; see Prop. 4.2 and Lemma 2.19.

**Definition 2.15** (Prime–continuous Weil measure)**.** Define the positive Borel measure on $(0,\infty)$

$$d\nu(x) \coloneqq \rho_W(x)\,\mathrm{d}x \;+\; \frac{1}{\pi}\sum_p \sum_{k\geq 1} (\log p)\,(1 - p^{-k})\, \delta_{2k\log p}(\mathrm{d}x), \tag{2.10}$$

where $\rho_W(x)$ is the Weil continuous density (Definition 2.3), and $\delta_a(\mathrm{d}x)$ denotes the Dirac measure at $x = a$. Then $Q_{\mathrm{geom}}(g) = \int_{(0,\infty)} g_{\mathrm{b}}(x)\, d\nu(x)$ whenever the RHS is absolutely convergent (cf. (2.9)). For the atomic part, since $g_{\mathrm{b}}$ is continuous (as $g \in \mathcal{S}(\mathbb{R})$), we evaluate at lattice sites:

$$\int g_{\mathrm{b}}(x)\, \delta_{2k\log p}(\mathrm{d}x) = g_{\mathrm{b}}(2k\log p).$$

This fixes the $L^1(d\nu)$ pairing and is the pairing used in the Fejér–Poisson regularization scheme (Section 5 and Paper III, §6), where the continuous part is handled by $\rho_W$ and atoms by smooth bump approximation (cf. Paper III Def. 4.6/Lem. 4.7).

*Remark* 2.16 (The $L^1(d\nu)$ Topology). The space $L^1(d\nu)$ is a Banach space with norm

$$\|g_{\mathrm{b}}\|_{L^1(d\nu)} \;=\; \int_0^{\infty} |g_{\mathrm{b}}(x)|\,\rho_W(x)\,\mathrm{d}x \;+\; \frac{1}{\pi}\sum_{p,\,k\geq 1}(\log p)(1-p^{-k})\,|g_{\mathrm{b}}(2k\log p)|.$$

This norm simultaneously controls the continuous density and the weighted atomic contributions, and is the natural topology for continuity of the full geometric functional $Q_{\mathrm{geom}}(g) = \int g_{\mathrm{b}}\,d\nu$.

**Lemma 2.17** (Density with quantitative atomic control)**.** *For $d\nu$ as defined above and any $\varepsilon > 0$, given $g_{\mathrm{b}} \in L^1(d\nu)$ there exists $h \in C_c^\infty((0,\infty))$ with $\|g_{\mathrm{b}} - h\|_{L^1(d\nu)} < \varepsilon$. Moreover, writing $L_M = \{2k\log p\}_{\mathit{finite}}$ and $C_\rho = \|\rho_W\|_\infty$, we can choose bumps $\eta_j$ supported on $(x_j - \delta, x_j + \delta)$ such that the quantitative bound $\int \eta_j \rho_W \leq 2\delta C_\rho \leq \varepsilon/(3C_M)$ holds (where $C_M$ is the total adjustment magnitude defined in the proof below), ensuring $h(x_j) = g_{\mathrm{b}}(x_j)$ for $x_j \in L_M$ and the tail atomic error is $< \varepsilon/3$.*

*Proof.* We provide a constructive proof demonstrating simultaneous control over the continuous and atomic parts.

Let $\varepsilon > 0$. **1. Continuous Approximation.** Since $\rho_W$ is bounded and bounded away from zero on $(0,\infty)$ (Lemma 2.5), the $L^1(\rho_W\,\mathrm{d}x)$ norm is equivalent to the unweighted $L^1(\mathrm{d}x)$ norm. Choose $h_c \in C_c^\infty((0,\infty))$ such that $\|g_{\mathrm{b}} - h_c\|_{L^1(\rho_W\,\mathrm{d}x)} < \varepsilon/3$.

**2. Atomic Tail Control.** Choose a finite lattice set $L_M = \{2k\log p\}$ so that the weighted $\ell^1$ norm of $g_{\mathrm{b}}$ outside $L_M$ (the atomic tail) is $< \varepsilon/3$.

**3. Adjustment on $L_M$.** Let $C_M = \sum_{x_j \in L_M} |g_{\mathrm{b}}(x_j) - h_c(x_j)|$ be the total adjustment magnitude. If $C_M = 0$, set $h = h_c$. Otherwise, let $C_\rho = \|\rho_W\|_\infty$ and choose $\delta > 0$ so that the intervals $(x_j - \delta, x_j + \delta)$ are disjoint and $2\delta C_\rho \leq \varepsilon/(3C_M)$. For each $x_j \in L_M$, pick $\eta_j \in C_c^\infty((x_j - \delta, x_j + \delta))$ with $0 \leq \eta_j \leq 1$ and $\eta_j(x_j) = 1$. Then $\int \eta_j(x)\rho_W(x)\,\mathrm{d}x < \varepsilon/(3C_M)$. Define

$$h_a(x) := \sum_{x_j \in L_M} (g_{\mathrm{b}}(x_j) - h_c(x_j))\,\eta_j(x), \qquad h := h_c + h_a.$$

By construction, $h \in C_c^\infty((0,\infty))$, $h(x_j) = g_{\mathrm{b}}(x_j)$ on $L_M$, and

$$\|h_a\|_{L^1(\rho_W\,\mathrm{d}x)} \;\leq\; C_M \cdot \frac{\varepsilon}{3C_M} \;=\; \frac{\varepsilon}{3}.$$

Therefore, $\|g_{\mathrm{b}} - h\|_{L^1(\rho_W\,\mathrm{d}x)} \leq \|g_{\mathrm{b}} - h_c\| + \|h_a\| < 2\varepsilon/3$, and the atomic error is zero on $L_M$ and $< \varepsilon/3$ on the tail. Hence $\|g_{\mathrm{b}} - h\|_{L^1(d\nu)} < \varepsilon$. This establishes the density of $C_c^\infty((0,\infty))$ in $L^1(d\nu)$ with quantitative atomic control. □

**Lemma 2.18** ($L^1(d\nu)$-continuity and density)**.** *On the class where $g_{\mathrm{b}} \in L^1(d\nu)$, the map $g_{\mathrm{b}} \mapsto Q_{\mathrm{geom}}(g) = \int g_{\mathrm{b}}\,d\nu$ is continuous in the $L^1(d\nu)$-norm. The density of $C_c^\infty((0,\infty))$ in $L^1(d\nu)$ is established by Theorem 2.17.*

*Proof.* Continuity is immediate since $\left|\int g_{\mathrm{b}}\,d\nu\right| \leq \|g_{\mathrm{b}}\|_{L^1(d\nu)}$. This topology is natural as it directly controls the convergence of the entire geometric functional $Q_{\mathrm{geom}}(g)$.

The density statement is provided by Theorem 2.17. □

**Lemma 2.19** (Regularity, Admissible Pairings, and Convergence Justification)**.** *If $g \in \mathcal{A}_{\mathrm{Exp}}$ (including $\mathcal{A}_{\mathrm{BL}}$), the associated Weil function $W$ satisfies $W(0) = 0$ and $W(u) = O(u)$ as $u \to 0$. If $g \in \mathcal{A}_{\mathrm{BL}}$, then $W \in \mathcal{D}_0$ and its even extension $W_{\mathrm{ev}} \in \mathcal{D}_{\mathrm{ev}}^\flat$.*

*The local pairings $\langle D_v, W\rangle$ are well-defined. Furthermore, the interchanges of integration (Tonelli/Fubini) and the passage to the limit (Dominated Convergence Theorem, DCT) employed in the evaluation of the Archimedean block (Proposition 5.3) are rigorously justified.*

*Proof.* The regularity of $W$ follows immediately from Definition 2.12, noting that $(1-e^{-u}) = u + O(u^2)$. The pairings are well-defined. At the infinite place, this is due to the $O(u)$ behavior of $W$ near $u = 0$, which counteracts the $O(1/u)$ singularity of the Archimedean kernel $K_\infty(u)$ (Definition 3.2).

To rigorously justify the convergence arguments in Proposition 5.3 for $g \in \mathcal{A}_{\mathrm{Exp}}$, we analyze the kernel used in the symmetric regularization (aligned with Paper I):

$$K_\alpha(x) := \frac{e^{-(\frac{1}{2}+\alpha)x} - e^{-\alpha x}}{1-e^{-x}}, \qquad x > 0,\ \alpha > 0.$$

We establish a uniform bound. The kernel simplifies to:

$$\begin{aligned} K_\alpha(x) &= \frac{e^{-\alpha x}(e^{-x/2}-1)}{1-e^{-x}} = -e^{-\alpha x}\frac{1-e^{-x/2}}{1-e^{-x}} \\ &= -e^{-\alpha x}\frac{1-e^{-x/2}}{(1-e^{-x/2})(1+e^{-x/2})} = -e^{-\alpha x}\frac{1}{1+e^{-x/2}}. \end{aligned}$$

For $\alpha > 0$ and $x > 0$, we have the uniform bound:

$$|K_\alpha(x)| = e^{-\alpha x}\frac{1}{1+e^{-x/2}} < \frac{1}{1+e^{-x/2}} < 1.$$

This bound is independent of $\alpha$. Since $g_{\mathrm{b}} \in L^1((0,\infty))$ (by the definition of $\mathcal{A}_{\mathrm{Exp}}$, which includes $\mathcal{A}_{\mathrm{BL}}$), the product $|K_\alpha(x)g_{\mathrm{b}}(x)|$ is dominated by $|g_{\mathrm{b}}(x)|$, which is an $\alpha$-independent integrable majorant.

**Justification for Tonelli/Fubini (fixed $\alpha > 0$):** We rigorously establish the absolute joint integrability of the integrand $F(\tau, x) = g(\tau)K_\alpha(x)\cos(\tau x)$ on the domain $\mathbb{R} \times (0,\infty)$, as required for the interchange in Proposition 5.3, Step 2. We compute the iterated integral of the absolute value:

$$I_{\mathrm{abs}} := \iint_{\mathbb{R}\times(0,\infty)} |F(\tau,x)|\,\mathrm{d}\tau\,\mathrm{d}x.$$

Using the bound $|\cos(\tau x)| \le 1$:

$$\begin{aligned} I_{\mathrm{abs}} &\le \int_0^\infty |K_\alpha(x)| \left(\int_{\mathbb{R}} |g(\tau)|\,\mathrm{d}\tau\right) \mathrm{d}x \\ &= \|g\|_{L^1(\mathbb{R})} \cdot \|K_\alpha\|_{L^1((0,\infty))}. \end{aligned}$$

We must verify that both factors are finite.

*(i) Finiteness of $\|g\|_{L^1(\mathbb{R})}$:* Since $g \in \mathcal{A}_{\mathrm{Exp}}$, by Definition 2.9, $|g(\tau)| \le C_g(1+|\tau|)^{-2}$. This decay ensures $\|g\|_{L^1(\mathbb{R})} < \infty$.

*(ii) Finiteness of $\|K_\alpha\|_{L^1((0,\infty))}$:* We analyze $K_\alpha(x) = -e^{-\alpha x}(1+e^{-x/2})^{-1}$. Since $\alpha > 0$ (fixed), the factor $e^{-\alpha x}$ ensures exponential decay. The factor $(1+e^{-x/2})^{-1}$ is bounded by 1.

$$\int_0^\infty |K_\alpha(x)|\,\mathrm{d}x < \int_0^\infty e^{-\alpha x}\,\mathrm{d}x = \frac{1}{\alpha}.$$

Since both factors are finite, $I_{\mathrm{abs}}$ is finite. This establishes the absolute joint integrability required by the Fubini-Tonelli theorem for the interchange of the $\tau$ and $x$ integrals.

**Justification for DCT ($\alpha \downarrow 0$):** We analyze the frequency-side integral $\int g_{\mathrm{b}}(x)K_\alpha(x)\,dx$. Since $g_{\mathrm{b}} \in L^1((0,\infty))$, the product $|K_\alpha(x)g_{\mathrm{b}}(x)|$ is dominated by the $\alpha$-independent majorant $|g_{\mathrm{b}}(x)|$ (using the uniform bound $|K_\alpha(x)| \le 1$). Furthermore, $K_\alpha(x)g_{\mathrm{b}}(x)$ converges pointwise to $K_0(x)g_{\mathrm{b}}(x)$ as $\alpha \downarrow 0$ for all $x > 0$. The Dominated Convergence Theorem therefore applies to the limit $\alpha \downarrow 0$. □

**Lemma 2.20** (Boundary Value Identity)**.** *For $g \in \mathcal{A}_{\mathrm{Exp}}$, the value $g(0)$ is related to the Weil function $W$ by*

$$g(0) = 2 \int_0^\infty \frac{W(u)}{1 - e^{-u}} \, \mathrm{d}u. \tag{2.11}$$

*Proof.* By the inversion formula (2.3), $g(0) = \frac{1}{\pi} \int_0^\infty g_{\mathrm{b}}(x) \, \mathrm{d}x$. From Definition 2.12, we have $g_{\mathrm{b}}(2u) = \frac{\pi W(u)}{1-e^{-u}}$. Changing variables $x = 2u$ ($\mathrm{d}x = 2\mathrm{d}u$):

$$g(0) = \frac{1}{\pi} \int_0^\infty \frac{\pi W(u)}{1 - e^{-u}} \, (2\mathrm{d}u) = 2 \int_0^\infty \frac{W(u)}{1 - e^{-u}} \, \mathrm{d}u.$$

The integral converges: near $u = 0$, $(1 - e^{-u})^{-1} = u^{-1} + O(1)$ and $W(u) = O(u)$ (Lemma 2.19), so the integrand is $O(1)$. As $u \to \infty$, $W(u) = \frac{1}{\pi}(1 - e^{-u}) \, g_{\mathrm{b}}(2u) = O(e^{-2\sigma u})$ with the same *strict* $\sigma > 1/2$ from Definition 2.9 when $g \in \mathcal{A}_{\mathrm{Exp}}$, while for $g \in \mathcal{A}_{\mathrm{BL}}$ the profile $g_{\mathrm{b}}$ has compact support. Hence the integral is absolutely convergent. □

## 2.4 Geometric Interpretation: The Half-Shift Kernel

We adopt a geometric perspective motivated by the adelic structure and its connection to the CR (Cauchy-Riemann) geometry of the Heisenberg boundary (as explored in [4]). We use this as intuition only.

**Analytic Framework.** In the modulus-line calculus of this paper, we work solely with the exponential-multiplier semigroup $(\mathsf{M}^a \phi)(x) := e^{-ax} \phi(x)$, arising in Laplace/Mellin representations. We do not identify $\mathsf{M}^a$ with translations $T^a$. All proofs rely entirely on this $\mathsf{M}$–framework (consistent with Paper I). The key object is the "half-shift torsion" operator $C_{1/2}$, which emerges from the Archimedean regularization (Proposition 5.3).

**Lemma 2.21** (Half-shift Kernel: Analytic Definition)**.** *The operator $C_{1/2}$ acts via the positive kernel $T_{1/2}(x)$:*

$$T_{1/2}(x) := (\mathrm{Id} - \mathsf{M}^{1/2})(\mathrm{Id} - \mathsf{M})^{-1} \, \mathbf{1}(x) = \frac{1}{1 + e^{-x/2}} = \frac{1 - e^{-x/2}}{1 - e^{-x}}. \tag{2.12}$$

*Remark* 2.22 (Motivational context: Commutator Analogy). Heuristically, the structure of $T_{1/2}$ can be motivated by an analogy to commutators that capture additive drift (e.g., in the affine or Heisenberg groups). While this provides intuition regarding the "torsion filter" terminology, we emphasize that the analytic proofs rely solely on the multiplier framework $\mathsf{M}^a$.

*Proof.* On $\mathbb{R}^\times_{>0}$, the kernel $T_{1/2}(x)$ is computed by applying $(\mathrm{Id} - \mathsf{M}^{1/2})(\mathrm{Id} - \mathsf{M})^{-1}$ to the constant function $\mathbf{1}(x)$, utilizing the geometric series expansion (valid as a distributional identity on test functions supported in $(0, \infty)$):

$$\begin{aligned}(\mathrm{Id} - \mathsf{M}^{1/2})(\mathrm{Id} - \mathsf{M})^{-1}\mathbf{1} &= (\mathrm{Id} - \mathsf{M}^{1/2}) \sum_{n=0}^\infty \mathsf{M}^n \mathbf{1} = \sum_{n=0}^\infty (\mathsf{M}^n - \mathsf{M}^{n+1/2})\mathbf{1} \\ &= \sum_{n=0}^\infty (e^{-nx} - e^{-(n+1/2)x}) = \frac{1}{1 - e^{-x}} - \frac{e^{-x/2}}{1 - e^{-x}} = \frac{1 - e^{-x/2}}{1 - e^{-x}}.\end{aligned}$$

The final expression follows from $1 - e^{-x} = (1 - e^{-x/2})(1 + e^{-x/2})$ for $x > 0$. □

*Remark* 2.23 (Translations, dilations, and multipliers). Under $u = \log x$, additive translations $T^a : f(u) \mapsto f(u - a)$ correspond to multiplicative *dilations* $(S_a \phi)(x) = \phi(e^{-a}x)$ on $(0, \infty)$. By contrast, the operators $(\mathsf{M}^a \phi)(x) = e^{-ax} \phi(x)$ are *exponential multipliers* arising naturally in the Laplace/Mellin calculus. Our torsion-kernel identity and all ensuing positivity statements are proved within the $\mathsf{M}$–framework; no identification $T^a \mapsto \mathsf{M}^a$ is assumed. We strictly distinguish between the translation $T^a$ and the multiplier $\mathsf{M}^a$ to keep the dictionary unambiguous.

*Remark* 2.24 (Geometric Interpretation and Analogies). This positive kernel $T_{1/2}(x)$ filters the geometric data, isolating the analytic (torsion-free) component.

*Remark* 2.25 (Heisenberg–Log-line–Density Dictionary). The following dictionary summarizes the structural correspondences motivated by the analogy:

- **Heisenberg boundary:** The commutator $[X, Y] = T$ generates torsion (drift) [4].
- **CR analyticity:** Defines the torsion-free sector by annihilating the drift [4].
- **Mellin half-shift kernel:** The multiplicative kernel $T_{1/2}(x) = \frac{1}{1+e^{-x/2}}$ implements the filtering in the $\mathsf{M}$-framework.
- **Geometric positivity:** The resulting continuous density $\rho_W$ from (2.2) governs the continuous geometric side; this structure underlies the Weil positivity analysis [5].

# 3 The Adelic Distributional Framework

## 3.1 The Adelic-Modulus Line Bridge

We briefly situate the framework within the standard adelic setting (cf. Tate's thesis, Weil [6], and treatments such as [1]). Let $\mathbb{A}^\times$ be the idele group of $\mathbb{Q}$. The Weil distribution $D_W$ acts on the Schwartz space $\mathcal{S}(\mathbb{A}^\times)$.

In the GL(1) case, this action is realized via the Mellin transform, mapping functions on $\mathbb{A}^\times$ to functions on the modulus line $\mathbb{R}^\times_{>0}$ (or its additive counterpart $\mathbb{R}$). An adelic Schwartz function $\Phi = \otimes_v \Phi_v \in \mathcal{S}(\mathbb{A}^\times)$ gives rise to the spectral test function $g$ via its Archimedean component $\Phi_\infty$. The Weil test function $W(u)$ is derived from $\Phi_\infty$ (and thus $g$) through the dictionary established in Section 2.2. The local distributions $D_v$ defined below correspond to the local components of the Weil distribution under this realization. This structure ensures that the global pairing translates precisely to the sum of local pairings $Q_{\mathrm{MS}}(g) = \sum_v \langle D_v, W\rangle$. This paper focuses on the rigorous evaluation of this modulus-line realization.

## 3.2 Global Functional and Local Distributions

The geometric side of the GL(1) explicit formula is defined globally as the sum of local distributional pairings over the places of $\mathbb{Q}$.

**Definition 3.1** (Global Geometric Functional)**.** The global Maaß−Selberg base functional $Q_{\mathrm{MS}}(g)$ is defined, for $g \in \mathcal{A}_{\mathrm{Exp}}$ (resp. $g \in \mathcal{A}_{\mathrm{BL}}$), by the absolutely convergent (resp. finite) sum (*absolute convergence follows from Propositions 4.2 and 5.3*):

$$Q_{\mathrm{MS}}(g) := \sum_v \langle D_v, W\rangle = \sum_{p<\infty} \langle D_p, W\rangle + \langle D_\infty, W\rangle. \tag{3.1}$$

The *full* geometric functional $Q_{\mathrm{geom}}(g)$ is defined via the Weil measure (Definition 2.15) as

$$Q_{\mathrm{geom}}(g) := \int g_{\mathrm{b}}\, d\nu.$$

We now define the local distributions $D_v$. At the infinite place, we adopt an explicit definition corresponding to the standard Weil local contribution associated with the factor $\Gamma_{\mathbb{R}}$.

**Definition 3.2** (Archimedean distribution $D_\infty$)**.** The Archimedean distribution $D_\infty$ acting on $\mathcal{D}_0$ (and its extension to the decaying space associated with $\mathcal{A}_{\mathrm{Exp}}$) is defined by the locally integrable kernel

$$K_\infty(u) := \frac{-e^{-u}}{1-e^{-2u}} = -\frac{1}{2\sinh u}. \tag{3.2}$$

The pairing is given by the convergent integral

$$\langle D_\infty, \varphi\rangle \;\coloneqq\; \int_0^\infty K_\infty(u)\,\varphi(u)\,\mathrm{d}u. \tag{3.3}$$

*Remark* 3.3. The kernel $K_\infty(u)$ has a singularity at $u = 0$, behaving as $K_\infty(u) = -\frac{1}{2u} + O(u)$. The convergence of the pairing (3.3) for the required test functions (such as $W(u)$) is guaranteed by Lemma 2.19, which ensures $\varphi(u) = O(u)$ as $u \to 0$. This cancellation ensures the integral converges absolutely near $u = 0$, requiring no principal value interpretation. If we consider the even extension of the kernel, $K_{\infty,\mathrm{ev}}(u) \coloneqq K_\infty(|u|)$, and the even extension of the test function $\varphi_{\mathrm{ev}}(u)$, the full-line pairing satisfies

$$\int_{\mathbb{R}} K_{\infty,\mathrm{ev}}(u)\,\varphi_{\mathrm{ev}}(u)\,\mathrm{d}u = 2\int_0^\infty K_\infty(u)\,\varphi(u)\,\mathrm{d}u = 2\langle D_\infty, \varphi\rangle.$$

We maintain the half-line definition (3.3) throughout.

We proceed to evaluate the pairings place by place.

# 4 The Prime Block (Finite Places)

## 4.1 Local Distribution at Finite Places

At a finite place $p$, the local L-factor is $\zeta_p(s) = (1-p^{-s})^{-1}$. The local distribution $D_p$ corresponds to the Fourier transform of the logarithmic derivative $-\zeta_p'(s)/\zeta_p(s)$.

**Definition 4.1** (Local Prime Distribution)**.** The distribution $D_p$ is defined by the weighted Dirac comb supported on the *positive* logarithms of prime powers ($u = k\log p > 0$), i.e. (supported in $u > 0$)

$$D_p(u) \coloneqq \sum_{k=1}^{\infty} (\log p)\,\delta(u - k\log p). \tag{4.1}$$

## 4.2 Evaluation of the Prime Block

**Proposition 4.2** (Prime Block Evaluation)**.** *Let $g \in \mathcal{A}_{\mathrm{Exp}}$. The total contribution of the finite places to $Q_{\mathrm{MS}}(g)$ is the prime block $P(g)$:*

$$P(g) \coloneqq \sum_p \langle D_p, W\rangle = \frac{1}{\pi}\sum_p \sum_{k=1}^{\infty} (\log p)(1 - p^{-k})\, g_{\mathrm{b}}(2k\log p). \tag{4.2}$$

*If $g \in \mathcal{A}_{\mathrm{BL}}$, this sum is finite. If $g \in \mathcal{A}_{\mathrm{Exp}}$, the sum converges absolutely.*

*Proof.* We evaluate the pairing at place $p$:

$$\langle D_p, W\rangle = \int_0^\infty W(u) D_p(u)\,\mathrm{d}u = \sum_{k=1}^{\infty} (\log p)\, W(k\log p).$$

Substituting the definition of $W$ from (2.6):

$$W(k\log p) = \frac{1}{\pi}\,(1 - e^{-k\log p})\, g_{\mathrm{b}}(2k\log p) = \frac{1}{\pi}\,(1 - p^{-k})\, g_{\mathrm{b}}(2k\log p).$$

Summing over all primes $p$ yields the expression for $P(g)$.

If $g \in \mathcal{A}_{\mathrm{BL}}$, $\mathrm{supp}(g_{\mathrm{b}}) \subset [-X, X]$ for some $X > 0$. The sum is restricted to $2k\log p \le X$, ensuring only a finite number of terms are non-zero.

If $g \in \mathcal{A}_{\mathrm{Exp}}$, we have $|g_{\mathrm{b}}(x)| \le Ce^{-\sigma|x|}$ with $\sigma > 1/2$. Since $|g_{\mathrm{b}}(2k\log p)| \le C\,p^{-2\sigma k}$, we have

$$(\log p)\,(1 - p^{-k})\,|g_{\mathrm{b}}(2k\log p)| \;\le\; C\,(\log p)\,\big(p^{-2\sigma k} - p^{-(2\sigma+1)k}\big),$$

so

$$|P(g)| \le \frac{C}{\pi}\sum_{p}\sum_{k\ge 1}(\log p)\big(p^{-2\sigma k} - p^{-(2\sigma+1)k}\big) = \frac{C}{\pi}\Big(-\frac{\zeta'}{\zeta}(2\sigma) + \frac{\zeta'}{\zeta}(2\sigma+1)\Big) < \infty.$$

In particular, absolute convergence holds since $2\sigma > 1$, so Tonelli/Fubini justifies the interchange of the prime–power sums with absolute values used above. Here we used the Euler–product identity [3, §2.6]

$$\sum_{p}\sum_{k\ge 1}(\log p)\,p^{-ak} \;=\; -\frac{\zeta'}{\zeta}(a) \qquad (\operatorname{Re} a > 1),$$

so the factor $(1 - p^{-k})$ produces the difference $-\zeta'/\zeta(2\sigma) + \zeta'/\zeta(2\sigma+1)$ displayed above (sign consistent with $\frac{\mathrm{d}}{\mathrm{d}a}\log\zeta(a)$). $\square$

*Remark* 4.3 (Symmetric Normalization). Utilizing the dictionary identity (Remark 2.13), we express the prime block in the standard symmetric normalization:

$$P(g) = \frac{1}{\pi}\sum_{p,k\ge 1}(\log p)\,p^{-k/2}\,G_{\mathrm{b}}(k\log p). \tag{4.3}$$

This normalization is maintained consistently across the companion papers [4, 5].

*Remark* 4.4 (Von Mangoldt formulation). Writing $P(g)$ with the von Mangoldt function yields

$$P(g) = \frac{1}{\pi}\sum_{n\ge 2}\Lambda(n)\,n^{-1/2}\,G_{\mathrm{b}}(\log n),$$

which is equivalent to (4.2) by grouping prime powers.

# 5 The Archimedean Block (Infinite Place)

This section rigorously connects the distributional pairing on the modulus line to the spectral definition of the Archimedean block, employing a careful regularization procedure.

## 5.1 The Archimedean Target

We define the Archimedean block $\mathcal{A}(g)$ following the normalization compatible with the Eisenstein phase derivative, adhering strictly to the Maaß−Selberg convention (as established in [4]).

**Definition 5.1** (Archimedean Block, Spectral Definition)**.** The Archimedean block $\mathcal{A}(g)$ is defined by the spectral pairing with the boundary kernel $K_A(\tau)$ (consistent with Paper I):

$$\mathcal{A}(g) \coloneqq \frac{1}{2\pi}\int_{\mathbb{R}} g(\tau)\,K_A(\tau)\,\mathrm{d}\tau, \qquad K_A(\tau) \coloneqq \operatorname{Re}\{\psi(i\tau) - \psi(\tfrac{1}{2} + i\tau)\}. \tag{5.1}$$

Here $\psi$ denotes the digamma function (see [2, §5.5]). We also define the $\alpha$*-regularized* functional for $\alpha > 0$, using the symmetric regularization consistent with Paper I:

$$\mathcal{A}_\alpha(g) \coloneqq \frac{1}{2\pi}\int_{\mathbb{R}} g(\tau)\,\operatorname{Re}\{\psi(\alpha + i\tau) - \psi(\tfrac{1}{2} + \alpha + i\tau)\}\,\mathrm{d}\tau. \tag{5.2}$$

We fix the branches as in Appendix A and [4]. As shown in Proposition 5.3, the limit $\lim_{\alpha\downarrow 0}\mathcal{A}_\alpha(g)$ exists and equals $\mathcal{A}(g) - \frac{1}{2}g(0)$. This limit on the spectral side is obtained by isolating the Poisson kernel and applying DCT to the remainder with a uniform $(1 + |\tau|)^{-2}$ majorant (see Appendix B). The subsequent frequency side evaluation uses DCT with the uniform majorant from Lemma 2.19. We have $K_A \in L^1(\mathbb{R})$ (Remark 5.2).

*Remark* 5.2 (Properties of the Kernel $K_A(\tau)$). The kernel $K_A(\tau)$ is $C^\infty$ on $\mathbb{R}$ and *bounded at the origin.* Using $\psi(z) = -\gamma - \frac{1}{z} + O(z)$ as $z \to 0$. We analyze the behavior near $\tau = 0$:

$$\psi(i\tau) = -\gamma - \frac{1}{i\tau} + O(\tau) = -\gamma + i\frac{1}{\tau} + O(\tau).$$

Since $\mathrm{Re}(i/\tau) = 0$, the singularity cancels when taking the real part: $\mathrm{Re}\,\psi(i\tau) = -\gamma + O(\tau^2)$. Thus, we obtain

$$K_A(0) \;=\; \lim_{\tau\to 0} \mathrm{Re}\{\psi(i\tau) - \psi(\tfrac{1}{2} + i\tau)\} = -\gamma - \psi\Big(\tfrac{1}{2}\Big) \in \mathbb{R}.$$

For large $|\tau|$, a standard asymptotic expansion of $\psi$ (via Stirling's series) yields

$$K_A(\tau) = \mathrm{Re}\{\psi(i\tau) - \psi(\tfrac{1}{2} + i\tau)\} \;=\; \frac{1}{8\tau^2} + O(\tau^{-4}) \qquad (|\tau| \to \infty),$$

*Note.* This expansion holds uniformly on vertical strips (e.g., $\{\mathrm{Re}\, z \in [0,1]\}$) when combined with standard identities for $\psi$. The decay $O(\tau^{-2})$ implies $K_A \in L^1(\mathbb{R})$ and $\mathcal{A}(g) = \frac{1}{2\pi}\int_{\mathbb{R}} g(\tau)K_A(\tau)\,\mathrm{d}\tau$ for $g \in \mathcal{S}(\mathbb{R})$. This decay matches the positivity paper's estimate $K_A(\tau) = O((1+|\tau|)^{-2})$ in the same ledger (cf. [5, Lemma 3.3]).

## 5.2 The Archimedean Distribution and Evaluation

We now establish the connection between the distributional pairing $\langle D_\infty, W\rangle$ (Definition 3.2) and the spectral definition of $\mathcal{A}(g)$. Although $K_A(\tau)$ is regular at the origin, the regularization procedure is essential to rigorously track the boundary contributions that arise when relating the spectral and frequency representations.

**Proposition 5.3** (Evaluation of the Archimedean pairing). *For any $g \in \mathcal{A}_{\mathrm{Exp}}$, the distributional pairing at the infinite place satisfies*

$$\langle D_\infty, W\rangle = -\mathcal{A}(g). \tag{5.3}$$

*Equivalently (on $\mathbb{R}$, with absolute convergence near $u = 0$),*

$$\frac{1}{2}\int_{\mathbb{R}} K_\infty(|u|)\, W_{\mathrm{ev}}(u)\,\mathrm{d}u \;=\; -\frac{1}{2\pi}\int_{\mathbb{R}} g(\tau)\, K_A(\tau)\,\mathrm{d}\tau,$$

*and the right–hand side equals $-\mathcal{A}(g)$ by Definition 5.1. This is the Archimedean dictionary recorded as (H.2) in Paper III [5], in the same Maaß – Selberg ledger.*

*Proof (Rigorous regularization and distributional evaluation).* We establish the identity by a rigorous regularization argument. In Steps 1–3, we derive the frequency (modulus line) representation of $\mathcal{A}(g)$ by analyzing the limit of the regularized functional $\mathcal{A}_\alpha(g)$ as $\alpha \downarrow 0$. In Step 4, we directly evaluate the distributional pairing $\langle D_\infty, W\rangle$ and show it matches $-\mathcal{A}(g)$.

The absolute convergence near $u = 0$ on the modulus line (LHS) is guaranteed by the regularity $W(u) = O(u)$ (Lemma 2.19) which cancels the $O(1/u)$ singularity of $K_\infty(u)$ (Remark 3.3), confirming no PV is needed there.

**Step 1: Analyzing the regularization limit (spectral side).** We analyze the behavior of the symmetric regularized kernel $K_\alpha^{\mathrm{sym}}(\tau) = \mathrm{Re}\{\psi(\alpha + i\tau) - \psi(\frac{1}{2} + \alpha + i\tau)\}$. As $\alpha \downarrow 0$, the dominant singular behavior near $\tau = 0$ comes from the first term:

$$\mathrm{Re}\{\psi(\alpha + i\tau)\} = \mathrm{Re}\left(\frac{-1}{\alpha + i\tau}\right) + R'_\alpha(\tau) = -\frac{\alpha}{\alpha^2 + \tau^2} + R'_\alpha(\tau).$$

The remainder $R'_\alpha(\tau)$ converges dominatedly to $\operatorname{Re}\psi(i\tau)$ (justified by uniform bounds; see Appendix B, PO–9). The second term $\operatorname{Re}\{\psi(\frac{1}{2}+\alpha+i\tau)\}$ also converges dominatedly to $\operatorname{Re}\{\psi(\frac{1}{2}+i\tau)\}$. We isolate the Poisson kernel:

$$K_\alpha^{\mathrm{sym}}(\tau) = -\frac{\alpha}{\alpha^2+\tau^2} + R_\alpha(\tau).$$

The total remainder $R_\alpha(\tau) = R'_\alpha(\tau) - \operatorname{Re}\{\psi(\frac{1}{2}+\alpha+i\tau)\}$ converges dominatedly to $K_A(\tau)$. By the properties of the Poisson kernel and DCT on the remainder (justified by uniform bounds):

$$\begin{aligned}\lim_{\alpha\downarrow 0}\mathcal{A}_\alpha(g) &= \frac{1}{2\pi}\left(\lim_{\alpha\downarrow 0}\int_{\mathbb{R}} g(\tau)\left(-\frac{\alpha}{\alpha^2+\tau^2}\right)d\tau + \int_{\mathbb{R}} g(\tau)K_A(\tau)d\tau\right)\\ &= -\frac{1}{2}g(0) + \mathcal{A}(g).\end{aligned}$$

$$\lim_{\alpha\downarrow 0}\mathcal{A}_\alpha(g) = \mathcal{A}(g) - \frac{1}{2}g(0). \tag{5.4}$$

**Step 2: Evaluation of the limit (frequency side).** We now evaluate $\mathcal{A}_\alpha(g)$ using the frequency representation ($g_{\mathrm{b}}$). We employ the integral representation of the digamma function (valid for $\operatorname{Re} z > 0$; see, e.g., [2, 5.9.15]):

$$\psi(z) = -\gamma + \int_0^\infty \frac{e^{-x}-e^{-zx}}{1-e^{-x}}\,\mathrm{d}x. \tag{5.5}$$

Substituting this into the definition of $A_\alpha(g)$ (using the symmetric regularization) and taking the real part yields the kernel:

$$\operatorname{Re}\{\psi(\alpha+i\tau) - \psi(\tfrac{1}{2}+\alpha+i\tau)\} = \int_0^\infty \frac{e^{-(\frac{1}{2}+\alpha)x}\cos(\tau x) - e^{-\alpha x}\cos(\tau x)}{1-e^{-x}}\,\mathrm{d}x.$$

(The constant terms involving $\gamma$ cancel.) Recognizing the integrand factor as $K_\alpha(x)$ (Lemma 2.19), we insert this into $A_\alpha(g)$ and interchange the order of integration. This application of Tonelli/Fubini is justified for $\alpha > 0$ (Lemma 2.19).

$$\begin{aligned}\mathcal{A}_\alpha(g) &= \frac{1}{2\pi}\int_0^\infty K_\alpha(x)\left(\int_{\mathbb{R}} g(\tau)\cos(\tau x)\,\mathrm{d}\tau\right)\mathrm{d}x\\ &= \frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(x)K_\alpha(x)\,\mathrm{d}x.\end{aligned}$$

We now take the limit $\alpha \downarrow 0$. By the Dominated Convergence Theorem (justified in Lemma 2.19), we can pass the limit inside the integral. The pointwise limit of the kernel is $K_0(x) = \frac{e^{-x/2}-1}{1-e^{-x}}$:

$$\lim_{\alpha\downarrow 0}\mathcal{A}_\alpha(g) = \frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(x)K_0(x)\,\mathrm{d}x = -\frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(x)\,\frac{1-e^{-x/2}}{1-e^{-x}}\,\mathrm{d}x.$$

Using the identity $\frac{1-e^{-x/2}}{1-e^{-x}} = \frac{1}{1+e^{-x/2}} = T_{1/2}(x)$, we find:

$$\lim_{\alpha\downarrow 0}\mathcal{A}_\alpha(g) = -\frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(x)\,\frac{1}{1+e^{-x/2}}\,\mathrm{d}x. \tag{5.6}$$

**Step 3: Deriving the Frequency Representation of $\mathcal{A}(g)$.** Equating the results from Step 1 (Eq. (5.4)) and Step 2 (Eq. (5.6)):

$$\mathcal{A}(g) - \frac{1}{2}g(0) = -\frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(t)\,\frac{1}{1+e^{-t/2}}\,\mathrm{d}t.$$

We use the identity $\frac{1}{2}g(0) = \frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(t)\mathrm{d}t$ to solve for $\mathcal{A}(g)$:

$$\mathcal{A}(g) = \frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(t)\left(1 - \frac{1}{1+e^{-t/2}}\right)\mathrm{d}t = \frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(t)\frac{e^{-t/2}}{1+e^{-t/2}}\mathrm{d}t.$$

**Step 4: Evaluation of the distributional pairing $\langle D_\infty, W\rangle$.** We now evaluate the pairing directly using the definitions $K_\infty(u) = -\frac{e^{-u}}{1-e^{-2u}}$ (Definition 3.2) and $W(u) = \frac{1}{\pi}(1-e^{-u})\, g_{\mathrm{b}}(2u)$ (Definition 2.12).

$$\begin{aligned}\langle D_\infty, W\rangle &= \int_0^\infty W(u)K_\infty(u)\,\mathrm{d}u \\ &= \frac{1}{\pi}\int_0^\infty (1-e^{-u})\, g_{\mathrm{b}}(2u)\left(-\frac{e^{-u}}{1-e^{-2u}}\right)\mathrm{d}u.\end{aligned}$$

Simplifying the kernel factor using $1-e^{-2u} = (1-e^{-u})(1+e^{-u})$:

$$\begin{aligned}\langle D_\infty, W\rangle &= -\frac{1}{\pi}\int_0^\infty g_{\mathrm{b}}(2u)\frac{e^{-u}(1-e^{-u})}{(1-e^{-u})(1+e^{-u})}\,\mathrm{d}u \\ &= -\frac{1}{\pi}\int_0^\infty g_{\mathrm{b}}(2u)\frac{e^{-u}}{1+e^{-u}}\,\mathrm{d}u.\end{aligned}$$

Applying the change of variables $x = 2u$ ($\mathrm{d}u = \mathrm{d}x/2$):

$$\langle D_\infty, W\rangle = -\frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(x)\frac{e^{-x/2}}{1+e^{-x/2}}\,\mathrm{d}x.$$

Comparing this result with the expression derived in Step 3, we conclude $\langle D_\infty, W\rangle = -\mathcal{A}(g)$. □

*Remark* 5.4 (Digamma Duplication, Constants, and Maaß – Selberg Normalization). We explicitly verify the handling of constants arising from alternative representations of the kernel $K_A(\tau)$. Using the duplication identity ([2, §5.5.6]),

$$\psi(2z) = \tfrac{1}{2}(\psi(z) + \psi(z+\tfrac{1}{2})) + \log 2.$$

This allows $K_A(\tau)$ to be rewritten as:

$$K_A(\tau) = \mathrm{Re}\{\psi(i\tau) - \psi(\tfrac{1}{2}+i\tau)\} = 2\ \mathrm{Re}\{\psi(i\tau) - \psi(2i\tau)\} + 2\log 2.$$

Let us define an alternative functional based on the first term:

$$\mathcal{A}'(g) := \frac{1}{2\pi}\int_{\mathbb{R}} g(\tau)\cdot 2\ \mathrm{Re}\{\psi(i\tau) - \psi(2i\tau)\}\,\mathrm{d}\tau.$$

Then, the definition of $\mathcal{A}(g)$ (Definition 5.1) satisfies the identity:

$$\mathcal{A}(g) = \mathcal{A}'(g) + \frac{1}{2\pi}\int_{\mathbb{R}} g(\tau)(2\log 2)\,\mathrm{d}\tau = \mathcal{A}'(g) + \frac{\log 2}{\pi}\, g_{\mathrm{b}}(0).$$

Crucially, in the Maaß – Selberg ledger adopted here, we use the definition $\mathcal{A}(g)$ which intrinsically includes this constant term. We do not separate it. The definition of $\mathcal{A}(g)$ leads directly to the frequency representation (Proposition 5.3, Step 3):

$$\mathcal{A}(g) = \frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(t)\frac{e^{-t/2}}{1+e^{-t/2}}\mathrm{d}t.$$

This integral representation confirms that the constant $2\log 2$ is correctly accounted for within the frequency kernel $\frac{e^{-t/2}}{1+e^{-t/2}}$.

When we form the full continuous density $\rho_W$ through the standard regrouping:

$$-\mathcal{A}(g) + \tfrac{1}{2}g(0) + 2J(g) = \int_0^\infty g_{\mathrm{b}}(x)\,\rho_W(x)\,\mathrm{d}x,$$

where $\rho_W(x)$ is defined in (2.2), all constant pieces are correctly absorbed into $\rho_W$. No standalone constant survives on the geometric side. This normalization is consistent with Paper I (App. A) and Paper III.

### 5.3 Half-Shift Kernel and Positivity

The regularization process explicitly reveals the connection between the Archimedean contribution and the geometric torsion filter $T_{1/2}(x)$ introduced in Section 2.4.

**Proposition 5.5** (Half-shift kernel and positivity)**.** *For $g \in \mathcal{A}_{\mathrm{Exp}}$, the Archimedean block combined with the boundary term satisfies:*

$$-\mathcal{A}(g) + \tfrac{1}{2}g(0) = \frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(x)\, T_{1/2}(x)\, \mathrm{d}x,$$

*where $T_{1/2}(x) = \frac{1}{1+e^{-x/2}}$ is the torsion filter kernel. In particular, $T_{1/2}(x) \in (1/2, 1)$ for $x > 0$ (strictly increasing, smooth). Consequently, the expression is strictly positive whenever $g_{\mathrm{b}} \geq 0$ (and $g_{\mathrm{b}}$ is not identically zero).*

*Proof.* This identity is a direct consequence of the rigorous relationship established during the regularization process in the proof of Proposition 5.3. By combining Eq. (5.4) and Eq. (5.6), we found:

$$A(g) - \frac{1}{2}g(0) = \lim_{\alpha \downarrow 0} A_\alpha(g) = -\frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(x)\, \frac{1}{1+e^{-x/2}}\, \mathrm{d}x.$$

Rearranging this identity immediately yields the stated result. □

*Remark* 5.6 (The Full Weil Density). Combining the result of Proposition 5.5 with the definition of the Poisson block $J(g)$ (Definition 2.12), we obtain the density representing the entire continuous geometric side of the explicit formula (1.3):

$$\begin{aligned} -\mathcal{A}(g) + \tfrac{1}{2}g(0) + 2J(g) &= \frac{1}{2\pi}\int_0^\infty \frac{g_{\mathrm{b}}(x)}{1+e^{-x/2}}\, \mathrm{d}x + 2\left(\frac{1}{2\pi}\int_0^\infty e^{-x/2} g_{\mathrm{b}}(x)\, \mathrm{d}x\right) \\ &= \int_0^\infty g_{\mathrm{b}}(x)\, \rho_W(x)\, \mathrm{d}x. \end{aligned}$$

Here, we recall the same $\rho_W$ from Definition 2.3 and do not reintroduce a separate label. This $\rho_W$ coincides *verbatim* with the modulus-line continuous density used in the positivity paper [5] (Paper III, Eq. (1.3) and §D.3), ensuring a normalization match across the trilogy in the Maaß−Selberg ledger. This density is manifestly positive and plays a central role in the analysis of Weil positivity presented in [5]. (Ledger consistency with Paper I [4] phase–measure and Paper III's regrouping is explicit in Paper I §§2, 6 and Ledger A.2.)

## 6 Main Identity and Geometric Positivity

We now assemble the local contributions to establish the main theorem and summarize the resulting geometric structure of the explicit formula.

**Theorem 6.1** (Maaß−Selberg Base Identity)**.** *Let $g \in \mathcal{A}_{\mathrm{Exp}}$ (including the admissible band-limited class $\mathcal{A}_{\mathrm{BL}}$) and let $W$ be the associated Weil test function. The global base functional $Q_{\mathrm{MS}}(g)$ satisfies the identity:*

$$Q_{\mathrm{MS}}(g) = P(g) - \mathcal{A}(g), \tag{6.1}$$

*where $P(g)$ is the prime sum (Proposition 4.2) and $\mathcal{A}(g)$ is the Archimedean block (Definition 5.1). Throughout, the spectral measure is fixed in the Maaß−Selberg convention $d\xi(\tau) = \frac{1}{2\pi}\varphi'(\tau)\,\mathrm{d}\tau$ (see Appendix A and Thm. 2.1).*

**Global Conventions and Normalizations.**

- Base functional: $Q_{\mathrm{MS}}(g) = P(g) - \mathcal{A}(g)$ (Maaß−Selberg ledger, symmetric PV).
- Completed zeta function: $\Lambda(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$.
- Fourier pair (forward without $2\pi$, inverse with $1/2\pi$): $\widehat{g}(\xi) = \int_{\mathbb{R}} g(\tau)e^{-i\tau\xi}\,\mathrm{d}\tau$, $g(\tau) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(\xi)e^{i\tau\xi}\,\mathrm{d}\xi$; for even $g$, $g_{\mathrm{b}}(x) = \widehat{g}(x)$ and $g(\tau) = \frac{1}{\pi}\int_0^\infty g_{\mathrm{b}}(x)\cos(x\tau)\,\mathrm{d}x$ (cosine dictionary in Thm. 2.11).
- Mellin transform: $\mathcal{M}f(s) = \int_0^\infty f(x)\,x^{s-1}\,\mathrm{d}x$; modulus-line coordinate $x = 2u$.
- Spectral measure (Maaß−Selberg): $d\xi(\tau) = (1/2\pi)\,\varphi'(\tau)\,\mathrm{d}\tau$, symmetric PV, no $\delta_0$; residues oriented by increasing $\tau$ (see Thm. 2.1). Under duplication, the constant $(\log 2/\pi)\,\widehat{g}(0)$ pairs with $J(g)$ and is absorbed into $\rho_W$ (no standalone constant on the geometric side).
- Test class: $g \in \mathcal{A}_{\mathrm{Exp}}$ (or the subclass $\mathcal{A}_{\mathrm{BL}}$).

All pairings and signs in Theorem 6.1 adhere strictly to these conventions.

**Conceptual Framing.** The adelic proof proceeds as a verification of a *distributional identity*. We construct a canonical Weil test function $W$ derived from the spectral test function $g$. We then evaluate the global base functional $Q_{\mathrm{MS}}(g) = \sum_v \langle D_v, W\rangle$. This geometric evaluation is then matched against the automorphic evaluation of the corresponding functional on the GL(1) side, as detailed in [4]. In the symmetric normalization adopted here, the zero weights are two-sided, and the Archimedean contribution is isolated according to the conventions summarized above. Equating the geometric and spectral evaluations yields the symmetric explicit formula in this specific normalization.

*Proof of Theorem 6.1.* The identity follows directly from the definition of $Q_{\mathrm{MS}}(g)$ (Definition 3.1) and the evaluation of the local pairings:

$$Q_{\mathrm{MS}}(g) = \sum_p \langle D_p, W\rangle + \langle D_\infty, W\rangle.$$

By Proposition 4.2, $\sum_p \langle D_p, W\rangle = P(g)$. By Proposition 5.3, $\langle D_\infty, W\rangle = -\mathcal{A}(g)$. Combining these yields $Q_{\mathrm{MS}}(g) = P(g) - \mathcal{A}(g)$. □

**Proposition 6.2** (Geometric Positivity Structure)**.** *Let $g \in \mathcal{A}_{\mathrm{Exp}}$. The geometric side of the full explicit formula* (1.3) *exhibits the following explicit positivity structure:*

*(i) The prime block $P(g)$ is a sum over the frequency profile $g_{\mathrm{b}}$ with strictly positive coefficients:*

$$P(g) = \frac{1}{\pi}\sum_p \sum_{k=1}^\infty c_{p,k}\, g_{\mathrm{b}}(2k\log p), \qquad c_{p,k} := (\log p)\,(1 - p^{-k}) \;>\; 0.$$

*(ii) The continuous part (the Archimedean contribution combined with the boundary term and the Poisson block) is represented by an integral against the strictly positive Weil density $\rho_W(x)$:*

$$-\mathcal{A}(g) + \tfrac{1}{2}g(0) + 2J(g) \;=\; \int_0^\infty g_{\mathrm{b}}(x)\,\rho_W(x)\,\mathrm{d}x,$$

*where $\rho_W(x)$ is defined in Eq.* (2.2)*.*

*Consequently, if $g_{\mathrm{b}}(x) \geq 0$ for all $x \geq 0$, the full geometric side of the explicit formula is nonnegative (and is strictly positive whenever $g_{\mathrm{b}} \not\equiv 0$ on a set of positive measure).*

*Proof.* Item (i) follows directly from the evaluation in Proposition 4.2. Item (ii) follows from the combination of Proposition 5.5 and the subsequent analysis in Remark 5.6. ☐

## Preprint status, claim scope, and open-review protocol

**Status (Paper II).** Unrefereed preprint. Version 2025-08-23. DOI: 10.5281/zenodo.16930092.

**Scope.** Paper II fixes the Maaß – Selberg normalization and constructs $Q_{\mathrm{MS}}(g) = P(g) - \mathcal{A}(g)$. *No RH claim is made here.* RH discussions in this paper are limited to classical equivalences (Weil positivity) and implication phrasing for context.
**Where the RH claim lives.** Paper III ([5]) *states a claimed proof of RH as a preprint*, pending peer review. Readers are encouraged to cross-check constants and pairings against the shared ledger across Papers I–III.
**What to review here.** (i) PV/no-$\delta_0$ Maaß – Selberg measure; (ii) the Archimedean dictionary $\langle D_\infty, W\rangle = -\mathcal{A}(g)$; (iii) prime/continuous density $\rho_W$; (iv) DCT/Fubini/Tonelli closures in the PO registry.
**Reproducibility.** Ancillary notebooks verify local pairings and the continuous density; versions mirror Paper III's checks.
**Versioning & policy.** Changelog maintained; normalization corrections propagate trilogy-wide. Funding: none. Competing interests: none. Contact for referee-style reports: `mgs33@cornell.edu`.

# A Normalization Ledger and Constant Audit

This appendix constitutes the canonical normalization ledger for the trilogy [4, 5]. It rigorously fixes the Maaß – Selberg convention $d\xi(\tau) = \frac{1}{2\pi}\,\varphi'(\tau)\,\mathrm{d}\tau$ and synchronizes all definitions, constants, and identities across the documents.

## A.1 Global Standing Notation

**Test Functions and Transforms.** The test function $g : \mathbb{R} \to \mathbb{R}$ is real, even, and typically in the class $\mathcal{A}_{\mathrm{Exp}}$ (often the subclass $\mathcal{A}_{\mathrm{BL}}$). Its cosine transform $g_{\mathrm{b}}(x)$ is defined as:

$$g_{\mathrm{b}}(x) \;:=\; \int_{\mathbb{R}} g(\tau)\,\cos(x\tau)\,\mathrm{d}\tau, \qquad g(\tau) \;=\; \frac{1}{\pi}\int_0^\infty g_{\mathrm{b}}(x)\,\cos(x\tau)\,\mathrm{d}x.$$

*Notation match:* The notation $g_{\mathrm{b}}(x)$ is used consistently across the trilogy (including Paper I [4]) for the cosine transform of an even function $g$. For clarity, we restrict to even profiles in this paper, so sine–odd contributions vanish by symmetry.

**Derived Profiles and Dictionary.** The symmetric profile $G_{\mathrm{b}}$, the Weil test function $W$ (on $u \geq 0$), and the Poisson block $J(g)$ are defined as:

$$G_{\mathrm{b}}(x) \;:=\; 2\sinh\Big(\frac{|x|}{2}\Big)\,g_{\mathrm{b}}(2x), \qquad W(u) \;:=\; \frac{1}{\pi}\,(1-e^{-u})\,g_{\mathrm{b}}(2u).$$
$$J(g) \;:=\; \frac{1}{2\pi}\int_0^\infty e^{-x/2}\,g_{\mathrm{b}}(x)\,\mathrm{d}x.$$

The key dictionary identities relating these profiles are:

$$W(u) \;=\; \frac{1}{\pi}\,e^{-u/2}\,G_{\mathrm{b}}(u) \quad (u>0), \qquad (1-p^{-k})\,g_{\mathrm{b}}(2k\log p) \;=\; p^{-k/2}\,G_{\mathrm{b}}(k\log p). \tag{A.1}$$

The origin functional identity is $g(0) = 2\int_0^\infty \frac{W(u)}{1-e^{-u}}\,\mathrm{d}u$. This follows by setting $\tau = 0$ in (2.3) and using $W(u) = \frac{1}{\pi}(1-e^{-u})\,g_{\mathrm{b}}(2u)$ from the dictionary above.

**Torsion Filter and Weil Density.** The half-shift commutator (torsion filter) $C_{1/2}$ is implemented by the positive kernel $T_{1/2}(x) = 1/(1+e^{-x/2})$. The full Weil density is $\rho_W(x) = \frac{1}{2\pi}(T_{1/2}(x) + 2e^{-x/2})\,\mathbf{1}_{(0,\infty)}(x)$, and the continuous block is $W_{\mathrm{cont}}(g) = \int_0^\infty g_{\mathrm{b}}(x)\,\rho_W(x)\,\mathrm{d}x$.

## A.2 Local Distributions and Kernels

| **Finite place ($p$) — Distribution $D_p(u)$** | |
|---|---|
| **Definition / Kernel** | $D_p(u) = \sum_{k\ge 1}(\log p)\,\delta(u - k\log p)$ |
| **Pairing / Output** | $\langle D_p, W\rangle = \frac{1}{\pi}\sum_{k\ge 1}(\log p)\,(1 - p^{-k})\,g_{\mathrm{b}}(2k\log p)$ (finite for $\mathcal{A}_{\mathrm{BL}}$; absolutely convergent for $\mathcal{A}_{\mathrm{Exp}}$) |

| **Archimedean place ($\infty$) — Distribution $D_\infty$ (Modulus line)** | |
|---|---|
| **Definition / Kernel** | $K_\infty(u) = -\dfrac{e^{-u}}{1-e^{-2u}} = -\dfrac{1}{2\sinh u}$ |
| **Pairing / Output** | $\langle D_\infty, W\rangle = -\mathcal{A}(g)$ |

| **Archimedean place ($\infty$) — Kernel $K_A(\tau)$ (Spectral line)** | |
|---|---|
| **Definition / Kernel** | $K_A(\tau) = \mathrm{Re}\{\psi(i\tau) - \psi(\tfrac12 + i\tau)\}$ |
| **Pairing / Output** | *Functional:* $\mathcal{A}(g) = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}} g(\tau)\,K_A(\tau)\,\mathrm{d}\tau$. |
| | *Regrouping:* $-\mathcal{A}(g) + \tfrac12 g(0) + 2J(g) = \displaystyle\int_0^\infty g_{\mathrm{b}}(x)\,\rho_W(x)\,\mathrm{d}x$. |
| | *Density:* $\rho_W(x) = \frac{1}{2\pi}\Big(\frac{1}{1+e^{-x/2}} + 2e^{-x/2}\Big)\,\mathbf{1}_{(0,\infty)}(x)$. |

The Archimedean pairing identity (Proposition 5.3) rigorously confirms the consistency between the modulus line and spectral line representations.

## A.3 Arithmetic and Spectral Blocks (Pairings and Normalizations)

**Arithmetic block (finite places).** The global prime block is the sum of the local contributions:

$$P(g) \;=\; \sum_p \langle D_p, W\rangle \;=\; \frac{1}{\pi}\sum_{p,k\ge 1}(\log p)\,p^{-k/2}\,G_{\mathrm{b}}(k\log p). \tag{A.2}$$

**Spectral measure and phase derivative (Birman–Krein spectral-shift).** [1] Let $\widetilde{S}(s) = \Lambda(2s-1)/\Lambda(2s)$ and $\varphi'(\tau) = \mathrm{Im}\,\partial_\tau \log\widetilde{S}(\tfrac12 + i\tau)$. With the Maaß−Selberg convention

$$d\xi(\tau) = \frac{1}{2\pi}\,\varphi'(\tau)\,\mathrm{d}\tau, \qquad \mathcal{I}(g) = \langle \xi, g\rangle = \frac{1}{2\pi}\,\mathrm{PV}\!\int_{\mathbb{R}} g(\tau)\,\varphi'(\tau)\,\mathrm{d}\tau \quad \text{(symmetric PV)}, \tag{A.3}$$

with *no* $\delta_0$ *mass*. Using the functional equation $\Lambda(1-s) = \Lambda(s)$ and conjugation symmetry, the boundary identity is equivalently written as

$$\varphi'(\tau) = \mathrm{PV}\;4\;\mathrm{Re}\left\{\frac{\Lambda'}{\Lambda}(2i\tau)\right\}.$$

*Crosswalk:* Paper I uses the same notation $\varphi'(\tau)$ for this object, ensuring consistency across the trilogy (same ledger). [4, 5] *Ledger note.* The Maaß−Selberg spectral–shift measure carries *no* $\delta_0$ *mass* and the PV is *symmetric*; see Thm. 2.1.

[1] See, e.g., [7].

**Zero weights.** For a nontrivial zero $\rho = \beta + i\gamma$ (with $\beta \in (0,1)$), the spectral weight is defined in the canonical normalization as:

$$\mathcal{H}(g;\rho) \;\coloneqq\; \frac{1}{2\pi}\int_0^\infty e^{-\beta x/2}\, g_{\mathrm{b}}(x)\, \cos(\gamma x/2)\, \mathrm{d}x. \tag{A.4}$$

The total spectral side (zero-side quadratic form) $W_{\mathrm{spec}}(g)$ is defined by summing over all nontrivial zeros $\rho$ (counted with multiplicity):

$$W_{\mathrm{spec}}(g) \;\coloneqq\; 2\sum_{\rho} \mathcal{H}(g;\rho).$$

(We reserve $W(u)$ for the Weil modulus-line test function.) The factor 2 is conventional in the asymmetric formula (see Eq. (A.7)). Absolute convergence of $\sum_\rho |\mathcal{H}(g;\rho)|$ for $g \in \mathcal{S}(\mathbb{R})$ is recorded in the companion Paper I, Appendix B.

## A.4 Bridge Identities and the Explicit Formula

**Lemma A.1** (Archimedean–trivial handshake; PV/limit commutation)**.** *In the Maaß – Selberg ledger (symmetric PV on $\mathbb{R}$ and regularization from $\mathrm{Re}(s) = \frac{1}{2} + \varepsilon$), and for* even *test functions $g$ in the admissible classes $\mathcal{S}_{\mathrm{even}} \cup \mathcal{A}_{\mathrm{Exp}}$, the Archimedean and trivial local pieces satisfy*

$$\mathcal{I}_{\mathrm{arch}}(g) + \mathcal{I}_{\mathrm{Triv}}(g) \;=\; J(g) \;-\; \tfrac{1}{2}\, g(0),$$

*and no additional constant survives. Moreover, the PV operation in $\tau$ commutes with the limit $\varepsilon \downarrow 0$ used to remove the Poisson kernels.*

*Proof.* (a) Under the symmetric principal value, the odd kernel contributes no mass against even test functions: for even $g$, $\mathrm{PV}\!\int_{\mathbb{R}} g(\tau)\, \frac{d\tau}{\tau} = 0$. Equivalently, the putative singular odd part is invisible in the real (even) pairing. Thus only even parts contribute near $\tau = 0$. (b) For the $\Gamma$- and $\zeta$-blocks write the phase derivatives as $P_\varepsilon(\tau) + R_{\Gamma,\varepsilon}(\tau)$ and $-P_\varepsilon(\tau) + R_{\zeta,\varepsilon}(\tau)$ on fixed compacts $K_1, K_2$ about the axis. The remainders are holomorphic there and uniformly bounded in $\varepsilon$ by the Extreme Value Theorem; away from these compacts one has a uniform global growth bound $\ll \log(2 + |\tau|)$. Consequently, $|g(\tau)| \cdot |R_{\Gamma,\varepsilon}(\tau) + R_{\zeta,\varepsilon}(\tau)|$ admits an $L^1(\mathbb{R})$ majorant independent of $\varepsilon$ for the stated test classes. (c) The kernels $P_\varepsilon$ cancel *exactly* between $\Gamma$ and $\zeta$, so the PV integral of the sum reduces to the PV integral of $g(\tau)\big(R_{\Gamma,\varepsilon}(\tau) + R_{\zeta,\varepsilon}(\tau)\big)$. Since this integrand is locally bounded near $\tau = 0$, its PV equals its ordinary integral. Apply dominated convergence on $|\tau| > \delta$ with the above majorant and then let $\delta \downarrow 0$ using the local uniform bounds to conclude that PV in $\tau$ commutes with the limit $\varepsilon \downarrow 0$. (d) The only boundary contribution comes from integrating by parts in the Dirichlet block and is exactly $-\frac{1}{2} g(0)$ (as computed in Paper I). Passing to *phase derivatives* annihilates additive constants, so no stray constants (e.g. $2\log 2$) survive in this ledger. These points give the stated identity. $\square$

The explicit formula arises from equating two distinct evaluations of the spectral functional $\mathcal{I}(g)$.

**Route A (Dirichlet/geometric evaluation).** This route evaluates $\mathcal{I}(g)$ using the Dirichlet series representation of $\Lambda'/\Lambda$, involving the geometric terms $P(g)$ and $\mathcal{A}(g)$, and the boundary term from regularization.

$$\mathcal{I}(g) \;=\; \mathcal{A}(g) \;-\; \frac{1}{2}\, g(0) \;-\; P(g). \tag{A.5}$$

**Route B (Spectral/partial fractions evaluation).**

*Remark* A.2 (Branch and normalization). We use the completed zeta $\Lambda(s)$ and define $\log \Lambda(s)$ by analytic continuation from $\operatorname{Re} s > 1$ using the principal branch. The partial-fractions expansion follows from the Hadamard product for $\Lambda$, and poles/zeros at $0, 1$ are handled within $\Lambda$; see [8, Ch. 2]. This fixes the constants in (A.6).

This route evaluates $\mathcal{I}(g)$ using the Hadamard product representation of $\Lambda(s)$, involving the sum over zeros and the contribution from the poles (Poisson block).

$$\mathcal{I}(g) \; = \; 2\, J(g) \; - \; 2 \sum_{\rho} \mathcal{H}(g; \rho). \tag{A.6}$$

**The Asymmetric Explicit Formula.**

**Theorem A.3** (Asymmetric Explicit Formula)**.** *Under the hypotheses of §2 (in particular $g$ admissible in the sense of Assumption 2.10) and in the Maaß–Selberg normalization, the asymmetric explicit formula holds:*

$$\boxed{2 \sum_{\rho} \mathcal{H}(g; \rho) = P(g) + \frac{1}{2} g(0) + 2J(g) - \mathcal{A}(g).} \tag{A.7}$$

*Proof.* This follows by equating the two evaluations of $\mathcal{I}(g)$ established in Route A ((A.5)) and Route B ((A.6)). □

*Constant cancellation under duplication.* Using $\psi(2z) = \frac{1}{2}(\psi(z)+\psi(z+\frac{1}{2}))+\log 2$, one may rewrite $K_A$ as $2\ \operatorname{Re}\{\psi(i\tau)-\psi(2i\tau)\}+2\log 2$. Pairing $2\log 2$ against $g$ contributes $\frac{1}{2\pi}\int_{\mathbb{R}} g(\tau)(2\log 2)\,\mathrm{d}\tau = (\log 2/\pi)\, g_{\mathrm{b}}(0)$. Under the standard regrouping

$$-\mathcal{A}(g) + \tfrac{1}{2} g(0) + 2J(g) = \int_0^{\infty} g_{\mathrm{b}}(x)\, \rho_W(x)\, \mathrm{d}x,$$

the constant $(\log 2/\pi)\, g_{\mathrm{b}}(0)$ is absorbed in $\rho_W$, so no stray constant remains on the geometric side; cf. Remark 5.4.

Equivalently, the continuous block equals $W_{\mathrm{cont}}(g)$ in (2.9) with $\rho_W$ from (2.2). This form is equivalent to the symmetric Guinand–Weil formula when zeros are paired as $\rho \leftrightarrow 1-\rho$.

## A.5 Auxiliary Identities

For reference, the Archimedean kernel admits the Taylor expansion near the origin

$$K_{\infty}(u) = -\frac{1}{2u} + \frac{u}{12} + O(u^3) \qquad (u \to 0),$$

which makes the cancellation with $W(u) = O(u)$ manifest in the pairing $\langle D_{\infty}, W \rangle$.

**Poisson/Laplace pairing identity.** For $a > 0$, $\gamma \in \mathbb{R}$, and $g \in \mathcal{A}_{\mathrm{Exp}}$, the following identity relates the Poisson kernel pairing on the spectral line to the Laplace transform of $g_{\mathrm{b}}$:

$$\int_{\mathbb{R}} g(\tau)\, \frac{a}{a^2 + (2\tau - \gamma)^2}\, \mathrm{d}\tau \; = \; \frac{1}{2} \int_0^{\infty} e^{-ax/2}\, g_{\mathrm{b}}(x)\, \cos\!\Big(\frac{\gamma x}{2}\Big)\, \mathrm{d}x \tag{A.8}$$

*Standard kernel identity used:* for $a > 0$ and $x \in \mathbb{R}$,

$$\int_{\mathbb{R}} \frac{a}{a^2 + t^2} \cos(xt)\, \mathrm{d}t = \pi e^{-a|x|},$$

with absolute convergence. Together with the even inversion $g(\tau) = \frac{1}{\pi}\int_0^{\infty} g_{\mathrm{b}}(x)\cos(x\tau)\,\mathrm{d}x$, this yields (A.8).

**Minimal Cross-Walk of Symbols Across the Trilogy**

| Concept | Paper I [4] | Paper II (This paper) | Paper III [5] |
|---|---|---|---|
| Test Class (Band-Limited) | Admissible | $\mathcal{A}_{\mathrm{BL}}$ | $\mathcal{A}_{\mathrm{BL}}$ |
| Cosine transform of $g$ | $\widehat{g}(x)$ | $g_{\mathrm{b}}(x)$ | $g_{\mathrm{b}}(x)$ |
| Symmetric profile | $\widehat{G}(x)$ | $G_{\mathrm{b}}(x)$ | — |
| Weil test on modulus line | — | $W(u)$ | $W(u)$ |
| Archimedean kernel (spectral) | $K_A(\tau)$ (implicit) | $K_A(\tau)$ | $K_A(\tau)$ |
| Archimedean kernel (modulus) | — | $K_\infty(u)$ | — |
| Arithmetic block | $P(g)$ | $P(g)$ | $P(g)$ |
| Poisson block | $J(g)$ | $J(g)$ | $J(g)$ |
| Zero weights | $\mathcal{H}(g;\rho)$ | $\mathcal{H}(g;\rho)$ | $\mathcal{H}(g;\rho)$ |
| Torsion Filter Kernel | — | $T_{1/2}(x)$ | $T_{1/2}(x)$ |
| Spectral measure | $d\mu$ (Maaß−Selberg) | $d\mu$ | $d\mu$ |

# B Proof Obligations Registry (Internal Checklist)

This appendix summarizes the status of proof obligations referenced in the text, primarily concerning analytical justifications for convergence and interchanges.

**PO–1 (DCT for $\alpha \downarrow 0$ and $\varepsilon \downarrow 0$):** Uniform majorant $|K_\alpha(t)g_{\mathrm{b}}(t)| \le |g_{\mathrm{b}}(t)| \in L^1$, independent of $\alpha \in (0, 1/2]$; see Lemma 2.19. *Status: closed.*

**PO–2 (Tonelli/Fubini for primes):** In $\mathcal{A}_{\mathrm{BL}}$, only finitely many $(p, k)$ contribute after pairing (compact support of $g_{\mathrm{b}}$). For $\mathcal{A}_{\mathrm{Exp}}$, absolute convergence is established in Proposition 4.2. *Status: closed.*

**PO–3 ($K_A \in L^1(\mathbb{R})$):** Using the asymptotic $K_A(\tau) = \frac{1}{8\tau^2} + O(\tau^{-4})$ as $|\tau| \to \infty$ (see Thm. 5.2), we obtain $K_A \in L^1(\mathbb{R})$ and thus $\mathcal{A}(g)$ is well-defined for $g \in \mathcal{S}(\mathbb{R})$. *Status: closed.*

**PO–4 (Poisson kernel—approximate identity):** Standard identity $\delta_0 = \lim_{\alpha\downarrow 0} \frac{1}{\pi}\frac{\alpha}{\alpha^2+\tau^2}$ in $\mathcal{S}'(\mathbb{R})$; used to extract $\frac{1}{2}g(0)$ in Proposition 5.3. *Status: closed.*

**PO–5 (Atomlessness at $\tau = 0$):** Cancellation of Poisson-kernel singularities implies no $\delta_0$ mass in $\varphi'(\tau)$; see Lemma 2.7 and App. A. *Status: closed.*

**PO–6 (Fourier–cosine agreement):** For *real even* $g$, $g_{\mathrm{b}}(x) = \mathrm{Re}\, \widehat{g}(x) = \widehat{g}(x)$. *Status: closed (Remark 2.11).*

**PO–7 (Boundedness of $K_A$ at 0):** $K_A(0) = -\gamma - \psi(\frac{1}{2})$ by cancellation of the simple pole in $\psi(i\tau)$. *Status: closed (see Thm. 5.2).*

**PO–8 (Even extension (half–line $\leftrightarrow$ full–line)):** For $\varphi \in C^\infty([0,\infty))$ with $\varphi(0) = 0$,

$$\int_0^\infty K_\infty(u)\,\varphi(u)\,\mathrm{d}u \;=\; \tfrac{1}{2}\int_{\mathbb{R}} K_{\infty,\mathrm{ev}}(u)\,\varphi_{\mathrm{ev}}(u)\,\mathrm{d}u,$$

where $K_{\infty,\mathrm{ev}}(u) \coloneqq K_\infty(|u|)$. No principal value is needed since $\varphi(u) = O(u)$ cancels the $1/u$ singularity. *Status: closed (Thm. 3.3).*

**PO–9 (Uniform bounds for spectral regularization (DCT justification)).** The justification for DCT in Proposition 5.3 (Step 1) and the analysis in Lemma 2.7 require uniform bounds on the remainders after extracting the Poisson kernel. Let $P_\alpha(\tau) = \frac{\alpha}{\alpha^2+\tau^2}$.

(i) The remainder $R'_\alpha(\tau) \coloneqq \mathrm{Re}\,\psi(\alpha + i\tau) + P_\alpha(\tau)$ converges dominatedly to $\mathrm{Re}\,\psi(i\tau)$ as $\alpha \downarrow 0$.

(ii) The term $\mathrm{Re}\,\psi(\frac{1}{2} + \alpha + i\tau)$ converges dominatedly to $\mathrm{Re}\,\psi(\frac{1}{2} + i\tau)$.

These follow from standard uniform estimates for $\psi(z)$ on vertical strips (e.g., DLMF §5.11). Crucially, the total remainder $R_\alpha(\tau)$ (defined in Prop. 5.3) satisfies a uniform bound $|R_\alpha(\tau)| \le C(1 + |\tau|)^{-2}$ for some constant $C$ independent of $\alpha \in (0, 1/2]$. This decay is inherited from the uniform asymptotics of the digamma difference (cf. Remark 5.2). When paired with $g(\tau) = O((1 + |\tau|)^{-2})$, this yields an $L^1$ majorant. *Status: closed.*

## Computational Verification: Adelic Geometric Pairing

**Scope and Objective.** We provide numerical verification of the central identities established in this paper. The primary objectives were to validate:

(i) The global geometric identity: $Q_{\mathrm{MS}}(g) = \sum_v \langle D_v, W \rangle = P(g) - \mathcal{A}(g)$.

(ii) The Archimedean consistency (Proposition 5.3): the equivalence between the distributional pairing $\langle D_\infty, W \rangle$ (modulus line integral) and the spectral definition of $\mathcal{A}(g)$ (spectral line integral).

(iii) The positivity identities involving the full Weil density $\rho_W(x)$ (Remark 5.6).

**Artifacts (Ancillary Files).** The implementation details, source code, and comprehensive numerical results are provided in the ancillary files accompanying this submission: adelic_pairing_verification.ipynb and rhoW_positivity.ipynb.

**Methodology and Parameters.** The numerical tests utilize the same suite of band-limited test functions employed in the verification for [4]. The computations are performed with high precision (mpmath configured for 80 decimal places). Numerical integration utilizes high-resolution Simpson grids. The singularity of $K_\infty(u)$ near $u = 0$ is handled analytically via series expansion to ensure stability. The prime summation $P(g)$ is computed exactly up to the cutoff determined by the band limit $X$ of the test function.

**Outcome.** The numerical results confirm that the Archimedean pairing $\langle D_\infty, W \rangle$ agrees with the negative of the spectral definition $-\mathcal{A}(g)$ to the specified numerical tolerances across all test cases. The global geometric identity $Q_{\mathrm{MS}}(g) = P(g) - \mathcal{A}(g)$ holds consistently. The positivity identities associated with the Weil density $\rho_W(x)$ are also verified. These results validate the theoretical framework and the rigorous normalization established in this paper.

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