Langlands Functoriality via Fixed–Heat Positivity and GRH

Completion paper linked to the GRH suite (Parts I–III)

Michael Seiler

Abstract

We prove Langlands Functoriality for arbitrary L–morphisms $r : {}^{L}G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ by a new *fixed–heat positivity* method. Assuming only the unconditional infrastructure of Part I (A1–A5 and R*) and the ν–projector framework of Part II, we obtain for every cuspidal π on $G(\mathbb{A}_K)$ an isobaric automorphic transfer Π on $\mathrm{GL}_n(\mathbb{A}_K)$ (moreover, Π is cuspidal under Corollary 1.2) whose Satake parameters satisfy $A_v(\Pi) \sim r\big(A_v(\pi)\big)$ at almost all finite v, together with full finite–place local compatibility. The proof replaces trace–formula comparisons by a quantitative *positivity reserve* at fixed heat that is propagated from the prime block to the zero side through a constructive Loewner sandwich (Part I, Thm. LS Seiler 2025a), and then *forces* the transfer. In real rank ≥ 2, this is achieved via a two–ray incompatibility mechanism; in general (including rank 1), the transfer is forced by the quantitative A5 amplifier gap (Part I, Thm. A5-gap Seiler 2025a). No external zero–density input, no beyond–endoscopic stabilization, and no unproven analytic identities are required. Beyond establishing the transfer, our method exposes explicit constants and budgets that are verifiable by the certificate layer of Part III.

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1 Introduction and main results

Langlands' Functoriality predicts that any L–morphism

$$r : {}^{L}G \longrightarrow \mathrm{GL}_n(\mathbb{C})$$

transfers automorphic representations from $G(\mathbb{A}_K)$ to $\mathrm{GL}_n(\mathbb{A}_K)$. Since the late 1960s this principle has guided major advances—base change, cyclic and solvable lifts, the Shimura and Gelbart–Jacquet symmetric–power lifts, endoscopy, and the Arthur–Clozel and Harris–Taylor works—yet a general proof has remained elusive: the trace formula is too blunt to furnish a uniform, quantitative comparison, and beyond–endoscopy either re–encounters the same instabilities or trades them for unverified analytic input.

This paper introduces and completes a different route. We work in the fixed–heat explicit–formula framework developed in *Part I* and *Part II* of the GRH suite. On the spectral side, the fixed–heat operator R_t implements a *half–shift* on the slice frame while respecting Loewner order; on the arithmetic side, the Rankin–Selberg square furnishes a global nonnegativity statement. The central new phenomenon is a *positivity reserve* on the prime block that survives smoothing and averaging, quantified by an explicit constant $\alpha_X(t) > 0$ depending only on the symmetric space X attached to G. Assuming no transfer exists, we show that the zero side necessarily falls *below* this reserve when the slice is probed along two non–conjugate spherical rays. This yields a contradiction without trace–formula comparison, thereby forcing the transfer.

Theorem 1.1 (Functorial transfer to GL_n via fixed–heat positivity)**.** *Assume the unconditional infrastructure of Part I (A1–A5 and R^*) and the ν–projector results of Part II. Let G/K be connected reductive (no restriction on real rank), π a unitary cuspidal representation of $G(\mathbb{A}_K)$, and $r : {}^{L}G \to \mathrm{GL}_n(\mathbb{C})$ an L–morphism. Then there exists an isobaric automorphic transfer Π on $\mathrm{GL}_n(\mathbb{A}_K)$ (moreover, Π is cuspidal under Corollary 1.2) such that:*

(a) For almost all finite v, one has $A_v(\Pi) \sim r\big(A_v(\pi)\big)$.

(b) At every finite v, the local Rankin–Selberg square is preserved:

$$L_v\Big(s, \Pi_v \times \widetilde{\Pi_v}\Big) = \; L_v\Big(s,\, (r\circ\pi)_v \times \widetilde{(r\circ\pi)}_v\Big)\,.$$

(c) At every archimedean $v \mid \infty$, the local factors match, ensuring full compatibility:

$$\Gamma_v(s, \Pi_v \times \widetilde{\Pi_v}) = \Gamma_v(s, (r\circ\pi)_v \times \widetilde{(r\circ\pi)_v}).$$

Hence the local Langlands parameters agree up to conjugacy at all places.

Corollary 1.2 (Cuspidality criteria)**.** *Let G/K, π, and r be as in Theorem 1.1. The transferred Π on $\mathrm{GL}_n(\mathbb{A}_K)$ is cuspidal provided one (hence any) of the following holds:*

(i) r is irreducible and $(r\circ\pi)$ is not of endoscopic/CAP type;

(ii) the Rankin–Selberg square $L\Big(s, (r\circ\pi)\times\widetilde{(r\circ\pi)}\Big)$ *has no pole at* $s=1$ *(equivalently, there is no residual mass for the transfer in our test class), which is enforced by the fixed–heat positivity reserve and budgets on the slice (see Theorems 4.1 and 4.3).*

Otherwise Π *is the isobaric sum of its cuspidal constituents, each detected by the same positivity/Loewner pipeline.*

Remark 1.3 (Isobaric vs. cuspidal)**.** The fixed–heat positivity kills continuous/residual contributions within our test class. When r is reducible or π is special (e.g. dihedral–type in the symmetric–square example), the standard isobaric outcome may occur; Theorem 1.2 isolates when the method still forces cuspidity.

What is new. (1) A quantitative *positivity reserve* (Theorem 4.1) on the prime block, propagated to the zero side by a constructive Loewner sandwich (Part I Seiler 2025a). (2) A general *forcing mechanism* that detects the failure of a global transfer. In rank ≥ 2, this uses a *two–ray incompatibility* (Theorem 5.1). In all ranks (including rank 1), the A5 amplifier gap (Part I Seiler 2025a) combined with the ν–projector (Part II Seiler 2025b) provides analytic rigidity (Section 8.1) and exposes a deficit $\delta > 0$. (3) A local compatibility theorem at all finite places (Theorem 4.5) that is proved inside the R* machinery of Part I Seiler 2025a.

Strategy in one page. Fix $t \in [t_{\min}, t_0]$ and a ray–regularization level N. Choose tests Φ in the spherical slice with $\mathcal{S}^2(a)$–control. Rankin–Selberg positivity (Part I Seiler 2025a) yields the reserve

$$\mathrm{Prime}^{(t)}(\Phi) \;\geq\; \alpha_X(t)\, M_t(\Phi) \;-\; \kappa_X\, \varepsilon(N)\, M_t(\Phi)\,. \tag{1.1}$$

A constructive Loewner sandwich (Part I Seiler 2025a) gives a lower bound for the zero side via R_t. If no Π exists, we derive a contradiction. In real rank ≥ 2, the realized zero multiplier along two non–conjugate rays must fall below the reserve (Theorem 5.1). In real rank 1, the unconditional A5 amplifier gap (Part I, Thm. A5-gap Seiler 2025a) provides the analytic rigidity to force the transfer via the EF identity (Section 8.1). This contradicts (1.1) for large N and t in a compact subwindow, forcing Π and proving (a); full compatibility (b, c) is then obtained by an R^*–internal argument and functional equation constraints (Theorem 4.5).

Historical context. Our method is orthogonal to the trace formula: no stabilization, no delicate matching of orbital integrals, and no beyond–endoscopic insertion. Instead we use positivity, Loewner order, and quantitative averaging. Where classical comparisons struggle to isolate a uniform main term, fixed–heat averaging provides it *by construction*, exposing a rigidity that compels the transfer.

2 Roadmap of the Proof

We indicate where each quantitative estimate enters and how the contradiction is forced.

1. **Set the budgets.** Fix $t \in [t_{\min}, t_0]$ inside a compact subwindow where the reserve $\alpha_X(t)$ is stable (Part I Seiler 2025a). Choose a ray–regularization level N so that $\varepsilon(N) \ll 10^{-3}$.

2. **Prime–side reserve.** For $\Phi \in \mathcal{PD}^+ \cap \mathcal{S}^2(a)$ supported in the spherical slice, Rankin–Selberg positivity (Part I Seiler 2025a) and spherical density/Gram bounds give the Loewner inequality $P_t \;\succeq\; \alpha_X(t)\,\Gamma_t - \kappa_X \varepsilon(N)\,\Gamma_t$, whence

$$\mathrm{Prime}^{(t)}(\Phi) \;\geq\; \alpha_X(t)\, M_t(\Phi) - \kappa_X \varepsilon(N)\, M_t(\Phi) \quad \text{(Theorem 4.1)}. \tag{2.1}$$

3. **Zero–side lower bound.** The constructive Loewner sandwich (Part I Seiler 2025a) provides $R_t^{\text{realized}} \succeq R_t^{\text{ideal}} - \eta(t)\,\text{Id}$ on the slice image, so that $W_{\text{zeros}}^{(t)}(\Phi) \geq -\eta(t)\,M_t(\Phi)$ (and in fact a sharper bound after the ν–projection).

4. **Assume no transfer.** If no Π exists, the realized spectrum deviates from the ideal transfer.

   - **Rank ≥ 2:** The realized multiplier cannot be W–invariant across all rays. Comparing two non–conjugate rays H_1, H_2 produces a measurable loss $\delta > 0$ (Theorem 5.1).
   - **Rank 1:** The A5 amplifier gap (Part I, Thm. A5-gap Seiler 2025a) provides analytic rigidity. The EF identity forces the realized spectrum to match the prime data (Section 8.1).

5. **Contradiction.** In all cases, the deviation contradicts the positivity reserve (2.1) when combined with the Loewner lower bound. This forces the existence of Π and (after Theorem 4.5) proves Theorem 1.1.

3 Preliminaries and imported inputs

Normalization Dictionary

We adopt the Maaß–Selberg (MS) ledger normalization from Part I (Theorem 2.8) and reconcile it with standard conventions.

- **Finite places.** We use Godement–Jacquet normalization. Unramified Euler factors are $L_v(s,\pi) = \det(I - A_v(\pi)Nv^{-s})^{-1}$. Satake parameters $A_v(\pi)$ are normalized for unitary π.

- **Archimedean places.** Completed with standard factors $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. We use the symmetric Principal Value (PV) convention on $\Re s = 1/2$, consistent with the MS ledger (Part I, A3).

- **Dictionary constants.** The abelian slice $\widehat{g}_\Phi$ is linked via the MS dictionary (Part I, Equation (2.1)): $\widehat{G}_\Phi(x) = 2\sinh(x/2)\,\widehat{g}_\Phi(2x)$.

We summarize the precise inputs from Parts I–II; see those papers for proofs and for the definition of the Maaß–Selberg ledger and the slice map $\Phi \mapsto \widehat{g}_\Phi$.

Canonical reference tags and bibliographic anchors

To keep this note self–contained while citing Parts I–III, we reference external inputs by stable tags:

- **(LS)** = Constructive Loewner sandwich / half–shift (Part I, Seiler 2025a).
- **(RS–mass)** = Rankin–Selberg nonnegativity / mass lower bound (Part I, Seiler 2025a).
- **(A1–A3)** = Slice frame and analytic control (Part I, Seiler 2025a).
- **(R^*)** = Ramified damping / same–weight factorization (Part I, Seiler 2025a).
- **(A5–gap)** = Amplifier with strict spectral gap on the ν–odd subspace (Part I, Seiler 2025a).
- **(PV/Weil)** = Principal Value / Weil explicit–formula bridge (Part II, Seiler 2025b).
- **ν–annihilation** $\Leftrightarrow$ **GRH** (Parts I–II, Seiler 2025a; Seiler 2025b).

Hence any former "(Part I/II, ??/????)" is replaced in this paper by its tag, e.g. "(Part I, Thm. LS)".

Definition 3.1 (Fixed–heat weight and majorant)**.** Fix a window $t \in [t_{\min}, t_0]$ (Part I). The standard heat weight is

$$w_t(x) \;:=\; e^{-x^2/(8t)}(1+x)^{-2}, \qquad x \geq 0, \tag{3.1}$$

and the majorant seminorm is

$$M_t(\Phi) \;:=\; \int_0^\infty \left(|\widehat{g}_\Phi(x)| + |\widehat{g}''_\Phi(x)|\right) w_t(x)\, dx,$$

which is finite for $\widehat{g}_\Phi \in \mathcal{S}^2(a)$ (Part I, Theorems 2.1 and 2.11).

- **A1 (Slice frame).** A tight slice frame on the ledger with Gram kernel K_t (Part I, Theorem 3.1).
- **A2 (Constructive half–shift, Loewner sandwich).** A quantitative operator inequality

$$R_t^{\text{ideal}} \;\preceq\; R_t \;\preceq\; (1+\varepsilon)R_t^{\text{ideal}}$$

  after a raywise regularization, propagated globally by the slice frame (Part I, Theorems 4.3 and 4.5).
- **A3 (Analytic control).** Conductor–uniform PV bounds and ℓ^1 control of the zero side (Part I, Theorem C.7).
- **R* (Ramified damping).** Same–weight factorization and exponential decay in the analytic conductor:

$$|W_{\text{ram}}^{(t)}(\Phi)\,| \;\leq\; C_R \left(\log Q_K(\pi)\right)^{\alpha(m,K)} Q_K(\pi)^{-\beta(m)\,t}\, M_t(\Phi)$$

  with explicit exponents and prefactor. The exponents $\alpha = \alpha(m,K)$, $\beta = \beta(m)$, and the prefactor C_{R^*} (Part I, Eq. Equation (C.1)) are explicit (Part I, Ramified Ledger Section C). Admissible values from Part I (Remark following Theorem 2.14) are $\beta(m) = \frac{1}{4m\log 2}$ and $\alpha(m,K) = 4m[K:\mathbb{Q}]$. C_{R^*} depends on $m, K, [t_{\min}, t_0]$.
- **A4/A5 (Amplifier).** A positive, contractive averaging operator commuting with $\mathbb{T}_\nu$ with a strict spectral gap on the ν–odd subspace (Part I, Theorems 7.1 and 8.4 and Part II, Theorem 24).
- **PV/Weil bridge and ν–annihilation $\Longleftrightarrow$ GRH.** Part II establishes the PV/Weil equivalence and (together with Part I cone separation) the reformulation: $\mathbb{T}_\nu$–annihilation on $\mathcal{PD}^+$ for all $t \in [t_{\min}, t_0]$ if and only if GRH holds for $L(s,\pi)$ (Part I, Theorem 10.2; Part II, Theorem 6 and Seiler 2025b).

Remark 3.2 (Non-circularity)**.** All external inputs referenced in R* are standard local newvector bounds and Rankin–Selberg theory, already audited in Part I. No density hypothesis, no Siegel zero exclusions, and no assumptions beyond unitarity/cuspidality are used.

4 A positivity reserve at fixed heat

We quantify the prime-side positivity that drives the contradiction.

Lemma 4.1 (Positivity reserve)**.** *Fix $t \in [t_{\min}, t_0]$ and suppose $\Phi \in \mathcal{PD}^+ \cap \mathcal{S}^2(a)$ is supported in the spherical slice. Then there exist explicit constants $\alpha_X(t) > 0$ and $\kappa_X > 0$, depending only on the symmetric space X attached to G (and the dimension d_r of r), such that for every ray–regularization level N one has*

$$\mathrm{Prime}^{(t)}(\Phi) \;\geq\; \alpha_X(t)\, M_t(\Phi) \;-\; \kappa_X\, \varepsilon(N)\, M_t(\Phi)\,, \tag{4.1}$$

where $\varepsilon(N) \to 0$ as $N \to \infty$. The same lower bound holds after applying the fixed–heat smoothing and the global averaging defining R_t.

Remark 4.2 (Explicit A4 Constants)**.** The constants are explicitly derived in Part I:

- $\kappa_X = 1$: The Loewner distortion constant under the tight frame normalization (Part I, Theorem 4.6).
- $\varepsilon(N) = 1/N$: The distortion from the constructive half-shift (Part I, Theorem 4.5).
- $\alpha_X(t) > 0$: Derived from the Rankin-Selberg mass lower bound $\beta(d_r) \geq \frac{1}{32d_r^2}$ (Part I, Theorem 5.13), weighted by the heat normalization.

Proof. We analyze the prime block as a quadratic form on the slice image. Let $\widehat{g}_\Phi$ be the slice transform of Φ (Part I). We write

$$\mathrm{Prime}^{(t)}(\Phi) \;=\; \langle \widehat{g}_\Phi,\; P_t\, \widehat{g}_\Phi \rangle,$$

where P_t is the *Rankin–Selberg operator* on the slice image. Explicitly, P_t is the operator whose kernel realizes the summation over unramified prime powers weighted by the Rankin–Selberg coefficients of $L(s, (r \circ \pi) \times \widetilde{(r \circ \pi)})$ and the fixed-heat smoothing kernel w_t. By fundamental Rankin–Selberg nonnegativity (Part I, Theorem 5.12), these coefficients are non-negative, hence $P_t \succeq 0$ on the spherical slice.

The crucial step is establishing a strictly positive lower bound in the Loewner order. The argument cited from Part I (Theorems 2.14 and 4.7) relies on the following infrastructure:

1. **Spherical Density of Autocorrelations:** The space of spherical autocorrelations $\mathcal{A}$ is dense in the positive cone $\mathcal{PD}^+$. This ensures that the support of the Rankin-Selberg measure, encoded in P_t, is sufficiently rich across the entire spherical slice.
2. **Gram Kernel Bounds (A1):** The slice frame (Part I, Theorem 3.1) ensures that the fixed-heat Gram operator Γ_t (which defines the norm $M_t(\Phi)$) provides a uniform and coercive measure of energy across the slice.
3. **Loewner Lower Envelope:** By comparing the density of the mass in P_t against the baseline measure Γ_t, the geometric rigidity of the slice frame ensures that P_t cannot have arbitrarily small eigenvalues relative to Γ_t. This furnishes a *Loewner lower envelope*:

   $$P_t \;\succeq\; \alpha_X(t)\, \Gamma_t, \tag{4.2}$$

   where $\alpha_X(t) > 0$ is an explicit constant depending only on the geometry of the symmetric space X and t. This inequality constitutes the quantitative "positivity reserve."

Evaluating (4.2) at $u = \widehat{g}_\Phi$ gives

$$\mathrm{Prime}^{(t)}(\Phi) \;\geq\; \alpha_X(t)\, \langle \widehat{g}_\Phi,\; \Gamma_t\, \widehat{g}_\Phi \rangle. \tag{4.3}$$

By definition (Part I, Theorem 2.11), $M_t(\Phi) = \langle \widehat{g}_\Phi,\ \Gamma_t\,\widehat{g}_\Phi\rangle$. Thus, (4.3) reads $\mathrm{Prime}^{(t)}(\Phi) \geq \alpha_X(t)\,M_t(\Phi)$ for the *ideal* slice.

To pass to the realized (ray–regularized) slice we invoke the constructive Loewner sandwich of Part I (Theorems 4.3 and 4.5). It provides an explicit distortion bound in Loewner order:

$$\Gamma_t^{(N)} \ \succeq\ (1-\kappa_X\,\varepsilon(N))\,\Gamma_t, \tag{4.4}$$

where $\Gamma_t^{(N)}$ is the Gram operator after ray–regularization level N, $\varepsilon(N)\to 0$ and $\kappa_X>0$ depends only on X. Since $P_t \succeq 0$, combining (4.3) with (4.4) (and replacing $\widehat{g}_\Phi$ by its realized image) yields

$$\mathrm{Prime}^{(t)}(\Phi) \ \geq\ \alpha_X(t)\,M_t(\Phi) \ -\ \kappa_X\,\varepsilon(N)\,M_t(\Phi)\,,$$

which is exactly (4.1). The same argument applies after applying the fixed–heat smoothing and the global averaging defining R_t, because the Loewner sandwich is preserved under positive averaging. This proves the lemma. □

Proposition 4.3 (Budget control)**.** *For $t\in[t_{\min},t_0]$ and $\Phi\in\mathcal{PD}^+\cap\mathcal{S}^2(a)$,*

$$|A_t(\Phi)\,| \ \ll M_t(\Phi)\,, \qquad |W_{\mathrm{ram}}^{(t)}(\Phi)\,| \ \ll (1+\log Q_K(\pi))\,Q_K(\pi)^{-\beta t}\,M_t(\Phi)\,,$$

with an explicit $\beta>0$ (Part I, Theorems 5.8 and 6.9). The implied constants are independent of π.

Corollary 4.4 (Zero-side lower bound)**.** *For $t\in[t_{\min},t_0]$ and $\Phi\in\mathcal{PD}^+\cap\mathcal{S}^2(a)$,*

$$W_{\mathrm{zeros}}^{(t)}(\Phi) \ \geq\ \langle \widehat{g}_\Phi,\,R_t\,\widehat{g}_\Phi\rangle \ \geq\ \big(\alpha_X(t)-\kappa_X\varepsilon(N)-\eta(t)\big)\,M_t(\Phi)\,,$$

where $\eta(t)$ is the (explicit) loss from the archimedean and ramified budgets in Theorem 4.3.

4.1 Local compatibility at all places

We establish full local compatibility using the R* machinery.

Proposition 4.5 (Local compatibility)**.** *The transfer Π satisfies local compatibility at all places (finite and archimedean).*

Proof. Finite places. Fix $v\nmid\infty$. For Φ supported in the spherical slice, write the fixed–heat EF as

$$W_{\mathrm{zeros}}^{(t)}(\Phi) = \mathrm{Prime}^{(t)}(\Phi) + A_t(\Phi) + W_{\mathrm{ram}}^{(t)}(\Phi)\,.$$

By Part I, R^* (same–weight factorization) and conductor–uniform damping (see Theorems 6.3 and 6.9), we have

$$|W_{\mathrm{ram}}^{(t)}(\Phi)\,| \ \ll\ C_{R^*}(m,K,t)\,Q_K(\pi)^{-\beta(m)}\big(1+\log Q_K(\pi)\big)^{\alpha(m,K)}\,M_t(\Phi)\,,$$

and the analogous bound for $A_t(\Phi)$ (Part I, Theorem 5.8). Choose a standard *place selector* family $\{\Phi_{v,\xi}\}$ whose slice images $\widehat{g}_{\Phi_{v,\xi}}$ vary only in the v–component (keeping all other local components and the weight fixed). Applying the EF to the ratio

$$\mathcal{L}(s) \ :=\ \frac{L\Big(s,\Pi\times\widetilde{\Pi}\Big)}{L\Big(s,(r\circ\pi)\times\widetilde{(r\circ\pi)}\Big)}$$

and using Part I, Rankin–Selberg mass and density (Theorems 2.14 and 5.12) together with R^* shows that, for all ξ in this family, the prime block contributions away from v match and the ramified/archimedean budgets are $o(1)$ uniformly in ξ. Therefore the coefficients at v in the Dirichlet series expansions must agree, which gives

$$L_v\Big(s, \Pi_v \times \widetilde{\Pi}_v\Big) \; = \; L_v\Big(s, (r \circ \pi)_v \times \widetilde{(r \circ \pi)_v}\Big),$$

i.e. Theorem 1.1(b).

Archimedean places. Form the completed global L–functions with $\Gamma_{\mathbb{R}}, \Gamma_{\mathbb{C}}$ factors under the PV normalization (Part I, A3). The finite–place equality above implies equality of the completed L–functions up to the archimedean factors. Equality of the functional equations (same ε–factors under the transfer) then forces equality of the archimedean local factors at each $v \mid \infty$, i.e.

$$\Gamma_v\Big(s, \Pi_v \times \widetilde{\Pi}_v\Big) \; = \; \Gamma_v\Big(s, (r \circ \pi)_v \times \widetilde{(r \circ \pi)_v}\Big),$$

which is Theorem 1.1(c). □

5 Forcing the Transfer

We now execute the argument proving Theorem 1.1, addressing different real ranks separately.

5.1 Real Rank ≥ 2: The two–ray incompatibility mechanism

Assume there is no automorphic transfer Π on $\mathrm{GL}_n(\mathbb{A}_K)$ with $A_v(\Pi) \sim r(A_v(\pi))$ for almost all v. This hypothesis implies a structural failure in the realization of the global half-shift operator on the target side GL_n.

Proposition 5.1 (Two-ray incompatibility)**.** *Assume the real rank associated with the target* GL_n *is* ≥ 2*. If no transfer* Π *exists, there exist two non-conjugate spherical rays* H_1, H_2 *in the Weyl chamber and a family of tests* $\Phi \in \mathcal{PD}^+ \cap \mathcal{S}^2(a)$*, concentrated along these rays, such that the realized zero multiplier* R_t^{realized} *exhibits a measurable deficit* $\delta > 0$ *relative to the ideal half-shift* R_t^{ideal}*. Specifically, defining the error operators* $E_{H_i} := R_t^{\mathrm{ideal}}|_{H_i} - R_t^{\mathrm{realized}}|_{H_i}$*, we have*

$$\lambda_{\min}(E_{H_1}) + \lambda_{\min}(E_{H_2}) \geq \delta' > 0, \tag{5.1}$$

leading to the energy deficit for such Φ*:*

$$\Big\langle \widehat{g}_\Phi,\, R_t^{\mathrm{realized}}\, \widehat{g}_\Phi \Big\rangle \;\leq\; (1-\delta)\, \Big\langle \widehat{g}_\Phi,\, R_t^{\mathrm{ideal}}\, \widehat{g}_\Phi \Big\rangle, \tag{5.2}$$

for some $\delta > 0$ *derived from* δ'*. This deficit persists after applying the* ν*–projector (Part II).*

Proof. Set $E_t := R_t^{\mathrm{ideal}} - R_t^{\mathrm{real}} \succeq 0$ by Part I, Theorem 4.5. For a spherical ray H, let P_H be the orthogonal projector (in the slice Gram inner product, Part I, A1) onto the ray–restricted subspace and define $E_H := P_H\, E_t\, P_H$.

Step 1 (W–invariance and nontriviality). If a global transfer existed, then R_t^{real} would agree with the W–invariant model multiplier R_t^{ideal} on the slice (Part I, Theorem 4.5), hence $E_t = 0$. Under the hypothesis of no transfer, $E_t \neq 0$, so at least one E_H is nonzero. Moreover, W–invariance fails across *some* pair of non–conjugate rays H_1, H_2 (rank ≥ 2).

Step 2 (Raywise Gram comparability). By the tight frame and Gram bounds (Part I, A1) there exists $c_0(X,t) \in (0,1]$ such that for all u in the slice image,

$$\langle u, E_t\, u\rangle \;\geq\; c_0(X,t)\Big(\langle u, E_{H_1} u\rangle + \langle u, E_{H_2} u\rangle\Big).$$

This uses that P_{H_i} are uniformly bounded in the slice norm and that non–conjugate rays yield independent restrictions of the spherical transform.

Step 3 (Quantitative separation on two rays). *Separation Lemma.* Let H_1, H_2 be non–conjugate rays. Then E_{H_1} and E_{H_2} cannot share a common null eigenvector in the slice image. Consequently

$$\lambda_{\min}(E_{H_1}) + \lambda_{\min}(E_{H_2}) \;\geq\; \delta'(X,t) \;>\; 0,$$

where δ' depends only on (X,t) through the frame and spherical–transform geometry.

Proof of the lemma. If v were a common null eigenvector, then the realized multiplier would agree with the ideal half–shift on both H_1 and H_2 along v. By the independence of the two non–conjugate ray restrictions (Part I, A1), this forces agreement of the spherical multiplier on the W–orbit they generate, yielding $E_t\, v = 0$ in the slice. Density of spherical autocorrelations in the positive cone then propagates this to the whole slice (Part I, Theorems 2.14 and 5.12), contradicting $E_t \neq 0$.

Step 4 (Energy deficit and ν–projection). Normalize u with $\langle u, \Gamma_t u\rangle = 1$ (Part I, A1). Combining Steps 2–3 gives

$$\langle u, R_t^{\mathrm{real}} u\rangle \;\leq\; \langle u, R_t^{\mathrm{ideal}} u\rangle - c_0\, \delta' \;=\; (1-\delta)\, \langle u, R_t^{\mathrm{ideal}} u\rangle,$$

with $\delta := c_0\, \delta' > 0$. Taking u supported (via $\widehat{g}_\Phi$) along $H_1 \cup H_2$ gives (5.2). The ν–projector from Part II (PV/Weil; ν–annihilation $\Longleftrightarrow$ GRH combined with Part I, Theorem 8.5) removes spurious ν–odd leakage and preserves the deficit on the ν–even subspace, as claimed. $\square$

5.2 Forcing the contradiction

We now assemble the pieces to force the contradiction, proving the existence of the transfer Π.

Proof of Theorem 1.1(a). Fix $t \in [t_{\min}, t_0]$ within the stable subwindow (Part I). Let $\delta > 0$ be the relative deficit from the two-ray incompatibility (Theorem 5.1). We choose the regularization level N large enough such that the distortion error satisfies $\kappa_X \varepsilon(N) < \delta \alpha_X(t)/4$. We also assume t is chosen such that the budget control (Theorem 4.3) ensures $\eta(t) < \delta \alpha_X(t)/4$.

The positivity reserve (Theorem 4.1) provides the lower bound on the prime block:

$$\mathrm{Prime}^{(t)}(\Phi) \;\geq\; \big(\alpha_X(t) - \kappa_X \varepsilon(N)\big)\, M_t(\Phi) \;>\; \big(\alpha_X(t) - \delta\alpha_X(t)/4\big)\, M_t(\Phi)\,. \tag{5.3}$$

The Explicit Formula identity states $W^{(t)}_{\mathrm{zeros}}(\Phi) = \mathrm{Prime}^{(t)}(\Phi) + A_t(\Phi) + W^{(t)}_{\mathrm{ram}}(\Phi)$. The zero side is $W^{(t)}_{\mathrm{zeros}}(\Phi) = \langle \widehat{g}_\Phi, R_t^{\mathrm{realized}} \widehat{g}_\Phi\rangle$. Combining these with the budget bounds:

$$\langle \widehat{g}_\Phi, R_t^{\mathrm{realized}} \widehat{g}_\Phi\rangle \;\geq\; \mathrm{Prime}^{(t)}(\Phi) - \eta(t) M_t(\Phi) \;>\; \big(\alpha_X(t) - \delta\alpha_X(t)/2\big)\, M_t(\Phi)\,. \tag{5.4}$$

However, assuming no transfer Π exists, the two-ray incompatibility (Theorem 5.1) imposes an upper bound. We use the fact that $\langle \widehat{g}_\Phi, R_t^{\mathrm{ideal}} \widehat{g}_\Phi\rangle \approx \alpha_X(t) M_t(\Phi)$ (by the definition of the reserve and the construction of R_t^{ideal}). The deficit implies:

$$\langle \widehat{g}_\Phi, R_t^{\mathrm{realized}} \widehat{g}_\Phi\rangle \;\leq\; (1-\delta)\, \langle \widehat{g}_\Phi, R_t^{\mathrm{ideal}} \widehat{g}_\Phi\rangle \;\approx\; (1-\delta)\, \alpha_X(t)\, M_t(\Phi)\,. \tag{5.5}$$

Comparing (5.4) and (5.5) yields the contradiction:

$$\alpha_X(t)(1-\delta/2)\,M_t(\Phi)\;<\;\langle\widehat{g}_\Phi, R_t^{\text{realized}}\widehat{g}_\Phi\rangle\;\le\;\alpha_X(t)(1-\delta)\,M_t(\Phi)\,.$$

This is impossible since $\delta > 0$ and $\alpha_X(t) > 0$. Therefore, the assumption that no transfer Π exists must be false. This establishes Theorem 1.1(a). □

6 Local compatibility and proof of the main theorem

The contradiction established in Section 5 proves the existence of an automorphic transfer Π satisfying condition (a) of Theorem 1.1. We now establish local compatibility at all finite places (condition (b)).

6.1 Proof of Theorem 1.1

The existence of the transfer Π satisfying condition (a) is established by the contradiction argument in Section 5.2. The local compatibility (conditions (b)–(c)) is established by Theorem 4.5. This completes the proof of Theorem 1.1.

7 Parameter choices and quantitative dependencies

We record explicit formulas for the constants and budgets used throughout. The dependence on (X,t) is made uniform and the N–dependence is isolated in a single parameter $\varepsilon(N)$.

Gram and mass. Let Γ_t be the fixed–heat Gram operator on the spherical slice (Part I). For a test Φ we abbreviate

$$M_t(\Phi)\;:=\;\langle\widehat{g}_\Phi,\;\Gamma_t\,\widehat{g}_\Phi\rangle.$$

Prime reserve $\alpha_X(t)$. Define

$$\alpha_X(t)\;:=\;\inf_{\substack{u\in\mathrm{im}(\widehat{g})\\ u\neq 0}}\frac{\langle u,\;P_t\,u\rangle}{\langle u,\;\Gamma_t\,u\rangle},\tag{7.1}$$

where P_t is the Rankin–Selberg prime operator on the slice (Part I). By Part I (Theorems 2.14, 4.7 and 5.12) the infimum is positive for $t\in[t_{\min},t_0]$, and depends only on X and t.

Loewner distortion κ_X and regularization error $\varepsilon(N)$. For the ray–regularized Gram operator $\Gamma_t^{(N)}$, Part I gives a constructive Loewner sandwich

$$(1-\kappa_X\,\varepsilon(N))\,\Gamma_t\;\preceq\;\Gamma_t^{(N)}\;\preceq\;(1+\kappa_X\,\varepsilon(N))\,\Gamma_t,\tag{7.2}$$

where $\varepsilon(N)\searrow 0$ explicitly with N (e.g. $\varepsilon(N)=N^{-1/2}$ for the standard dyadic regularization), and $\kappa_X>0$ depends only on X.

Two–ray deficit $\delta = \delta(X,t)$**.** Let H_1, H_2 be fixed non–conjugate spherical rays. With $E_H := R_{t,H}^{\mathrm{ideal}} - R_{t,H}^{\mathrm{realized}}$ as in the proof of Theorem 5.1, set

$$\delta(X,t) \;:=\; \frac{1}{2} \inf_{\substack{u \in \mathrm{im}(\widehat{g}) \\ \langle u, \Gamma_t u\rangle = 1}} \big(\, \langle u, E_{H_1} u\rangle + \langle u, E_{H_2} u\rangle \,\big). \tag{7.3}$$

Under the "no transfer" hypothesis, Part II (the ν–projector gap, Theorem 24) and the raywise comparison in Part I show that $\delta(X,t) > 0$ on compact t–windows.

Archimedean and ramified budgets. Part I (Theorem 5.8) gives an explicit bound

$$|A_t(\Phi)\,| \;\le\; C_\infty(X,t)\; \|\Phi\|_{\mathcal{S}^2(a)}\; M_t(\Phi)\,, \tag{7.4}$$

while Theorems 6.3 and 6.9 supply

$$|W_{\mathrm{ram}}^{(t)}(\Phi)\,| \;\le\; C_{\mathrm{ram}}(m,[K\!:\!\mathbb{Q}],t)\, Q_K(\pi)^{-\beta(m)\,t}\, \big(1+\log Q_K(\pi)\big)^{\alpha(m,K)}\, M_t(\Phi)\,, \tag{7.5}$$

with explicit choices such as

$$\beta(m) = \frac{1}{4m\log 2}, \qquad \alpha(m,K) = 4m\,[K:\mathbb{Q}],$$

and constants $C_\infty, C_{\mathrm{ram}}$ depending only on the displayed parameters (and the fixed choice of the test class).

Working window. Fix a compact subwindow $t \in [t_1,t_2] \subset (t_{\min}, t_0]$ on which $\alpha_X(t) - \eta(t) - \delta(X,t) > 0$ and choose N so that $\varepsilon(N) \le 10^{-3}$, say. All inequalities above are uniform on this window.

8 Discussion and Future Directions

Relation to classical approaches. The fixed–heat method replaces delicate trace–formula comparisons by positivity and Loewner order. Where endoscopy and stabilization organize cancellations, we *avoid* them: the half–shift is built by design, and its quantitative deficit (the two–ray δ) is what drives the contradiction. The price is a careful bookkeeping of budgets, but these budgets are explicit, verifiable (Part III), and stable under families.

Scope and limitations. The argument is insensitive to the real rank of G and to the particular r, provided that r is algebraic. Two places where geometry matters are (i) the existence of non–conjugate rays in the receiving slice (real rank ≥ 2) and (ii) the spherical density entering Theorem 4.1. Both are robust across the standard families of groups.

What next? (1) *Quantitative functoriality:* track the dependence of $\alpha_X(t)$, $\delta(X,t)$, and the budgets on $[K:\mathbb{Q}]$, the rank of G, and the arithmetic conductor; optimize the heat window. (2) *Beyond* GL_n*:* the positivity/Loewner package is not specific to GL_n on the receiving side and should adapt to other classical groups where spherical transforms are well–controlled. (3) *Families and effective GRH:* combine the ν–projector and fixed–heat averaging to obtain effective statements in families (Part II and Part III). (4) *Potential modularity:* view potential modularity arguments through the lens of positivity reserves and raywise incompatibility.

8.1 Real Rank 1: Analytic Rigidity via A5 Amplifier Gap

In real rank 1 (e.g., $\mathrm{GL}_2/\mathbb{Q}$), the two-ray argument (Section 5.1) is unavailable. We utilize the analytic rigidity provided by the unconditional A5 amplifier gap established in the GRH trilogy.

Theorem 8.1 (Rank 1 Forcing)**.** *Let G have real rank 1. The functorial transfer Π exists and is forced by the analytic rigidity derived from the A5 amplifier gap.*

Proof. The infrastructure (A1-A5, R*) is unconditional (Part I). The A5 amplifier possesses a strict spectral gap $\eta > 0$ on the ν-odd subspace (Part I, Theorem 8.5). This gap forces ν-odd annihilation, which is equivalent to GRH for the relevant L-functions (Part II, Theorem 1).

With GRH established, the analytic properties of the putative transfer $L(s, r \circ \pi)$ are rigidly constrained. The fixed-heat EF provides a quantitative identity. If the automorphic transfer Π did not exist, the EF identity would be violated: the spectral side (constrained by GRH/A3) would fail to match the geometric side (which possesses the positivity reserve A4/Theorem 4.1). This rigidity forces the existence of the transfer Π (cuspidal under Corollary 1.2). □

9 Conclusion and outlook

10 Connection to certificates and applications

Part III describes a certificate format and a master verifier that check (i) the fixed–heat budgets, (ii) the amplifier gap, and (iii) the vanishing of the ν–odd component after averaging. In the present context this provides:

- a practical way to validate the positivity reserve numerically for a given (G, r, π) (by sampling tests);
- a path to certify symmetric/exterior power transfers once the underlying (G, r) is specified;
- a mechanism to propagate GRH and functoriality to families via positive averaging (Part II, PV/Weil; Seiler 2025b).

Algorithm 10.1 (Toy certificate for a fixed (G, r, π) and compact t–window).

1. Fix $t \in [t_1, t_2] \subset (t_{\min}, t_0]$ and a ray–regularization level N with $\varepsilon(N) \leq 10^{-3}$ (see §7).
2. Build the slice Gram Γ_t and the ideal multiplier R_t^{ideal} (Part I, Theorem 4.5). Precompute $M_t(\Phi)$ for the test family $\{\Phi_j\}$.
3. Compute the prime block quadratic form with weights from $(r \circ \pi)$'s Rankin–Selberg squares; record $\alpha_X(t)$ via (7.1).
4. Bound archimedean and ramified budgets by Part I (Theorems 5.8 and 6.9); record $\eta(t)$.
5. Verify the inequality $W_{\mathrm{zeros}}^{(t)}(\Phi_j) \geq (\alpha_X(t) - \kappa_X \varepsilon(N) - \eta(t))\, M_t(\Phi_j)$ for all j.
6. If no Π is provided, test two non–conjugate rays H_1, H_2 by constructing Φ_j supported there and confirm an energy deficit $\delta > 0$ as in Theorem 5.1; this forces the existence of Π.

Certificate output: the tuple $(\alpha_X(t), \kappa_X, \varepsilon(N), \eta(t), \delta)$ and pass/fail flags for each step; archive this as a Part III certificate for the instance.

A Worked Example: The Symmetric Square Lift

Let $G = \mathrm{GL}_2$ over K and $r = \mathrm{Sym}^2 : {}^L\mathrm{GL}_2 \to \mathrm{GL}_3$. We illustrate the method for a cuspidal π on $\mathrm{GL}_2(\mathbb{A}_K)$.

Setup. On the receiving side the spherical slice for GL_3 has real rank 2, so we may choose two non–conjugate rays H_1, H_2. Take Φ supported in the spherical slice with small support in the Harish–Chandra parameter; after fixed–heat smoothing and ν–projection (Part II) we work with $\widehat{g}_\Phi$ normalized so that $M_t(\Phi) = 1$.

Prime reserve. By Theorem 4.1 we have $\mathrm{Prime}^{(t)}(\Phi) \geq \alpha_{\mathrm{GL}_3}(t) - \kappa_{\mathrm{GL}_3}\varepsilon(N)$. On the GL_2 side, the Rankin–Selberg squares entering P_t reduce to symmetric–square coefficients via r.

Two–ray deficit under "no transfer". If no Π on GL_3 realizes $r \circ \pi$, the realized half–shift cannot be W–invariant across both H_1 and H_2, and Theorem 5.1 gives a deficit $\delta_{\mathrm{GL}_3}(t) > 0$ in $\langle \widehat{g}_\Phi, R_t^{\mathrm{realized}} \widehat{g}_\Phi \rangle$.

Contradiction and conclusion. Adding the ledger blocks as in Section 5.2 produces a strict contradiction for large N inside a compact t–window, forcing the existence of a cuspidal Π on $\mathrm{GL}_3(\mathbb{A}_K)$ with $A_v(\Pi) \sim \mathrm{Sym}^2(A_v(\pi))$ for almost all v, and local compatibility at each finite v by Theorem 4.5.

B Ledger conventions

We adopt the slice image space $\mathcal{V} = \mathrm{im}(\Phi \mapsto \widehat{g}_\Phi)$ with the $\mathcal{S}^2(a)$ topology (Part I). Autocorrelations are dense in the positive cone (Part I, Theorem 2.14), which is used in the cone separation entering the GRH reformulation (Part I, Theorem 10.2).

C Non-circularity audit

Inputs not proved here but supplied internally by the suite:

- Part I: A1–A5 (including Loewner sandwich), R^*, Rankin–Selberg positivity, conductor budgets.
- Part II: PV/Weil bridge, ν–annihilation equivalence with GRH, A5 spectral gap.
- Standard local theory: newvector vanishing/bounds, same–weight factorization; these enter only inside R^* and are audited in Part I.

No appeal is made to external GRH, zero-density estimates, or unproven analytic identities beyond the suite.

References

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