An Adelic Distributional Framework for the Symmetric Explicit Formula on a Band-Limited Class*

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Scope (Paper II). Fixes the Maaß–Selberg normalization and constructs the adelic geometric functional $Q_{\mathrm{MS}}(g) = P(g) - \mathcal{A}(g)$ on a band-limited class. *No Riemann Hypothesis (RH) claim is asserted in this paper.*
Trilogy crosswalk. Paper I (10.5281/zenodo.16930061): automorphic derivation of the asymmetric explicit formula; *no RH claim*. Paper III (10.5281/zenodo.16930095): *states a claimed proof of RH as a preprint, pending peer review*, within the same ledger.

Abstract

We construct the geometric side of the GL(1) symmetric explicit formula, $Q_{\mathrm{MS}}(g) = P(g) - \mathcal{A}(g)$ (the Maaß–Selberg base functional), within an adelic distributional framework, focusing on a band-limited test class. We rigorously fix the constants according to the Maaß–Selberg normalization, ensuring compatibility with the automorphic derivation presented in the companion paper [3]. The Archimedean term $\mathcal{A}(g)$ is evaluated through a careful regularization procedure, connecting the distributional pairing on the modulus line to the spectral definition involving the digamma function. The finite-place contribution yields the prime block $P(g)$, a weighted sum over prime powers. The local distributional contributions D_v are explicitly calculated and summed to the global functional $Q_{\mathrm{MS}}(g)$. This construction reveals the underlying positivity structure of the geometric side, which is governed by a positive kernel (the half-shift kernel $T_{1/2}$). A comprehensive normalization ledger synchronizes conventions across this series of papers. All constructions are carried out without assuming the Riemann Hypothesis.

Contributions. (i) A self-contained adelic distributional construction of the geometric side on a band-limited class; (ii) a rigorous identification, enforced by the normalization, of the Archimedean boundary kernel as the half-shift kernel $T_{1/2}$; (iii) a two-route evaluation (Dirichlet/geometric and spectral/partial fractions) whose equality yields the symmetric explicit formula on this class, with all analytic interchanges justified and without assuming RH.

Keywords: Explicit formula, adelic distributions, Riemann zeta function, Weil distribution, GL(1), Heisenberg/CR boundary, Maaß–Selberg normalization, half-shift kernel.
MSC 2020: Primary 11M06; Secondary 11F72, 11R42.

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Reader's map. §2–§4 fix the pairing spaces and the Maaß–Selberg ledger; §5 evaluates the Archimedean block; §6–§7 assemble the global identity $Q_{\mathrm{MS}} = P - A$. RH appears only as *classical* cross-references (Weil positivity), not as a claim in this paper.

Contents

1 Introduction

Claim boundaries. Throughout this paper we *prove identities and normalizations* for the geometric side $Q_{\mathrm{MS}}(g) = P(g) - \mathcal{A}(g)$ in the Maaß–Selberg ledger and *do not* assert a proof of RH. Any RH-related statement here is strictly *classical equivalence or implication phrasing* (e.g., Weil's criterion) and is used for context only. The *preprint* *claim* regarding RH appears solely in Paper III (10.5281/zenodo.16930095) and is offered for community review.

The classical explicit formulas, originating with Riemann and developed by Guinand and Weil, establish a profound duality between the spectral properties of the Riemann zeta function (encoded in the distribution of its nontrivial zeros) and its arithmetic properties (encoded in the distribution of prime numbers). This paper provides a rigorous, self-contained derivation of the geometric side of the symmetric explicit formula within an adelic framework for GL(1). Our primary objective is to establish the fundamental identity

$$Q_{\mathrm{MS}}(g) = P(g) - \mathcal{A}(g), \tag{1.1}$$

for a specific band-limited test class (Definition 2.8). Here, $Q_{\mathrm{MS}}(g)$ denotes the Maaß–Selberg base geometric functional, $P(g)$ the arithmetic (prime) block, and $\mathcal{A}(g)$ the Archimedean block.

Remark 1.1 (Terminology: Symmetric vs. Asymmetric)*.* The title refers to the "Symmetric Explicit Formula" because the construction focuses on the geometric side derived from the symmetric adelic framework for GL(1), inherently utilizing even test functions g. This emphasizes the underlying structure of the geometric side. When this side is equated to the spectral side (the zeros), the resulting identity (e.g., (1.3) and Section A) utilizes one-sided Laplace weights $\mathcal{H}(g;\rho)$. This form is termed the "Asymmetric Explicit Formula" (as derived in Paper I [3]). The two forms are equivalent due to the symmetry of the zeros ($\rho \leftrightarrow 1-\rho$).

Analytic Setup and Adelic Framework. *Notation.* We operate within the adelic framework established by Tate and Weil. The global *base* functional $Q_{\mathrm{MS}}(g)$ is defined as the sum of local distributional pairings over all places v of $\mathbb{Q}$: *Normalization pointer.* Haar- and cosine-transform conventions (and the Maaß–Selberg ledger) are fixed in Section 2 and Section A.

$$Q_{\mathrm{MS}}(g) \coloneqq \sum_{v} \langle D_v, W \rangle. \tag{1.2}$$

We systematically evaluate the local distributions D_v and their pairings with the associated Weil test function W. At finite places p, the distribution D_p corresponds to the logarithmic derivative of the local zeta factor, yielding the prime block $P(g)$. At the infinite place ∞, the distribution D_∞ yields the Archimedean contribution $-\mathcal{A}(g)$.

A crucial aspect of this work is the rigorous justification of the Archimedean pairing. This requires a careful regularization procedure (Section 5) to handle the interplay between the kernel singularities and the test functions. This procedure ensures precise alignment with the Maaß–Selberg normalization utilized in the automorphic derivation of the full explicit formula presented in the companion paper [3].

Geometric Positivity and the Riemann Hypothesis. A key outcome of this construction is the explicit revelation of the positivity structure inherent in the geometric side. The *Full Geometric Functional*, $Q_{\mathrm{geom}}(g)$, combines the base functional $Q_{\mathrm{MS}}(g)$ with boundary terms arising from regularization ($\frac{1}{2}g(0)$) and the Poisson block ($2J(g)$; see Definition 2.11):

$$Q_{\mathrm{geom}}(g) \coloneqq Q_{\mathrm{MS}}(g) + \frac{1}{2}g(0) + 2J(g) = P(g) - \mathcal{A}(g) + \frac{1}{2}g(0) + 2J(g).$$

We demonstrate that $Q_{\mathrm{geom}}(g)$ is governed by a manifestly positive kernel arising from a mechanism we term the "torsion filter" $C_{1/2}$ (Section 2.4).

This structure is central to the connection with the Riemann Hypothesis (RH) via the full explicit formula established in [3], which equates the spectral side to the full geometric functional:

$$2\sum_{\rho}\mathcal{H}(g;\rho) \;=\; Q_{\mathrm{geom}}(g). \tag{1.3}$$

(See Appendix A for the definition of the spectral weights $\mathcal{H}$). The results herein demonstrate that the entire geometric side (the RHS of (1.3)) is represented by positive coefficients (for $P(g)$) and a strictly positive density (for the continuous part), as summarized in Proposition 6.2.

Proposition 1.2 (Weil positivity in this ledger: implication and criterion crosswalk)**.** *Within the Maaß–Selberg normalization fixed here:*

(i) ***(RH $\Rightarrow$)*** *If all nontrivial zeros lie on $\Re s = \frac{1}{2}$, then for every* autocorrelation *test $g_\Phi = \Phi * \Phi^\vee$ with $\Phi \in \mathcal{S}(\mathbb{A})$ the full geometric functional*

$$P(g_\Phi) - \mathcal{A}(g_\Phi) + \tfrac{1}{2}g_\Phi(0) + 2J(g_\Phi)$$

is nonnegative.

(ii) ***(Classical criterion)*** *The Riemann Hypothesis is* equivalent *to*

$$Q_{\mathrm{MS}}(g_\Phi) \;=\; P(g_\Phi) - \mathcal{A}(g_\Phi) \;\geq 0 \qquad \textit{for all autocorrelations } g_\Phi,$$

i.e., to Weil positivity for the Maaß–Selberg base functional.

Remark 1.3 (Sign conventions). The minus sign in $\langle D_\infty, W\rangle = -\mathcal{A}(g)$ reflects the Maaß–Selberg choice of spectral phase derivative and boundary pairing; it is consistent with $Q_{\mathrm{MS}}(g) = P(g) - \mathcal{A}(g)$ in (1.1).

Proof sketch (Classical arguments rephrased in the Maaß–Selberg ledger). (i) Assume RH. For an autocorrelation $g_\Phi = \Phi * \Phi^\vee$, g_Φ is positive definite. By Bochner's theorem, its cosine profile $g_{\flat\Phi}$ is non-negative. The full geometric functional is given by $Q_{\mathrm{geom}}(g_\Phi) = \int g_{\flat\Phi}\, d\nu$ (see Lemma 1.5). Since $d\nu$ is a positive measure (Definition 2.14), $Q_{\mathrm{geom}}(g_\Phi) \geq 0$. (Alternatively, under RH, the spectral side $2\sum_\rho \mathcal{H}(g;\rho)$ defines a positive definite distribution; pairing it with the positive definite function g_Φ yields a non-negative result, which equals $Q_{\mathrm{geom}}(g_\Phi)$ by (1.3).)

(ii) This is the classical Weil criterion [5]. The criterion states that RH is equivalent to the positive definiteness of the Weil distribution, which in this ledger corresponds to $Q_{\mathrm{MS}}(g_\Phi) = \sum_v \langle D_v, W_\Phi\rangle \geq 0$ for all autocorrelations g_Φ. □

Remark 1.4 (Adelic-Spectral Dictionary). The spectral function g_Φ associated with an adelic autocorrelation $\Phi * \Phi^\vee$ (where $\Phi \in \mathcal{S}(\mathbb{A})$) arises from the adelic Poisson summation framework. For GL(1), the structure ensures that g_Φ is even. Crucially, the cosine profile $g_{\flat\Phi}$ inherits rapid decay from the Schwartz conditions on Φ. Specifically, $g_\Phi \in \mathcal{S}(\mathbb{R})$ and $g_{\flat\Phi}$ exhibits rapid decay (faster than any required exponential $e^{-\sigma|x|}$), thus $g_\Phi \in \mathcal{A}_{\mathrm{Exp}}$. This guarantees $g_{\flat\Phi} \in L^1(d\nu)$ and justifies the application of the derived identities to autocorrelations, as well as the continuity arguments in Lemma 1.5.

Lemma 1.5 (Bridge to autocorrelations and closure in the Maaß–Selberg ledger)**.** *Let $\Phi \in \mathcal{S}(\mathbb{A})$ and $g_\Phi := \Phi * \Phi^\vee$ be its autocorrelation on the spectral line. Let $d\nu$ be the prime–continuous Weil measure (Definition 2.14). Assume the cosine profile $g_{\flat\Phi} \in L^1(d\nu)$. Then:*

(i) (Adelic–spectral agreement) In the Maaß–Selberg normalization fixed here,

$$Q_{\mathrm{MS}}(g_\Phi) \;=\; \sum_v \langle D_v, W_\Phi\rangle, \qquad Q_{\mathrm{geom}}(g_\Phi) \;=\; \int_{(0,\infty)} g_{\mathrm{b}\Phi}(x)\,\mathrm{d}\nu(x),$$

and, equivalently,

$$Q_{\mathrm{geom}}(g_\Phi) \;=\; Q_{\mathrm{MS}}(g_\Phi) + \tfrac{1}{2} g_\Phi(0) + 2J(g_\Phi).$$

Here W_Φ is the Weil modulus-line test attached to g_Φ by (2.6).

(ii) (Closure/continuity) If $g_{\mathrm{b},n} \to g_{\mathrm{b}\Phi}$ in $L^1(d\nu)$ with each $g_n \in \mathcal{A}_{\mathrm{BL}}$, then

$$\lim_{n\to\infty} Q_{\mathrm{geom}}(g_n) \;=\; Q_{\mathrm{geom}}(g_\Phi), \qquad \lim_{n\to\infty} \int g_{\mathrm{b},n}\,\mathrm{d}\nu \;=\; \int g_{\mathrm{b}\Phi}\,\mathrm{d}\nu.$$

In particular, since $d\nu$ is a positive measure, $Q_{\mathrm{geom}}(g_\Phi) \geq 0$ whenever $g_{\mathrm{b}\Phi} \geq 0$ a.e.

Proof. For (i), the identity $Q_{\mathrm{MS}}(g_\Phi) = \sum_v \langle D_v, W_\Phi\rangle$ is Definition 3.1, evaluated as $P(g_\Phi) - \mathcal{A}(g_\Phi)$ in Sections 4 and 5. The identity $Q_{\mathrm{geom}}(g_\Phi) = \int g_{\mathrm{b}\Phi}\,\mathrm{d}\nu$ follows from Definition 2.14 (the Weil measure), which combines the prime block $P(g_\Phi)$ and the continuous block (Remark 2.12). The equivalence $Q_{\mathrm{geom}}(g_\Phi) = Q_{\mathrm{MS}}(g_\Phi) + \frac{1}{2} g_\Phi(0) + 2J(g_\Phi)$ follows from the regrouping identity (Remark 5.6), where the continuous block equals $-\mathcal{A}(g_\Phi) + \frac{1}{2} g_\Phi(0) + 2J(g_\Phi)$. The Maaß–Selberg normalization ensures consistency (Proposition 5.3).

For (ii), use Lemma 2.15 and the continuity of the linear functional $g_{\mathrm{b}} \mapsto \int g_{\mathrm{b}}\,\mathrm{d}\nu$ in the $L^1(d\nu)$ topology. □

Remark 1.6 (Scope and Context). *No new RH criterion is proposed here.* Proposition 1.2 simply restates the classical Weil positivity equivalence in our fixed Maaß–Selberg normalization and on the test classes used here. The contribution of this paper is to construct $Q_{\mathrm{MS}}(g) = P(g) - \mathcal{A}(g)$ rigorously, fix constants, and expose the continuous positivity structure (via the torsion filter) as input to Paper III [4]. Any *claimed* RH result is confined to Paper III (10.5281/zenodo.16930095) and is *pending peer review.*

Remark 1.7 (Test Class Properties). *Crosswalk.* The band-limited class $\mathcal{A}_{\mathrm{BL}}$ (also denoted $\mathcal{S}_{\mathrm{BL}}$ in the trilogy) is used as the primary class for derivations, as it simplifies convergence arguments (e.g., finite prime sums). The results are then extended to the exponentially damped class $\mathcal{A}_{\mathrm{Exp}}$ (Theorem 2.8) and autocorrelations (Theorem 1.2, Theorem 1.5) via continuity arguments.

The relevant topology for this extension is the $L^1(d\nu)$ topology (Theorem 2.14). Since the profiles of $\mathcal{A}_{\mathrm{BL}}$ are dense in $L^1(d\nu)$ (Theorem 2.15), and the functional $Q_{\mathrm{geom}}(g) = \int g_{\mathrm{b}}\,d\nu$ is continuous in this topology, the identities established on $\mathcal{A}_{\mathrm{BL}}$ hold throughout $L^1(d\nu)$. This topology naturally controls both the continuous density (ρ_W) and the atomic prime lattice.

2 Conventions, Test Functions, and Analytical Prerequisites

2.1 Analytic Objects and Normalizations

We denote the completed Riemann zeta function and the associated real gamma factor as:

$$\Lambda(s) := \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s), \qquad \Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big). \tag{2.1}$$

The logarithmic derivative involves the digamma function $\psi(s) = \Gamma'(s)/\Gamma(s)$.

Remark 2.1 (Phase measure vs. Plancherel). *Normalization lock.* We pair throughout against the Maaß–Selberg (Birman–Kreĭn) spectral–shift measure $\mathrm{d}\xi(\tau) = \frac{1}{2\pi}\,\varphi'(\tau)\,\mathrm{d}\tau$ in the *symmetric* principal–value sense, and there is no δ_0 mass at $\tau = 0$. We never use the Selberg Plancherel density $\mathrm{d}\tau/(4\pi)$ in pairings. *Scattering note.* For reference, the Eisenstein scattering coefficient is $\widetilde{S}(s) = \Lambda(2s-1)/\Lambda(2s)$, and $\varphi'(\tau) = \Im\,\partial_\tau \log \widetilde{S}(\frac{1}{2} + i\tau)$ under this ledger (see also Lemma 2.5). *Ledger crosswalk.* This lock matches the density ρ_W in the positivity paper [4] and the phase–measure ledger in the automorphic paper [3]; see Paper III (continuous density) and Paper I (App. A).

Normalization pointer. The PV/no-δ_0 convention is proved locally in Lemma 2.6. For a single canonical ledger of constants and pairings used across the trilogy, see Section A.

Definition 2.2 (The Weil continuous density ρ_W)**.** The continuous geometric density on the modulus line $(0,\infty)$ is

$$\rho_W(x) \coloneqq \frac{1}{2\pi}\left(\frac{1}{1+e^{-x/2}} + 2e^{-x/2}\right)\,\mathbf{1}_{(0,\infty)}(x). \tag{2.2}$$

This density encodes the Archimedean, boundary, and Poisson contributions in the Maaß–Selberg ledger (derived in Section 5, Remark 5.6).

Definition 2.3 (Spectral functional in Maaß–Selberg normalization)**.** We pair throughout against the Birman–Krein spectral–shift measure in the *symmetric* principal–value sense with no atom at $\tau = 0$:

$$\mathcal{I}(g) \coloneqq \frac{1}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}} g(\tau)\,\varphi'(\tau)\,\mathrm{d}\tau \quad (\textit{symmetric PV}).$$

Symmetric PV convention: $\mathrm{PV}\int_{-\infty}^{\infty} f(\tau)\,\mathrm{d}\tau \coloneqq \lim_{T\to\infty}\int_{-T}^{T} f(\tau)\,\mathrm{d}\tau$. In the Maaß–Selberg ledger there is no δ_0 atom (Lemma 2.6).

Lemma 2.4 (Pointwise bounds for ρ_W)**.** *The Weil density $\rho_W(x)$ (Definition 2.2) satisfies the following bounds for all $x > 0$:*

$$\frac{1}{2\pi} < \rho_W(x) < \frac{5}{4\pi}.$$

Proof. Put $y = e^{-x/2} \in (0,1)$. Then $\rho_W(x) = \frac{1}{2\pi}(\frac{1}{1+y} + 2y)$. The function $f(y) = \frac{1}{1+y} + 2y$ is strictly increasing on $(0,1)$. We have $f'(y) = -\frac{1}{(1+y)^2} + 2$. Since $1 < (1+y)^2 < 4$ for $y \in (0,1)$, we have $\frac{1}{4} < \frac{1}{(1+y)^2} < 1$. Thus $f'(y) = 2 - \frac{1}{(1+y)^2} > 2 - 1 = 1 > 0$. Thus $\inf_{y\in(0,1)} f = \lim_{y\to 0^+} f(y) = 1$ and $\sup_{y\in(0,1)} f = \lim_{y\to 1^-} f(y) = \frac{1}{2} + 2 = \frac{5}{2}$. This yields the stated strict bounds. $\square$

Consequence. In particular, $\rho_W \in L^1((0,\infty)) \cap L^\infty((0,\infty))$, so the pairing $g_{\mathrm{b}} \mapsto \int_0^\infty g_{\mathrm{b}}(x)\,\rho_W(x)\,\mathrm{d}x$ is well defined for $g_{\mathrm{b}} \in L^1$ and bounded by $\|\rho_W\|_\infty\,\|g_{\mathrm{b}}\|_{L^1}$.

Lemma 2.5 (Boundary phase derivative)**.** *In $\mathcal{S}'(\mathbb{R})$, taken in the* symmetric *principal–value sense, one has*

$$\varphi'(\tau) = \mathrm{PV}\ 4\,\Re\Big(\frac{\Lambda'}{\Lambda}(2i\tau)\Big), \qquad \Lambda(s) = \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s).$$

There is no δ_0 atom under the Maaß–Selberg normalization (see Thm. 2.1 and Paper I App. A). Moreover, $\varphi'(\tau) = O(\log(2+|\tau|))$ as $|\tau| \to \infty$ (cf. Paper I, Lemma 1.11), hence φ' defines a tempered distribution.

Lemma 2.6 (No δ_0 atom: local proof)**.** *The potential δ_0 mass arises from the regularization limit $\varepsilon \downarrow 0$ (with $s = \frac{1}{2} + \varepsilon + i\tau$). The phase derivative involves terms $\Re\psi(s - \frac{1}{2})$ and $-2\Re\frac{\zeta'}{\zeta}(2s)$. As $\varepsilon \downarrow 0$, these develop canceling Poisson kernel singularities:*

$$\Re\psi(\varepsilon + i\tau) \sim -\frac{\varepsilon}{\varepsilon^2 + \tau^2}, \qquad -2\Re\frac{\zeta'}{\zeta}(1 + 2\varepsilon + 2i\tau) \sim +\frac{\varepsilon}{\varepsilon^2 + \tau^2}.$$

This exact cancellation ensures that no δ_0 mass persists in the limit (see Paper I, Prop. 2.8 and 4.1). Consequently, the boundary distribution is given by the locally integrable function

$$\varphi'(\tau) = \mathrm{PV}\ 4\,\Re\Big(\frac{\Lambda'}{\Lambda}(2i\tau)\Big).$$

Local integrability at $\tau = 0$ is confirmed by analyzing the components of $\Lambda'/\Lambda(2i\tau)$. The potential singularity near $\tau = 0$ is governed by the digamma function component. The Laurent series $\psi(z) = -1/z - \gamma + O(z)$ implies

$$\psi(i\tau) = -\frac{1}{i\tau} - \gamma + O(\tau) = \frac{i}{\tau} - \gamma + O(\tau).$$

The singular term i/τ is purely imaginary, so $\Re(i/\tau) = 0$ (for $\tau \neq 0$). Taking the real part gives $\Re\psi(i\tau) = -\gamma + O(\tau^2)$. Since $\zeta'/\zeta(2i\tau)$ is analytic at $\tau = 0$, the measure $d\xi(\tau) = \frac{1}{2\pi}\varphi'(\tau)\,\mathrm{d}\tau$ has no *δ_0 atom in the Maaß–Selberg normalization.*

Remark 2.7 (Measure Convention). We conduct integration on the modulus line $\mathbb{R}^{\times}_{>0}$ using the additive coordinate $u = \log q$. We employ the additive Haar measure $\mathrm{d}u$ on $\mathbb{R}$, which under $q = e^u$ corresponds to the multiplicative Haar measure $\mathrm{d}^{\times}q = \mathrm{d}q/q$ on $\mathbb{R}^{\times}_{>0}$.
Dictionary Jacobian. Passing to the symmetric profile via $x = 2u$ gives $\mathrm{d}x = 2\,\mathrm{d}u$; this explains the consistent factors of 2 in G_{b} and W, removes half–factor ambiguity in pairings, and fixes the *prime lattice* at $x = 2k\log p$.

2.2 Test Functions and Transforms

We lock the cosine and profile dictionary *before* any local pairings, so that later Tonelli/DCT interchanges cite a single source of domination. The class and all labels in this subsection are canonical across the trilogy.

Definition 2.8 (Band-limited class and exponentially–damped extension $\mathcal{A}_{\mathrm{Exp}}$)**.** A real, even $g : \mathbb{R} \to \mathbb{R}$ belongs to the *band-limited class* ($\mathcal{A}_{\mathrm{BL}}$, also denoted $\mathcal{S}_{\mathrm{BL}}$ in the trilogy) if its cosine transform

$$g_{\mathrm{b}}(x) \;\coloneqq\; \int_{\mathbb{R}} g(\tau)\cos(x\tau)\,\mathrm{d}\tau$$

lies in $C_c^{\infty}(\mathbb{R})$, and the even inversion holds:

$$g(\tau) \;=\; \frac{1}{\pi}\int_0^{\infty} g_{\mathrm{b}}(x)\,\cos(x\tau)\,\mathrm{d}x \qquad (\tau \in \mathbb{R}). \tag{2.3}$$

Normalization. We use the unnormalized cosine transform with Lebesgue measure; factors of π appear explicitly in inversion and in the Archimedean block. Since g is even, g_{b} is also even. As $g_{\mathrm{b}} \in C_c^{\infty}(\mathbb{R})$, there exists $X > 0$ with $\mathrm{supp}(g_{\mathrm{b}}) \subseteq [-X, X]$. The inversion in (2.3) (integrating over $[0, \infty)$) holds in the usual L^1 (and distributional) sense.

The *exponentially-damped class* $\mathcal{A}_{\rm Exp}$ consists of even $g \in C^\infty(\mathbb{R})$ satisfying $|g(\tau)| \le C_g(1+|\tau|)^{-2}$ (sufficient for pairing against tempered distributions) such that its cosine transform $g_{\rm b}$ satisfies

$$|g_{\rm b}(x)| \;\le\; C\, e^{-\sigma|x|} \quad \text{for some } C>0 \text{ and } \textit{strict } \sigma > 1/2.$$

Note on decay: Since $\varphi'(\tau) = O(\log|\tau|)$ (Lemma 2.5), the decay $|g(\tau)| = O((1+|\tau|)^{-2})$ ensures $|g(\tau)\varphi'(\tau)| \in L^1(\mathbb{R})$, so the spectral pairing $\mathcal{I}(g)$ (Definition 2.3) converges absolutely.

Assumption 2.9 (Admissible test class)**.** Throughout, g denotes a real, even test function that is either band-limited ($g \in \mathcal{A}_{\rm BL}$) or exponentially damped ($g \in \mathcal{A}_{\rm Exp}$).

Why strict? The same strict $\sigma > 1/2$ supplies: (i) absolute convergence of $P(g)$ via $\sum_{p,k}(\log p)\,|g_{\rm b}(2k\log p)| < \infty$, and (ii) an α-independent L^1 majorant on $K_\alpha \cdot g_{\rm b}$ in the Archimedean limits (DCT/Tonelli in §5).

Remark 2.10 (Fourier pair and cosine specialization)*.*

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(\tau)\, e^{-i\tau\xi}\, \mathrm{d}\tau, \qquad f(\tau) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\xi)\, e^{i\tau\xi}\, \mathrm{d}\xi.$$

For even g we use the cosine specialization:

$$\widehat{g}(x) = \int_{\mathbb{R}} g(\tau)\cos(x\tau)\,\mathrm{d}\tau, \quad g(\tau) = \frac{1}{\pi}\int_0^\infty \widehat{g}(x)\cos(x\tau)\,\mathrm{d}x.$$

For a *real even* g we denote its Fourier transform by $g_{\rm b}(x)$ (i.e. $g_{\rm b}(x) = \widehat{g}(x)$), and we reserve $g_{\rm b}$ (never a bare hat) in the prime/continuous blocks to prevent normalization drift. We will informally refer to

$$\mathrm{d}\mu_g(x) = \frac{1}{2\pi}\, g_{\rm b}(|x|)\,\mathrm{d}x \qquad (\text{here } g_{\rm b} = \widehat{g} \text{ in the cosine dictionary}) \tag{2.4}$$

as the *Bochner profile measure*. For a general even g this is a (possibly signed) measure; when $g = \Phi * \Phi^\vee$ (autocorrelation) it is positive. All phase-measure pairings use the *symmetric* PV in the Maaß–Selberg ledger; see Thm. 2.1.

Definition 2.11 (Derived profiles and the Poisson block)**.** For $g \in \mathcal{A}_{\rm Exp}$ (hence for $\mathcal{A}_{\rm BL}$) set:

(i) **Symmetric profile** (even on $\mathbb{R}$):

$$G_{\rm b}(x) \;\coloneqq\; 2\sinh\Big(\frac{|x|}{2}\Big)\, g_{\rm b}(2|x|). \tag{2.5}$$

If $\mathrm{supp}(g_{\rm b}) \subset [-X, X]$, then $\mathrm{supp}(G_{\rm b}) \subset [-X/2, X/2]$, so $2k\log p \le X \iff k\log p \le X/2$.

(ii) **Weil test on the modulus half–line** ($u \ge 0$):

$$W(u) \;\coloneqq\; \frac{1}{\pi}\,(1 - e^{-u})\, g_{\rm b}(2u). \tag{2.6}$$

(iii) **Poisson block** (pole of ζ at $s = 1$):

$$J(g) \;\coloneqq\; \frac{1}{2\pi}\int_0^\infty e^{-x/2}\, g_{\rm b}(x)\,\mathrm{d}x \;=\; \frac{1}{\pi}\int_0^\infty e^{-y}\, g_{\rm b}(2y)\,\mathrm{d}y \text{ via the substitution } y = x/2. \tag{2.7}$$

Remark 2.12 (Dictionary and continuous density locks). For $u > 0$ we have the basic dictionary

$$W(u) \;=\; \frac{1}{\pi}\, e^{-u/2}\, G_{\mathrm{b}}(u), \qquad (1-p^{-k})\, g_{\mathrm{b}}(2k\log p) \;=\; p^{-k/2}\, G_{\mathrm{b}}(k\log p). \tag{2.8}$$

Combining the Archimedean block with the boundary and Poisson terms yields the continuous block:

$$W_{\mathrm{cont}}(g) \;=\; \int_0^\infty g_{\mathrm{b}}(x)\, \rho_W(x)\, \mathrm{d}x, \tag{2.9}$$

where $\rho_W(x)$ is the Weil continuous density (Definition 2.2). These identities are in the same Maaß–Selberg normalization as the automorphic paper. *All* later uses of Tonelli/DCT on W_{cont} and $P(g)$ rely on the strict $\sigma > 1/2$ tilt recorded in Thm. 2.8.

2.3 Analytical Justifications and Test Spaces

We establish the prerequisite lemmas justifying the distributional pairings and the analytical maneuvers required for the evaluation of the Archimedean block.

Definition 2.13 (Distributional Test Spaces)**.** We define the required regularity properties for test functions on the modulus line:

- *Boundary-safe (half-line)* $\mathcal{D}_0$: Functions $\varphi \in C^\infty((0,\infty))$ such that all one-sided derivatives exist and are continuous at $u = 0$ (denoted $\varphi \in C^\infty([0,\infty))$), satisfying the boundary condition $\varphi(0) = 0$.
- *Adapted even (full line)* $\mathcal{D}^\flat_{\mathrm{ev}}$: $\Phi \in C^\infty(\mathbb{R})$ such that Φ is even and $\Phi(0) = 0$.

We utilize the following subspaces characterized by support/decay, corresponding to the image of the spectral test classes under the mapping $g \mapsto W$:

(i) **Compact support (**$\mathcal{D}_{0,c}$**)**: The subspace of $\mathcal{D}_0$ consisting of functions with compact support. If $g \in \mathcal{A}_{\mathrm{BL}}$, then $W \in \mathcal{D}_{0,c}$.

(ii) **Exponential decay (**$\mathcal{D}_0^{(\sigma)}$**)**: The subspace of $\mathcal{D}_0$ consisting of functions with decay $O(e^{-2\sigma u})$. If $g \in \mathcal{A}_{\mathrm{Exp}}$ (with parameter $\sigma > 1/2$), then $W \in \mathcal{D}_0^{(\sigma)}$.

These ensure W and all kernels pair absolutely; see Prop. 4.2 and Lemma 2.16.

Definition 2.14 (Prime–continuous Weil measure)**.** Define the positive Borel measure on $(0,\infty)$

$$d\nu(x) \coloneqq \rho_W(x)\, \mathrm{d}x \;+\; \frac{1}{\pi}\sum_p \sum_{k\ge 1} (\log p)\, (1-p^{-k})\, \delta_{2k\log p}, \tag{2.10}$$

where $\rho_W(x)$ is the Weil continuous density (Definition 2.2). Then $Q_{\mathrm{geom}}(g) = \int_{(0,\infty)} g_{\mathrm{b}}(x)\, d\nu(x)$ whenever the RHS is absolutely convergent (cf. (2.9)). For the atomic part, since g_{b} is continuous (as $g \in \mathcal{S}(\mathbb{R})$), we evaluate at lattice sites:

$$\int g_{\mathrm{b}}\, \delta_{2k\log p} = g_{\mathrm{b}}(2k\log p).$$

This fixes the $L^1(d\nu)$ pairing and is the pairing used in the Fejér–Poisson regularization scheme (Section 5 and Paper III, §6), where the continuous part is handled by ρ_W and atoms by smooth bump approximation (cf. Paper III Def. 4.6/Lem. 4.7).

Lemma 2.15 ($L^1(d\nu)$-continuity and density). *On the class where $g_{\mathrm{b}} \in L^1(d\nu)$, the map $g_{\mathrm{b}} \mapsto Q_{\mathrm{geom}}(g) = \int g_{\mathrm{b}}\, d\nu$ is continuous in the $L^1(d\nu)$-norm. Moreover, the space $C_c^\infty((0,\infty))$, which corresponds to the profiles of the band-limited class $\mathcal{A}_{\mathrm{BL}}$ restricted to $(0,\infty)$, is dense in $L^1(d\nu)$. In particular, $L^1(d\nu)$ convergence forces convergence at the atomic sites $x = 2k\log p$ in the* weighted ℓ^1 *sense (tails beyond a finite lattice set have total weight $< \varepsilon$ for any $\varepsilon > 0$).*

Proof. Continuity is immediate since $|\int g_{\mathrm{b}}\, d\nu| \le \|g_{\mathrm{b}}\|_{L^1(d\nu)}$. This topology is natural as it directly controls the convergence of the entire geometric functional $Q_{\mathrm{geom}}(g)$.

The $L^1(d\nu)$ norm is explicitly:

$$\|g_{\mathrm{b}}\|_{L^1(d\nu)} = \int_0^\infty |g_{\mathrm{b}}(x)|\rho_W(x)\,\mathrm{d}x + \frac{1}{\pi}\sum_{p,k}(\log p)(1-p^{-k})|g_{\mathrm{b}}(2k\log p)|.$$

For density, we first note that ρ_W is bounded above and below by positive constants on $(0,\infty)$ (Lemma 2.4): $\frac{1}{2\pi} < \rho_W(x) < \frac{5}{4\pi}$ (using the strict bounds from Lemma 2.4). Consequently, the $L^1(\rho_W\,\mathrm{d}x)$ norm is equivalent to the unweighted $L^1(\mathrm{d}x)$ norm, ensuring $C_c^\infty((0,\infty))$ is dense in the continuous part of $L^1(d\nu)$.

To handle the atomic part, we construct an approximation $h \in C_c^\infty((0,\infty))$. Fix $\varepsilon > 0$.

(i) **Continuous approximation:** Choose $h_c \in C_c^\infty((0,\infty))$ such that the continuous error $\|g_{\mathrm{b}} - h_c\|_{L^1(\rho_W\,\mathrm{d}x)} < \varepsilon/3$.

(ii) **Tail truncation:** Choose a finite lattice set $\mathcal{L}_M = \{2k\log p\}$ such that the weighted ℓ^1 norm of g_{b} on the tail (outside $\mathcal{L}_M$) is $< \varepsilon/3$.

(iii) **Atomic adjustment:** We construct an adjustment $h_a \in C_c^\infty((0,\infty))$ so that $h = h_c + h_a$ matches g_{b} exactly on $\mathcal{L}_M$. Let $C_M := \sum_{x_j\in\mathcal{L}_M} |g_{\mathrm{b}}(x_j) - h_c(x_j)|$ be the total magnitude of the required adjustments.

If $C_M = 0$, set $h_a = 0$. If $C_M > 0$, for each $x_j \in \mathcal{L}_M$, choose disjoint smooth bumps $\eta_j \in C_c^\infty((0,\infty))$ such that $\eta_j(x_j) = 1$, $0 \le \eta_j \le 1$, and their supports are small enough that $\int \eta_j(x)\rho_W(x)\,\mathrm{d}x < \frac{\varepsilon}{3C_M}$. Define the adjustment function:

$$h_a(x) := \sum_{x_j\in\mathcal{L}_M} \left(g_{\mathrm{b}}(x_j) - h_c(x_j)\right)\eta_j(x).$$

The added continuous error is bounded by:

$$\begin{aligned}\|h_a\|_{L^1(\rho_W\,\mathrm{d}x)} &\le \sum_{x_j\in\mathcal{L}_M} |g_{\mathrm{b}}(x_j) - h_c(x_j)| \int \eta_j(x)\rho_W(x)\,\mathrm{d}x \\ &< \sum_{x_j\in\mathcal{L}_M} |g_{\mathrm{b}}(x_j) - h_c(x_j)| \cdot \frac{\varepsilon}{3C_M} = C_M\cdot\frac{\varepsilon}{3C_M} = \frac{\varepsilon}{3}.\end{aligned}$$

The total error $\|g_{\mathrm{b}} - h\|_{L^1(d\nu)}$ is the sum of the continuous error and the atomic error. The continuous error is bounded by the triangle inequality:

$$\|g_{\mathrm{b}} - h\|_{L^1(\rho_W\,\mathrm{d}x)} = \|g_{\mathrm{b}} - (h_c + h_a)\| \le \|g_{\mathrm{b}} - h_c\| + \|h_a\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.$$

The atomic error is zero on $\mathcal{L}_M$ (by construction) and $< \varepsilon/3$ on the tail (by choice of M). The total error is $< \varepsilon$.

Note on scope. This density result ensures that the functional Q_{geom} is continuous on $L^1(d\nu)$ and that identities proven on the dense subclass $\mathcal{A}_{\mathrm{BL}}$ extend to the whole space. It does *not* assert the density of the subclass of profiles arising from autocorrelations (the positive-definite cone). We invoke this lemma only for continuity and approximation of general profiles, not to transfer positivity arguments from autocorrelations to all nonnegative g_{b}. □

Lemma 2.16 (Regularity, Admissible Pairings, and Convergence Justification)**.** *If $g \in \mathcal{A}_{\mathrm{Exp}}$ (including $\mathcal{A}_{\mathrm{BL}}$), the associated Weil function W satisfies $W(0) = 0$ and $W(u) = O(u)$ as $u \to 0$. If $g \in \mathcal{A}_{\mathrm{BL}}$, then $W \in \mathcal{D}_0$ and its even extension $W_{\mathrm{ev}} \in \mathcal{D}^{\flat}_{\mathrm{ev}}$.*

The local pairings $\langle D_v, W\rangle$ are well-defined. Furthermore, the interchanges of integration (Tonelli/Fubini) and the passage to the limit (Dominated Convergence Theorem, DCT) employed in the evaluation of the Archimedean block (Proposition 5.3) are rigorously justified.

Proof. The regularity of W follows immediately from Definition 2.11, noting that $(1 - e^{-u}) = u + O(u^2)$. The pairings are well-defined. At the infinite place, this is due to the $O(u)$ behavior of W near $u = 0$, which counteracts the $O(1/u)$ singularity of the Archimedean kernel $K_\infty(u)$ (Definition 3.2).

To rigorously justify the convergence arguments in Proposition 5.3 for $g \in \mathcal{A}_{\mathrm{Exp}}$: we analyze the kernel

$$K_\alpha(t) := \frac{e^{-t/2} - e^{-\alpha t}}{1 - e^{-t}}, \qquad t > 0,\ \alpha \in (0, 1/2].$$

We establish a uniform bound. Since $\alpha \le 1/2$ and $t > 0$, we have $e^{-\alpha t} \ge e^{-t/2}$, so $K_\alpha(t) \le 0$. We analyze the absolute value:

$$|K_\alpha(t)| = \frac{e^{-\alpha t} - e^{-t/2}}{1 - e^{-t}}.$$

For fixed $t > 0$, the function $\alpha \mapsto e^{-\alpha t}$ is decreasing. Therefore, $|K_\alpha(t)|$ attains its supremum as $\alpha \downarrow 0$:

$$\sup_{\alpha \in (0,1/2]} |K_\alpha(t)| = \frac{1 - e^{-t/2}}{1 - e^{-t}} = \frac{1}{1 + e^{-t/2}} \le 1.$$

This bound is independent of α. Since $g_{\mathrm{b}} \in L^1((0,\infty))$ (by the definition of $\mathcal{A}_{\mathrm{Exp}}$, which includes $\mathcal{A}_{\mathrm{BL}}$), the product $|K_\alpha(t) g_{\mathrm{b}}(t)|$ is dominated by $|g_{\mathrm{b}}(t)|$, which is an α-independent integrable majorant. The Dominated Convergence Theorem applies to the limit $\alpha \downarrow 0$ (PO–1 closed).

Justification for Tonelli/Fubini (fixed $\alpha > 0$): We analyze the joint integrability required for Proposition 5.3, Step 2.

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}} |g(\tau)\, K_\alpha(t) \cos(\tau t)|\, \mathrm{d}\tau\, \mathrm{d}t \;&\le\; \int_0^\infty |K_\alpha(t)| \left(\int_{\mathbb{R}} |g(\tau)|\, \mathrm{d}\tau \right) \mathrm{d}t \\ &= \|g\|_{L^1(\mathbb{R})} \int_0^\infty |K_\alpha(t)|\, \mathrm{d}t.\end{aligned}$$

Since $K_\alpha(t)$ decays exponentially as $t \to \infty$ ($|K_\alpha(t)| = O(e^{-\alpha t})$), we check behavior near $t = 0$. Using Taylor expansion (or L'Hôpital's rule), $K_\alpha(t) = (\frac{1}{2} - \alpha) + O(t)$, confirming it is bounded near the origin. Thus, the integral is finite for $\alpha > 0$. Since $g \in \mathcal{A}_{\mathrm{Exp}}$ satisfies the decay $|g(\tau)| \le C_g(1+|\tau|)^{-2}$ (Theorem 2.8), $\|g\|_{L^1(\mathbb{R})}$ is finite. This establishes absolute joint integrability and justifies the interchange.

Justification for DCT ($\alpha \downarrow 0$): Since $g_{\mathrm{b}} \in L^1((0,\infty))$, the product $|K_\alpha(t) g_{\mathrm{b}}(t)|$ is dominated by $|g_{\mathrm{b}}(t)|$ (using the uniform bound $|K_\alpha(t)| \le 1$), which is an α-independent integrable majorant. Furthermore, $K_\alpha(t) g_{\mathrm{b}}(t)$ converges pointwise as $\alpha \downarrow 0$ for all $t > 0$. The Dominated Convergence Theorem therefore applies to the limit $\alpha \downarrow 0$. □

Lemma 2.17 (Boundary Value Identity)**.** *For $g \in \mathcal{A}_{\mathrm{Exp}}$, the value $g(0)$ is related to the Weil function W by*

$$g(0) = 2\int_0^\infty \frac{W(u)}{1-e^{-u}}\,\mathrm{d}u. \tag{2.11}$$

Proof. By the inversion formula (2.3), $g(0) = \frac{1}{\pi}\int_0^\infty g_{\mathrm{b}}(x)\,\mathrm{d}x$. From Definition 2.11, we have $g_{\mathrm{b}}(2u) = \frac{\pi W(u)}{1-e^{-u}}$. Changing variables $x = 2u$ ($\mathrm{d}x = 2\mathrm{d}u$):

$$g(0) = \frac{1}{\pi}\int_0^\infty \frac{\pi W(u)}{1-e^{-u}}\,(2\mathrm{d}u) = 2\int_0^\infty \frac{W(u)}{1-e^{-u}}\,\mathrm{d}u.$$

The integral converges: near $u = 0$, $(1-e^{-u})^{-1} = u^{-1} + O(1)$ and $W(u) = O(u)$ (Lemma 2.16), so the integrand is $O(1)$. As $u \to \infty$, $W(u) = \frac{1}{\pi}(1-e^{-u})\,g_{\mathrm{b}}(2u) = O(e^{-2\sigma u})$ with the same *strict* $\sigma > 1/2$ from Definition 2.8 when $g \in \mathcal{A}_{\mathrm{Exp}}$, while for $g \in \mathcal{A}_{\mathrm{BL}}$ the profile g_{b} has compact support. Hence the integral is absolutely convergent. □

2.4 Geometric Interpretation: The Half-Shift Kernel

We adopt a geometric perspective motivated by the adelic structure and its connection to the CR (Cauchy-Riemann) geometry of the Heisenberg boundary (as explored in [3]). We use this as intuition only.

Analytic Framework. In the modulus-line calculus of this paper, we work solely with the exponential-multiplier semigroup $(\mathsf{E}^a\phi)(x) := e^{-ax}\phi(x)$, arising in Laplace/Mellin representations. We do not identify E^a with translations T^a. All proofs rely entirely on this E–framework. The key object is the "half-shift torsion" operator $C_{1/2}$, which emerges from the Archimedean regularization (Proposition 5.3).

Lemma 2.18 (Half-shift Kernel: Analytic Definition)**.** *The operator $C_{1/2}$ acts via the positive kernel $T_{1/2}(x)$:*

$$T_{1/2}(x) := (\mathrm{Id} - \mathsf{E}^{1/2})(\mathrm{Id} - \mathsf{E})^{-1}\,\mathbf{1}(x) = \frac{1}{1+e^{-x/2}} = \frac{1-e^{-x/2}}{1-e^{-x}}. \tag{2.12}$$

Remark 2.19 (Motivational context: Affine commutator). Heuristically, the structure of $T_{1/2}$ can be motivated by an analogy to the affine group commutator involving translations, which captures additive drift. On $\mathbb{R}$, let $R : u \mapsto -u$ (reflection) and $T^a : u \mapsto u - a$ (translation). The commutator $[R, T^{1/2}] = T^{-1}$. While this provides intuition, we emphasize that the translation operators T^a are *not* used in the analytic proofs, which rely solely on the multiplier framework E^a.

Proof. On $\mathbb{R}^\times_{>0}$, the kernel $T_{1/2}(x)$ is computed by applying $(\mathrm{Id} - \mathsf{E}^{1/2})(\mathrm{Id} - \mathsf{E})^{-1}$ to the constant function $\mathbf{1}(x)$, utilizing the geometric series expansion (valid as a distributional identity on test functions supported in $(0,\infty)$):

$$\begin{aligned}(\mathrm{Id} - \mathsf{E}^{1/2})(\mathrm{Id} - \mathsf{E})^{-1}\mathbf{1} &= (\mathrm{Id} - \mathsf{E}^{1/2})\sum_{n=0}^\infty \mathsf{E}^n\mathbf{1} = \sum_{n=0}^\infty(\mathsf{E}^n - \mathsf{E}^{n+1/2})\mathbf{1} \\ &= \sum_{n=0}^\infty(e^{-nx} - e^{-(n+1/2)x}) = \frac{1}{1-e^{-x}} - \frac{e^{-x/2}}{1-e^{-x}} = \frac{1-e^{-x/2}}{1-e^{-x}}.\end{aligned}$$

The final expression follows from $1 - e^{-x} = (1-e^{-x/2})(1+e^{-x/2})$ for $x > 0$. □

Remark 2.20 (Translations, dilations, and multipliers). Under $u = \log x$, additive translations $T^a : f(u) \mapsto f(u-a)$ correspond to multiplicative *dilations* $(S_a\phi)(x) = \phi(e^{-a}x)$ on $(0,\infty)$. By contrast, the operators $(\mathsf{E}^a\phi)(x) = e^{-ax}\phi(x)$ are *exponential multipliers* arising naturally in the Laplace/Mellin calculus. Our torsion-kernel identity and all ensuing positivity statements are proved within the E–framework; no identification $T^a \mapsto \mathsf{E}^a$ is assumed. We strictly distinguish between the translation T^a and the multiplier E^a to keep the dictionary unambiguous.

Remark 2.21 (Geometric Interpretation and Analogies). The commutator captures the failure of reflection and translation to commute, resulting in a net drift. Under $u = \log x$, this additive drift manifests as the multiplicative weighting kernel $T_{1/2}(x)$. This positive kernel filters the geometric data, isolating the analytic (torsion-free) component.

This mechanism offers an analogy to the Heisenberg commutator $[X, Y] = T$ (torsion) and the role of CR analyticity ($(X + iY)f = 0$) in isolating the torsion-free sector (see [3]). In this setting, $C_{1/2}$ acts similarly to a Szegő projector. We stress that this analogy is purely interpretive; all analytic results rely strictly on the E-framework defined in Lemma 2.18.

Remark 2.22 (Heisenberg–Log-line–Density Dictionary). The following dictionary summarizes the structural correspondences:

- **Heisenberg boundary:** The commutator $[X, Y] = T$ generates torsion (drift in the T direction) [3].
- **CR analyticity:** The condition $(X + iY)f = 0$ defines the torsion-free sector by annihilating the T-drift [3].
- **Log-line avatar:** The half-shift commutator $(R\,T^{1/2})\,(T^{1/2}R)^{-1} = T^{-1}$ encodes one unit of additive drift.
- **Mellin half-shift kernel:** Under $u = \log x$, this drift corresponds to the multiplicative kernel $T_{1/2}(x) = \frac{1}{1+e^{-x/2}}$.
- **Geometric positivity:** The resulting continuous density ρ_W from (2.2) governs the Archimedean, boundary, and Poisson blocks; this structure underlies the Weil positivity analysis [4].

3 The Adelic Distributional Framework

3.1 The Adelic-Modulus Line Bridge

We briefly situate the framework within the standard adelic setting (cf. Tate's thesis and Weil [5]). Let $\mathbb{A}^\times$ be the idele group of $\mathbb{Q}$. The Weil distribution D_W acts on the Schwartz space $\mathcal{S}(\mathbb{A}^\times)$.

In the GL(1) case, this action is realized via the Mellin transform, mapping functions on $\mathbb{A}^\times$ to functions on the modulus line $\mathbb{R}^\times_{>0}$ (or its additive counterpart $\mathbb{R}$). The local distributions D_v defined below correspond to the local components of the Weil distribution under this realization. The Weil test function $W(u)$ arises from the Archimedean component, ensuring that the global pairing translates precisely to the sum of local pairings $Q_{\mathrm{MS}}(g) = \sum_v \langle D_v, W\rangle$, where g is the associated spectral test function. This paper focuses on the rigorous evaluation of this modulus-line realization.

3.2 Global Functional and Local Distributions

The geometric side of the GL(1) explicit formula is defined globally as the sum of local distributional pairings over the places of $\mathbb{Q}$.

Definition 3.1 (Global Geometric Functional)**.** The global Maaß–Selberg base functional $Q_{\mathrm{MS}}(g)$ is defined, for $g \in \mathcal{A}_{\mathrm{Exp}}$ (resp. $g \in \mathcal{A}_{\mathrm{BL}}$), by the absolutely convergent (resp. finite) sum (*absolute convergence follows from Propositions 4.2 and 5.3*):

$$Q_{\mathrm{MS}}(g) := \sum_{v} \langle D_v, W \rangle = \sum_{p<\infty} \langle D_p, W \rangle + \langle D_\infty, W \rangle. \tag{3.1}$$

The *full* geometric functional $Q_{\mathrm{geom}}(g)$ is defined via the Weil measure (Definition 2.14) as

$$Q_{\mathrm{geom}}(g) := \int g_{\mathrm{b}}\, d\nu.$$

We now define the local distributions D_v. At the infinite place, we adopt an explicit definition corresponding to the standard Weil local contribution associated with the factor $\Gamma_{\mathbb{R}}$.

Definition 3.2 (Archimedean distribution D_∞)**.** The Archimedean distribution D_∞ acting on $\mathcal{D}_0$ (and its extension to the decaying space associated with $\mathcal{A}_{\mathrm{Exp}}$) is defined by the locally integrable kernel

$$K_\infty(u) := \frac{-e^{-u}}{1-e^{-2u}} \;=\; -\frac{1}{2\sinh u}. \tag{3.2}$$

The pairing is given by the convergent integral

$$\langle D_\infty, \varphi \rangle \;:=\; \int_0^\infty K_\infty(u)\, \varphi(u)\, \mathrm{d}u. \tag{3.3}$$

Remark 3.3. The kernel $K_\infty(u)$ has a singularity at $u = 0$, behaving as $K_\infty(u) = -\frac{1}{2u} + O(u)$. The convergence of the pairing (3.3) for the required test functions (such as $W(u)$) is guaranteed by Lemma 2.16, which ensures $\varphi(u) = O(u)$ as $u \to 0$. This cancellation ensures the integral converges absolutely near $u = 0$, requiring no principal value interpretation. If we consider the even extension of the kernel, $K_{\infty,\mathrm{ev}}(u) := K_\infty(|u|)$, and the even extension of the test function $\varphi_{\mathrm{ev}}(u)$, the full-line pairing satisfies

$$\int_{\mathbb{R}} K_{\infty,\mathrm{ev}}(u)\, \varphi_{\mathrm{ev}}(u)\, \mathrm{d}u = 2\int_0^\infty K_\infty(u)\, \varphi(u)\, \mathrm{d}u = 2\langle D_\infty, \varphi \rangle.$$

We maintain the half-line definition (3.3) throughout.

We proceed to evaluate the pairings place by place.

4 The Prime Block (Finite Places)

4.1 Local Distribution at Finite Places

At a finite place p, the local L-factor is $\zeta_p(s) = (1-p^{-s})^{-1}$. The local distribution D_p corresponds to the Fourier transform of the logarithmic derivative $-\zeta_p'(s)/\zeta_p(s)$.

Definition 4.1 (Local Prime Distribution)**.** The distribution D_p is defined by the weighted Dirac comb supported on the *positive* logarithms of prime powers ($u = k\log p > 0$), i.e. (supported in $u > 0$)

$$D_p(u) := \sum_{k=1}^{\infty} (\log p)\, \delta(u - k\log p). \tag{4.1}$$

4.2 Evaluation of the Prime Block

Proposition 4.2 (Prime Block Evaluation)**.** *Let $g \in \mathcal{A}_{\mathrm{Exp}}$. The total contribution of the finite places to $Q_{\mathrm{MS}}(g)$ is the prime block $P(g)$:*

$$P(g) := \sum_p \langle D_p, W\rangle = \frac{1}{\pi}\sum_p \sum_{k=1}^{\infty} (\log p)(1 - p^{-k})\, g_{\mathrm{b}}(2k\log p). \tag{4.2}$$

If $g \in \mathcal{A}_{\mathrm{BL}}$, this sum is finite. If $g \in \mathcal{A}_{\mathrm{Exp}}$, the sum converges absolutely.

Proof. We evaluate the pairing at place p:

$$\langle D_p, W\rangle = \int_0^{\infty} W(u) D_p(u)\,\mathrm{d}u = \sum_{k=1}^{\infty} (\log p)\, W(k\log p).$$

Substituting the definition of W from (2.6):

$$W(k\log p) = \frac{1}{\pi}\left(1 - e^{-k\log p}\right) g_{\mathrm{b}}(2k\log p) = \frac{1}{\pi}\left(1 - p^{-k}\right) g_{\mathrm{b}}(2k\log p).$$

Summing over all primes p yields the expression for $P(g)$.

If $g \in \mathcal{A}_{\mathrm{BL}}$, $\mathrm{supp}(g_{\mathrm{b}}) \subset [-X, X]$ for some $X > 0$. The sum is restricted to $2k\log p \le X$, ensuring only a finite number of terms are non-zero.

If $g \in \mathcal{A}_{\mathrm{Exp}}$, we have $|g_{\mathrm{b}}(x)| \le Ce^{-\sigma|x|}$ with $\sigma > 1/2$. Since $|g_{\mathrm{b}}(2k\log p)| \le C\, p^{-2\sigma k}$, we have

$$(\log p)\,(1 - p^{-k})\,|g_{\mathrm{b}}(2k\log p)| \;\le\; C\,(\log p)\,\left(p^{-2\sigma k} - p^{-(2\sigma+1)k}\right),$$

so

$$|P(g)| \le \frac{C}{\pi}\sum_p \sum_{k\ge 1} (\log p)\left(p^{-2\sigma k} - p^{-(2\sigma+1)k}\right) = \frac{C}{\pi}\Big(-\frac{\zeta'}{\zeta}(2\sigma) + \frac{\zeta'}{\zeta}(2\sigma+1)\Big) < \infty.$$

In particular, absolute convergence holds since $2\sigma > 1$, so Tonelli/Fubini justifies the interchange of the prime–power sums with absolute values used above. Here we used the Euler–product identity [2, §2.6]

$$\sum_p \sum_{k\ge 1} (\log p)\, p^{-ak} \;=\; -\frac{\zeta'}{\zeta}(a) \qquad (\Re a > 1),$$

so the factor $(1 - p^{-k})$ produces the difference $-\zeta'/\zeta(2\sigma) + \zeta'/\zeta(2\sigma+1)$ displayed above (sign consistent with $\frac{\mathrm{d}}{\mathrm{d}a}\log\zeta(a)$). □

Remark 4.3 (Symmetric Normalization). Utilizing the dictionary identity (Remark 2.12), we express the prime block in the standard symmetric normalization:

$$P(g) = \frac{1}{\pi}\sum_{p,k\ge 1} (\log p)\, p^{-k/2}\, G_{\mathrm{b}}(k\log p). \tag{4.3}$$

This normalization is maintained consistently across the companion papers [3, 4].

Remark 4.4 (Von Mangoldt formulation). Writing $P(g)$ with the von Mangoldt function yields

$$P(g) = \frac{1}{\pi}\sum_{n\ge 2} \Lambda(n)\, n^{-1/2}\, G_{\mathrm{b}}(\log n),$$

which is equivalent to (4.2) by grouping prime powers.

5 The Archimedean Block (Infinite Place)

This section rigorously connects the distributional pairing on the modulus line to the spectral definition of the Archimedean block, employing a careful regularization procedure.

5.1 The Archimedean Target

We define the Archimedean block $\mathcal{A}(g)$ following the normalization compatible with the Eisenstein phase derivative, adhering strictly to the Maaß–Selberg convention (as established in [3]).

Definition 5.1 (Archimedean Block, Spectral Definition)**.** The Archimedean block $\mathcal{A}(g)$ is defined by the spectral pairing with the boundary kernel $K_A(\tau)$ (consistent with Paper I):

$$\mathcal{A}(g) := \frac{1}{2\pi}\int_{\mathbb{R}} g(\tau)\,K_A(\tau)\,\mathrm{d}\tau, \qquad K_A(\tau) := \Re\{\psi(i\tau) - \psi(\tfrac{1}{2}+i\tau)\}. \tag{5.1}$$

Here ψ denotes the digamma function (see [1, §5.5]). We also define the *α-regularized* functional for $\alpha > 0$:

$$\mathcal{A}_\alpha(g) := \frac{1}{2\pi}\int_{\mathbb{R}} g(\tau)\,\Re\{\psi(\alpha+i\tau) - \psi(\tfrac{1}{2}+i\tau)\}\,\mathrm{d}\tau. \tag{5.2}$$

We fix the branches as in Appendix A and [3]. As shown in Proposition 5.3, the limit $\lim_{\alpha\downarrow 0}\mathcal{A}_\alpha(g)$ exists and equals $\mathcal{A}(g) - \frac{1}{2}g(0)$. This limit on the spectral side is obtained by isolating the Poisson kernel and applying DCT to the remainder with a uniform $(1+|\tau|)^{-2}$ majorant (PO–9). The subsequent frequency-side evaluation uses DCT with the uniform majorant from Lemma 2.16 ($|K_\alpha(t)\,g_{\mathrm{b}}(t)| \le |g_{\mathrm{b}}(t)|$; PO–1). We have $K_A \in L^1(\mathbb{R})$ (PO–3).

Remark 5.2 (Properties of the Kernel $K_A(\tau)$). The kernel $K_A(\tau)$ is C^∞ on $\mathbb{R}$ and *bounded at the origin.* Using $\psi(z) = -\gamma - \frac{1}{z} + O(z)$ as $z \to 0$. We analyze the behavior near $\tau = 0$:

$$\psi(i\tau) = -\gamma - \frac{1}{i\tau} + O(\tau) = -\gamma + i\frac{1}{\tau} + O(\tau).$$

Since $\Re(i/\tau) = 0$, the singularity cancels when taking the real part: $\Re\psi(i\tau) = -\gamma + O(\tau^2)$. Thus, we obtain

$$K_A(0) \;=\; \lim_{\tau\to 0}\Re\{\psi(i\tau) - \psi(\tfrac{1}{2}+i\tau)\} = -\gamma - \psi\Big(\tfrac{1}{2}\Big) \in \mathbb{R}.$$

For large $|\tau|$, a standard asymptotic expansion of ψ (via Stirling's series) yields

$$K_A(\tau) = \Re\{\psi(i\tau) - \psi(\tfrac{1}{2}+i\tau)\} \;=\; \frac{1}{8\tau^2} + O(\tau^{-4}) \qquad (|\tau| \to \infty),$$

Note. This expansion holds uniformly on vertical strips (e.g., $\{\Re z \in [0,1]\}$) when combined with standard identities for ψ. The decay $O(\tau^{-2})$ implies $K_A \in L^1(\mathbb{R})$ and $\mathcal{A}(g) = \frac{1}{2\pi}\int_{\mathbb{R}} g(\tau)K_A(\tau)\,\mathrm{d}\tau$ for $g \in \mathcal{S}(\mathbb{R})$. This decay matches the positivity paper's estimate $K_A(\tau) = O((1+|\tau|)^{-2})$ in the same ledger. *(PO–7 closed; PO–3 reaffirmed; PO–9 used.)* [4, Lemma 3.3]

5.2 The Archimedean Distribution and Evaluation

We now establish the connection between the distributional pairing $\langle D_\infty, W\rangle$ (Definition 3.2) and the spectral definition of $\mathcal{A}(g)$. Although $K_A(\tau)$ is regular at the origin, the regularization procedure is essential to rigorously track the boundary contributions that arise when relating the spectral and frequency representations.

Proposition 5.3 (Evaluation of the Archimedean pairing)**.** *For any $g \in \mathcal{A}_{\mathrm{Exp}}$, the distributional pairing at the infinite place satisfies*

$$\langle D_\infty, W \rangle = -\mathcal{A}(g). \tag{5.3}$$

Equivalently (on $\mathbb{R}$, with absolute convergence near $u = 0$),

$$\frac{1}{2} \int_{\mathbb{R}} K_\infty(|u|)\, W_{\mathrm{ev}}(u)\, \mathrm{d}u \;=\; -\frac{1}{2\pi} \int_{\mathbb{R}} g(\tau)\, K_A(\tau)\, \mathrm{d}\tau,$$

and the right–hand side equals $-\mathcal{A}(g)$ by Definition 5.1. This is the Archimedean dictionary recorded as (H.2) in Paper III [4], in the same Maaß–Selberg ledger.

Proof (Rigorous regularization and distributional evaluation). We establish the identity by analyzing the limit of the regularized Archimedean block $\mathcal{A}_\alpha(g)$ (defined in (5.2)) as $\alpha \downarrow 0$.

Step 1: Analyzing the regularization limit (spectral side). We analyze the behavior near the singularity. As $\alpha \downarrow 0$, we use the expansion $\psi(z) \approx -1/z$ near $z = 0$. The dominant singular behavior comes from the first term:

$$\Re\{\psi(\alpha + i\tau)\} = -\frac{\alpha}{\alpha^2 + \tau^2} + R'_\alpha(\tau),$$

where $R'_\alpha(\tau)$ is regular near $\tau = 0$. We isolate this delta-convergent piece (the Poisson kernel) explicitly in the regularized kernel:

$$\Re\{\psi(\alpha + i\tau)\} - \Re\{\psi(\tfrac{1}{2} + i\tau)\} = -\frac{\alpha}{\alpha^2 + \tau^2} + R_\alpha(\tau).$$

The total remainder $R_\alpha(\tau) = R'_\alpha(\tau) - \Re\{\psi(\frac{1}{2} + i\tau)\}$ is continuous in (α, τ). By the duplication/shift identities for ψ together with Stirling's asymptotics (see PO–9), $R_\alpha(\tau)$ satisfies the *uniform* bound

$$|R_\alpha(\tau)| \ll (1 + |\tau|)^{-2} \qquad \text{uniformly for } \alpha \in (0, \tfrac{1}{2}].$$

Hence

$$\frac{1}{2\pi} \int_{\mathbb{R}} g(\tau) \Big(-\frac{\alpha}{\alpha^2 + \tau^2} \Big)\, \mathrm{d}\tau \;\to\; -\frac{1}{2} g(0) \qquad (\alpha \downarrow 0),$$

by the Poisson–kernel approximate identity, and

$$\frac{1}{2\pi} \int_{\mathbb{R}} g(\tau)\, R_\alpha(\tau)\, \mathrm{d}\tau \;\to\; \frac{1}{2\pi} \int_{\mathbb{R}} g(\tau)\, K_A(\tau)\, \mathrm{d}\tau \qquad (\alpha \downarrow 0)$$

by DCT with the $(1 + |\tau|)^{-2}$ majorant. Therefore

$$\lim_{\alpha \downarrow 0} \mathcal{A}_\alpha(g) = \mathcal{A}(g) - \frac{1}{2} g(0). \tag{5.4}$$

Step 2: Evaluation of the limit (frequency side). Having established $\lim_{\alpha \downarrow 0} \mathcal{A}_\alpha(g) = \mathcal{A}(g) - \frac{1}{2} g(0)$, we now evaluate $\mathcal{A}_\alpha(g)$ using the frequency representation (g_{b}). We employ the integral representation of the digamma function (valid for $\Re z > 0$; see, e.g., [1, 5.9.15]):

$$\psi(z) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-zt}}{1 - e^{-t}}\, \mathrm{d}t. \tag{5.5}$$

Substituting this into the definition of $A_\alpha(g)$ and taking the real part yields the kernel:

$$\Re\{\psi(\alpha+i\tau)-\psi(\tfrac{1}{2}+i\tau)\} = \int_0^\infty \frac{e^{-t/2}\cos(\tau t)-e^{-\alpha t}\cos(\tau t)}{1-e^{-t}}\,\mathrm{d}t.$$

(The constant terms involving γ cancel.) We insert this into $A_\alpha(g)$ and interchange the order of integration. This application of Tonelli/Fubini is justified for $\alpha>0$ by Lemma 2.16.

$$\begin{aligned}\mathcal{A}_\alpha(g) &= \frac{1}{2\pi}\int_0^\infty \frac{e^{-t/2}-e^{-\alpha t}}{1-e^{-t}}\left(\int_{\mathbb{R}} g(\tau)\cos(\tau t)\,\mathrm{d}\tau\right)\mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(t)\left(\frac{e^{-\frac{1}{2}t}-e^{-\alpha t}}{1-e^{-t}}\right)\mathrm{d}t.\end{aligned}$$

We now take the limit $\alpha\downarrow 0$. By the Dominated Convergence Theorem (justified in Lemma 2.16), we can pass the limit inside the integral:

$$\lim_{\alpha\downarrow 0}\mathcal{A}_\alpha(g) = \frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(t)\left(\frac{e^{-\frac{1}{2}t}-1}{1-e^{-t}}\right)\mathrm{d}t = -\frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(t)\,\frac{1-e^{-t/2}}{1-e^{-t}}\,\mathrm{d}t.$$

Using the identity $\frac{1-e^{-t/2}}{1-e^{-t}} = \frac{1}{1+e^{-t/2}}$, we find:

$$\lim_{\alpha\downarrow 0}\mathcal{A}_\alpha(g) = -\frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(t)\,\frac{1}{1+e^{-t/2}}\,\mathrm{d}t. \tag{5.6}$$

Step 3: Deriving the Frequency Representation of $\mathcal{A}(g)$. Equating the results from Step 1 (Eq. (5.4)) and Step 2 (Eq. (5.6)):

$$\mathcal{A}(g)-\frac{1}{2}g(0) = -\frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(t)\,\frac{1}{1+e^{-t/2}}\,\mathrm{d}t.$$

We use the identity $\frac{1}{2}g(0)=\frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(t)\mathrm{d}t$ to solve for $\mathcal{A}(g)$:

$$\mathcal{A}(g) = \frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(t)\left(1-\frac{1}{1+e^{-t/2}}\right)\mathrm{d}t = \frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(t)\frac{e^{-t/2}}{1+e^{-t/2}}\mathrm{d}t.$$

Step 4: Evaluation of the distributional pairing $\langle D_\infty, W\rangle$. We now evaluate the pairing directly using the definitions $K_\infty(u) = -\frac{e^{-u}}{1-e^{-2u}}$ (Definition 3.2) and $W(u)=\frac{1}{\pi}(1-e^{-u})\,g_{\mathrm{b}}(2u)$ (Definition 2.11).

$$\begin{aligned}\langle D_\infty, W\rangle &= \int_0^\infty W(u)K_\infty(u)\,\mathrm{d}u \\ &= \frac{1}{\pi}\int_0^\infty (1-e^{-u})\,g_{\mathrm{b}}(2u)\left(-\frac{e^{-u}}{1-e^{-2u}}\right)\mathrm{d}u.\end{aligned}$$

Simplifying the kernel factor using $1-e^{-2u}=(1-e^{-u})(1+e^{-u})$:

$$\begin{aligned}\langle D_\infty, W\rangle &= -\frac{1}{\pi}\int_0^\infty g_{\mathrm{b}}(2u)\frac{e^{-u}(1-e^{-u})}{(1-e^{-u})(1+e^{-u})}\,\mathrm{d}u \\ &= -\frac{1}{\pi}\int_0^\infty g_{\mathrm{b}}(2u)\frac{e^{-u}}{1+e^{-u}}\,\mathrm{d}u.\end{aligned}$$

Applying the change of variables $t=2u$ ($\mathrm{d}u=\mathrm{d}t/2$):

$$\langle D_\infty, W\rangle = -\frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(t)\frac{e^{-t/2}}{1+e^{-t/2}}\,\mathrm{d}t.$$

Comparing this result with the expression derived in Step 3, we conclude $\langle D_\infty, W\rangle = -\mathcal{A}(g)$. □

Remark 5.4 (Digamma Duplication and Maaß–Selberg Normalization). Using the duplication identity ([1], §5.5.6),

$$\psi(2z) = \tfrac{1}{2}\big(\psi(z) + \psi(z+\tfrac{1}{2})\big) + \log 2.$$

This allows $K_A(\tau)$ to be rewritten as:

$$K_A(\tau) = \Re\{\psi(i\tau) - \psi(\tfrac{1}{2} + i\tau)\} = 2\,\Re\{\psi(i\tau) - \psi(2i\tau)\} + 2\log 2.$$

If $\mathcal{A}(g)$ were evaluated using this form, the constant $2\log 2$ would contribute:

$$\frac{1}{2\pi}\int_{\mathbb{R}} g(\tau)(2\log 2)\,\mathrm{d}\tau = \frac{\log 2}{\pi}\,g_{\mathrm{b}}(0).$$

Crucially, in the Maaß–Selberg ledger adopted here, this constant is *not* treated separately. It is automatically incorporated into the continuous density ρ_W through the standard regrouping:

$$-\mathcal{A}(g) + \tfrac{1}{2}g(0) + 2J(g) = \int_0^\infty g_{\mathrm{b}}(x)\,\rho_W(x)\,\mathrm{d}x,$$

where $\rho_W(x)$ is defined in (2.2). This ensures the geometric side is invariant under equivalent presentations of the kernel $K_A(\tau)$. All constant pieces are absorbed into ρ_W; no standalone constant survives on the geometric side. This normalization is consistent with Paper I (App. A) and Paper III.

5.3 Half-Shift Kernel and Positivity

The regularization process explicitly reveals the connection between the Archimedean contribution and the geometric torsion filter $T_{1/2}(x)$ introduced in Section 2.4.

Proposition 5.5 (Half-shift kernel and positivity)**.** *For $g \in \mathcal{A}_{\mathrm{Exp}}$, the Archimedean block combined with the boundary term satisfies:*

$$-\mathcal{A}(g) + \tfrac{1}{2}g(0) = \frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(x)\,T_{1/2}(x)\,\mathrm{d}x,$$

where $T_{1/2}(x) = \frac{1}{1+e^{-x/2}}$ is the torsion filter kernel. In particular, $T_{1/2}(x) \in (1/2, 1)$ for $x > 0$ (strictly increasing, smooth). Consequently, the expression is strictly positive whenever $g_{\mathrm{b}} \geq 0$ (and g_{b} is not identically zero).

Proof. This identity is a direct consequence of the rigorous relationship established during the regularization process in the proof of Proposition 5.3. By combining Eq. (5.4) and Eq. (5.6), we found:

$$A(g) - \frac{1}{2}g(0) = \lim_{\alpha\downarrow 0} A_\alpha(g) = -\frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(x)\,\frac{1}{1+e^{-x/2}}\,\mathrm{d}x.$$

Rearranging this identity immediately yields the stated result. □

Remark 5.6 (The Full Weil Density). Combining the result of Proposition 5.5 with the definition of the Poisson block $J(g)$ (Definition 2.11), we obtain the density representing the entire continuous geometric side of the explicit formula (1.3):

$$\begin{aligned} -\mathcal{A}(g) + \tfrac{1}{2}g(0) + 2J(g) &= \frac{1}{2\pi}\int_0^\infty \frac{g_{\mathrm{b}}(x)}{1+e^{-x/2}}\,\mathrm{d}x + 2\left(\frac{1}{2\pi}\int_0^\infty e^{-x/2}g_{\mathrm{b}}(x)\,\mathrm{d}x\right) \\ &= \int_0^\infty g_{\mathrm{b}}(x)\,\rho_W(x)\,\mathrm{d}x. \end{aligned}$$

Here, we recall the same ρ_W from Definition 2.2 and do not reintroduce a separate label. This ρ_W coincides *verbatim* with the modulus-line continuous density used in the positivity paper [4] (Paper III, Eq. (1.3) and §D.3), ensuring a normalization match across the trilogy in the Maaß–Selberg ledger. This density is manifestly positive and plays a central role in the analysis of Weil positivity presented in [4]. (Ledger consistency with Paper I [3] phase–measure and Paper III's regrouping is explicit in Paper I §§2, 6 and Ledger A.2.)

6 Main Identity and Geometric Positivity

We now assemble the local contributions to establish the main theorem and summarize the resulting geometric structure of the explicit formula.

Theorem 6.1 (Maaß–Selberg Base Identity)**.** *Let $g \in \mathcal{A}_{\mathrm{Exp}}$ (including the admissible band-limited class $\mathcal{A}_{\mathrm{BL}}$) and let W be the associated Weil test function. The global base functional $Q_{\mathrm{MS}}(g)$ satisfies the identity:*

$$Q_{\mathrm{MS}}(g) = P(g) - \mathcal{A}(g), \tag{6.1}$$

where $P(g)$ is the prime sum (Proposition 4.2) and $\mathcal{A}(g)$ is the Archimedean block (Definition 5.1). Throughout, the spectral measure is fixed in the Maaß–Selberg convention $d\xi(\tau) = \frac{1}{2\pi}\,\varphi'(\tau)\,\mathrm{d}\tau$ (see Appendix A and Thm. 2.1).

Global Conventions and Normalizations.

- Base functional: $Q_{\mathrm{MS}}(g) = P(g) - \mathcal{A}(g)$ (Maaß–Selberg ledger, symmetric PV).
- Completed zeta function: $\Lambda(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$.
- Fourier pair (forward without 2π, inverse with $1/2\pi$): $\widehat{g}(\xi) = \int_{\mathbb{R}} g(\tau)e^{-i\tau\xi}\,\mathrm{d}\tau$, $g(\tau) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{g}(\xi)e^{i\tau\xi}\,\mathrm{d}\xi$; for even g, $g_{\mathrm{b}}(x) = \widehat{g}(x)$ and $g(\tau) = \frac{1}{\pi}\int_0^\infty g_{\mathrm{b}}(x)\cos(x\tau)\,\mathrm{d}x$ (cosine dictionary in Thm. 2.10).
- Mellin transform: $\mathcal{M}f(s) = \int_0^\infty f(x)\,x^{s-1}\,\mathrm{d}x$; modulus-line coordinate $x = 2u$.
- Spectral measure (Maaß–Selberg): $d\xi(\tau) = (1/2\pi)\,\varphi'(\tau)\,\mathrm{d}\tau$, symmetric PV, no δ_0; residues oriented by increasing τ (see Thm. 2.1). Under duplication, the constant $(\log 2/\pi)\,\widehat{g}(0)$ pairs with $J(g)$ and is absorbed into ρ_W (no standalone constant on the geometric side).
- Test class: $g \in \mathcal{A}_{\mathrm{Exp}}$ (or the subclass $\mathcal{A}_{\mathrm{BL}}$).

All pairings and signs in Theorem 6.1 adhere strictly to these conventions.

Conceptual Framing. The adelic proof proceeds as a verification of a *distributional identity*. We construct a canonical Weil test function W derived from the spectral test function g. We then evaluate the global base functional $Q_{\mathrm{MS}}(g) = \sum_v \langle D_v, W\rangle$. This geometric evaluation is then matched against the automorphic evaluation of the corresponding functional on the GL(1) side, as detailed in [3]. In the symmetric normalization adopted here, the zero weights are two-sided, and the Archimedean contribution is isolated according to the conventions summarized above. Equating the geometric and spectral evaluations yields the symmetric explicit formula in this specific normalization.

Proof of Theorem 6.1. The identity follows directly from the definition of $Q_{\mathrm{MS}}(g)$ (Definition 3.1) and the evaluation of the local pairings:

$$Q_{\mathrm{MS}}(g) = \sum_p \langle D_p, W\rangle + \langle D_\infty, W\rangle.$$

By Proposition 4.2, $\sum_p \langle D_p, W\rangle = P(g)$. By Proposition 5.3, $\langle D_\infty, W\rangle = -\mathcal{A}(g)$. Combining these yields $Q_{\mathrm{MS}}(g) = P(g) - \mathcal{A}(g)$. □

Proposition 6.2 (Geometric Positivity Structure)**.** *Let $g \in \mathcal{A}_{\mathrm{Exp}}$. The geometric side of the full explicit formula* (1.3) *exhibits the following explicit positivity structure:*

(i) The prime block $P(g)$ is a sum over the frequency profile g_{b} with strictly positive coefficients:

$$P(g) = \frac{1}{\pi}\sum_p \sum_{k=1}^{\infty} c_{p,k}\, g_{\mathrm{b}}(2k\log p), \qquad c_{p,k} := (\log p)\,(1 - p^{-k}) \ > \ 0.$$

(ii) The continuous part (the Archimedean contribution combined with the boundary term and the Poisson block) is represented by an integral against the strictly positive Weil density $\rho_W(x)$:

$$-\mathcal{A}(g) + \tfrac{1}{2}g(0) + 2J(g) \ = \ \int_0^\infty g_{\mathrm{b}}(x)\,\rho_W(x)\,\mathrm{d}x,$$

where $\rho_W(x)$ is defined in Eq. (2.2)*.*

Consequently, if $g_{\mathrm{b}}(x) \geq 0$ for all $x \geq 0$, the full geometric side of the explicit formula is nonnegative (and is strictly positive whenever $g_{\mathrm{b}} \not\equiv 0$ on a set of positive measure).

Proof. Item (i) follows directly from the evaluation in Proposition 4.2. Item (ii) follows from the combination of Proposition 5.5 and the subsequent analysis in Remark 5.6. □

Preprint status, claim scope, and open-review protocol

Status (Paper II). Unrefereed preprint. Version 2025-08-23. DOI: DOI: 10.5281/zenodo.16930093.

Scope. Paper II fixes the Maaß–Selberg normalization and constructs $Q_{\mathrm{MS}}(g) = P(g) - \mathcal{A}(g)$. *No RH claim is made here.* RH discussions in this paper are limited to classical equivalences (Weil positivity) and implication phrasing for context.
Where the RH claim lives. Paper III (10.5281/zenodo.16930095) *states a claimed proof of RH as a preprint*, pending peer review. Readers are encouraged to cross-check constants and pairings against the shared ledger across Papers I–III.
What to review here. (i) PV/no-δ_0 Maaß–Selberg measure; (ii) the Archimedean dictionary $\langle D_\infty, W\rangle = -\mathcal{A}(g)$; (iii) prime/continuous density ρ_W; (iv) DCT/Fubini/Tonelli closures in the PO registry.
Reproducibility. Ancillary notebooks verify local pairings and the continuous density; versions mirror Paper III's checks.
Versioning & policy. Changelog maintained; normalization corrections propagate trilogy-wide.
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A Normalization Ledger and Constant Audit

This appendix constitutes the canonical normalization ledger for the trilogy [3, 4]. It rigorously fixes the Maaß–Selberg convention $d\xi(\tau) = \frac{1}{2\pi}\,\varphi'(\tau)\,\mathrm{d}\tau$ and synchronizes all definitions, constants, and identities across the documents.

A.1 Global Standing Notation

Test Functions and Transforms. The test function $g : \mathbb{R} \to \mathbb{R}$ is real, even, and typically in the class $\mathcal{A}_{\mathrm{Exp}}$ (often the subclass $\mathcal{A}_{\mathrm{BL}}$). Its cosine transform $g_{\mathrm{b}}(x)$ is defined as:

$$g_{\mathrm{b}}(x) \;:=\; \int_{\mathbb{R}} g(\tau)\,\cos(x\tau)\,\mathrm{d}\tau, \qquad g(\tau) \;=\; \frac{1}{\pi}\int_0^\infty g_{\mathrm{b}}(x)\,\cos(x\tau)\,\mathrm{d}x.$$

Notation match: In the notation of [3] (Paper I), $\widehat{g}(x)$ corresponds to $g_{\mathrm{b}}(x)$ here. For clarity, we restrict to even profiles in this paper, so sine–odd contributions vanish by symmetry.

Derived Profiles and Dictionary. The symmetric profile G_{b}, the Weil test function W (on $u \ge 0$), and the Poisson block $J(g)$ are defined as:

$$G_{\mathrm{b}}(x) \;:=\; 2\sinh\Big(\frac{|x|}{2}\Big)\,g_{\mathrm{b}}(2|x|), \qquad W(u) \;:=\; \frac{1}{\pi}\,(1-e^{-u})\,g_{\mathrm{b}}(2u).$$
$$J(g) \;:=\; \frac{1}{2\pi}\int_0^\infty e^{-x/2}\,g_{\mathrm{b}}(x)\,\mathrm{d}x.$$

The key dictionary identities relating these profiles are:

$$W(u) \;=\; \frac{1}{\pi}\,e^{-u/2}\,G_{\mathrm{b}}(u) \quad (u>0), \qquad (1-p^{-k})\,g_{\mathrm{b}}(2k\log p) \;=\; p^{-k/2}\,G_{\mathrm{b}}(k\log p). \tag{A.1}$$

The origin functional identity is $g(0) = 2\int_0^\infty \frac{W(u)}{1-e^{-u}}\,\mathrm{d}u$. This follows by setting $\tau = 0$ in (2.3) and using $W(u) = \frac{1}{\pi}(1-e^{-u})\,g_{\mathrm{b}}(2u)$ from the dictionary above.

Torsion Filter and Weil Density. The half-shift commutator (torsion filter) $C_{1/2}$ is implemented by the positive kernel $T_{1/2}(x) = 1/(1+e^{-x/2})$. The full Weil density is $\rho_W(x) = \frac{1}{2\pi}(T_{1/2}(x) + 2e^{-x/2})\,\mathbf{1}_{(0,\infty)}(x)$, and the continuous block is $W_{\mathrm{cont}}(g) = \int_0^\infty g_{\mathrm{b}}(x)\,\rho_W(x)\,\mathrm{d}x$.

A.2 Local Distributions and Kernels

Finite place (p) — Distribution $D_p(u)$	
Definition / Kernel	$D_p(u) = \sum_{k\ge1}(\log p)\,\delta(u-k\log p)$
Pairing / Output	$\langle D_p, W\rangle = \frac{1}{\pi}\sum_{k\ge1}(\log p)\,(1-p^{-k})\,g_{\mathrm{b}}(2k\log p)$ (finite for $\mathcal{A}_{\mathrm{BL}}$; absolutely convergent for $\mathcal{A}_{\mathrm{Exp}}$)

Archimedean place (∞) — Distribution D_∞ (Modulus line)	
Definition / Kernel	$K_\infty(u) = -\dfrac{e^{-u}}{1-e^{-2u}} = -\dfrac{1}{2\sinh u}$
Pairing / Output	$\langle D_\infty, W\rangle = -\mathcal{A}(g)$

Archimedean place (∞) — Kernel $K_A(\tau)$ (Spectral line)	
Definition / Kernel	$K_A(\tau) = \Re\{\psi(i\tau) - \psi(\tfrac{1}{2} + i\tau)\}$
Pairing / Output	*Functional:* $\mathcal{A}(g) = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}} g(\tau)\, K_A(\tau)\, \mathrm{d}\tau.$
	Regrouping: $-\mathcal{A}(g) + \tfrac{1}{2}g(0) + 2J(g) = \displaystyle\int_0^\infty g_{\mathrm{b}}(x)\, \rho_W(x)\, \mathrm{d}x.$
	Density: $\rho_W(x) = \tfrac{1}{2\pi}\Big(\tfrac{1}{1+e^{-x/2}} + 2e^{-x/2}\Big)\, \mathbf{1}_{(0,\infty)}(x).$

The Archimedean pairing identity (Proposition 5.3) rigorously confirms the consistency between the modulus line and spectral line representations.

A.3 Arithmetic and Spectral Blocks (Pairings and Normalizations)

Arithmetic block (finite places). The global prime block is the sum of the local contributions:

$$P(g) \;=\; \sum_p \langle D_p, W\rangle \;=\; \frac{1}{\pi}\sum_{p,k\geq 1} (\log p)\, p^{-k/2}\, G_{\mathrm{b}}(k\log p). \tag{A.2}$$

Spectral measure and phase derivative (Birman–Krein spectral-shift). [1] Let $\widetilde{S}(s) = \Lambda(2s-1)/\Lambda(2s)$ and $\varphi'(\tau) = \Im\, \partial_\tau \log \widetilde{S}(\tfrac{1}{2} + i\tau)$. With the Maaß–Selberg convention

$$d\xi(\tau) = \frac{1}{2\pi}\, \varphi'(\tau)\, \mathrm{d}\tau, \qquad \mathcal{I}(g) = \langle \xi, g\rangle = \frac{1}{2\pi}\, \mathrm{PV}\!\int_{\mathbb{R}} g(\tau)\, \varphi'(\tau)\, \mathrm{d}\tau \quad \text{(symmetric PV)}, \tag{A.3}$$

with *no* δ_0 *mass*. Using the functional equation $\Lambda(1-s) = \Lambda(s)$ and conjugation symmetry, the boundary identity is equivalently written as

$$\varphi'(\tau) = \mathrm{PV}\ 4\,\Re\left\{\frac{\Lambda'}{\Lambda}(2i\tau)\right\}.$$

Crosswalk: Paper I uses the same notation $\varphi'(\tau)$ for this object, ensuring consistency across the trilogy (same ledger). [3, 4] *Ledger note.* The Maaß–Selberg spectral–shift measure carries *no* δ_0 *mass* and the PV is *symmetric*; see Thm. 2.1.

Zero weights. For a nontrivial zero $\rho = \beta + i\gamma$ (with $\beta \in (0,1)$), the spectral weight is defined in the canonical normalization as:

$$\mathcal{H}(g;\rho) \;:=\; \frac{1}{2\pi}\int_0^\infty e^{-\beta x/2}\, g_{\mathrm{b}}(x)\, \cos(\gamma x/2)\, \mathrm{d}x. \tag{A.4}$$

The total spectral side (zero-side quadratic form) $W_{\mathrm{spec}}(g)$ is defined by summing over all nontrivial zeros ρ (counted with multiplicity):

$$W_{\mathrm{spec}}(g) \;:=\; 2\sum_\rho \mathcal{H}(g;\rho).$$

(We reserve $W(u)$ for the Weil modulus-line test function.) The factor 2 is conventional in the asymmetric formula (see Eq. (A.7)). Absolute convergence of $\sum_\rho |\mathcal{H}(g;\rho)|$ for $g \in \mathcal{S}(\mathbb{R})$ is recorded in the companion Paper I, Appendix B.

[1] See, e.g., [6].

A.4 Bridge Identities and the Explicit Formula

The explicit formula arises from equating two distinct evaluations of the spectral functional $\mathcal{I}(g)$.

Route A (Dirichlet/geometric evaluation). This route evaluates $\mathcal{I}(g)$ using the Dirichlet series representation of Λ'/Λ, involving the geometric terms $P(g)$ and $\mathcal{A}(g)$, and the boundary term from regularization.

$$\mathcal{I}(g) \;=\; \mathcal{A}(g) \;-\; \frac{1}{2}\,g(0) \;-\; P(g). \tag{A.5}$$

Route B (Spectral/partial fractions evaluation).

Remark A.1 (Branch and normalization). We use the completed zeta $\Lambda(s)$ and define $\log \Lambda(s)$ by analytic continuation from $\Re s > 1$ using the principal branch. The partial-fractions expansion follows from the Hadamard product for Λ, and poles/zeros at $0, 1$ are handled within Λ; see [7, Ch. 2]. This fixes the constants in (A.6).

This route evaluates $\mathcal{I}(g)$ using the Hadamard product representation of $\Lambda(s)$, involving the sum over zeros and the contribution from the poles (Poisson block).

$$\mathcal{I}(g) \;=\; 2\,J(g) \;-\; 2\sum_{\rho} \mathcal{H}(g;\rho). \tag{A.6}$$

The Asymmetric Explicit Formula.

Theorem A.2 (Asymmetric Explicit Formula)**.** *Under the hypotheses of §2 (in particular g admissible in the sense of Assumption 2.9) and in the Maaß–Selberg normalization, the asymmetric explicit formula holds:*

$$\boxed{2\sum_{\rho} \mathcal{H}(g;\rho) = P(g) + \frac{1}{2}g(0) + 2J(g) - \mathcal{A}(g).} \tag{A.7}$$

Proof. This follows by equating the two evaluations of $\mathcal{I}(g)$ derived in Route A ((A.5)) and Route B ((A.6)), utilizing $\varphi'(\tau)$ from (A.3). $\square$

Constant cancellation under duplication. Using $\psi(2z) = \frac{1}{2}(\psi(z) + \psi(z + \frac{1}{2})) + \log 2$, one may rewrite K_A as $2\,\Re\{\psi(i\tau) - \psi(2i\tau)\} + 2\log 2$. Pairing $2\log 2$ against g contributes $\frac{1}{2\pi}\int_{\mathbb{R}} g(\tau)(2\log 2)\,\mathrm{d}\tau = (\log 2/\pi)\,g_{\mathrm{b}}(0)$. Under the standard regrouping

$$-\mathcal{A}(g) + \tfrac{1}{2}g(0) + 2J(g) = \int_0^\infty g_{\mathrm{b}}(x)\,\rho_W(x)\,\mathrm{d}x,$$

the constant $(\log 2/\pi)\,g_{\mathrm{b}}(0)$ is absorbed in ρ_W, so no stray constant remains on the geometric side; cf. Remark 5.4.

Equivalently, the continuous block equals $W_{\mathrm{cont}}(g)$ in (2.9) with ρ_W from (2.2). This form is equivalent to the symmetric Guinand–Weil formula when zeros are paired as $\rho \leftrightarrow 1 - \rho$.

A.5 Auxiliary Identities

For reference, the Archimedean kernel admits the Taylor expansion near the origin

$$K_\infty(u) = -\frac{1}{2u} + \frac{u}{12} + O(u^3) \qquad (u \to 0),$$

which makes the cancellation with $W(u) = O(u)$ manifest in the pairing $\langle D_\infty, W\rangle$.

Poisson/Laplace pairing identity. For $a > 0$, $\gamma \in \mathbb{R}$, and $g \in \mathcal{A}_{\mathrm{Exp}}$, the following identity relates the Poisson kernel pairing on the spectral line to the Laplace transform of g_{b}:

$$\int_{\mathbb{R}} g(\tau)\, \frac{a}{a^2 + (2\tau - \gamma)^2}\, \mathrm{d}\tau \;=\; \frac{1}{2}\int_0^\infty e^{-ax/2}\, g_{\mathrm{b}}(x)\, \cos\Big(\frac{\gamma x}{2}\Big)\, \mathrm{d}x \tag{A.8}$$

Standard kernel identity used: for $a > 0$ and $x \in \mathbb{R}$,

$$\int_{\mathbb{R}} \frac{a}{a^2 + t^2} \cos(xt)\, \mathrm{d}t = \pi e^{-a|x|},$$

with absolute convergence. Together with the even inversion $g(\tau) = \frac{1}{\pi}\int_0^\infty g_{\mathrm{b}}(x)\cos(x\tau)\,\mathrm{d}x$, this yields (A.8).

Minimal Cross-Walk of Symbols Across the Trilogy

Concept	**Paper I [3]**	**Paper II (This paper)**	**Paper III [4]**
Test Class (Band-Limited)	Admissible	$\mathcal{A}_{\mathrm{BL}}$	$\mathcal{A}_{\mathrm{BL}}$
Cosine transform of g	$\widehat{g}(x)$	$g_{\mathrm{b}}(x)$	$g_{\mathrm{b}}(x)$
Symmetric profile	$\widehat{G}(x)$	$G_{\mathrm{b}}(x)$	—
Weil test on modulus line	—	$W(u)$	$W(u)$
Archimedean kernel (spectral)	$K_A(\tau)$ (implicit)	$K_A(\tau)$	$K_A(\tau)$
Archimedean kernel (modulus)	—	$K_\infty(u)$	—
Arithmetic block	$P(g)$	$P(g)$	$P(g)$
Poisson block	$J(g)$	$J(g)$	$J(g)$
Zero weights	$\mathcal{H}(g;\rho)$	$\mathcal{H}(g;\rho)$	$\mathcal{H}(g;\rho)$
Torsion Filter Kernel	—	$T_{1/2}(x)$	$T_{1/2}(x)$
Spectral measure	$d\mu$ (Maaß–Selberg)	$d\mu$	$d\mu$

B Proof Obligations Registry (Internal Checklist)

This appendix summarizes the status of proof obligations referenced in the text, primarily concerning analytical justifications for convergence and interchanges.

PO–1 (DCT for $\alpha \downarrow 0$ and $\varepsilon \downarrow 0$): Uniform majorant $|K_\alpha(t)g_{\mathrm{b}}(t)| \le |g_{\mathrm{b}}(t)| \in L^1$, independent of $\alpha \in (0, 1/2]$; see Lemma 2.16. *Status: closed.*

PO–2 (Tonelli/Fubini for primes): In $\mathcal{A}_{\mathrm{BL}}$, only finitely many (p, k) contribute after pairing (compact support of g_{b}). For $\mathcal{A}_{\mathrm{Exp}}$, absolute convergence is established in Proposition 4.2. *Status: closed.*

PO–3 ($K_A \in L^1(\mathbb{R})$): Using the asymptotic $K_A(\tau) = \frac{1}{8\tau^2} + O(\tau^{-4})$ as $|\tau| \to \infty$ (see Thm. 5.2), we obtain $K_A \in L^1(\mathbb{R})$ and thus $\mathcal{A}(g)$ is well-defined for $g \in \mathcal{S}(\mathbb{R})$. *Status: closed.*

PO–4 (Poisson kernel—approximate identity): Standard identity $\delta_0 = \lim_{\alpha\downarrow 0} \frac{1}{\pi}\frac{\alpha}{\alpha^2+\tau^2}$ in $\mathcal{S}'(\mathbb{R})$; used to extract $\frac{1}{2}g(0)$ in Proposition 5.3. *Status: closed.*

PO–5 (Atomlessness at $\tau = 0$): Cancellation of Poisson-kernel singularities implies no δ_0 mass in $\varphi'(\tau)$; see Lemma 2.6 and App. A. *Status: closed.*

PO–6 (Fourier–cosine agreement): For *real even* g, $g_{\mathrm{b}}(x) = \Re\,\widehat{g}(x) = \widehat{g}(x)$. *Status: closed (Remark 2.10).*

PO–7 (Boundedness of K_A at 0): $K_A(0) = -\gamma - \psi(\frac{1}{2})$ by cancellation of the simple pole in $\psi(i\tau)$. *Status: closed (see Thm. 5.2).*

PO–8 (Even extension (half–line $\leftrightarrow$ full–line)): For $\varphi \in C^\infty([0,\infty))$ with $\varphi(0) = 0$,

$$\int_0^\infty K_\infty(u)\,\varphi(u)\,\mathrm{d}u \;=\; \tfrac{1}{2}\int_{\mathbb{R}} K_{\infty,\mathrm{ev}}(u)\,\varphi_{\mathrm{ev}}(u)\,\mathrm{d}u,$$

where $K_{\infty,\mathrm{ev}}(u) := K_\infty(|u|)$. No principal value is needed since $\varphi(u) = O(u)$ cancels the $1/u$ singularity. *Status: closed (Thm. 3.3).*

PO–9 (Uniform digamma remainder bound). For $\alpha \in (0, \frac{1}{2}]$ the remainder

$$R_\alpha(\tau) := \Re\{\psi(\alpha + i\tau) - \psi(\tfrac{1}{2} + i\tau)\} + \frac{\alpha}{\alpha^2 + \tau^2}$$

satisfies the uniform bound $|R_\alpha(\tau)| \ll (1 + |\tau|)^{-2}$. This is standard, obtained by combining the digamma duplication formula (DLMF §5.5.6) with the uniform large-$|z|$ expansion of $\psi(z)$ on vertical strips (DLMF §5.9(ii)), and elementary bounds for $\Re\psi$ near 0; see also the integral representation (5.5). *Status: closed.*

Computational Verification: Adelic Geometric Pairing

Scope and Objective. We provide numerical verification of the central identities established in this paper. The primary objectives were to validate:

(i) The global geometric identity: $Q_{\mathrm{MS}}(g) = \sum_v \langle D_v, W\rangle = P(g) - \mathcal{A}(g)$.

(ii) The Archimedean consistency (Proposition 5.3): the equivalence between the distributional pairing $\langle D_\infty, W\rangle$ (modulus line integral) and the spectral definition of $\mathcal{A}(g)$ (spectral line integral).

(iii) The positivity identities involving the full Weil density $\rho_W(x)$ (Remark 5.6).

Artifacts (Ancillary Files). The implementation details, source code, and comprehensive numerical results are provided in the ancillary files accompanying this submission: adelic_pairing_verification.ipynb and rhoW_positivity.ipynb.

Methodology and Parameters. The numerical tests utilize the same suite of band-limited test functions employed in the verification for [3]. The computations are performed with high precision (mpmath configured for 80 decimal places). Numerical integration utilizes high-resolution Simpson grids. The singularity of $K_\infty(u)$ near $u = 0$ is handled analytically via series expansion to ensure stability. The prime summation $P(g)$ is computed exactly up to the cutoff determined by the band limit X of the test function.

Outcome. The numerical results confirm that the Archimedean pairing $\langle D_\infty, W\rangle$ agrees with the negative of the spectral definition $-\mathcal{A}(g)$ to the specified numerical tolerances across all test cases. The global geometric identity $Q_{\mathrm{MS}}(g) = P(g) - \mathcal{A}(g)$ holds consistently. The positivity identities associated with the Weil density $\rho_W(x)$ are also verified. These results validate the theoretical framework and the rigorous normalization established in this paper.

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