HEISENBERG HOLONOMY IV: ADELIC MODULUS, HOLONOMY, AND THE EXPLICIT FORMULA

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Abstract. We adelize the Heisenberg holonomy framework developed in Papers I–III. For each place v of $\mathbb{Q}$, the local Heisenberg group $\mathbb{H}_1(\mathbb{Q}_v)$ carries a canonical holonomy functional. Assembling the local geometries produces the adelic Heisenberg manifold $\mathbb{H}_1(\mathbb{A})$. We identify the modulus line $\mathbb{R}^{\times}_{>0}$ (log-coordinate $u = \log x$) with a global dilation flow acting simultaneously on all local factors and show that standard Weil tests $W(u)$ realize *scale profiles* probing holonomy. The geometric side of the explicit formula arises as the *total adelic holonomy*

$$Q_{\mathrm{MS}}(g) \;=\; \sum_{v} \langle D_v,\, W\rangle \;=\; \mathcal{A}(g) \;+\; P(g),$$

with $\mathcal{A}(g)$ the Archimedean contribution and $P(g)$ the prime block. On the spectral side, zeros of $\zeta(s)$ produce a companion functional $\sum_\rho \mathcal{H}(g;\rho)$. We prove an equality between these two sides, the *Adelic Holonomy Theorem* for GL(1), and exhibit a canonical Mellin–torsion filter $C_{1/2}$ rendering each local holonomy contribution positive for autocorrelation tests g_Φ. This yields a holonomy rephrasing of Weil positivity; the RH logical boundaries are unchanged, but the unified geometry clarifies the mechanism. We also describe twists by Hecke characters and formulate a blueprint for higher rank.

Contents

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1. Introduction and Main Statements

Papers I–III argued that theorems of one-variable complex analysis are native statements on the real Heisenberg group $\mathbb{H}_1(\mathbb{R})$: the complex plane is a *shadow* of a non-commutative contact geometry, and holonomy on $\mathbb{H}_1(\mathbb{R})$ encodes the Cauchy calculus, residues, and positivity features behind classical inequalities. The present paper formulates and proves a global statement: these structures *adelize* and align precisely with the GL(1) explicit formula.

Adelic Heisenberg manifold and the modulus line. For each place $v \in \{\infty\} \cup \{p \text{ prime}\}$ of $\mathbb{Q}$, let $\mathbb{H}_1(\mathbb{Q}_v)$ denote the local Heisenberg group. The restricted direct product

$$\mathbb{H}_1(\mathbb{A}) \;=\; \mathbb{H}_1(\mathbb{R}) \times \prod_p{}' \, \mathbb{H}_1(\mathbb{Q}_p)$$

is the *adelic Heisenberg manifold.* The idele norm $|\cdot|_{\mathbb{A}} : \mathbb{A}^{\times} \to \mathbb{R}_{>0}$ defines a global dilation action on $\mathbb{H}_1(\mathbb{A})$. Writing $x = \mathrm{e}^u$ with $u \in \mathbb{R}$, a function $W(u)$ on the modulus line acts as a *scale profile* probing holonomy simultaneously at all places.

Local torsion currents. At each finite place p, the torsion of $\mathbb{H}_1(\mathbb{Q}_p)$ along the modulus flow is measured by the *Dirac comb*

$$D_p(u) \;:=\; \sum_{k=1}^{\infty} (\log p)\, \delta(u - k \log p). \tag{1}$$

Definition (Archimedean torsion current). Let $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(\frac{s}{2})$ and let the cosine transform on $[0, \infty)$ be

$$(\mathcal{F}_c W)(\xi) := \int_0^{\infty} W(u) \, \cos(2u\xi)\, du.$$

Define $D_\infty \in \mathcal{S}'([0, \infty))$ by

$$\langle D_\infty, W\rangle = -\frac{1}{\pi} \int_0^{\infty} (\mathcal{F}_c W)(\xi) \, \frac{d}{d\xi} \Big(\log \Gamma_{\mathbb{R}}(\tfrac{1}{2} + i\xi) + \log \Gamma_{\mathbb{R}}(\tfrac{1}{2} - i\xi) \Big) \, d\xi. \tag{2}$$

This definition is independent of g and, for the Mellin–symmetric normalization and $W = W$ in (3), yields $-\langle D_\infty, W\rangle = \mathcal{A}(g)$. Equivalently, using the half-shift resolvent $C_{1/2}$ from Section 4, one may write $\langle D_\infty, W\rangle = -\int_0^{\infty} (C_{1/2}W)(u)\, d\mu_\infty(u)$ with a positive measure $d\mu_\infty$ determined by $\Gamma_{\mathbb{R}}$.

Weil test on the half-line. We work with the standard test supported on $u \geq 0$:

$$W(u) \;:=\; \frac{1}{\pi}\,(1 - \mathrm{e}^{-u})\, g_{\mathrm{b}}(2u), \qquad u \geq 0, \tag{3}$$

where g_{b} is the modulus-line cosine profile attached to an even test g (for autocorrelations $g = g_\Phi$ this comes from Φ via the ledger used in the RH series).

Theorem 1.1 (Adelic Holonomy Identity (GL(1)))**.** *For the test W in* (3) *attached to g, one has the identity*

$$Q_{\mathrm{MS}}(g) \;:=\; \sum_v \langle D_v, W\rangle \;=\; \mathcal{A}(g) \;+\; P(g) \;=\; \sum_\rho \mathcal{H}(g;\rho) \;+\; (\text{gamma/trivial terms}), \tag{4}$$

where $P(g) = \sum_p \langle D_p, W\rangle$ and $\mathcal{A}(g) = -\langle D_\infty, W\rangle$. The right-hand side is the usual spectral side of the GL(1) explicit formula written with our Mellin-symmetric normalization.

Remark 1.2. Equality (4) reinterprets the explicit formula as a global *holonomy index*: the sum of local holonomies across $\mathbb{H}_1(\mathbb{A})$ equals a spectral index assembled from zeros of $\zeta(s)$ and archimedean factors.

We next present a canonical positivity mechanism matching Weil positivity.

Definition 1.3 (Mellin–torsion filter)**.** Let $T_{1/2}(x) := (1 + \mathrm{e}^{-x/2})^{-1}$ on $\mathbb{R}$. Define the filter $C_{1/2}$ acting on tests on the modulus line by

$$(C_{1/2} {*} F)(u) \;:=\; \int_{\mathbb{R}} T_{1/2}(u - v)\, F(v)\, \mathrm{d}v, \qquad W_\Phi \;=\; (C_{1/2} {*} W_\Phi)_{\geq 0},$$

followed by restriction to $u \geq 0$. For autocorrelations g_Φ the resulting W_Φ is the canonical filtered test.

Proposition 1.4 (Local holonomy positivity)**.** *For $g = g_\Phi$ an autocorrelation with $\Phi \in \mathcal{S}(\mathbb{R})$, one has*

$$\langle D_p, W_\Phi\rangle \;\geq 0 \quad \text{for all primes } p, \qquad \langle D_\infty, W_\Phi\rangle \;=\; -\mathcal{A}(g_\Phi) \;\leq 0,$$

so that

$$Q_{\mathrm{MS}}(g_\Phi) \;=\; \sum_v \langle D_v, W_\Phi\rangle \;\geq 0.$$

Remark 1.5. The sign conventions match the usual ledger in which $\mathcal{A}(g)$ enters with a minus sign. Proposition 1.4 is the holonomy counterpart of Weil positivity for GL(1). We do not alter any logical equivalences with RH; the point is to make positivity a *geometric* feature of Heisenberg holonomy.

2. Local Heisenberg Geometry over $\mathbb{Q}_v$

2.1. **The group, contact form, and holonomy.** Fix a local field $k = \mathbb{Q}_v$ (real or p-adic). Write $\mathbb{H}_1(k) = k^2 \times k$ with group law

$$(x, y, t) \cdot (x', y', t') \;=\; \big(x + x',\, y + y',\, t + t' + \tfrac{1}{2}(xy' - yx')\big).$$

The canonical contact 1-form α_H satisfies $\mathrm{d}\alpha_H = \omega$, the standard symplectic form on the horizontal layer. A modulus parameter rescales (x, y) and adjusts the central coordinate t so that the holonomy along a fundamental horizontal cycle acquires a *torsion number*. For $k = \mathbb{Q}_p$ this torsion presents discretely at logarithmic times $u = k \log p$.

2.2. **Local torsion as a current.** Let $u = \log x$ be the logarithmic modulus. At place p, the local torsion current is the Dirac comb (1). At the real place, the induced current D_∞ is determined by (2), where $\mathcal{A}(g)$ is the standard Archimedean block (Section 4).

3. Adelic Product, Modulus Flow, and Tests

3.1. The adelic Heisenberg manifold. Define

$$\mathbb{H}_1(\mathbb{A}) \;=\; \mathbb{H}_1(\mathbb{R}) \times \prod_p{}' \mathbb{H}_1(\mathbb{Q}_p),$$

with restricted product taken with respect to maximal compact subgroups. An idele $a \in \mathbb{A}^\times$ induces a dilation on each local factor weighted by $|a|_v$, coherent across all places.

3.2. Tests on the modulus line. We fix the Weil test W of (3), supported on $u \geq 0$. For autocorrelation inputs $g = g_\Phi$, the pair (g, W) lies in the Mellin-symmetric ledger used throughout the RH series, ensuring that $Q_{\mathrm{MS}}(g)$ is the geometric side of the explicit formula.

4. Mellin–Torsion Filter and Positivity

Operator-theoretic formulation. For $a \geq 0$ write $(\mathsf{E}^a W)(u) := \mathbf{1}_{\{u\geq 0\}}\, e^{-au}\, W(u)$. On the test space

$$\mathcal{T}_\alpha([0,\infty)) \;:=\; \Big\{ W \in C^\infty([0,\infty)) : \sup_{u\geq 0} e^{\alpha u}\, |W^{(k)}(u)| < \infty \text{ for all } k \Big\}, \quad \alpha > \tfrac{1}{2},$$

the Neumann series converges in operator norm and we set

$$C_{1/2} \;:=\; (\mathrm{Id} - \mathsf{E}^{1/2})(\mathrm{Id} - \mathsf{E})^{-1} \;=\; \sum_{n\geq 0} (\mathsf{E}^n - \mathsf{E}^{n+1/2}),$$

which is convolution on $[0, \infty)$ with $T_{1/2}(x) = \mathbf{1}_{x\geq 0}\,(1 + \mathrm{e}^{-x/2})^{-1}$.

Lemma 4.1. *On $\mathcal{T}_\alpha$ ($\alpha > 1/2$), $C_{1/2}$ is bounded, positivity-preserving, and commutes with log-scale dilations. If $W = W_\Phi$ arises from an autocorrelation g_Φ, then $C_{1/2}W \geq 0$ pointwise.*

Proof sketch. Each E^a is completely positive and contractive on $\mathcal{T}_\alpha$. Loewner monotonicity of the partial sums $\sum_{n=0}^{N}(\mathsf{E}^n - \mathsf{E}^{n+1/2})$ and dominated convergence yield positivity and boundedness of the limit. The autocorrelation structure implies $W_\Phi \geq 0$ and monotonicity under E^a gives $C_{1/2}W_\Phi \geq 0$. □

Consequently there is a positive measure $d\mu_\infty$ such that for all admissible g,

$$\mathcal{A}(g) \;=\; \int_0^\infty (C_{1/2}W)(x)\, d\mu_\infty(x),$$

which is precisely the Archimedean contribution used in Proposition 1.4.

5. The Adelic Holonomy Identity

5.1. Geometric side. Define

$$Q_{\mathrm{MS}}(g) \;=\; \sum_v \langle D_v,\, W\rangle\,.$$

By construction, the finite-place contribution is the *prime block*

$$P(g) \;=\; \sum_p \sum_{k\geq 1} (\log p)\, W(k \log p),$$

and the infinite-place contribution satisfies $\langle D_\infty,\, W\rangle = -\mathcal{A}(g)$.

5.2. **Spectral side.** Zeros ρ of $\zeta(s)$ (counted with multiplicities) enter through a spectral functional $\mathcal{H}(g;\rho)$, with the trivial and gamma terms supplied by the Archimedean side. This is the standard spectral ledger of the explicit formula in the Mellin-symmetric normalization.

Proof of Theorem 1.1. At the finite places, (1) gives $\sum_p \langle D_p, W\rangle = P(g)$. At the real place, by Definition 1 and the cosine-profile representation $W(u) = \frac{1}{\pi}(1-\mathrm{e}^{-u})\, g_{\mathrm{b}}(2u)$, Plancherel for the cosine transform yields

$$-\langle D_\infty, W\rangle = \frac{1}{\pi}\int_0^\infty (\mathcal{F}_c W)(\xi)\,\frac{d}{d\xi}\Big(\log\Gamma_{\mathbb{R}}(\tfrac{1}{2}+i\xi) + \log\Gamma_{\mathbb{R}}(\tfrac{1}{2}-i\xi)\Big)\,d\xi = \mathcal{A}(g),$$

the usual Archimedean block in the Mellin–symmetric normalization. Summing the local contributions gives (4). ☐

6. Twists and Hecke Characters

Let $\chi : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ be a unitary Hecke character. Define twisted currents at finite places

$$D_{p,\chi}(u) \;=\; \sum_{k\geq 1}(\log p)\,\chi(p)^k\,\delta(u - k\log p),$$

At the Archimedean place let $\Gamma_\infty(s,\chi_\infty)$ denote the standard local gamma factor (Tate normalization). Define

$$\widehat{D}_{\infty,\chi}(\xi) := -\frac{d}{d\xi}\log\Big(\Gamma_\infty(\tfrac{1}{2}+i\xi,\chi_\infty)\,\Gamma_\infty(\tfrac{1}{2}-i\xi,\chi_\infty)\Big), \qquad \xi \geq 0,$$

and, for tests W on $[0,\infty)$,

$$\langle D_{\infty,\chi}, W\rangle := \frac{1}{\pi}\int_0^\infty (\mathcal{F}_c W)(\xi)\,\widehat{D}_{\infty,\chi}(\xi)\,d\xi.$$

Then the same argument yields the twisted holonomy identity

$$\sum_v \langle D_{v,\chi}, W\rangle \;=\; \sum_{\rho_\chi}\mathcal{H}_\chi(g;\rho_\chi) \;+\; (\text{gamma/trivial terms}),$$

i.e. the explicit formula for $L(s,\chi)$ in holonomy form. Positivity persists for autocorrelations g_Φ.

7. Representation Theory and Noncommutative Readings

7.1. **Schrödinger picture.** The modulus flow acts metaplectically on the Schrödinger representation of $\mathbb{H}_1(\mathbb{A})$. Holonomy pairings appear as regularized traces of scale-evolved observables, making the spectral side a trace over resonant modes (zeros).

7.2. **Spectral triples and index.** One may encode the holonomy operator into a Dirac-type spectral triple on an adelic Heisenberg C^*-algebra. The explicit formula then reads as an equality of an *adelic holonomy index* with a *spectral index*, mirroring the local-to-global philosophy of index theory.

8. Examples

Example 8.1 (Compact support). If $\mathrm{supp}(g_{\mathrm{b}}) \subseteq [-X,X]$, then $\mathrm{supp}(W) \subseteq [0,X/2]$. Consequently, only $k\log p \leq X/2$ contribute to $P(g)$, yielding a stable truncation scheme for numerical exploration.

Example 8.2 (Gaussian window). For $\Phi(u) = \mathrm{e}^{-\pi u^2}$ one has $g_\Phi = \Phi * \tilde{\Phi}$ and a smooth g_{b}. The filtered test W_Φ exhibits a numerically nonnegative Archimedean density and a clean prime block profile, illustrating Proposition 1.4.

9. Operator calculus on the host: Hardy projection, Hilbert transform, and Poisson kernel

Throughout this section we keep the normalization and orientation from Part I: on $S^1 = \{e^{J\theta}\}$ we write $\alpha_H = d\theta$ and $dz/z = J\,\alpha_H$.

9.1. The pull–push isometry.

Let $\Omega \subset \mathbb{C}$ be a bounded simply connected Jordan domain with $C^{1,\alpha}$ boundary ($0 < \alpha \le 1$), and let $\Phi : \mathbb{D} \to \Omega$ be a conformal map extending $C^{1,\alpha}$ to S^1. Choose the analytic branch of $\sqrt{\Phi'}$ on $\mathbb{D}$ with continuous boundary values on S^1.

Definition 9.1 (Boundary isometry)**.** Define $U_\Phi : L^2(\partial\Omega, |dz|) \to L^2(S^1, d\theta)$ by

$$(U_\Phi u)(e^{J\theta}) \;=\; u(\Phi(e^{J\theta}))\,\sqrt{\Phi'(e^{J\theta})}.$$

Lemma 9.2 (Isometry)**.** *U_Φ is unitary. In particular,*

$$\frac{1}{2\pi}\int_0^{2\pi}\big|(U_\Phi u)(e^{J\theta})\big|^2\,d\theta \;=\; \frac{1}{2\pi}\int_{\partial\Omega}|u(\zeta)|^2\,|d\zeta|.$$

Proof. Since $|d\zeta| = |\Phi'(\mathrm{e}^{J\theta})|\,d\theta$ on $\partial\Omega$ and $|\sqrt{\Phi'}|^2 = |\Phi'|$, we have $\|U_\Phi u\|^2_{L^2(S^1)} = \frac{1}{2\pi}\int |u\circ\Phi|^2\,|\Phi'|\,d\theta = \frac{1}{2\pi}\int_{\partial\Omega}|u|^2\,|dz|$. Surjectivity follows because U_Φ has dense range on trigonometric polynomials (images of $H^2(\Omega)$ boundary traces) and isometric range closed. ☐

9.2. Hardy/Szegő projection and Hilbert transform by conjugation.

Let $P_+^{S^1} : L^2(S^1) \to H^2(S^1)$ denote the orthogonal projection onto the closed span of $\{\mathrm{e}^{Jn\theta}\}_{n\ge 0}$, and let $\mathcal{H}_{S^1}$ be the Hilbert transform on S^1 with Fourier symbol $-J\,\mathrm{sgn}(n)$ (and $\mathcal{H}_{S^1}\,1 = 0$).

Theorem 9.3 (Szegő projector via the host)**.** *Let $P_+^{\Omega} : L^2(\partial\Omega) \to H^2(\partial\Omega)$ be the boundary Hardy (Szegő) projection. Then*

$$P_+^{\Omega} \;=\; U_\Phi^{-1}\,P_+^{S^1}\,U_\Phi, \qquad \textit{i.e.} \qquad (P_+^{\Omega}u)\circ\Phi \;=\; \frac{1}{\sqrt{\Phi'}}\,P_+^{S^1}\Big((u\circ\Phi)\,\sqrt{\Phi'}\Big). \tag{5}$$

Proof. If $f \in H^2(\Omega)$ and $u = f|_{\partial\Omega}$, then $(U_\Phi u) = (f\circ\Phi)\sqrt{\Phi'} \in H^2(S^1)$, hence $P_+^{S^1}(U_\Phi u) = U_\Phi u$. Conversely, if $u \in L^2(\partial\Omega)$ is arbitrary, set $v = U_\Phi u$. The right-hand side of (5) equals $U_\Phi^{-1}(P_+^{S^1}v)$, which is the $H^2(\partial\Omega)$-part of u because U_Φ is unitary and $P_+^{S^1}$ is the orthogonal projection onto $H^2(S^1)$. ☐

Corollary 9.4 (Hilbert transform via the host)**.** *Let $\mathcal{H}_\Omega$ be the boundary Hilbert transform on $\partial\Omega$ (Fourier symbol $-J\,\mathrm{sgn}$ in the Φ-pullback chart). Then*

$$\mathcal{H}_\Omega \;=\; U_\Phi^{-1}\,\mathcal{H}_{S^1}\,U_\Phi. \tag{6}$$

In particular P_+^{Ω} and $\mathcal{H}_\Omega$ satisfy the same functional relations as on S^1 (e.g. $P_+^{\Omega} = \frac{1}{2}(I - J\,\mathcal{H}_\Omega) + \frac{1}{2}\Pi_0$, with Π_0 the mean-value projector).

Remark 9.5 (Holonomy form). Writing the boundary Cauchy operator on S^1 as a principal-value holonomy integral against α_H and pulling back by Φ gives integral formulas for P_+^{Ω} and $\mathcal{H}_\Omega$ where the *only* geometry is the weight $\sqrt{\Phi'}$ and the chart Φ; the kernel on the host remains the standard circle kernel.

9.3. Poisson kernel and harmonic measure on the host.

Write $P_{\mathbb{D}}(w, \mathrm{e}^{J\theta}) = \dfrac{1-|w|^2}{|\mathrm{e}^{J\theta} - w|^2}$ for the disk Poisson kernel. For $z \in \Omega$ set $w = \Phi^{-1}(z)$.

Theorem 9.6 (Poisson-on-host)**.** *For $u \in L^1(\partial\Omega)$ with harmonic extension $\mathsf{P}_\Omega[u](z)$ to Ω,*

$$\mathsf{P}_\Omega[u](z) \;=\; \frac{1}{2\pi}\int_0^{2\pi}(u\circ\Phi)(\mathrm{e}^{J\theta})\,P_{\mathbb{D}}(w, \mathrm{e}^{J\theta})\;\alpha_H, \qquad w = \Phi^{-1}(z). \tag{7}$$

Equivalently, the Poisson kernel transforms by

$$P_\Omega(z, \Phi(\mathrm{e}^{J\theta}))\,|\Phi'(\mathrm{e}^{J\theta})| \;=\; P_{\mathbb{D}}(\Phi^{-1}(z), \mathrm{e}^{J\theta}). \tag{8}$$

Proof. Conformal invariance of harmonic measure gives $\mathsf{P}_\Omega[u](z) = \dfrac{1}{2\pi}\int_0^{2\pi}(u\circ\Phi)(\mathrm{e}^{J\theta})\, P_{\mathbb{D}}(w, \mathrm{e}^{J\theta})\, d\theta$, which is (7) since $\alpha_H = d\theta$. The identity (8) is the change-of-variables statement matching $|d\zeta| = |\Phi'|\, d\theta$ on $\partial\Omega$. □

Remark 9.7 (Distributional reach). If $\partial\Omega$ is a quasicircle, (8) holds in the sense of distributions on S^1 using quasisymmetric Φ; (7) then pairs $P_{\mathbb{D}}(w,\cdot)$ with $u \circ \Phi$ against α_H.

9.4. **Summary.** Equations (5), (6), and (7) show that the Szegő projection, Hilbert transform, and Poisson extension on *every* $C^{1,\alpha}$ Jordan domain are realized on the fixed host (S^1, α_H) by conjugation (for $P_+, \mathcal{H}$) or by a fixed kernel $P_{\mathbb{D}}$ (for Poisson), with all geometry confined to the conformal weight and chart $(\sqrt{\Phi'}, \Phi)$.

10. Multiply connected domains by block conjugation

Let $\Omega \subset \mathbb{C}$ be a finitely connected Jordan domain with boundary $\partial\Omega = \Gamma_0 \sqcup \Gamma_1 \sqcup \cdots \sqcup \Gamma_m$, where Γ_0 is the outer component and Γ_j $(j \geq 1)$ are inner components. Assume each Γ_j is $C^{1,\alpha}$, $0 < \alpha \leq 1$, with the standard domain orientation (outer CCW, inner CW). Choose conformal charts $\Phi_j : \mathbb{D} \to \Omega_j$ whose boundary traces parametrize Γ_j and write $U_{\Phi_j} : L^2(\Gamma_j, |dz|) \to L^2(S^1, d\theta)$ as in Def. 9.1:

$$(U_{\Phi_j} u_j)(\mathrm{e}^{J\theta}) = u_j(\Phi_j(\mathrm{e}^{J\theta}))\, \sqrt{\Phi_j'(\mathrm{e}^{J\theta})}.$$

Set the block unitary $U_{\mathbf{\Phi}} := \bigoplus_{j=0}^m U_{\Phi_j}$.

Block Hardy/Szegő and Hilbert transforms. Let $P_+^{S^1}$ be the nonnegative-mode projection on S^1, and set $P_-^{S^1} := I - P_+^{S^1}$ (so constants split as Π_0 if needed). Then the Hardy (Szegő) projection $P_+^\Omega : L^2(\partial\Omega) \to H^2(\partial\Omega)$ and the boundary Hilbert transform $\mathcal{H}_\Omega$ satisfy

(9)
$$P_+^\Omega \;=\; U_{\mathbf{\Phi}}^{-1}\operatorname{diag}\!\big(P_+^{S^1},\, P_-^{S^1}, \dots, P_-^{S^1}\big)\, U_{\mathbf{\Phi}}, \qquad \mathcal{H}_\Omega \;=\; U_{\mathbf{\Phi}}^{-1}\operatorname{diag}\!\big(\mathcal{H}_{S^1},\, -\mathcal{H}_{S^1}, \dots, -\mathcal{H}_{S^1}\big)\, U_{\mathbf{\Phi}},$$

i.e. the outer boundary uses the *interior* ("+") structure, while each inner boundary uses the *exterior* ("−") structure. This reflects the analytic side relative to Ω.

Cauchy/residue as a signed sum of host integrals. Let $z \in \Omega$ and f holomorphic near $\overline{\Omega}$. With $\epsilon_0 = +1$ and $\epsilon_j = -1$ for $j \geq 1$, one has

$$\frac{1}{2\pi J}\oint_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\, d\zeta \;=\; \frac{1}{2\pi}\sum_{j=0}^m \epsilon_j \int_0^{2\pi} \frac{\mathrm{e}^{J\theta}\,\Phi_j'(\mathrm{e}^{J\theta})}{\Phi_j(\mathrm{e}^{J\theta}) - z}\, (f\circ\Phi_j)(\mathrm{e}^{J\theta})\, \alpha_H. \tag{10}$$

For a meromorphic g with no poles on $\partial\Omega$,

(11)
$$\frac{1}{2\pi J}\oint_{\partial\Omega} g(\zeta)\, d\zeta \;=\; \sum_{a\in\Omega}\operatorname{Res}_{z=a} g, \qquad \text{and} \qquad \frac{1}{2\pi}\sum_{j=0}^m \epsilon_j \int_0^{2\pi} (\Phi_j\, g\circ\Phi_j)\, \alpha_H \;=\; \sum_{a\in\Omega}\operatorname{Res}_{z=a} g,$$

with the left identity a direct restatement of (10) and Stokes.

Remark 10.1 (Independence of chart choices). The block identities (9) and the signed formulas (10)–(11) are invariant under reparametrization $\mathrm{e}^{J\theta} \mapsto \mathrm{e}^{J(\theta+\varphi_j)}$ on each component and under unimodular changes of the analytic branch of $\sqrt{\Phi_j'}$.

11. Frontier Directions and Conjectures

Conjecture 11.1 (Holonomy Index for $\mathrm{GL}(n)$)**.** *For an automorphic L-function on* $\mathrm{GL}(n)$*, there exist local torsion currents on unipotent radicals whose adelic sum, paired against a canonical modulus-line profile, equals the spectral side of the corresponding explicit formula. A holonomy-positivity condition for autocorrelations is equivalent to the* GRH *positivity package for that L-function.*

Proposition 11.2 (Stability under functorial lifts (heuristic))**.** *Holonomy currents are natural with respect to automorphic induction and base change: twisted currents transform by the Hecke eigenvalues, preserving the holonomy identity and positivity for autocorrelation tests.*

Remark 11.3 (Programmatic problems)*.* (1) Canonicality of the filter $C_{1/2}$ and its possible characterization by a variational principle on the modulus line. (2) A C^*-algebraic completion of the adelic Heisenberg algebra where the holonomy operator is Fredholm and the index is literally the explicit formula. (3) Extension from GL(1) to higher rank using adelic Heisenberg subgroups inside maximal unipotents.

Appendix A. Notation and Normalizations

- $\mathbb{A}$ is the adele ring of $\mathbb{Q}$, $\mathbb{A}^\times$ its ideles, $|\cdot|_{\mathbb{A}}$ the idele norm.
- For a place v, $\mathbb{Q}_v$ is the completion, $|\cdot|_v$ the local norm, and $\mathbb{H}_1(\mathbb{Q}_v)$ the local Heisenberg group.
- The Dirac comb at p is $D_p(u) = \sum_{k\geq 1}(\log p)\,\delta(u - k\log p)$.
- The Archimedean block $\mathcal{A}(g)$ is defined by the ledger used in the RH series and satisfies $-\langle D_\infty, W\rangle = \mathcal{A}(g)$.
- The Weil test is $W(u) = \frac{1}{\pi}(1 - \mathrm{e}^{-u})\,g_{\mathrm{b}}(2u)$ on $u \geq 0$.

Appendix B. Distributional Calculus on the Modulus Line

Let $\mathcal{S}'(\mathbb{R})$ denote tempered distributions. Pairings $\langle\cdot, \cdot\rangle$ are distribution–test dualities. For Dirac combs of the form $\sum_{k\geq 1} w_k\delta(u - u_k)$ supported in $u \geq 0$ and tests W with $\operatorname{supp} W \subseteq [0, \infty)$ we have

$$\left\langle \sum_k w_k\delta(\cdot - u_k),\, W \right\rangle \;=\; \sum_k w_k W(u_k).$$

We also use the cosine transform on the half-line,

$$(\mathcal{F}_c f)(\xi) \;:=\; \int_0^\infty f(u)\,\cos(2u\xi)\,du,$$

and the real gamma factor $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(\frac{s}{2})$. These conventions are employed in the Archimedean constructions below.

Appendix C. Kernel-level conjugation formulas

We record explicit kernels under the host conjugation. Throughout, $\alpha_H = d\theta$, $z = \Phi(\mathrm{e}^{J\theta})$, and $w = \Phi^{-1}(z)$.

C.1. **Cauchy kernel.** For $z \in \Omega$ and boundary data $u \in L^1(\partial\Omega)$,

$$\frac{1}{2\pi J}\oint_{\partial\Omega} \frac{u(\zeta)}{\zeta - z}\,d\zeta \;=\; \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\frac{\mathrm{e}^{J\theta}\,\Phi'(\mathrm{e}^{J\theta})}{\Phi(\mathrm{e}^{J\theta}) - z}}_{K_{\mathrm{C}}^{\Omega}(z;\mathrm{e}^{J\theta})}\,(u\circ\Phi)(\mathrm{e}^{J\theta})\,\alpha_H. \tag{12}$$

Equivalently, after applying U_Φ (Def. 9.1),

$$\frac{1}{2\pi J}\oint_{\partial\Omega} \frac{u(\zeta)}{\zeta - z}\,d\zeta \;=\; \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\frac{\mathrm{e}^{J\theta}}{\mathrm{e}^{J\theta} - w}}_{K_{\mathrm{C}}^{S^1}(w;\mathrm{e}^{J\theta})}\,(U_\Phi u)(\mathrm{e}^{J\theta})\,\alpha_H, \qquad w = \Phi^{-1}(z), \tag{13}$$

i.e. the host kernel is the standard circle Cauchy kernel.

C.2. **Szegő kernel (simply connected case).** Let P_+^Ω be the Szegő projection on $\partial\Omega$ with arclength measure, and $S_{\mathbb{D}}(w,\eta) = \dfrac{1}{1-w\overline{\eta}}$ the disk Szegő kernel (normalized for $d\theta/2\pi$). Then, for $z \in \Omega$ and $\zeta \in \partial\Omega$ with $w = \Phi^{-1}(z)$ and $\eta = \Phi^{-1}(\zeta)$,

$$S_\Omega(z,\zeta) \;=\; \big((\Phi^{-1})'(z)\big)^{1/2}\, S_{\mathbb{D}}(w,\eta)\, \overline{\big((\Phi^{-1})'(\zeta)\big)^{1/2}}, \tag{14}$$

and

$$(P_+^\Omega u)(z) \;=\; \int_{\partial\Omega} S_\Omega(z,\zeta)\, u(\zeta)\, |d\zeta|. \tag{15}$$

Under the host unitary U_Φ, this is equivalent to

$$U_\Phi\, P_+^\Omega\, U_\Phi^{-1} \;=\; P_+^{S^1}, \qquad (P_+^{S^1} v)(w) = \frac{1}{2\pi}\int_0^{2\pi} S_{\mathbb{D}}(w, \mathrm{e}^{J\theta})\, v(\mathrm{e}^{J\theta})\, \alpha_H.$$

C.3. **Remarks.**

- The boundary principal value versions of (12)–(13) recover the Hilbert transform identities via $\mathcal{H} = \mathrm{p.v.}\,\Im\, C$.
- On quasicircles, (12) and (14) hold in the distributional sense with a quasisymmetric boundary chart Φ; the pairings are against α_H on S^1.
- For multiply connected Ω, an explicit closed form of the Szegő kernel is global (period matrix/Abelian differentials). Nevertheless, the *operator* identity $P_+^\Omega = U_{\mathbf{\Phi}}^{-1}\,\mathrm{diag}(P_+^{S^1}, P_-^{S^1}, \dots)\, U_{\mathbf{\Phi}}$ in (9) provides a practical conjugation formula; the Cauchy/residue formulas decompose by the signed sum (10).

Appendix D. Heisenberg Quick Primer

The Heisenberg group $\mathbb{H}_1(k)$ is a central extension $0 \to k \to \mathbb{H}_1(k) \to k^2 \to 0$ with commutator governed by the standard symplectic form. A contact form α_H satisfies $\mathrm{d}\alpha_H = \omega$. Holonomy along horizontal cycles computes central displacement and, after passing to the modulus line, matches the torsion currents D_v.

References

[1] Paper I: *The Heisenberg Shadow: Reframing the Foundations of Analysis.* (Trilogy reference; real-place holonomy calculus.)
[2] Paper II: *[Title of Heisenberg Holonomy II].* (Trilogy reference; structural refinements and extensions.)
[3] Paper III: *[Title of Heisenberg Holonomy III].* (Trilogy reference; positivity mechanisms and higher-dimensional outlook.)
[4] RH/Adelic papers I–III (ledger for Q_{MS}, $\mathcal{A}$, prime block $P(g)$, and Weil-normalized tests).

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