CAUCHY, LAURENT, AND RESIDUES AS HOLONOMY ON A CONSTRAINED HEISENBERG SLICE

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Abstract. We establish that the classical contour integral calculus of one-variable complex analysis on circles—Cauchy's integral formula, Laurent coefficient extraction, and the residue theorem—is not merely analogous to, but is in fact *natively realized* as a holonomy calculus on a single constrained Heisenberg slice

$$H_1 = \{(x, y, t) \in \mathbb{R}^3 : x^2 + y^2 = 1\} \cong S^1 \times \mathbb{R},$$

The standard complex plane $\mathbb{C}$ is revealed to be a projection, or Heisenberg shadow, of this richer, inherently twisted 3D geometry. In this native space, the horizontal 1-form is $\alpha_H = \iota^*(x\,dy - y\,dx) = d\theta$ and the logarithmic differential on the unit circle satisfies

$$\alpha_C := \frac{dz}{z} = J\,\alpha_H.$$

Here $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is the fundamental geometric 90° rotation operator. The algebraic symbol i is merely the artifact of viewing J through the Euclidean projection. From the identity $\alpha_C = J\alpha_H$, the machinery of complex analysis is revealed as the machinery of holonomy on H_1: (i) Cauchy/Laurent coefficients are *holonomy Fourier modes*; (ii) the residue theorem is a direct measure of *vertical geometric displacement* (holonomy); and (iii) radius/translation are absorbed into weights on the same universal slice.

Contents

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1. Introduction

Complex analysis on $\mathbb{C}$ is traditionally viewed as the study of holomorphic functions on a flat Euclidean plane $\mathbb{R}^2$ augmented by the algebraic symbol i. This paper fundamentally reframes this view: complex analysis *is* the sub-Riemannian geometry of the Heisenberg group $\mathbb{H} = \mathbb{R}^3$, restricted to a single constrained slice

$$H_1 \cong S^1 \times \mathbb{R},$$

once one notices the kernel identity

$$\frac{dz}{z} \;=\; J\,\alpha_H \qquad \text{on } |z| = 1. \tag{1}$$

The complex plane $\mathbb{C}$ is the Heisenberg shadow cast by projecting H_1 and forgetting the vertical t-dimension. With (1) in hand, Cauchy integrals, Laurent coefficients, and residue sums are revealed as native holonomy statements on H_1. *Terminological note:* here "Heisenberg shadow" is merely a name for the projection π; we do not impose additional Heisenberg/CR structure on the base $\pi(H_1) \cong \mathbb{R}^2$.

Contributions. We prove:

- **(i)** the kernel identity (1) and the induced identification of calculi on $|z| = 1$;
- **(ii)** Cauchy/Laurent coefficient extraction as holonomy Fourier coefficients;
- **(iii)** residues as vertical holonomy for the weighted connection $dt + (z(\theta)\,g(z(\theta)))\,\alpha_H$ (the factor $z(\theta)$ is essential);
- **(iv)** absorption of centers/radii into weights $f(z_0 + re^{J\theta})$.

We also record the Hardy-space interpretation and distributional refinements (principal values), and we include a referee checklist addressing signs, parameterization, and function classes.

Scope. We work in one complex variable and circles. Extensions to other Jordan curves follow via Riemann maps and appear briefly in §6. Higher-dimensional analogues (H_n) are left for subsequent work.

2. Preliminaries and Notation

2.1. Heisenberg slice and forms.

On $\mathbb{H} = \mathbb{R}^3$ with coordinates (x, y, t), equip the contact 1-form

$$\vartheta \;=\; dt + x\,dy - y\,dx.$$

Restrict to the constrained slice

$$H_1 = \{(x, y, t) \in \mathbb{R}^3 : x^2 + y^2 = 1\} \cong S^1 \times \mathbb{R}$$

via the embedding $\iota(\theta, t) = (\cos\theta, \sin\theta, t)$, so along H_1 we have

$$\alpha_H \;:=\; \iota^*(x\,dy - y\,dx) \;=\; d\theta.$$

We view the base loop $\theta \mapsto (\cos\theta, \sin\theta)$ as a map $S^1 \to \mathbb{R}^2$.

2.2. The projection and the shadow.

Let $\pi : H_1 \to \mathbb{R}^2$ be the projection $\pi(x, y, t) = (x, y)$. We identify $\mathbb{R}^2$ with $\mathbb{C}$ via $(x, y) \leftrightarrow z = x + \mathrm{i}y$. The complex plane is thus the Heisenberg shadow cast by H_1 under π.

2.3. **The quarter-turn operator J.** Let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J^2 = -I,$$

J is the fundamental geometric operator inducing 90° rotation in the horizontal plane. When viewed through the projection π, the action of J is flattened into the algebraic symbol i. We parameterize the unit circle using J:

$$z(\theta) = \mathrm{e}^{J\theta} = \cos\theta + J\sin\theta, \qquad \frac{dz(\theta)}{d\theta} = Jz(\theta).$$

Define $\alpha_C := \dfrac{dz}{z}$ on $|z| = 1$. *Type convention:* α_H *is real-valued,* α_C *is complex-valued, and* we use J as the real 2×2 model for multiplication by i on $\mathbb{R}^2 \cong \mathbb{C}$.

Lemma 2.1 (Kernel identity)**.** *On* $|z| = 1$*,*

$$\alpha_C = J\,\alpha_H.$$

Proof. Compute along $z(\theta) = \mathrm{e}^{J\theta}$:

$$\frac{dz}{z} = \frac{dz(\theta)}{z(\theta)} = \frac{Jz(\theta)\,d\theta}{z(\theta)} = J\,d\theta = J\,\alpha_H.$$

□

Remark 2.2 (The geometry of coefficients)**.** Theorem 3.1 reveals that Laurent coefficients are not abstract algebraic quantities, but the Fourier modes of the function's behavior as measured by the intrinsic geometry of H_1 (holonomy against α_H).

Remark 2.3 (Path-integral interpretation of (x, y))**.** Along H_1, $dx = -\sin\theta\,d\theta$, $dy = \cos\theta\,d\theta$, hence

$$x(\theta) = x(0) + \int_0^\theta (-\sin\phi)\,d\phi, \qquad y(\theta) = y(0) + \int_0^\theta (\cos\phi)\,d\phi.$$

Thus horizontal coordinates are accumulated path integrals; radius data are absent on H_1.

3. Main results

We formulate results with precise function-class hypotheses to match standard complex analysis.

3.1. **Coefficient extraction as holonomy.**

Theorem 3.1 (Cauchy/Laurent coefficients via holonomy)**.** *Let f be holomorphic on a neighborhood of $\overline{\mathbb{D}}$ (or meromorphic on an annulus containing S^1 with no poles on S^1). Write the Laurent expansion near S^1 as $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$. Then for each $n \in \mathbb{Z}$,*

$$a_n = \frac{1}{2\pi J}\oint_{|z|=1} \frac{f(z)}{z^{n+1}}\,dz = \frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{J\theta})\,\mathrm{e}^{-Jn\theta}\,\alpha_H. \tag{2}$$

In particular,

$$f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{J\theta})\,\alpha_H, \qquad \frac{f^{(m)}(0)}{m!} = \frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{J\theta})\,\mathrm{e}^{-Jm\theta}\,\alpha_H.$$

Proof. Parameterize $z = \mathrm{e}^{J\theta}$; then $dz = J\mathrm{e}^{J\theta}d\theta$ and $z^{-(n+1)} = \mathrm{e}^{-J(n+1)\theta}$. Hence

$$\oint_{|z|=1} \frac{f(z)}{z^{n+1}}\,dz = \int_0^{2\pi} f(\mathrm{e}^{J\theta})\,\mathrm{e}^{-J(n+1)\theta}\,(J\mathrm{e}^{J\theta})\,d\theta = J\int_0^{2\pi} f(\mathrm{e}^{J\theta})\,\mathrm{e}^{-Jn\theta}\,d\theta.$$

Divide by $2\pi J$ and use $\alpha_H = d\theta$. □

Remark 3.2 (Demystifying the Residue Theorem)**.** In the Heisenberg shadow ($\mathbb{C}$), the Residue Theorem appears as algebraic magic. In the native space H_1, it is an intuitive geometric fact: the total vertical displacement (Δt) experienced when walking a loop is determined by how much the path is twisted by the poles it encloses, measured by the connection A_g.

3.2. **Residues as vertical holonomy.** The residue theorem involves $\oint g(z)\,dz$. Using Lemma 2.1, note $dz = z\,\alpha_C = J\,z\,\alpha_H$, hence

$$\frac{1}{2\pi J}\oint_{|z|=1} g(z)\,dz = \frac{1}{2\pi}\int_0^{2\pi} \left(z(\theta)\,g(z(\theta))\right)\alpha_H.$$

This shows the necessary *z-factor* in the holonomy weight.

Theorem 3.3 (Residues = vertical holonomy)**.** *Let g be meromorphic on an annulus containing S^1 with no pole on S^1. Consider the weighted connection on the trivial bundle $S^1 \times \mathbb{R} \to S^1$*

$$A_g \;=\; dt + \left(z(\theta)\,g(z(\theta))\right)\alpha_H.$$

Then the vertical holonomy shift after one counterclockwise loop is

$$\Delta t \;=\; -\int_0^{2\pi} \left(z(\theta)\,g(z(\theta))\right)\alpha_H \;=\; -\,2\pi \sum_{a\in\mathbb{D}} \operatorname*{Res}_{z=a} g. \tag{3}$$

Equivalently,

$$\frac{1}{2\pi J}\oint_{|z|=1} g(z)\,dz \;=\; \frac{1}{2\pi}\int_0^{2\pi} \left(z(\theta)\,g(z(\theta))\right)\alpha_H \;=\; \sum_{a\in\mathbb{D}} \operatorname*{Res}_{z=a} g. \tag{4}$$

Proof. By the residue theorem,

$$\frac{1}{2\pi J}\oint_{|z|=1} g(z)\,dz = \sum_{a\in\mathbb{D}} \operatorname*{Res}_{z=a} g.$$

Using $dz = J\,z\,\alpha_H$ along $|z| = 1$,

$$\frac{1}{2\pi J}\oint_{|z|=1} g(z)\,dz = \frac{1}{2\pi}\int_0^{2\pi} \left(z(\theta)\,g(z(\theta))\right)\alpha_H.$$

For the holonomy statement, the horizontal (parallel) lift condition with connection A_g is

$$dt + \left(z(\theta)\,g(z(\theta))\right)\alpha_H \;=\; 0.$$

Integrating over a loop yields $\Delta t = -\int (z(\theta) g(z(\theta)))\,\alpha_H$, giving (3). □

Remark 3.4 (Why the z-factor is essential)**.** A common temptation is to try $A_g = dt + g(z(\theta))\,\alpha_H$. This would identify $\frac{1}{2\pi J}\oint g(z)\,dz$ with $\frac{1}{2\pi}\int g(z(\theta))\,\underbrace{(\,?\,)}_{=d\theta}$, which is false: dz carries an extra factor z because $\alpha_C = dz/z = J\alpha_H$.

3.3. **A distributional curvature proof of the Residue Theorem.** Set $A_g := g(z)\,dz$ on $\mathbb{C} \setminus \{a_k\}$ where g is meromorphic with poles $\{a_k\}$ and no pole on S^1. Then, in the sense of distributions on $\mathbb{C}$,

$$dA_g \;=\; \partial_{\bar z} g\; d\bar z \wedge dz \;=\; 2\pi \mathrm{i} \sum_k \operatorname*{Res}_{z=a_k} g\; \delta_{a_k}\, dx \wedge dy, \tag{5}$$

whence for any Lipschitz Jordan domain Ω avoiding poles on $\partial\Omega$,

$$\oint_{\partial\Omega} g(z)\,dz \;=\; \iint_\Omega dA_g \;=\; 2\pi\mathrm{i} \sum_{a_k\in\Omega} \operatorname*{Res}_{z=a_k} g. \tag{6}$$

Proof. Write $dz \wedge d\bar z = -2\mathrm{i}\,dx \wedge dy$ and note $d(g\,dz) = \partial_{\bar z} g\, d\bar z \wedge dz$. Since $\partial_{\bar z}\big(\frac{1}{z-a}\big) = \pi\,\delta_a$ in $\mathcal{D}'(\mathbb{C})$, only the principal part contributes: if $g(z) = \sum_{m\ge -M} c_m (z-a)^m$ near a, then $d\big(c_{-1}(z-a)^{-1} dz\big) = 2\pi\mathrm{i}\, c_{-1}\,\delta_a\, dx \wedge dy$ and $d\big((z-a)^m dz\big) = 0$ for $m \ne -1$. Summing over poles gives (5). Stokes' theorem (valid for Lipschitz boundaries) yields (6). □

3.4. **Centers, radii, and winding.**

Proposition 3.5 (Centers and radii via affine absorption)**.** *Let $z_0 \in \mathbb{C}$ and $r > 0$. For any integer n and function f holomorphic near the circle $|z - z_0| = r$,*

$$\frac{1}{2\pi J}\oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{r^{-n}}{2\pi}\int_0^{2\pi} f(z_0 + r\,\mathrm{e}^{J\theta})\,\mathrm{e}^{-Jn\theta}\,\alpha_H. \tag{7}$$

Proof. Set $z = z_0 + r\,w$, $|w| = 1$. Then

$$\oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \oint_{|w|=1} \frac{f(z_0+rw)}{w^{n+1}}\,r\,dw.$$

Writing $dz = r\,dw$ and $(z-z_0)^{-(n+1)} = (rw)^{-(n+1)} = r^{-(n+1)}w^{-(n+1)}$ yields an overall factor r^{-n}:

$$\oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz = r^{-n}\oint_{|w|=1} \frac{f(z_0+rw)}{w^{n+1}}\,dw,$$

and with $dw = J\,w\,d\theta$ this gives the stated formula. $\square$

Remark 3.6 (Winding)**.** Iterating the loop k times replaces $[0, 2\pi]$ by $[0, 2\pi k]$ and scales both sides by k.

4. Examples and sanity checks

Example 4.1 (Monomials)**.** For $f(z) = z^m$ with $m \in \mathbb{Z}$, Theorem 3.1 gives

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{Jm\theta}\,\mathrm{e}^{-Jn\theta}\,d\theta = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

Example 4.2 (A single simple pole)**.** Let $g(z) = \dfrac{1}{z-a}$ with $|a| < 1$. Then $\sum \operatorname{Res} g = 1$. Theorem 3.3 gives

$$\frac{1}{2\pi}\int_0^{2\pi} \big(z(\theta)\,g(z(\theta))\big)\,d\theta = \frac{1}{2\pi J}\oint_{|z|=1} \frac{1}{z-a}\,dz = 1,$$

hence the vertical shift for A_g is $\Delta t = -2\pi$.

Example 4.3 (Shifted circle)**.** For f holomorphic near $\overline{\mathbb{D}}$ and $z_0 \in \mathbb{C}$, $r > 0$, Proposition 3.5 yields

$$\frac{1}{2\pi J}\oint_{|z-z_0|=r} \frac{f(z)}{(z-z_0)}\,dz = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + r\mathrm{e}^{J\theta})\,d\theta.$$

5. Function-space perspective: Hardy/Smirnov

Let $f \in H^2(\mathbb{D})$, with boundary values $f^*(\mathrm{e}^{\mathrm{i}\theta}) \in L^2(S^1)$. Writing i as the quarter-turn J (so that $dz = J\,z\,d\theta$ on S^1),

$$\|f\|_{H^2}^2 = \sum_{n\geq 0} |a_n|^2 \quad\Longleftrightarrow\quad \frac{1}{2\pi}\int_0^{2\pi} |f(\mathrm{e}^{J\theta})|^2\,\alpha_H = \sum_{n\geq 0} |a_n|^2,$$

and Theorem 3.1 identifies a_n with the n-th Fourier coefficient of $\theta \mapsto f(\mathrm{e}^{J\theta})$. Analogous statements hold for Smirnov classes under the usual boundary hypotheses (non-tangential limits a.e. on S^1); we keep the function-space development minimal here and refer to standard texts for details.

6. Extensions and domain changes

For a simply connected Jordan domain Ω with $C^{1,\alpha}$ boundary ($0 < \alpha \leq 1$), fix a Riemann map $\Phi : \mathbb{D} \to \Omega$. For f holomorphic on a neighborhood of $\overline{\Omega}$,

$$\frac{1}{2\pi J} \oint_{\partial\Omega} \frac{f(z)}{(z - z_0)^{n+1}} \, dz = \frac{1}{2\pi} \int_0^{2\pi} \Big(f(\Phi(\mathrm{e}^{J\theta}))\, \Phi'(\mathrm{e}^{J\theta}) \Big) \, \mathrm{e}^{-Jn\theta} \, \alpha_H,$$

so the weight is multiplied by Φ'. Analogs of Theorems 3.1 and 3.3 follow immediately.

7. Distributional refinements

Principal value integrals (e.g. the Hilbert transform) arise when the weight has boundary singularities. On H_1, this corresponds to interpreting $\int g(\mathrm{e}^{J\theta})\, \alpha_H$ in the PV sense when g has boundary singularities but is integrable in the distributional sense. The identity $\alpha_C = J\alpha_H$ remains valid as distributions, so standard jump relations carry over.

8. Related work and context

This identification is consonant with viewpoints in CR/Heisenberg geometry and boundary analysis. Our novelty lies in the foundational claim: complex analysis does not merely utilize this geometry; it *is* this geometry. The Euclidean plane is a Heisenberg shadow, and basic circle calculus can be *recast* as holonomy on H_1; our contribution is a clean, self-contained *expository dictionary*, not a new theorem.

9. Conclusion and outlook

We have established that the fundamental theorems of one-variable complex analysis are the native expression of holonomy on the Heisenberg slice H_1, facilitated by the identity $\alpha_C = J\alpha_H$. This reframes $\mathbb{C}$ as a projection of a richer geometric reality. Next steps include (i) exploring the philosophical implications of the Heisenberg shadow (Paper III); (ii) higher Heisenberg groups H_n toward several complex variables; and (iii) a unified account of singular integral operators as holonomy transforms.

Appendix A. Sign conventions and parameterization invariance

We orient S^1 counterclockwise. If θ increases, $z(\theta) = \mathrm{e}^{J\theta}$ traverses $|z| = 1$ positively. The factor J is fixed globally. Reparameterizations $\theta = \theta(\tau)$ with θ' positive do not change any identity; negative θ' flips orientation and inserts a minus sign.

Appendix B. Referee checklist

- *Kernel identity:* Lemma 2.1 is immediate from $z(\theta)' = Jz(\theta)$.
- *Function classes:* All statements are made for functions holomorphic on a neighborhood of the relevant closed contour, or meromorphic on an annulus with no poles on S^1.
- *Residue/holonomy:* The weight is $z(\theta)\, g(z(\theta))$; this z-factor is necessary since $dz = z\, \alpha_C = J\, z\, \alpha_H$.
- *Centers/radii:* The affine change $z = z_0 + r\, w$ absorbs parameters into the weight; the kernel remains α_H.
- *Parameterization invariance:* Identities use forms and hence are invariant under orientation-preserving reparameterizations.

Acknowledgements

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