# An Automorphic Derivation of the *Asymmetric* Explicit Formula via the Eisenstein Phase*

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**Scope (Paper I).** Automorphic derivation of the asymmetric explicit formula using the BK measure derived via Maaß–Selberg relations. *No RH claim is made in Paper I.*
**Companion papers (Zenodo DOIs).** Paper II (adelic framework, no RH claim): 10.5281/zenodo.16930093. Paper III (positivity argument): 10.5281/zenodo.16930095. *Paper III presents a* claimed *proof of RH; we believe the argument may be correct, but treat it as a claim pending peer review and invite detailed scrutiny.*

**Abstract**

We present an automorphic derivation of the asymmetric explicit formula, utilizing the Birman–Kreĭn (BK) spectral-shift measure derived from the scattering phase of the Eisenstein series via the Maaß–Selberg relations. The analysis isolates a positive Archimedean kernel, realized via a half-shift resolvent operator $C_{1/2}$ (interpreted motivationally as a torsion filter), and identifies the analytic setting where the prime and Archimedean blocks naturally pair with one-sided (Laplace) zero weights. All steps are executed without assuming the Riemann Hypothesis (RH). Consequences for Weil positivity and the symmetry of the critical line are formulated precisely in terms of these pairings and are developed in related work.

*Preprint status:* This preprint fixes normalizations and analytic scaffolding for Paper III and invites open review. *Repository note.* Deposited on Zenodo at 10.5281/zenodo.16930061. A HAL mirror may be posted; the Zenodo DOI is the canonical record.

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**Artifacts.** Computational notebooks and numeric diagnostics appear with the companion positivity paper (Paper III, App. E) and are referenced there for reproduction.For Paper I specifically, no computation is required: all results are analytic.

## Contents



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# 1 Introduction

This paper develops an automorphic derivation of the canonical explicit formula, the fundamental analytic framework connecting the distribution of the zeros of the Riemann zeta function $\zeta(s)$ to the distribution of prime numbers. A guiding perspective in this work is that the analytic continuation of $\zeta(s)$ reflects profound structural constraints on the underlying geometry. Specifically, extending $\zeta(s)$ beyond its initial half-plane of absolute convergence can be viewed as enforcing a *torsion-free* alignment.

This perspective motivates the structure of our derivation, suggesting that such constraints are related to Weil positivity and the localization of the nontrivial zeros on the critical line $\mathrm{Re}(s) = 1/2$. This geometric intuition is further developed through adelic and CR-analytic frameworks in the companion papers [8, 9].

**Roadmap.** In Section 2, we fix PV/measure conventions and introduce the mechanism we interpret as a *torsion filter*—the half-shift resolvent operator $C_{1/2}$—which implements the geometric constraint via a positive Archimedean kernel. We prove the domination bounds needed to justify termwise operations. In Section 3, we give the local decomposition of the phase derivative and define the arithmetic functional $P(g)$. In Section 4, we evaluate the spectral side (Route B) via the partial fraction expansion of $\zeta'/\zeta$ with a symmetric PV prescription. In Section 5, we perform the geometric evaluation (Route A) using the Maaß–Selberg ledger and the regularization scheme. Finally, Sections 6 and 6.2 state the asymmetric explicit formula and its geometric interpretation, and Section A records the normalization crosswalk.

*Remark* 1.1 (Torsion Filter Intuition (Motivational)). The "torsion filter" language is *purely motivational.* It refers to the filtering of a non-commutative structure ("additive drift") on the modulus line. This drift is associated with the interplay between reflection ($R$, acting as $(R\phi)(x) = \phi(-x)$) and the half-shift multiplier $\mathsf{M}^{1/2}$ (defined later in the Conventions). These operators do not commute: $(R\mathsf{M}^{1/2}\phi)(x) = e^{x/2}\phi(-x)$, while $(\mathsf{M}^{1/2}R\phi)(x) = e^{-x/2}\phi(-x)$. The resolvent $C_{1/2}$ (see Thm. 6.3) effectively filters this non-commutativity. All proofs and constants come from the analytic route described here; no differential geometric torsion is invoked. Precise positivity enters via the explicit multiplier $T_{1/2}(x) = (1 + e^{-x/2})^{-1}$ in Section 6.2. **Disclaimer:** The analytical rigor of this paper relies strictly on the $\mathsf{M}$-calculus and the properties of the kernel $T_{1/2}(x)$, independent of this geometric analogy.

**Definition 1.2** (Extended test class $\mathcal{A}_{\mathrm{Exp}}$)**.** We define the extended class $\mathcal{A}_{\mathrm{Exp}}$ as the space of even functions $g$ satisfying two conditions:

(i) Spectral decay: $|g(\tau)| \le C(1+|\tau|)^{-2}$ (sufficient for pairing against the tempered distribution $\varphi'(\tau)$).

(ii) Exponential tilt: The cosine transform $g_{\mathrm{b}}$ (defined below in the Notation conventions) is smooth and obeys $|g_{\mathrm{b}}(x)| \le C'\, e^{-\sigma|x|}$ for some $\sigma > \frac{1}{2}$.

These conditions imply $g$ is analytic in the strip $|\operatorname{Im}(\tau)| < \sigma$. We write $\mathcal{A}_{\mathrm{Exp}}$ for this exponentially damped transform class, in which the prime and Archimedean blocks are absolutely convergent and analytic interchanges (like DCT on the spectral side, see Prop. 2.6) are justified. The strict threshold $\sigma > \frac{1}{2}$ is exactly what gives absolute convergence of $P(g)$ and an $\alpha$-independent $L^1$ majorant for the Archimedean DCT (cf. [8], Def. 2.3 and Prop. 4.2).

**Definition 1.3** (Admissible test class)**.** We call an even test function $g$ *admissible* if either $g \in \mathcal{A}_{\mathrm{Exp}}$ with tilt parameter $\sigma > \frac{1}{2}$, or if $g$ belongs to the band-limited Schwartz class $\mathcal{S}_{\mathrm{BL}}$ (i.e., its cosine transform $g_{\mathrm{b}}$ is compactly supported; see Def. 1.13). All statements in this paper are made for admissible even $g$.

Our derivation is rooted in the theory of Eisenstein scattering and the Maaß–Selberg relations. We evaluate a specific spectral integral $\mathcal{I}(g)$ via two independent routes and equate the outcomes; these two routes supply the validating machinery for the geometric constraint. The resulting explicit formula is structured to manifest this geometry directly.

We specifically adopt the Maaß–Selberg (MS) normalization, utilizing the Birman–Kreĭn spectral-shift measure derived directly from the scattering phase. This choice naturally aligns the spectral side with the geometric interpretation of the Archimedean block.

- **Route A (Geometric):** We evaluate the integral by regularization in the half-plane where the Dirichlet series for $\zeta'/\zeta$ converges absolutely. Analytic continuation yields the identity
$$\mathcal{I}(g) = \mathcal{A}(g) - P(g) - \frac{1}{2}g(0),$$
connecting the spectral integral to an Archimedean functional $\mathcal{A}(g)$, the prime functional $P(g)$, and a boundary term arising from the regularization limit.

- **Route B (Spectral):** We evaluate the integral directly on the critical line utilizing the canonical partial-fraction expansion of $\zeta'/\zeta$. A crucial identity demonstrates that the contribution from the trivial zeros combines with the Archimedean functional to produce the Poisson kernel associated with the pole of $\zeta(s)$ at $s = 1$, such that $\mathcal{I}_{\mathrm{arch}}(g) + \mathcal{I}_{\mathrm{Triv}}(g) = J(g)$. Consequently,
$$\mathcal{I}(g) = 2J(g) - 2\sum_{\rho} \mathcal{H}(g;\rho), \tag{1.1}$$
where the sum runs over the nontrivial zeros $\rho$, and $\mathcal{H}(g;\rho)$ denotes the one-sided Laplace weight.

**The Asymmetric Formula.** Equating the results of Route A and Route B yields the *asymmetric* explicit formula:

$$2\sum_{\rho}\mathcal{H}(g;\rho) \;=\; P(g) \;+\; \frac{1}{2}g(0) \;+\; 2J(g) \;-\; \mathcal{A}(g). \tag{1.2}$$

In our Maaß–Selberg ledger, the continuous part $\frac{1}{2}g(0)-\mathcal{A}(g)+2J(g)$ corresponds to the positive density $\rho_W$ on the modulus line (see Thm. 4.2 and (A.3)), in agreement with the companion papers ([8] App. A; [9] §§1,4).

*Remark* 1.4 (Scope and Context). This paper neither proposes a new criterion for RH nor draws any RH conclusion. Any classical equivalences (Weil positivity) are referenced only to orient the reader; their justification and RH implications are developed in the companion positivity paper [9]. Our contribution here is the rigorous construction of the geometric identity in this normalization, a canonical ledger aligned with [8], and the isolation of the positive Archimedean density arising from the torsion filter. We assume no form of RH; cusp forms and the full Selberg trace formula are not utilized—only the constant term of the Eisenstein series is required. We utilize both band-limited ($\mathcal{S}_{\mathrm{BL}}$) and exponentially damped ($\mathcal{A}_{\mathrm{Exp}}$) test classes, ensuring transparent regularization and rigorous convergence arguments throughout (see Thms. 1.2 and 1.3). *Cross-paper cue:* the same MS pairing and continuous density $\rho_W$ are stated in [9] (Normalization Lock and Eq. (4.2)), confirming constants in that ledger.

**Proposition 1.5** (Weil positivity — band-limited restatement)**.** *Under the Maaß–Selberg normalization fixed here, the following classical equivalence (restated on our band-limited class) holds:*

1. *All nontrivial zeros $\rho$ of $\zeta(s)$ lie on the critical line* $\mathrm{Re}(s) = 1/2$ *(the Riemann Hypothesis).*
2. *The full geometric functional (the RHS of Eq.* (1.2)*) is nonnegative for all test functions $g$ in the* Weil cone*, which is the set of functions satisfying* $\{g_{\mathrm{b}} \geq 0,\ G_{\mathrm{b}} \geq 0\}$.

Note on the Weil cone. *Since* $G_{\mathrm{b}}(x) = 2\sinh(x/2)\,g_{\mathrm{b}}(2x)$ *and* $\sinh(x/2) > 0$ *for* $x > 0$*, the condition* $g_{\mathrm{b}} \geq 0$ *(for* $x > 0$*) implies* $G_{\mathrm{b}} \geq 0$ *(for* $x > 0$*). We retain both conditions for alignment with standard formulations.*

*Proof Overview (restatement).* This is a restatement of the classical Weil criterion in the normalization used across this trilogy; its justification and any RH implications are developed in [9]. No new criterion is asserted here, and no RH conclusion is drawn in this paper. (For the RH *claim* and the requested external verification, see [9].) The constants and notations synchronize with [8].

□

*Remark* 1.6 (Context in Scattering Theory). The structure of the derivation, utilizing the Eisenstein scattering coefficient $\widetilde{S}(s) = \Lambda(2s-1)/\Lambda(2s)$, aligns with a broad literature in scattering theory where phase-/spectral-shift identities naturally produce Γ- and ζ-factors (e.g., Birman–Kreĭn [1], Friedel [3], Buslaev–Faddeev [2], Selberg [7]). This automorphic approach on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ provides the framework for the Maaß–Selberg normalization adopted here.

## 1.1 Notation and Conventions

**Completed objects and scattering.** We fix

$$\Lambda(s) := \pi^{-s/2}\Gamma\Big(\tfrac{s}{2}\Big)\zeta(s), \qquad \Lambda(s) = \Lambda(1-s),$$

and the standard Eisenstein scattering coefficient

$$\widetilde{S}(s) := \frac{\Lambda(2s-1)}{\Lambda(2s)}.$$

*Remark* 1.7 (Notation overlap: $\Lambda$). Following standard usage, we write $\Lambda(s)$ for the *completed zeta function* (complex argument $s$) and $\Lambda(n)$ for the *von Mangoldt function* (integer argument $n$). The context and argument type disambiguate the meaning throughout.

Throughout, $\log\Lambda(\cdot)$ and $\arg\widetilde{S}(\cdot)$ are taken on verticals by boundary limit from $\operatorname{Re}s = \frac{1}{2}+\varepsilon$, with the branch continuous in $\tau$ and *symmetric principal value at* $\tau = 0$.

**Fourier pair (non–$2\pi$) and cosine specialization.**

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(\tau)\, e^{-i\tau\xi}\,\mathrm{d}\tau, \qquad f(\tau) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\xi)\, e^{i\tau\xi}\,\mathrm{d}\xi.$$

For even $g$ we use the cosine specialization:

$$g_{\mathrm{b}}(x) = \int_{\mathbb{R}} g(\tau)\cos(x\tau)\,\mathrm{d}\tau, \quad g(\tau) = \frac{1}{\pi}\int_0^\infty g_{\mathrm{b}}(x)\cos(x\tau)\,\mathrm{d}x.$$

This locks all $2\pi$ factors used later (cosine inversion at $\tau = 0$ gives $g(0) = \frac{1}{\pi}\int_0^\infty g_{\mathrm{b}}(x)\,\mathrm{d}x$).

**Phase derivative symbol and boundary identity.** We write the phase derivative as $\varphi'(\tau)$ throughout.

**Lemma 1.8** (Phase identity: Birman–Kreĭn vs. $\Lambda'/\Lambda$)**.** *Let $\widetilde{S}(s)$ denote the normalized scattering coefficient*

$$\widetilde{S}(s) \coloneqq \frac{\Lambda(2s-1)}{\Lambda(2s)} \qquad (\Lambda(s) = \pi^{-s/2}\Gamma(\tfrac{s}{2})\zeta(s)).$$

*We define the scattering phase $\varphi(\tau) \coloneqq \arg\widetilde{S}(\frac{1}{2}+i\tau)$, utilizing the branch conventions fixed previously (continuous in $\tau$, symmetric PV at $\tau = 0$). Then, along $\operatorname{Re}s = \frac{1}{2}$, the phase derivative $\varphi'(\tau)$ is given by*

$$\begin{aligned} i\varphi'(\tau) \;&=\; \partial_\tau \log\widetilde{S}\Big(\tfrac{1}{2}+i\tau\Big), \\ \varphi'(\tau) \;&=\; 4\,\operatorname{Re}\Big(\tfrac{\Lambda'}{\Lambda}(2i\tau)\Big) \quad \textit{(symmetric PV at } \tau = 0\textit{)}. \end{aligned}$$

*Consequently, the Maaß–Selberg spectral measure used in all pairings is*

$$\mathrm{d}\xi(\tau) \;=\; \frac{1}{2\pi}\,\varphi'(\tau)\,\mathrm{d}\tau.$$

*Proof.* Since $|\widetilde{S}(\frac{1}{2}+i\tau)| = 1$, we have $\log\widetilde{S}(\frac{1}{2}+i\tau) = i\varphi(\tau)$, so $i\varphi'(\tau) = \partial_\tau\log\widetilde{S}(\frac{1}{2}+i\tau)$. Let $L(s) = \Lambda'/\Lambda(s)$. We use the chain rule. Setting $s(\tau) = \frac{1}{2}+i\tau$, so $s'(\tau) = i$.

$$\begin{aligned} i\varphi'(\tau) &= \frac{d}{d\tau}\log\widetilde{S}(s(\tau)) = i\frac{d}{ds}\log\widetilde{S}(s)\Big|_{s=\frac{1}{2}+i\tau} \\ &= i\cdot 2\,(L(2s-1) - L(2s))\,\Big|_{s=\frac{1}{2}+i\tau} \\ &= 2i\,\Big(L(2i\tau) - L(1+2i\tau)\Big). \end{aligned}$$

We now relate $L(1+2i\tau)$ back to $L(2i\tau)$. First, we use the functional equation $\Lambda(s) = \Lambda(1-s)$, which implies $L(z) = -L(1-z)$. Setting $z = 1+2i\tau$:

$$L(1+2i\tau) = -L(1-(1+2i\tau)) = -L(-2i\tau).$$

Next, we use the Schwarz reflection principle, $L(\bar{s}) = \overline{L(s)}$. Since $\tau \in \mathbb{R}$, we have $-2i\tau = \overline{2i\tau}$, so

$$L(-2i\tau) = L(\overline{2i\tau}) = \overline{L(2i\tau)}.$$

Combining these gives $L(1+2i\tau) = -\overline{L(2i\tau)}$. Substituting this into the expression for $i\varphi'(\tau)$:

$$\begin{aligned} i\varphi'(\tau) &= 2i\big(L(2i\tau) - (-\overline{L(2i\tau)})\big) = 2i\big(L(2i\tau) + \overline{L(2i\tau)}\big) \\ &= 2i\big(2\,\mathrm{Re}\,L(2i\tau)\big) = 4i\ \mathrm{Re}(L(2i\tau)). \end{aligned}$$

Dividing by $i$ gives $\varphi'(\tau) = 4\ \mathrm{Re}(L(2i\tau))$. The interpretation via symmetric PV at $\tau = 0$ is necessary due to the pole of $\Lambda'/\Lambda(s)$ at $s = 0$. The existence of the PV and the absence of an atom at $\tau = 0$ are justified by the cancellation of Poisson kernels established below (see Lemma 2.3). □

**Corollary 1.9** (Strong no–atom, a.e. domination, and modes of convergence)**.** *In the Maaß–Selberg ledger the limit $\varphi'_\varepsilon \to \varphi'$ holds in $L^1_{\mathrm{loc}}$ on $\mathbb{R}\setminus\mathcal{P}$ (hence on every compact $K \Subset \mathbb{R}\setminus\mathcal{P}$), and in the sense of distributions when paired against any $g \in \mathcal{S}_{\mathrm{even}} \cup \mathcal{A}_{\mathrm{Exp}}$ (pairings taken in the canonical symmetric principal value sense). Moreover, for every such $g$ and each $\delta \in (0,1]$ there exists an $M_{g,\delta} \in L^1(\mathbb{R})$* independent of $\varepsilon$ *such that $|\varphi'_\varepsilon(\tau)\, g(\tau)| \le M_{g,\delta}(\tau)$ for a.e. $\tau$ with* $\mathrm{dist}(\tau,\mathcal{P}) \ge \delta$*. In particular, the limit measure carries no $\delta_0$ atom.*

*Proof.* Write
$$\varphi'_\varepsilon(\tau) = \big[P_\varepsilon(\tau) + R_{\Gamma,\varepsilon}(\tau)\big] + \big[-P_\varepsilon(\tau) + R_{\zeta,\varepsilon}(\tau)\big].$$
In the Maaß–Selberg normalization the Poisson kernels $P_\varepsilon$ cancel identically, so only the holomorphic remainders remain; hence $\tau = 0$ is *non–polar*.

Let $\mathcal{P}$ denote the (discrete) set of real poles (if any). Fix any compact $K \Subset \mathbb{R}\setminus\mathcal{P}$. Choose an open set $U$ with $K \Subset U \Subset \mathbb{C}\setminus\mathcal{P}$, independent of $\varepsilon$. For $0 < \varepsilon \le 1$ the families $\{R_{\Gamma,\varepsilon}\}$ and $\{R_{\zeta,\varepsilon}\}$ are holomorphic on $U$ and converge locally uniformly on $U$ as $\varepsilon \downarrow 0$. Extending by the limit at $\varepsilon = 0$ makes $(\varepsilon, z) \mapsto R_{\Gamma,\varepsilon}(z), R_{\zeta,\varepsilon}(z)$ continuous on $[0,1]\times U$. By compactness,

$$\sup_{\varepsilon\in[0,1]}\ \sup_{z\in\partial U} \big(|R_{\Gamma,\varepsilon}(z)| + |R_{\zeta,\varepsilon}(z)|\big) < \infty.$$

Cauchy's formula on the fixed contour $\partial U$ yields a constant $C_K < \infty$ with

$$\sup_{\tau\in K,\ \varepsilon\in(0,1]} \big(|R_{\Gamma,\varepsilon}(\tau)| + |R_{\zeta,\varepsilon}(\tau)|\big) \le C_K.$$

Hence $|\varphi'_\varepsilon(\tau)| \le C_K$ on $K$ for all $\varepsilon$, so $\sup_\varepsilon \|\varphi'_\varepsilon\|_{L^1(K)} \le C_K\,|K| < \infty$. Since the remainders converge locally uniformly on $U$ as $\varepsilon \downarrow 0$, we have $\varphi'_\varepsilon(\tau) \to \varphi'(\tau)$ for every $\tau \in K$; dominated convergence with dominator $C_K \mathbf{1}_K$ gives

$$\|\varphi'_\varepsilon - \varphi'\|_{L^1(K)} \longrightarrow 0 \qquad (\varepsilon \downarrow 0),$$

i.e., $L^1_{\mathrm{loc}}$ convergence on $\mathbb{R}\setminus\mathcal{P}$.

For $|\tau| > 1$ and $\tau \notin \mathcal{P}$ one has the uniform growth bound $|\varphi'_\varepsilon(\tau)| \ll \log(2+|\tau|)$, with an implied constant independent of $\varepsilon$; let $C_{\log}$ denote such a constant. For $\delta \in (0,1]$ define the truncated complement
$$E_\delta := \mathbb{R}\setminus \bigcup_{\tau_j\in\mathcal{P}} (\tau_j - \delta, \tau_j + \delta).$$
On $K_\delta := ([-1,1]\cap E_\delta)$ the same Cauchy argument on a fixed $U_\delta \Subset \mathbb{C}\setminus\mathcal{P}$ with $K_\delta \Subset U_\delta$ yields a constant $C_{K_\delta} < \infty$ such that $\sup_\varepsilon |\varphi'_\varepsilon(\tau)| \le C_{K_\delta}$ for $\tau \in K_\delta$. Therefore, for $g \in \mathcal{S}_{\mathrm{even}} \cup \mathcal{A}_{\mathrm{Exp}}$ the choice
$$M_{g,\delta}(\tau) = \mathbf{1}_{E_\delta\cap\{|\tau|\le1\}}\, C_{K_\delta}\, |g(\tau)| + \mathbf{1}_{|\tau|>1}\, C_{\log}\, |g(\tau)| \log(2+|\tau|)$$
lies in $L^1(\mathbb{R})$ and satisfies $|\varphi'_\varepsilon(\tau)\, g(\tau)| \le M_{g,\delta}(\tau)$ for a.e. $\tau \in E_\delta$.

All global pairings are taken in the canonical symmetric principal value sense (see the convention below). Since $\mathcal{P}$ is discrete and $M_{g,\delta} \in L^1(\mathbb{R})$, dominated convergence applies on $E_\delta$; letting first $\varepsilon \downarrow 0$ at fixed $\delta > 0$ and then $\delta \downarrow 0$ (principal value limit) yields

$$\lim_{\varepsilon \downarrow 0} \mathrm{PV}\int_{\mathbb{R}} \varphi_\varepsilon'(\tau)\, g(\tau)\, \mathrm{d}\tau = \mathrm{PV}\int_{\mathbb{R}} \varphi'(\tau)\, g(\tau)\, \mathrm{d}\tau,$$

i.e., distributional convergence against the stated test classes. If the limit carried a mass $c\,\delta_0$, testing with any $g \in \mathcal{S}_{\mathrm{even}}$ with $g(0) = 1$ would produce an extra $c$, contradicting the dominated (PV) limit; hence $c = 0$. □

**Spectral measure (MS lock) vs. Plancherel.** *Normalization lock.* We pair throughout against the *phase* (Birman–Kreĭn spectral–shift) measure; by "MS normalization" we mean using this phase measure in all pairings (and not the Selberg Plancherel density) $\mathrm{d}\xi(\tau) = \frac{1}{2\pi}\,\varphi'(\tau)\,\mathrm{d}\tau$ in the *symmetric* principal–value sense, and there is no $\delta_0$ mass at $\tau = 0$. We never use the Selberg Plancherel density $\mathrm{d}\tau/(4\pi)$ in pairings.

**Corollary 1.10** (Ledger lock cross-reference)**.** *The normalization used here is identical to that established formally in Paper III (the* Maaß–Selberg ledger lock; no $\delta_0$ atom *theorem) [9], which proves the absence of the $\delta_0$ atom via boundary cancellation (see also Lemma 2.3 herein).*

This locks the continuous density $\rho_W$ used in Papers II–III *verbatim* (same constants and Maaß–Selberg ledger) and ensures the modulus-line regrouping here coincides with the adelic construction there.

$$\rho_W(x) = \frac{1}{2\pi}\left(\frac{1}{1+e^{-x/2}} + 2e^{-x/2}\right), \qquad x > 0,$$

appears in the geometric regrouping on the modulus line.

**Regularization convention.** We work on $\mathrm{Re}(s) = \frac{1}{2}$ and regularize on $\mathrm{Re}(s) = \frac{1}{2} + \varepsilon$ with $\varepsilon > 0$, pass to the boundary $\varepsilon \downarrow 0$, and keep the same branches as above. When Route A uses Dirichlet series, we temporarily assume $\varepsilon > \frac{1}{2}$. *Test-class crosswalk:* The band-limited class $\mathcal{S}_{\mathrm{BL}}$ is identical in normalization.

**Cross-walk notes.**

**Normalization Ledger**

**Quick ledger (front-loaded).**

$$\begin{aligned}
2\sum_{\rho} \mathcal{H}(g;\rho) &= P(g) + \tfrac{1}{2}g(0) + 2J(g) - \mathcal{A}(g),\\
\mathrm{d}\xi &= \tfrac{1}{2\pi}\,\varphi'(\tau)\,\mathrm{d}\tau,\\
\rho_W(x) &= \tfrac{1}{2\pi}\Big(\tfrac{1}{1+e^{-x/2}} + 2e^{-x/2}\Big).
\end{aligned}$$

*Symbols (quick glossary):* $g_{\mathrm{b}}$ = cosine transform of $g$; $G_{\mathrm{b}}(x) = 2\sinh(x/2)\,g_{\mathrm{b}}(2x)$; $P(g)$ = prime functional;
$J(g) = \frac{1}{2\pi}\int_0^\infty e^{-x/2} g_{\mathrm{b}}(x)\,\mathrm{d}x$ (pole block);
$\mathcal{A}(g) = \frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(x)\,\frac{e^{-x/2}}{1+e^{-x/2}}\,\mathrm{d}x$ (archimedean remainder);
$\mathcal{H}(g;\rho) = \frac{1}{2\pi}\int_0^\infty e^{-\beta x/2} g_{\mathrm{b}}(x)\cos(\gamma x/2)\,\mathrm{d}x$ (one-sided Laplace weight; $\rho = \beta + i\gamma$).
(Phase measure is *not* Selberg Plancherel $\mathrm{d}\tau/(4\pi)$; all PV symmetric, no $\delta_0$ at $\tau = 0$.)

The Paley–Wiener extension uses a *strict* $\sigma > \frac{1}{2}$ tilt to guarantee absolute convergence and dominated interchanges. **Variables:** We use $x > 0$ as the primary Laplace/modulus variable. Note that $x$ is used both as the argument for the cosine transform $g_{\mathrm{b}}(x)$ and as the argument for the dictionary profile $G_{\mathrm{b}}(x)$, where $G_{\mathrm{b}}(x)$ relates to $g_{\mathrm{b}}(2x)$ (see Def. 1.15). The context disambiguates the scaling. The symbol $\varphi'(\tau)$ is fixed trilogy-wide. *Symbol crosswalk.* In cosine contexts we write $g_{\mathrm{b}}$ for the cosine transform; for even inputs this coincides with $\widehat{g}$.

**Exponential multiplier on the modulus line.** For $a \in \mathbb{R}$ we let $(\mathsf{M}^a f)(x) := e^{-ax} f(x)$ act by pointwise multiplication on modulus–line functions (log-scale). We emphasize that $\mathsf{M}^a$ denotes multiplication, distinct from the additive translation operator $f(x) \mapsto f(x+a)$. In particular, $(\mathsf{M}f)(x) = e^{-x}f(x)$ and $(\mathsf{M}^{1/2}f)(x) = e^{-x/2}f(x)$. We introduce the positive kernel $T_{1/2}(x) = (1+e^{-x/2})^{-1}$ and the corresponding multiplication operator $C_{1/2}$, defined by $(C_{1/2}f)(x) = T_{1/2}(x)f(x)$. This corresponds to the formal algebraic identity $C_{1/2} = (\mathrm{Id} + \mathsf{M}^{1/2})^{-1}$. Since $\mathsf{M} = (\mathsf{M}^{1/2})^2$, this is equivalent to the factorization

$$C_{1/2} = (\mathrm{Id} - \mathsf{M}^{1/2})(\mathrm{Id} - \mathsf{M})^{-1};$$

see Lemma 6.3.

*Remark* 1.11 (Counting zeros and principal value conventions). All sums over the nontrivial zeros $\rho$ of $\zeta(s)$ are taken over zeros counted with multiplicity, and pairings in $\tau$ use the symmetric principal value when indicated. On $\mathcal{S}_{\mathrm{BL}}$ the zero sums are finite; on $\mathcal{A}_{\mathrm{Exp}}$ (with $\sigma > \frac{1}{2}$) they are absolutely convergent (see §B). The symbol PV denotes the *symmetric* principal value in the spectral parameter $\tau$:

$$\mathrm{PV}\int_{\mathbb{R}} h(\tau)\,\mathrm{d}\tau \;=\; \lim_{T\to\infty}\int_{-T}^{T} h(\tau)\,\mathrm{d}\tau.$$

Distributional statements are in $\mathcal{S}'(\mathbb{R})$.

**Lemma 1.12** (Pairing with odd or purely imaginary kernels)**.** *Let $g \in \mathcal{S}(\mathbb{R})$ be real and even.*

*(i) If $K(\tau)$ is an odd distribution (e.g., the PV distribution associated with $1/\tau$), then $\langle K, g\rangle_{\mathrm{PV}} = 0$.*

*(ii) If $K$ is a purely imaginary distribution (e.g., the PV distribution associated with $i/\tau$), the pairing $\langle K, g\rangle_{\mathrm{PV}}$ is purely imaginary. Consequently, $\mathrm{Re}\langle K, g\rangle_{\mathrm{PV}} = 0$.*

## 1.2 Test Class and the Phase Derivative

**Definition 1.13** (Band-limited class $\mathcal{S}_{\mathrm{BL}}$ and cosine pair)**.** We define the band-limited Schwartz class $\mathcal{S}_{\mathrm{BL}}$ as the space of real even functions $g : \mathbb{R} \to \mathbb{R}$ whose cosine transform

$$g_{\mathrm{b}}(x) := \int_{\mathbb{R}} g(\tau)\cos(x\tau)\,\mathrm{d}\tau$$

lies in $C_c^\infty(\mathbb{R})$. The even inversion holds:

$$g(\tau) = \frac{1}{\pi}\int_0^\infty g_{\mathrm{b}}(x)\,\cos(x\tau)\,\mathrm{d}x \qquad (\tau \in \mathbb{R}), \tag{1.3}$$

which is the specialization of the Fourier pair fixed in the conventions block. By Paley–Wiener, $g$ extends to an entire function of exponential type; in particular $g \in \mathcal{S}(\mathbb{R})$ and $g_{\mathrm{b}} \in L^1(\mathbb{R})$.

**Definition 1.14** (Origin term)**.** From (1.3) at $\tau = 0$,

$$g(0) = \frac{1}{\pi}\int_0^\infty g_{\mathrm{b}}(x)\,\mathrm{d}x. \tag{1.4}$$

**Definition 1.15** (Dictionary profile)**.** For $x > 0$ define

$$G_{\mathrm{b}}(x) := 2\sinh\left(\frac{x}{2}\right)\,g_{\mathrm{b}}(2x). \tag{1.5}$$

*Remark* 1.16 (Exponential extension). We also use the class $\mathcal{A}_{\mathrm{Exp}}$ from Def. 1.2. The strict tilt parameter $\sigma > \frac{1}{2}$ gives absolute convergence for $P(g)$ and an $\alpha$-independent $L^1$ majorant in Archimedean limits (DCT). The band-limited class $\mathcal{S}_{\mathrm{BL}}$ (also called $\mathcal{A}_{\mathrm{BL}}$) is identical in normalization.

**Definition 1.17** (Spectral functional $I(g)$ in Maaß–Selberg normalization)**.** With $\widetilde{S}(s) = \Lambda(2s-1)/\Lambda(2s)$ and $s = \frac{1}{2} + \varepsilon + i\tau$,

$$\varphi'_\varepsilon(\tau) := 2\,\mathrm{Re}\left\{\frac{\Lambda'}{\Lambda}(2s-1) - \frac{\Lambda'}{\Lambda}(2s)\right\}, \qquad \varphi'(\tau) := 4\,\mathrm{Re}\left(\frac{\Lambda'}{\Lambda}(2i\tau)\right).$$

The expression for $\varphi'(\tau)$ is understood in the sense of the symmetric principal value at $\tau = 0$. The spectral functional is

$$\mathcal{I}_\varepsilon(g) := \frac{1}{2\pi}\int_{\mathbb{R}} g(\tau)\,\varphi'_\varepsilon(\tau)\,\mathrm{d}\tau, \qquad \mathcal{I}(g) := \frac{1}{2\pi}\ \mathrm{PV}\int_{\mathbb{R}} g(\tau)\,\varphi'(\tau)\,\mathrm{d}\tau,$$

and $\mathcal{I}_\varepsilon(g) \to \mathcal{I}(g)$ by DCT with an $\varepsilon$-independent $L^1$ majorant (see Thm. 2.6).

**Lemma 1.18** (Boundary identity and temperedness)**.** *In $\mathcal{S}'(\mathbb{R})$ one has* (with symmetric PV eliminating any atom at $\tau = 0$ for the gamma-factor difference)

$$\varphi'(\tau) = 4\,\mathrm{Re}\left(\frac{\Lambda'}{\Lambda}(2i\tau)\right) \quad \textit{(symmetric PV at } \tau = 0\textit{)}, \tag{1.6}$$

*with* no $\delta_0$ *mass. Moreover $\varphi'(\tau) = O(\log(2+|\tau|))$, hence $\mathrm{d}\xi = (1/2\pi)\varphi'(\tau)\mathrm{d}\tau$ is tempered.*

*Proof.* From $\Lambda(s) = \Lambda(1-s)$, $(\Lambda'/\Lambda)(1+z) = -(\Lambda'/\Lambda)(-z)$; with $z = 2i\tau$ and conjugation symmetry one obtains (1.6). Cancellation of the Poisson kernels at the boundary yields no atom at 0 (rigorously shown in Thm. 2.3; see also Thm. 5.1). Stirling's formula for $\Gamma$ and the standard bound $(\zeta'/\zeta)(1+it) = O(\log(2+|t|))$ imply $\mathrm{Re}(\Lambda'/\Lambda(2i\tau)) = O(\log(2+|\tau|))$ (see [5, Chap. 5]). □

**Lemma 1.19** (Reality and evenness)**.** *$\varphi'(\tau)$ is real and even; hence $\mathcal{I}(g)$, $\mathcal{A}(g)$, $J(g)$, and the weights $\mathcal{H}(g;\rho)$ are real for admissible $g$.*

*Proof.* Evenness follows from $\overline{(\Lambda'/\Lambda)(\bar{s})} = (\Lambda'/\Lambda)(s)$ and (1.6); reality is immediate since we take $\mathrm{Re}(\cdot)$ and $g$ is even.. □

*Remark* 1.20 (Symbol and class crosswalk). We lock the phase symbol to $\varphi'$ (single glyph across the trilogy). For cosine transforms we *only* use $g_{\mathrm{b}}$; any legacy synonyms (e.g. $\widehat{g}_c$) are avoided, and for even inputs $g_{\mathrm{b}} = \widehat{g}$ by definition. The band-limited class $\mathcal{S}_{\mathrm{BL}}$ (alias $\mathcal{A}_{\mathrm{BL}}$) is identical in normalization.

# 2 Regularization and Convergence Analysis

We collect the domination estimates, principal–value conventions, and boundary extraction used below. All statements are in the Maaß–Selberg (MS) ledger: symmetric PV on $\mathbb{R}$; no $\delta_0$ mass in $\varphi'$; pairings always against $\mathrm{d}\xi(\tau) = (1/2\pi)\varphi'(\tau)\mathrm{d}\tau$ (Plancherel is *not* used in pairings; see the Conventions block).

**Lemma 2.1** (Cancellation of $\log\pi$ terms)**.** *The phase derivative $\varphi'_\varepsilon(\tau)$ (Def. 1.17) depends only on the $\Gamma$ and $\zeta$ factors; the $\pi^{-s/2}$ terms in $\Lambda(s)$ cancel identically.*

*Proof.* Recall $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. The logarithmic derivative is

$$\frac{\Lambda'}{\Lambda}(s) = -\frac{1}{2}\log\pi + \frac{1}{2}\psi\left(\frac{s}{2}\right) + \frac{\zeta'}{\zeta}(s).$$

The phase derivative involves the difference $\frac{\Lambda'}{\Lambda}(2s-1) - \frac{\Lambda'}{\Lambda}(2s)$. The constant terms $(-\frac{1}{2}\log\pi)$ cancel, yielding:

$$\frac{\Lambda'}{\Lambda}(2s-1) - \frac{\Lambda'}{\Lambda}(2s) = \frac{1}{2}(\psi(s-\tfrac{1}{2}) - \psi(s)) + \left(\frac{\zeta'}{\zeta}(2s-1) - \frac{\zeta'}{\zeta}(2s)\right).$$

□

**Lemma 2.2** (Decomposition and uniform bounds)**.** Note: SBL $\Rightarrow$ finite prime sums; $\mathcal{A}_{\mathrm{Exp}}$ $(\sigma > \frac{1}{2})$ $\Rightarrow$ absolute convergence. Uniform $\varepsilon$-majorants are used below. *Let* $s = \frac{1}{2} + \varepsilon + i\tau$ *with* $0 < \varepsilon \le 1$*. In* $\mathcal{S}'(\mathbb{R})$*,*

$$\varphi_\varepsilon'(\tau) = \mathrm{Re}\Big[\psi(s-\tfrac{1}{2}) - \psi(s)\Big] + 2\ \mathrm{Re}\left\{\frac{\zeta'}{\zeta}(2s-1) - \frac{\zeta'}{\zeta}(2s)\right\}.$$

*Moreover:*

*(i) (*Local Poisson cancellation*). As* $\varepsilon \downarrow 0$*,*

$$\mathrm{Re}[\psi(s-\tfrac{1}{2}) - \psi(s)] = -\frac{\varepsilon}{\varepsilon^2+\tau^2} + O(1), \qquad -2\ \mathrm{Re}\frac{\zeta'}{\zeta}(2s) = +\frac{\varepsilon}{\varepsilon^2+\tau^2} + O(1),$$

*so the Poisson singularities cancel in the sum and no* $\delta_0$ *mass persists in the limit.*

*(ii) (*Uniform growth*). Uniformly for* $0 < \varepsilon \le 1$*,*

$$\varphi_\varepsilon'(\tau) = O\big(\log(2+|\tau|)\big) \qquad (|\tau| \to \infty),$$

*hence* $\varphi'(\tau) = \mathrm{PV}\ 4\ \mathrm{Re}(\Lambda'/\Lambda(2i\tau))$ *defines a tempered distribution and* $\mathrm{d}\xi$ *is tempered.*

**Lemma 2.3** (No $\delta_0$ at $\tau = 0$ on even Schwartz tests)**.** *Let* $\mathcal{S}_{\mathrm{even}} := \{g \in \mathcal{S}(\mathbb{R}) : g(\tau) = g(-\tau)\}$*. For* $0 < \varepsilon \le 1$ *set*

$$K_{\Gamma,\varepsilon}(\tau) = \mathrm{Re}\Big(\psi(\varepsilon + i\tau) - \psi(\tfrac{1}{2} + \varepsilon + i\tau)\Big), \qquad K_{\zeta,\varepsilon}(\tau) = 2\ \mathrm{Re}\Big(\tfrac{\zeta'}{\zeta}(2s-1) - \tfrac{\zeta'}{\zeta}(2s)\Big)_{s=\frac{1}{2}+\varepsilon+i\tau}.$$

*Then for all* $g \in \mathcal{S}_{\mathrm{even}}$ *(or* $g \in \mathcal{A}_{\mathrm{Exp}}$*),*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{2\pi}\int_{\mathbb{R}} g(\tau)\Big(K_{\Gamma,\varepsilon}(\tau) + K_{\zeta,\varepsilon}(\tau)\Big)\,\mathrm{d}\tau = \mathrm{PV}\ \frac{1}{2\pi}\int_{\mathbb{R}} g(\tau)\,\varphi'(\tau)\,\mathrm{d}\tau$$

*and the limiting distribution has* no *atom at* $\tau = 0$.

*Proof.* Let $P_\varepsilon(\tau) = \frac{\varepsilon}{\varepsilon^2+\tau^2}$ be the Poisson kernel. We analyze the behavior near $\tau = 0$ and establish uniform domination for $\varepsilon \in (0, 1]$.

**1. Gamma Part Analysis.** $K_{\Gamma,\varepsilon}(\tau) = \mathrm{Re}\left(\psi(\varepsilon+i\tau) - \psi(\tfrac{1}{2}+\varepsilon+i\tau)\right)$. We use the exact representation $\psi(z) = -1/z + \psi_{\mathrm{reg}}(z)$. The function $\psi_{\mathrm{reg}}(z) = \psi(z) + 1/z$ is holomorphic for $\mathrm{Re}(z) > -1$.

$$\mathrm{Re}\,\psi(\varepsilon+i\tau) = \mathrm{Re}\left(\frac{-1}{\varepsilon+i\tau}\right) + \mathrm{Re}\,\psi_{\mathrm{reg}}(\varepsilon+i\tau) = -P_\varepsilon(\tau) + R'_{\Gamma,\varepsilon}(\tau).$$

**Rigorous Uniform bound on $R'_{\Gamma,\varepsilon}$:** We define the compact domain $K_1 = \{\varepsilon + i\tau : \varepsilon \in [0,1], |\tau| \le 1\}$. Since $\psi_{\mathrm{reg}}$ is continuous on the compact set $K_1$, by the Extreme Value Theorem, it attains a maximum. Let $C_\Gamma = \sup_{z\in K_1} |\psi_{\text{reg}}(z)|$. This constant is finite and crucially *independent* of $\varepsilon$. Thus, $|R'_{\Gamma,\varepsilon}(\tau)| \le C_\Gamma$ uniformly for all $(\varepsilon,\tau)$ corresponding to $K_1$. Let $R_{\Gamma,\varepsilon}(\tau) = R'_{\Gamma,\varepsilon}(\tau) - \operatorname{Re}\psi(\frac{1}{2}+\varepsilon+i\tau)$. The second term is uniformly bounded on the compact set $[\frac{1}{2},\frac{3}{2}]+i[-1,1]$. Thus, $R_{\Gamma,\varepsilon}(\tau)$ is locally uniformly bounded, independent of $\varepsilon$.

**2. Zeta Part Analysis.** We analyze $-2\operatorname{Re}\frac{\zeta'}{\zeta}(1+2\varepsilon+2i\tau)$. We use $\frac{\zeta'}{\zeta}(s) = -1/(s-1) + Z_{\text{reg}}(s)$. The function $Z_{\text{reg}}(s)$ is holomorphic near $s=1$.

$$\begin{aligned}-2\operatorname{Re}\frac{\zeta'}{\zeta}(1+2\varepsilon+2i\tau) &= -2\operatorname{Re}\Big(\frac{-1}{2\varepsilon+2i\tau}\Big) - 2\operatorname{Re} Z_{\text{reg}}(1+2\varepsilon+2i\tau)\\ &= P_\varepsilon(\tau) + R_{\zeta,\text{pole},\varepsilon}(\tau).\end{aligned}$$

**Rigorous Uniform bound on $R_{\zeta,\text{pole},\varepsilon}$:** We define the compact domain $K_2 = \{1+2\varepsilon+2i\tau : \varepsilon\in[0,1], |\tau|\le 1\} = [1,3]+i[-2,2]$. $Z_{\text{reg}}$ is holomorphic (and thus continuous) on the compact set $K_2$. Let $C_\zeta = 2\sup_{s\in K_2}|Z_{\text{reg}}(s)|$. Thus, $|R_{\zeta,\text{pole},\varepsilon}(\tau)| \le C_\zeta$ uniformly. In particular, $C_\zeta$ is finite and independent of $\varepsilon$. Let $R_{\zeta,\varepsilon}(\tau)$ be the total zeta remainder.

**3. Cancellation and Domination.** The combined kernel is $K_{\Gamma,\varepsilon}(\tau) + K_{\zeta,\varepsilon}(\tau) = R_{\Gamma,\varepsilon}(\tau) + R_{\zeta,\varepsilon}(\tau) =: R_{\text{total},\varepsilon}(\tau)$. The Poisson kernels cancel exactly.

We establish a uniform $L^1$ majorant for $g(\tau)R_{\text{total},\varepsilon}(\tau)$.

**Local bound ($|\tau|\le 1$):** We established that $|R_{\text{total},\varepsilon}(\tau)| \le C_{\text{local}}$ uniformly for $0<\varepsilon\le 1$.

**Global bound ($|\tau|>1$):** We note that $R_{\text{total},\varepsilon}(\tau) = \varphi'_\varepsilon(\tau)$. By Lemma 2.2(ii) (which relies on standard estimates for $\psi$ via Stirling and bounds for $\zeta'/\zeta$ that are uniform across vertical strips away from poles; concretely, Stirling as in DLMF 5.11.2 and the bounds in [5, Thm 5.17]), we have the global uniform bound $|\varphi'_\varepsilon(\tau)| \le C_2\log(2+|\tau|)$ for $0<\varepsilon\le 1$.

**Conclusion (DCT):** We define the majorant $M(\tau) = |g(\tau)|\cdot\max(C_{\text{local}}, C_2\log(2+|\tau|))$. Since $g\in\mathcal{S}(\mathbb{R})$ (or $g\in\mathcal{A}_{\text{Exp}}$), $M(\tau)\in L^1(\mathbb{R})$ and is independent of $\varepsilon$. By DCT, the limit $\varepsilon\downarrow 0$ passes inside the integral. As the convergence is dominated (not merely weak-*), this rigorously confirms that no $\delta_0$ mass survives. □

## SBL vs. AExp: crosswalk and majorants

**Principal value convention (canonical)**

Throughout, the symmetric principal value is

$$\mathrm{PV}\int_{\mathbb{R}} h(\tau)\,\mathrm{d}\tau := \lim_{T\to\infty}\int_{-T}^{T} h(\tau)\,\mathrm{d}\tau,$$

and pairings with real-even $g\in\mathcal{S}(\mathbb{R})$ use this convention. Odd or purely imaginary kernels contribute no *real* part to such pairings.

**Lemma 2.4** (Uniform $L^1$ majorant for $\varphi'_\varepsilon$)**.** *For $0<\varepsilon\le 1$ and all $\tau\in\mathbb{R}$, $|\varphi'_\varepsilon(\tau)| \ll \log(2+|\tau|)$ and hence $|g(\tau)\varphi'_\varepsilon(\tau)| \le C(1+|\tau|)^{-2}\log(2+|\tau|)\in L^1(\mathbb{R})$ for $g\in\mathcal{S}(\mathbb{R})$.*

**Lemma 2.5** (Primes: SBL vs. AExp)**.** *If $g\in\mathcal{S}_{\text{BL}}$, only finitely many $(p,k)$ contribute to $P(g)$. If $g\in\mathcal{A}_{\text{Exp}}$ with strict $\sigma>\frac{1}{2}$, then $\sum_{n\ge 2}\Lambda(n)n^{-2\sigma}<\infty$, so $P(g)$ converges absolutely and Tonelli applies.*

**Proposition 2.6** (Vertical-line passage and branch synchronization)**.** *For any admissible $g$ (Def. 1.3),*

$$\mathcal{I}(g) = \mathrm{PV}\,\frac{1}{2\pi}\int_{\mathbb{R}} g(\tau)\,\varphi'(\tau)\,\mathrm{d}\tau = \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi}\int_{\mathbb{R}} g(\tau)\,\varphi'_\varepsilon(\tau)\,\mathrm{d}\tau.$$

*Moreover, with the* same *branch for* $\log\Lambda$ *in Route A and Route B, the Archimedean boundary extraction produces exactly the $-\frac{1}{2}g(0)$ term that recombines with $2J(g)$ in the MS ledger.*

*Proof.* **(i) DCT with an $\varepsilon$–independent $L^1$ majorant.** If $g \in \mathcal{S}_{\mathrm{BL}}$, $g$ is Schwartz, so $|g(\tau)|$ decays rapidly. If $g \in \mathcal{A}_{\mathrm{Exp}}$, by definition (Def. 1.2), $|g(\tau)| \le C(1+|\tau|)^{-2}$. In either case, by Lemma 2.2(ii),

$$|g(\tau)\varphi_\varepsilon'(\tau)| \le C(1+|\tau|)^{-2}\log(2+|\tau|) =: M(\tau), \qquad M \in L^1(\mathbb{R}),$$

uniform in $0 < \varepsilon \le 1$. Hence DCT gives the stated limit and the *symmetric* PV passes through the limit.

**(ii) Prime block (Tonelli).** If $g \in \mathcal{S}_{\mathrm{BL}}$, $\operatorname{supp}(g_{\mathrm{b}}) \subset [-X, X]$ and only finitely many $(p,k)$ satisfy $2k\log p \in \operatorname{supp}(g_{\mathrm{b}})$, so $\sum \leftrightarrow \int$ is trivial and the limit at $\varepsilon = 0$ is termwise. For $\mathcal{A}_{\mathrm{Exp}}$ (strict $\sigma > \frac{1}{2}$), absolute convergence follows from $\sum_{n\ge 2}\Lambda(n)n^{-2\sigma} < \infty$; Tonelli applies.

**(iii) Archimedean block (approximate identity).** With $K_\varepsilon(\tau) = \operatorname{Re}[\psi(\varepsilon + i\tau) - \psi(\frac{1}{2} + \varepsilon + i\tau)]$, we use the standard identity relating the difference of digamma functions on the spectral side to the geometric series kernel on the modulus side (cf. treatments of the explicit formula, e.g., [5, Chap. 5]). We apply cosine inversion and Fubini's theorem. **Justification for Fubini (fixed $\varepsilon > 0$):** We must verify the absolute joint integrability of the integrand $F(\tau, x) = g(\tau)K_{\mathrm{mod},\varepsilon}(x)\cos(\tau x)$. We require $\iint |F(\tau, x)|\, d\tau dx < \infty$. This integral is bounded by $\|g\|_{L^1(\mathbb{R})} \cdot \|K_{\mathrm{mod},\varepsilon}\|_{L^1((0,\infty))}$. Since $g \in \mathcal{A}_{\mathrm{Exp}}$, $\|g\|_{L^1(\mathbb{R})} < \infty$. The frequency kernel $K_{\mathrm{mod},\varepsilon}(x)$ (defined below) satisfies $|K_{\mathrm{mod},\varepsilon}(x)| < e^{-\varepsilon x}$. Thus $\|K_{\mathrm{mod},\varepsilon}\|_{L^1((0,\infty))} \le 1/\varepsilon < \infty$. Fubini's theorem applies for fixed $\varepsilon > 0$.

$$\frac{1}{2\pi}\int_{\mathbb{R}} g(\tau)K_\varepsilon(\tau)\,\mathrm{d}\tau = \frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(x)\,\frac{e^{-(\varepsilon+1/2)x} - e^{-\varepsilon x}}{1 - e^{-x}}\,\mathrm{d}x =: \mathcal{I}_{\mathrm{arch},\varepsilon}(g).$$

We split the modulus-line kernel by isolating the term $-e^{-\varepsilon x}$, which corresponds to the dominant Poisson singularity on the spectral side. We define the remainder $R_\varepsilon(x)$ as:

$$\frac{e^{-(\varepsilon+1/2)x} - e^{-\varepsilon x}}{1 - e^{-x}} \;=\; -e^{-\varepsilon x} \;+\; R_\varepsilon(x),$$

$$R_\varepsilon(x) := \frac{e^{-(\varepsilon+1/2)x} - e^{-\varepsilon x}}{1 - e^{-x}} + e^{-\varepsilon x}.$$

Combining terms over the denominator $1 - e^{-x}$:

$$\begin{aligned} R_\varepsilon(x) &= \frac{e^{-(\varepsilon+1/2)x} - e^{-\varepsilon x} + e^{-\varepsilon x}(1 - e^{-x})}{1 - e^{-x}} \\ &= e^{-\varepsilon x}\frac{e^{-x/2} - e^{-x}}{1 - e^{-x}}. \end{aligned}$$

Factoring the numerator as $e^{-x/2}(1 - e^{-x/2})$ and the denominator as $(1 - e^{-x/2})(1 + e^{-x/2})$ yields the manifestly positive form (see Section D for details):

$$R_\varepsilon(x) = e^{-\varepsilon x}\frac{e^{-x/2}}{1 + e^{-x/2}}.$$

The full kernel is thus $-e^{-\varepsilon x} + R_\varepsilon(x)$. The term $-e^{-\varepsilon x}$, when paired with $g_{\mathrm{b}}(x)/(2\pi)$, yields the boundary contribution $-\frac{1}{2}g(0)$ as $\varepsilon \downarrow 0$ by DCT (since $g_{\mathrm{b}} \in L^1$). **Justification for DCT ($\varepsilon \downarrow 0$):** The remainder $R_\varepsilon(x)$ converges pointwise to $R_0(x)$. Crucially, it is uniformly dominated: $|R_\varepsilon(x)| \le \frac{e^{-x/2}}{1+e^{-x/2}} < 1$ (independent of $\varepsilon$; see Section D). Since $g \in \mathcal{A}_{\mathrm{Exp}}$ (or $\mathcal{S}_{\mathrm{BL}}$), $g_{\mathrm{b}} \in L^1(\mathbb{R})$. The integrand $|g_{\mathrm{b}}(x)R_\varepsilon(x)|$ is dominated by the integrable majorant $M(x) = |g_{\mathrm{b}}(x)|$. DCT applies to the remainder integral. We define the Archimedean functional $\mathcal{A}(g)$ as the pairing with the limiting remainder $R_0(x)$:

$$\mathcal{A}(g) := \frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(x)R_0(x)\,\mathrm{d}x, \qquad R_0(x) = \frac{e^{-x/2}}{1 + e^{-x/2}}.$$

Thus, the limit of the Archimedean spectral contribution is $\lim_{\varepsilon\downarrow 0}\mathcal{I}_{\mathrm{arch},\varepsilon}(g) = \mathcal{A}(g) - \frac{1}{2}g(0)$.

**(iv) Branch synchronization.** We use the same vertical-line branches for $\log\Lambda$ as in Route B (boundary from $\operatorname{Re}s = \frac{1}{2}+\varepsilon$, continuous in $\tau$, symmetric PV at 0); this prevents spurious constants and aligns the $-\frac{1}{2}g(0)$ term with the $J(g)$ block fixed by the pole at $s=1$.

**(v) Temperedness and PV pairing.** By (ii) and (iii) the pieces are tempered; by (i) the PV limit exists and equals the boundary integral. □

**Proposition 2.7** (Continuous Spectral Measure)**.** *On the critical line $s = 1/2 + i\tau$, the phase derivative is $\varphi'(\tau) = \operatorname{Im}\partial_\tau \log \widetilde{S}(s)$. By Lemma 1.18 (functional equation + conjugation), $\varphi'(\tau) = 4\ \mathrm{PV}\ \mathrm{Re}(\Lambda'/\Lambda(2i\tau))$ with no atom at 0. Adopting the Maaß–Selberg convention, the (atomless) spectral–shift measure is*

$$\mathrm{d}\xi(\tau) \;:=\; \frac{1}{2\pi}\,\varphi'(\tau)\,\mathrm{d}\tau = \frac{1}{2\pi}\,\mathrm{d}\big(\arg\widetilde{S}(\tfrac{1}{2}+i\tau)\big),$$

*with the symmetric principal value at $\tau = 0$ and* no *$\delta_0$ mass (by Thm. 1.18 and Thm. 5.1). By contrast, the continuous Selberg Plancherel measure is $\mathrm{d}\tau/(4\pi)$. All pairings in this paper use $\mathrm{d}\xi$ (never Plancherel; see Section 1.1); this locks the constants in $\mathcal{I}(g)$ and matches the modulus-line regrouping with $\rho_W$ (cf. Thm. 4.2 and Section A).*

**Theorem 2.8** (Normalization Lock: The Maaß–Selberg Condition)**.** *As fixed in Section 1.1, we pair throughout against the Maaß–Selberg (Birman–Krein̆) spectral–shift measure $\mathrm{d}\xi(\tau) = \frac{1}{2\pi}\,\varphi'(\tau)\,\mathrm{d}\tau$ in the symmetric principal–value sense; there is no $\delta_0$ mass at $\tau = 0$, and we never use the Selberg Plancherel density $\mathrm{d}\tau/(4\pi)$ in pairings. This wording and choice are identical to [8] (Remark 2.1) and fix the same continuous density $\rho_W$ used there (see* (A.2)*).*

***Fixing the normalization.*** *The Maaß–Selberg relations give for the standard Eisenstein $E(z,s)$ on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ (with truncation $\Lambda_Y$)*

$$\big\|\Lambda_Y E(\cdot,\tfrac{1}{2}+i\tau)\big\|^2 \;=\; 2\log Y \;+\; \frac{1}{\pi}\,\varphi'(\tau) \;+\; O(Y^{-1}), \qquad Y\to\infty,$$

*with residues oriented by increasing $\tau$ (matching the PV convention). Hence the Birman–Krein̆ spectral–shift measure used throughout is*

$$\mathrm{d}\xi(\tau) \;=\; \frac{1}{2\pi}\,\varphi'(\tau)\,\mathrm{d}\tau \;=\; \frac{1}{2}\,\mathrm{d}\Big(\frac{1}{\pi}\arg\widetilde{S}(\tfrac{1}{2}+i\tau)\Big),$$

*distinct from the Selberg Plancherel density $\mathrm{d}\tau/(4\pi)$. In particular,*

$$\langle\xi, g\rangle = \frac{1}{2\pi}\,\mathrm{PV}\!\int_{\mathbb{R}} g(\tau)\,\varphi'(\tau)\,\mathrm{d}\tau.$$

*This also locks the boundary extraction in Route A: the Poisson kernel contributes $-\frac{1}{2}g(0)$, and regrouping with $2J(g)$ yields the continuous density*

$$\rho_W(x) = \frac{1}{2\pi}\left(\frac{1}{1+e^{-x/2}} + 2e^{-x/2}\right),$$

*in the Maaß–Selberg convention. Cross-check: [8] Appendix A (A.3) fixes $\mathrm{d}\xi = \frac{1}{2\pi}\varphi'(\tau)\,\mathrm{d}\tau$; [9] uses the same $\rho_W$ in $Q_{\mathrm{geom}}(g)$ (see §§1,4). We emphasize again that the Plancherel density is $\mathrm{d}\tau/(4\pi)$ and is* not *used in our pairings. The Maaß–Selberg relation above pins the working measure to $\mathrm{d}\xi = (1/2\pi)\,\varphi'(\tau)\,\mathrm{d}\tau$ (PV, no $\delta_0$), distinct from the Selberg Plancherel density $\mathrm{d}\tau/(4\pi)$; this is the convention used in all pairings and ledger identities.*

**Lemma 2.9** (Tempered Distribution)**.** *The continuous spectral distribution $\xi$ is a tempered distribution ($\xi \in \mathcal{S}'(\mathbb{R})$).*

*Proof.* Since $\varphi'(\tau) = O(\log(2+|\tau|))$ by Stirling for $\Gamma$ together with the classical bound $(\zeta'/\zeta)(1+it) = O(\log(2+|t|))$ (see [5, Chap. 5]), $\xi$ exhibits at most polynomial growth and hence defines an element of $\mathcal{S}'(\mathbb{R})$. □

# 3 Local Decompositions and the Dictionary Transform

We now decompose the phase derivative $\varphi'(\tau)$ into its local components—arithmetic and Archimedean—corresponding to the evaluation strategy of Route A.

## 3.1 The Arithmetic Block (Primes)

**Lemma 3.1** (The Arithmetic Block). *The contribution from the finite primes $p$ to the phase derivative, derived via regularization and subsequent passage to the limit $\varepsilon \downarrow 0$ after pairing, is given by the kernel:*

$$\varphi'_{primes}(\tau) = -2\sum_{p}\sum_{k\geq 1}(\log p)(1-p^{-k})\cos(2k\,\tau\log p). \tag{3.1}$$

*Consequently, the total contribution from the primes to the spectral integral is*

$$\mathcal{I}_{primes}(g) \;=\; -\frac{1}{\pi}\sum_{p}\sum_{k\geq 1}(\log p)\,(1-p^{-k})\,g_{\mathrm{b}}(2k\log p),$$

*where the* minus sign *originates from the Euler-product identity $\frac{\zeta'}{\zeta}(s) = -\sum_{p,k\geq 1}(\log p)\,p^{-ks}$.*

*Proof.* We analyze the component $2\ \mathrm{Re}\{\frac{\zeta'}{\zeta}(2s-1) - \frac{\zeta'}{\zeta}(2s)\}$. For $\varepsilon > \frac{1}{2}$, the Dirichlet series converges absolutely:

$$\begin{aligned}
\varphi'_{\mathrm{primes},\varepsilon}(\tau) &= 2\ \mathrm{Re}\left\{\frac{\zeta'}{\zeta}(2s-1) - \frac{\zeta'}{\zeta}(2s)\right\} \\
&= 2\ \mathrm{Re}\left\{-\sum_{p,k}(\log p)\,p^{-k(2s-1)} + \sum_{p,k}(\log p)\,p^{-k(2s)}\right\} \\
&= 2\ \mathrm{Re}\sum_{p,k}(\log p)\,\left(p^{-2ks} - p^{-k(2s-1)}\right) \\
&= 2\ \mathrm{Re}\sum_{p,k}(\log p)\,p^{-k(2s-1)}\,\underbrace{(p^{-k}-1)}_{<0}.
\end{aligned}$$

Substituting $s = 1/2 + \varepsilon + i\tau$, so $2s-1 = 2\varepsilon + 2i\tau$. Factoring out the negative sign:

$$\varphi'_{\mathrm{primes},\varepsilon}(\tau) = -2\sum_{p,k}(\log p)\,(1-p^{-k})\,p^{-2k\varepsilon}\cos(2k\tau\log p).$$

We now consider the pairing $\mathcal{I}_{\mathrm{primes},\varepsilon}(g)$. Since $g \in L^1(\mathbb{R})$ and the series converges absolutely (uniformly in $\tau$ for fixed $\varepsilon > \frac{1}{2}$), Fubini–Tonelli justifies the termwise integration:

$$\begin{aligned}
\mathcal{I}_{\mathrm{primes},\varepsilon}(g) &= \frac{1}{2\pi}\int_{\mathbb{R}} g(\tau)\,\varphi'_{\mathrm{primes},\varepsilon}(\tau)\,\mathrm{d}\tau \\
&= -\frac{1}{\pi}\sum_{p,k}(\log p)\,(1-p^{-k})\,p^{-2k\varepsilon}\,g_{\mathrm{b}}(2k\log p).
\end{aligned}$$

Taking the limit $\varepsilon \downarrow 0$ yields the result for $\mathcal{I}_{\mathrm{primes}}(g)$. The interchange of the limit and summation is justified by dominated convergence (for $\mathcal{A}_{\mathrm{Exp}}$ with $\sigma > 1/2$, see Prop. 3.6) or because the resulting sum is finite (for $\mathcal{S}_{\mathrm{BL}}$, see Remark 3.2). $\square$

*Remark* 3.2 (Finiteness of the Prime Sum). If $\mathrm{supp}(g_{\mathrm{b}}) \subset [-X, X]$, a nonzero contribution requires $|2k\log p| \leq X$, implying $p^k \leq e^{X/2}$. Thus, only finitely many pairs $(p,k)$ contribute.

### 3.2 The Dictionary Transform and the Arithmetic Functional

**Proposition 3.3** (Dictionary Identity). *For every prime $p$ and integer $k \geq 1$:*

$$(1 - p^{-k})\, g_{\mathrm{b}}(2k \log p) \;=\; p^{-k/2}\, G_{\mathrm{b}}(k \log p). \tag{3.2}$$

*Proof.* By (1.5) (with $x > 0$), $G_{\mathrm{b}}(x) = 2\sinh(x/2)\, g_{\mathrm{b}}(2x)$. Hence the RHS equals $p^{-k/2} \cdot 2\sinh(\frac{k \log p}{2}) \cdot g_{\mathrm{b}}(2k \log p) = (1 - p^{-k}) g_{\mathrm{b}}(2k \log p)$. This is the modulus-line avatar of the standard Weil dictionary; compare [8], (A.1)–(A.2). □

**Definition 3.4** (The Arithmetic Functional $P(g)$). We define the arithmetic functional $P(g)$ as

$$P(g) := \frac{1}{\pi} \sum_{p} \sum_{k \geq 1} (\log p)\, p^{-k/2}\, G_{\mathrm{b}}(k \log p). \tag{3.3}$$

The total contribution from the primes to the spectral integral is $\mathcal{I}_{\mathrm{primes}}(g) = -P(g)$.

**Proposition 3.5** (Von Mangoldt Formulation). *Let $\Lambda(n)$ denote the von Mangoldt function. Then*

$$P(g) = \frac{1}{\pi} \sum_{n \geq 2} \Lambda(n)\, n^{-1/2}\, G_{\mathrm{b}}(\log n). \tag{3.4}$$

**Proposition 3.6** (Extended Admissibility). *If $g \in \mathcal{A}_{\mathrm{Exp}}$ with decay parameter $\sigma > \frac{1}{2}$, the arithmetic functional $P(g)$ converges absolutely. For this extended class the Archimedean boundary extraction yields the same term $-\frac{1}{2} g(0)$.*

*Proof.* With $g_{\mathrm{b}}(2x) = O(e^{-2\sigma x})$ and $G_{\mathrm{b}}(x) = 2\sinh(x/2)\, g_{\mathrm{b}}(2x)$, we have $G_{\mathrm{b}}(\log n) = O(n^{-2\sigma + \frac{1}{2}})$. Hence

$$|P(g)| \;\ll\; \sum_{n \geq 2} \Lambda(n)\, n^{-1/2}\, |G_{\mathrm{b}}(\log n)| \;\ll\; \sum_{n \geq 2} \Lambda(n)\, n^{-2\sigma},$$

which converges for $\sigma > \frac{1}{2}$. Absolute convergence justifies termwise limits for $\varepsilon > \frac{1}{2}$ in Route A; the boundary extraction for the Archimedean block depends only on the local Poisson-kernel asymptotic and is unchanged, giving the same $-\frac{1}{2} g(0)$.

□

## 4 Route B: Spectral Evaluation via Partial Fractions

For the spectral evaluation (Route B), we compute $\mathcal{I}(g)$ directly on the critical line, $\mathrm{Re}(s) = 1/2$, utilizing the global meromorphic structure of the zeta function. This approach is independent of the Dirichlet series convergence arguments used in Route A.

**Definition 4.1** (Archimedean Spectral Contribution $\mathcal{I}_{\mathrm{arch}}(g)$). The contribution from the gamma factors to the spectral integral on the critical line is:

$$\mathcal{I}_{\mathrm{arch}}(g) := \frac{1}{2\pi}\, \mathrm{PV} \int_{\mathbb{R}} g(\tau)\, \mathrm{Re}\left[\psi(i\tau) - \psi(\tfrac{1}{2} + i\tau)\right] \mathrm{d}\tau. \tag{4.1}$$

*Note.* The pairing is understood in the symmetric principal value sense, consistent with the global convention (see Thm. 1.11).

**Definition 4.2** (The Archimedean Functional $\mathcal{A}(g)$). The renormalized Archimedean functional $\mathcal{A}(g)$ is defined via the positive remainder kernel $R_0(x)$ identified in the regularization analysis (Prop. 2.6):

$$\mathcal{A}(g) := \frac{1}{2\pi} \int_0^\infty g_{\mathrm{b}}(x)\, R_0(x)\, \mathrm{d}x, \qquad R_0(x) := \frac{e^{-x/2}}{1 + e^{-x/2}}. \tag{4.2}$$

By Prop. 2.6(iii), these are related by $\mathcal{I}_{\mathrm{arch}}(g) = \mathcal{A}(g) - \frac{1}{2} g(0)$.

*Remark* 4.3 (Ledger Note: Continuous Density $\rho_W$). In the Maaß–Selberg ledger, the Archimedean functional combines with the boundary term and the pole functional $J(g)$ to form the continuous density $\rho_W$. This regrouping (detailed in Cor. 6.7 and matching (A.2)) naturally absorbs constants (such as $2\log 2$ arising from Gamma duplication formulas) that may appear in other conventions:

$$-\mathcal{A}(g) + \tfrac{1}{2}g(0) + 2J(g) = \int_0^\infty g_{\mathrm{b}}(x)\,\rho_W(x)\,\mathrm{d}x,$$
$$\rho_W(x) = \frac{1}{2\pi}\left(\frac{1}{1+e^{-x/2}} + 2e^{-x/2}\right).$$

**Lemma 4.4** (Decomposition of the Phase Derivative)**.** *With the symmetric principal value at $\tau = 0$, the phase derivative $\varphi'(\tau)$ decomposes as $\varphi'(\tau) = \varphi'_{arch}(\tau) + \varphi'_\zeta(\tau)$, where*

$$\varphi'_{arch}(\tau) = \mathrm{Re}\left[\psi(i\tau) - \psi(1/2 + i\tau)\right] \quad \textit{(use } \psi(\sigma + i\tau) = \log|\tau| + O(1/|\tau|)\textit{; DLMF 5.11.2)},$$
$$\varphi'_\zeta(\tau) = 2\ \mathrm{Re}\left\{\frac{\zeta'}{\zeta}(2i\tau) - \frac{\zeta'}{\zeta}(1+2i\tau)\right\}.$$

*Proof.* This follows from the definition of $\Lambda(s)$ and the decomposition established in Thm. 2.2 by taking the boundary limit $\varepsilon \downarrow 0$. The spectral integral thus decomposes as $\mathcal{I}(g) = \mathcal{I}_{\mathrm{arch}}(g) + \mathcal{I}_\zeta(g)$, where $\mathcal{I}_{\mathrm{arch}}(g)$ is defined in Thm. 4.1 and $\mathcal{I}_\zeta(g)$ is the pairing of $g$ with $\varphi'_\zeta(\tau)$. ☐

## 4.1 Partial Fraction Expansion and Spectral Functionals

**Lemma 4.5** (Partial Fraction Expansion of $\zeta'/\zeta$)**.** *The logarithmic derivative of the Riemann zeta function admits the (Hadamard) expansion, valid for $s \in \mathbb{C} \setminus \{1, -2, -4, \dots\}$:*

$$\frac{\zeta'}{\zeta}(s) \;=\; C \;-\; \frac{1}{s-1} \;+\; \sum_\rho \Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big) \;+\; \sum_{n=1}^\infty \Big(\frac{1}{s+2n} - \frac{1}{2n}\Big), \tag{4.3}$$

*with $C \in \mathbb{R}$ an absolute constant. In the difference $\Delta(\tau) = \frac{\zeta'}{\zeta}(2i\tau) - \frac{\zeta'}{\zeta}(1+2i\tau)$ the constant and convergence terms cancel identically (see Thm. 4.6); cf. standard references (e.g., [5, Chap. 5]; see also [11, Chap. 12]).*

*Remark* 4.6 (Global Meromorphic Identity). When calculating the difference $\Delta(\tau) = \frac{\zeta'}{\zeta}(2i\tau) - \frac{\zeta'}{\zeta}(1+2i\tau)$, the constant $C$ and the convergence factors ($1/\rho$ and $1/(2n)$) cancel out. *Indeed*, inserting Eq. (4.3) gives

$$\Delta(\tau) = \Big[-\frac{1}{2i\tau - 1} + \frac{1}{2i\tau}\Big] \;+\; \sum_\rho \Big(\frac{1}{2i\tau - \rho} - \frac{1}{1+2i\tau-\rho}\Big) \;+\; \sum_{n\ge 1}\Big(\frac{1}{2i\tau+2n} - \frac{1}{1+2i\tau+2n}\Big),$$

so the constant $C$ and the convergence terms $\sum(1/\rho)$, $\sum(1/2n)$ drop from the difference.

**Definition 4.7** (One-sided Laplace Weight)**.** For a nontrivial zero $\rho = \beta + i\gamma$, we define the one-sided Laplace weight:

$$\mathcal{H}(g;\rho) \;\coloneqq\; \frac{1}{2\pi}\int_0^\infty e^{-\beta x/2}\, g_{\mathrm{b}}(x)\ \cos(\gamma x/2)\,\mathrm{d}x.$$

**Definition 4.8** (Pole Functional)**.** We define the functional $J(g)$ corresponding to the pole of $\zeta(s)$ at $s = 1$:

$$J(g) \;\coloneqq\; \mathcal{H}(g;1) = \frac{1}{2\pi}\int_0^\infty e^{-x/2}\, g_{\mathrm{b}}(x)\,\mathrm{d}x.$$

**Lemma 4.9** (Poisson Kernel Pairing)**.** *For $a > 0$, $\gamma \in \mathbb{R}$, and admissible even $g$:*

$$\frac{1}{2\pi} \int_{\mathbb{R}} g(\tau) \, \frac{2a}{a^2 + (2\tau - \gamma)^2} \, \mathrm{d}\tau = \mathcal{H}(g; a + i\gamma).$$

*Here and below the Laplace variable is $x > 0$, consistent with the modulus-line dictionary; the even extension is understood by reflection when needed (symmetric* PV *on $\mathbb{R}$).*

Consequence. *With $a = 1, \gamma = 0$, we have $\frac{1}{2\pi} \int_{\mathbb{R}} g(\tau) \frac{2}{1+4\tau^2} \mathrm{d}\tau = J(g)$.*

*Remark* 4.10 (DLMF references and domains)*.* We used the canonical series for $\psi$ (DLMF §§5.5.10–5.5.11) and the shift identity $\psi(z+1) = \psi(z) + 1/z$ (DLMF §5.5.2) to express the trivial-zero block. All manipulations occur for $\operatorname{Re} s > 0$ and then pass to the boundary via the symmetric PV. The Poisson calculation relies on the identity $\int_{\mathbb{R}} \frac{a}{a^2+t^2} \cos(yt) \mathrm{d}t = \pi e^{-a|y|}$.

*Proof.* We use the even inversion formula (Eq. (1.3)):

$$g(\tau) = \frac{1}{\pi} \int_0^\infty g_{\mathrm{b}}(x) \cos(x\tau) \, \mathrm{d}x.$$

This is valid since $g_{\mathrm{b}} \in L^1(\mathbb{R})$ for our classes. We analyze the double integral. The integrand is dominated by $|g_{\mathrm{b}}(x)| \cdot \frac{2a}{a^2+(2\tau-\gamma)^2}$. Since $g_{\mathrm{b}} \in L^1(0, \infty)$ and the Poisson kernel is in $L^1(\mathbb{R})$ (integrating to $\pi$), the double integral is absolutely convergent. Thus, Fubini's theorem applies:

$$\frac{1}{2\pi} \int_{\mathbb{R}} g(\tau) \, \frac{2a}{a^2 + (2\tau - \gamma)^2} \, \mathrm{d}\tau = \frac{1}{2\pi^2} \int_0^\infty g_{\mathrm{b}}(x) \left[ \int_{\mathbb{R}} \frac{2a}{a^2 + (2\tau - \gamma)^2} \cos(x\tau) \, \mathrm{d}\tau \right] \mathrm{d}x.$$

Let $t = 2\tau - \gamma$, so $\mathrm{d}\tau = \mathrm{d}t/2$. The inner integral becomes:

$$\int_{\mathbb{R}} \frac{a}{a^2 + t^2} \cos\!\left( \frac{x}{2}(t + \gamma) \right) \mathrm{d}t.$$

As $\tau$ ranges over $\mathbb{R}$, $t$ also ranges over $\mathbb{R}$. Expanding the cosine: $\cos(\frac{xt}{2} + \frac{x\gamma}{2}) = \cos(\frac{xt}{2}) \cos(\frac{x\gamma}{2}) - \sin(\frac{xt}{2}) \sin(\frac{x\gamma}{2})$. The term involving $\sin(xt/2)$ results in an integrand that is odd in $t$ (since $a/(a^2 + t^2)$ is even), so it integrates to 0 over $\mathbb{R}$. We use the classical cosine integral (for $x > 0$):

$$\int_{\mathbb{R}} \frac{a}{a^2 + t^2} \cos(xt/2) \, \mathrm{d}t = \pi e^{-ax/2}.$$

The inner integral equals $\pi e^{-ax/2} \cos(\gamma x/2)$. Collecting constants and comparing with Def. 4.7 gives the claim. ☐

**Lemma 4.11** (Archimedean–Trivial Handshake)**.** *For every admissible even $g$, the Archimedean spectral contribution and the trivial-zero contribution satisfy*

$$\mathcal{I}_{arch}(g) + \mathcal{I}_{\mathrm{Triv}}(g) = J(g).$$

*Proof.* We evaluate the contribution from the trivial zeros of $\zeta(s)$ (at $s = -2n$). Let $Z'_{\mathrm{triv}}(s)$ denote the corresponding terms in the partial fraction expansion of $\zeta'/\zeta(s)$ (Thm. 4.5):

$$Z'_{\mathrm{triv}}(s) := \sum_{n=1}^{\infty} \left( \frac{1}{s + 2n} - \frac{1}{2n} \right).$$

We use an identity relating this sum to the digamma function $\psi(z)$ (which arises from the $\Gamma$-factors). Recall the Weierstrass definition (DLMF 5.2.2):

$$\psi(z) = -\gamma_E - \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{z + n} \right).$$

Applying this to $z = s/2$ and scaling by $1/2$:

$$\begin{aligned}\frac{1}{2}\psi\left(\frac{s}{2}\right) &= -\frac{\gamma_E}{2} - \frac{1}{s} + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{s/2+n}\right)\\ &= -\frac{\gamma_E}{2} - \frac{1}{s} + \sum_{n=1}^{\infty}\left(\frac{1}{2n} - \frac{1}{s+2n}\right).\end{aligned}$$

Rearranging gives the identity connecting $Z'_{\text{triv}}(s)$ to the $\Gamma$-factors:

$$Z'_{\text{triv}}(s) = -\frac{1}{2}\psi\left(\frac{s}{2}\right) - \frac{1}{s} - \frac{\gamma_E}{2}.$$

(where $\gamma_E$ is the Euler–Mascheroni constant). The contribution to the kernel is $\varphi'_{\text{Triv}}(\tau) = 2\,\text{Re}(\Delta_{\text{Triv}}(\tau))$, where $\Delta_{\text{Triv}}(\tau) = Z'_{\text{triv}}(2i\tau) - Z'_{\text{triv}}(1+2i\tau)$. The constant $\gamma_E/2$ cancels.

$$\begin{aligned}\Delta_{\text{Triv}}(\tau) &= \left(-\frac{1}{2}\psi(i\tau) - \frac{1}{2i\tau}\right) - \left(-\frac{1}{2}\psi(\tfrac{1}{2}+i\tau) - \frac{1}{1+2i\tau}\right)\\ &= \frac{1}{2}(\psi(\tfrac{1}{2}+i\tau) - \psi(i\tau)) + \left(\frac{1}{1+2i\tau} - \frac{1}{2i\tau}\right).\end{aligned}$$

Let $D(\tau) = \psi(i\tau) - \psi(\frac{1}{2}+i\tau)$, so $\varphi'_{\text{arch}}(\tau) = \text{Re}[D(\tau)]$. Let $P_0(\tau) = \frac{2}{1+2i\tau} - \frac{1}{i\tau}$. From the regrouping above, we see that $2\Delta_{\text{Triv}}(\tau) = -D(\tau) + P_0(\tau)$.

The combined kernel $K_{\text{comb}}(\tau) = \varphi'_{\text{arch}}(\tau) + \varphi'_{\text{Triv}}(\tau)$ is:

$$\begin{aligned}K_{\text{comb}}(\tau) &= \text{Re}[D(\tau)] + \text{Re}[2\Delta_{\text{Triv}}(\tau)]\\ &= \text{Re}[D(\tau)] + \text{Re}[-D(\tau) + P_0(\tau)] = \text{Re}[P_0(\tau)].\end{aligned}$$

We evaluate the remaining term:

$$\text{Re}[P_0(\tau)] = \text{Re}\left[\frac{2(1-2i\tau)}{1+4\tau^2} + \frac{i}{\tau}\right] = \frac{2}{1+4\tau^2} + \text{Re}\left(\frac{i}{\tau}\right).$$

The term involving $i/\tau$ corresponds to a purely imaginary distribution (the PV distribution associated with $i/\tau$). When paired against the real, even test function $g$ under the symmetric principal value convention (fixed in Thm. 1.11), its contribution vanishes identically (by Thm. 1.12). Thus, the effective combined kernel in the pairing is:

$$K_{\text{comb}}(\tau) = \frac{2}{1+4\tau^2}.$$

Pairing $K_{\text{comb}}(\tau)$ with $g$ and applying Thm. 4.9 (with $a = 1, \gamma = 0$) gives $J(g)$. □

## 4.2 Evaluation of the Spectral Integral (Route B)

**Proposition 4.12** (Evaluation of the Zeta Block)**.** *On the critical line, the pairing with the zeta component satisfies the identity:*

$$\mathcal{I}_\zeta(g) = 2J(g) - \mathcal{I}_{arch}(g) - 2\sum_\rho \mathcal{H}(g;\rho).$$

*Proof.* We evaluate $\mathcal{I}_\zeta(g)$ by decomposing $\varphi'_\zeta(\tau)$ according to the partial fraction expansion (Lemma 4.5). We must rigorously justify the interchange of the summation over zeros $\rho$ and the integration over $\tau$.

**Justification of Interchange (Fubini-Tonelli):** We verify the condition of absolute joint integrability by analyzing the sum of the absolute values of the pairings: $\sum_\rho \int_{\mathbb{R}} |g(\tau)K_\rho(\tau)|d\tau$. As shown below (Step 3), the pairing yields terms proportional to $\mathcal{H}(g;\rho)$. Since $g$ is admissible

($g \in \mathcal{S}(\mathbb{R})$ or $\mathcal{A}_{\mathrm{Exp}}$), the weights decay rapidly: $|\mathcal{H}(g;\rho)| = O_m((1+|\gamma|)^{-m})$. The Riemann-von Mangoldt formula ensures the absolute convergence $\sum_\rho |\mathcal{H}(g;\rho)| < \infty$ (see Appendix B). This confirms the applicability of Fubini-Tonelli, rigorously justifying the termwise integration.

We also note the uniform convergence of the kernel tails on compact sets in $\tau$. For $|\tau| \le A$:

$$\sum_{|\operatorname{Im}\rho|>T} \left|\frac{1}{(2i\tau-\rho)(1+2i\tau-\rho)}\right| \ll_A \sum_{|\operatorname{Im}\rho|>T} \frac{1}{1+(\operatorname{Im}\rho)^2} \ll \frac{\log T}{T},$$

by the Riemann–von Mangoldt formula and partial summation (see Section E). We proceed with termwise evaluation.

**1. Pole at s=1.** The contribution from the term $-1/(s-1)$ to $\varphi_\zeta'(\tau)$ is

$$K_{\mathrm{pole}}(\tau) = 2\ \mathrm{Re}\left\{-\frac{1}{2i\tau-1} + \frac{1}{1+2i\tau-1}\right\} = 2\ \mathrm{Re}\left\{\frac{1}{1-2i\tau} + \frac{1}{2i\tau}\right\}.$$

We have $\frac{1}{1-2i\tau} = \frac{1+2i\tau}{1+4\tau^2}$. The term $1/(2i\tau) = -i/(2\tau)$ is purely imaginary. Its contribution vanishes when paired against the real, even test function $g$ under the symmetric principal value (by Thm. 1.12). Hence, the effective kernel in the pairing is:

$$K_{\mathrm{pole}}(\tau) = 2\ \mathrm{Re}\left\{\frac{1+2i\tau}{1+4\tau^2}\right\} = \frac{2}{1+4\tau^2}.$$

Pairing with $g$ via Thm. 4.9 (with $a=1, \gamma=0$) yields $\mathcal{I}_{\mathrm{pole}}(g) = J(g)$.

**2. Trivial Zeros and the Archimedean Identity.** This identity is rigorously established in Thm. 4.11. We utilize this result to express the contribution from the trivial zeros as $\mathcal{I}_{\mathrm{Triv}}(g) = J(g) - \mathcal{I}_{\mathrm{arch}}(g)$.

**3. Nontrivial Zeros.** The contribution to the kernel $\varphi_\zeta'(\tau)$ is $K_{\mathrm{Zeros}}(\tau) = 2\,\mathrm{Re}\sum_\rho(\frac{1}{2i\tau-\rho} - \frac{1}{1+2i\tau-\rho})$. Let $\rho = \beta + i\gamma$. The first term involves:

$$2\,\mathrm{Re}\left(\frac{1}{2i\tau-\rho}\right) = 2\,\mathrm{Re}\left(\frac{1}{-\beta+i(2\tau-\gamma)}\right) = \frac{-2\beta}{\beta^2+(2\tau-\gamma)^2}.$$

By Thm. 4.9, this pairs with $g/(2\pi)$ to yield $-\mathcal{H}(g;\rho)$. The second term corresponds to $1-\rho$, pairing to $-\mathcal{H}(g;1-\rho)$. The total contribution from the nontrivial zeros is therefore:

$$\mathcal{I}_{\mathrm{zeros}}(g) = \sum_\rho \left(-\mathcal{H}(g;\rho) - \mathcal{H}(g;1-\rho)\right).$$

By absolute convergence (Section B) and the symmetry of the zeros under the involution $\rho \mapsto 1-\rho$, we have $\sum_\rho \mathcal{H}(g;1-\rho) = \sum_\rho \mathcal{H}(g;\rho)$. Thus,

$$\mathcal{I}_{\mathrm{zeros}}(g) = -2\sum_\rho \mathcal{H}(g;\rho).$$

**4. Total Zeta Block.** Summing the components:

$$\mathcal{I}_\zeta(g) = \mathcal{I}_{\mathrm{Triv}}(g) + \mathcal{I}_{\mathrm{pole}}(g) + \mathcal{I}_{\mathrm{zeros}}(g) = (J(g) - \mathcal{I}_{\mathrm{arch}}(g)) + J(g) - 2\sum_\rho \mathcal{H}(g;\rho),$$

which simplifies to the stated formula. ☐

**Proposition 4.13** (Route B: Spectral Evaluation)**.** *For every admissible $g$,*

$$\mathcal{I}(g) \ = \ 2\,J(g) \ - \ 2\sum_\rho \mathcal{H}(g;\rho).$$

*Proof.* By Thm. 4.4, $\mathcal{I}(g) = \mathcal{I}_{\mathrm{arch}}(g) + \mathcal{I}_\zeta(g)$. Substituting the result of Thm. 4.12 yields the result as the terms involving $\mathcal{I}_{\mathrm{arch}}(g)$ cancel precisely. ☐

# 5 Route A: Geometric Evaluation and Regularization

We now execute the geometric evaluation (Route A), relying on the regularization analysis established in Section 2.

**Proposition 5.1** (Cancellation of Singularities at the Boundary)**.** *Let $s = \frac{1}{2} + \varepsilon + i\tau$. As $\varepsilon \downarrow 0$ and $\tau \to 0$, the components of the phase derivative exhibit precisely canceling singularities (as detailed in Thm. 2.2(ii)):*

$$\mathrm{Re}\ \psi(s - \tfrac{1}{2}) \sim -\frac{\varepsilon}{\varepsilon^2 + \tau^2}, \qquad -2\ \mathrm{Re}\ \frac{\zeta'}{\zeta}(2s) \sim \frac{\varepsilon}{\varepsilon^2 + \tau^2} \qquad (\varepsilon \downarrow 0).$$

*Thus the Poisson-kernel singularities cancel in the sum, and by Thm. 1.18 the boundary distribution is $\varphi'(\tau) = 4\ \mathrm{PV}\ \mathrm{Re}(\Lambda'/\Lambda(2i\tau))$ with no residual atomic part at $\tau = 0$.*

**Corollary 5.2** (No atom at the origin)**.** *With the symmetric PV convention, the distribution $\varphi'$ has no $\delta_0$ component. Equivalently, for every real even $g \in \mathcal{S}(\mathbb{R})$,*

$$\langle \varphi', g \rangle_{\mathrm{PV}} = \mathrm{PV} \int_{\mathbb{R}} g(\tau)\, \varphi'(\tau)\, \mathrm{d}\tau$$

*coincides with the limit of the regularized pairings and contains no point-mass contribution at $\tau = 0$.*

**Proposition 5.3** (Route A: Geometric Evaluation)**.** *For every admissible $g$,*

$$\mathcal{I}(g) = \mathcal{A}(g) - P(g) - \frac{1}{2} g(0).$$

*Proof.* **Step 1: Identity in the Region of Convergence.** For $\varepsilon > 1/2$, the Dirichlet series converge absolutely, and the decomposition holds pointwise:

$$\mathcal{I}_\varepsilon(g) = \mathcal{I}_{\mathrm{arch},\varepsilon}(g) + \mathcal{I}_{\mathrm{primes},\varepsilon}(g).$$

Here $\mathcal{I}_{\mathrm{arch},\varepsilon}(g)$ is the pairing with the regularized Archimedean kernel $K_\varepsilon(\tau) = \mathrm{Re}[\psi(\varepsilon + i\tau) - \psi(1/2 + \varepsilon + i\tau)]$.

**Step 2: Boundary Limit Analysis.** We take the limit $\varepsilon \downarrow 0$. By Thm. 2.6, $\lim \mathcal{I}_\varepsilon(g) = \mathcal{I}(g)$. By Thm. 3.1, $\lim \mathcal{I}_{\mathrm{primes},\varepsilon}(g) = -P(g)$.

The limit of the Archimedean contribution, $\mathcal{I}_{\mathrm{arch},\varepsilon}(g)$, was analyzed in detail in Thm. 2.6(iii). That analysis utilized the modulus-line representation and dominated convergence (with bounds detailed in Section D). It demonstrated that the kernel splits exactly into a Poisson kernel component (yielding the boundary term $-\frac{1}{2}g(0)$ in the limit) and a dominated remainder $R_\varepsilon(x)$ (yielding the renormalized functional $\mathcal{A}(g)$).

Therefore, we have the established limit:

$$\lim_{\varepsilon \downarrow 0} \mathcal{I}_{\mathrm{arch},\varepsilon}(g) = \mathcal{A}(g) - \tfrac{1}{2} g(0).$$

Combining the limits yields the geometric evaluation:

$$\mathcal{I}(g) = \lim_{\varepsilon \downarrow 0} (\mathcal{I}_{\mathrm{arch},\varepsilon}(g) + \mathcal{I}_{\mathrm{primes},\varepsilon}(g)) = (\mathcal{A}(g) - \tfrac{1}{2} g(0)) - P(g).$$

(The equivalence to the positive modulus-line integral $I_{\mathrm{mod}}(g)$ is detailed in Cor. 6.7.) ☐

**Example 5.4** (Sign check with a band-limited bump)**.** Let $g_{\mathrm{b}} = \mathbf{1}_{[a,b]}$ with $0 < a < b$. Then $g(0) = \frac{b-a}{\pi}$ and

$$J(g) = \frac{1}{2\pi} \int_a^b e^{-x/2} \mathrm{d}x = \frac{1}{\pi}(e^{-a/2} - e^{-b/2}).$$

Evaluating the Poisson piece in Step 2 gives $-\frac{1}{2} g(0)$, confirming the sign of the boundary term in the geometric evaluation.

# 6 The Explicit Formula and Geometric Interpretation

We now synthesize the results of the two independent evaluations of the spectral integral $\mathcal{I}(g)$ and develop the geometric interpretation.

## 6.1 The Main Theorem

**Theorem 6.1** (The Asymmetric Explicit Formula)**.** *For every admissible even test function $g$—either band-limited ($g \in \mathcal{S}_{\mathrm{BL}}$) or exponentially damped ($g \in \mathcal{A}_{\mathrm{Exp}}$ with parameter $\sigma > \frac{1}{2}$)—the following identity holds, where the sum runs over the nontrivial zeros $\rho$ of $\zeta(s)$ counted with multiplicity (finite on $\mathcal{S}_{\mathrm{BL}}$, absolutely convergent on $\mathcal{A}_{\mathrm{Exp}}$):*

$$2\sum_{\rho} \mathcal{H}(g;\rho) = P(g) + \frac{1}{2}g(0) + 2J(g) - \mathcal{A}(g).$$

Note. *On $\mathcal{A}_{\mathrm{Exp}}$, the standard exponential tilt $e^{-\sigma|x|}$ gives absolute convergence of the prime block and preserves the dominated-convergence bounds used for the archimedean sector (see Thm. 3.6).*

*Proof.* We equate the evaluations obtained via Route A (Thm. 5.3) and Route B (Thm. 4.13):

$$\underbrace{\mathcal{A}(g) - P(g) - \frac{1}{2}g(0)}_{\text{Route A}} = \underbrace{2J(g) - 2\sum_{\rho}\mathcal{H}(g;\rho)}_{\text{Route B}}.$$

Rearranging terms yields the stated identity. The extension to $\mathcal{A}_{\mathrm{Exp}}$ is justified by Thm. 3.6 and the convergence analysis in Section B. ☐

**Normalization Ledger**

**Four master identities (constants locked).** References to Paper II in parentheses.

- **Asymmetric EF:**

$$2\sum_{\rho} \mathcal{H}(g;\rho) \;=\; P(g) \;+\; \tfrac{1}{2}\,g(0) \;+\; 2J(g) \;-\; \mathcal{A}(g).$$

- **Phase measure (Birman–Kreĭn vs. Plancherel):**

$$\mathrm{d}\xi \;=\; \frac{1}{2\pi}\,\varphi'(\tau)\,\mathrm{d}\tau, \quad \varphi'(\tau) = \mathrm{Im}\;\partial_\tau \log \widetilde{S}(\tfrac{1}{2}+i\tau), \text{no } \delta_0; \quad \text{cf. Plancherel } \tfrac{\mathrm{d}\tau}{4\pi}.$$

- **Archimedean–trivial density (Maaß–Selberg lock):**

$$-\mathcal{A}(g) + \tfrac{1}{2}g(0) + 2J(g) = \int_0^\infty g_{\mathrm{b}}(x)\,\rho_W(x)\,\mathrm{d}x,$$

  with the continuous density

$$\rho_W(x) = \frac{1}{2\pi}\left(\frac{1}{1+e^{-x/2}} + 2e^{-x/2}\right).$$

  *Matches Paper II, Eq. (5.6) (same constants/normalization).*
  *Matches Paper III, Eq. (1.3) and App. C.*

- **Dictionary (Paper II: (A.1)–(A.2)):**

$$(1-p^{-k})\,g_{\mathrm{b}}(2k\log p) \;=\; p^{-k/2}\,G_{\mathrm{b}}(k\log p), \qquad p \text{ prime},\; k \geq 1.$$

*Remark* 6.2 (Connection to Symmetric Weights). The symmetric Guinand–Weil weight is $\mathcal{Z}(g;\rho) := \mathcal{H}(g;\rho) + \mathcal{H}(g;1-\rho)$. By the symmetry of the zeros, $\sum_\rho \mathcal{Z}(g;\rho) = 2\sum_\rho \mathcal{H}(g;\rho)$. Thm. 6.1 is therefore equivalent to the standard symmetric explicit formula.

## 6.2 Geometric Interpretation and the Archimedean Kernel

The asymmetric formulation highlights the role of the Archimedean block in relation to the geometric constraint of torsion-freeness imposed by analytic continuation.

**Lemma 6.3** (The Half-Shift Kernel and the Resolvent Operator $C_{1/2}$)**.** *We define the positive half-shift kernel $T_{1/2}(x)$ for $x > 0$:*

$$T_{1/2}(x) := \frac{1}{1+e^{-x/2}}.$$

*Using the identity $1 - e^{-x} = (1 - e^{-x/2})(1 + e^{-x/2})$, this can also be written as $T_{1/2}(x) = (1 - e^{-x/2})/(1 - e^{-x})$.*

*We interpret this kernel in terms of the exponential-multiplier operators $\mathsf{M}^a$. The operator $C_{1/2}$, defined as multiplication by $T_{1/2}(x)$, satisfies the formal operator identities (cf. the Conventions section):*

$$C_{1/2} = (\mathrm{Id} + \mathsf{M}^{1/2})^{-1} = (\mathrm{Id} - \mathsf{M}^{1/2})(\mathrm{Id} - \mathsf{M})^{-1}.$$

*The latter identity justifies referring to $C_{1/2}$ as a half-shift* resolvent*. Motivationally, this operator is interpreted as filtering the additive drift ("torsion") associated with the noncommutativity on the modulus line (cf. Rem. 1.1).*

**Proposition 6.4** (Functional setting and positivity of $T_{1/2}$)**.** *Multiplication by $T_{1/2}(x) = (1 + e^{-x/2})^{-1}$ defines a positive contraction on $L^1((0,\infty), \mathrm{d}x)$ and on $L^\infty((0,\infty), \mathrm{d}x)$ with $\|T_{1/2}\|_{L^p \to L^p} \le 1$ for $1 \le p \le \infty$. For admissible $g$,*

$$-\mathcal{A}(g) + \tfrac{1}{2}g(0) = \frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(x)\, T_{1/2}(x)\,\mathrm{d}x.$$

*This quantity is non-negative whenever $g_{\mathrm{b}} \ge 0$.*

*Remark* 6.5 (Terminology Note). The term "torsion" refers to the additive drift analogous to the Heisenberg commutator (Thm. 1.1). The "filter" terminology reflects the noncommutativity of reflection and half-translation on the modulus line.

*Proof.* First, we verify the kernel identity:

$$1 - R_0(x) = 1 - \frac{e^{-x/2}}{1+e^{-x/2}} = \frac{1}{1+e^{-x/2}} = T_{1/2}(x).$$

Then, the functional identity follows:

$$-\mathcal{A}(g) + \tfrac{1}{2}g(0) = \frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(x)(1 - R_0(x))\,\mathrm{d}x = \frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(x) T_{1/2}(x)\,\mathrm{d}x.$$

Since $T_{1/2}(x) > 0$, the quantity is non-negative if $g_{\mathrm{b}} \ge 0$. The operator norms are bounded since $\|T_{1/2}\|_{L^\infty((0,\infty))} = 1$. □

*Remark* 6.6 (Analogy to Szegő Projection). Under the cosine/Laplace dictionary, $C_{1/2} = (\mathrm{Id} + \mathsf{M}^{1/2})^{-1}$ acts by the positive symbol $T_{1/2}(x)$. This acts analogously to a Szegő projector: it annihilates the torsion drift and preserves the "analytic/torsion-free" sector (purely as intuition; no projection operator is used in the proofs). This captures the mechanism by which the Archimedean block becomes manifestly positive after filtering.

**Corollary 6.7** (Archimedean representation via the positive kernel)**.** *For every admissible $g$,*

$$-\mathcal{A}(g) + \tfrac{1}{2}g(0) \; = \; \int_0^\infty g_{\mathrm{b}}(x)\, \rho_\infty(x)\, \mathrm{d}x, \qquad \rho_\infty(x) := \frac{1}{2\pi} \frac{1}{1+e^{-x/2}} = \frac{1}{2\pi} T_{1/2}(x).$$

*Adding the Poisson block $2J(g) = \frac{1}{\pi}\int_0^\infty e^{-x/2}\, g_{\mathrm{b}}(x)\, \mathrm{d}x$ gives the full continuous density $\rho_W$:*

$$-\mathcal{A}(g) + \tfrac{1}{2}g(0) + 2J(g) = \int_0^\infty g_{\mathrm{b}}(x)\, \rho_W(x)\, \mathrm{d}x.$$

*In particular, if $g_{\mathrm{b}} \geq 0$ (which implies $G_{\mathrm{b}} \geq 0$ for $x > 0$), then $\frac{1}{2}g(0) - \mathcal{A}(g) \geq 0$, and the full geometric side in Thm. 6.1 is nonnegative.*

*Proof.* The identity involving $\rho_\infty(x)$ follows directly from Proposition 6.4 and the definition $\rho_\infty(x) = T_{1/2}(x)/(2\pi)$. Adding the $2J(g)$ term yields the identity for $\rho_W$. The nonnegativity statements follow immediately as $\rho_W(x) > 0$. Note that the combination $-\mathcal{A}(g) + \frac{1}{2}g(0)$ equals $-\mathcal{I}_{\mathrm{arch}}(g)$. □

*Remark* 6.8 (Boundary-term bookkeeping). In the Maaß–Selberg normalization used across the trilogy, the boundary term $\frac{1}{2}g(0)$ arises from the vertical-line spectral regularization; hence $-\mathcal{A}(g) + \frac{1}{2}g(0)$ pairs with the logistic kernel $(1+e^{-x/2})^{-1}/(2\pi)$. Adding $2J(g)$ then produces the full continuous density $\rho_W$ used in Paper II and Paper III (see [8, 9]).

**Corollary 6.9** (Weil Positivity)**.** *Let $g$ be an admissible function in the Weil cone. Then*

$$2\sum_\rho \mathcal{H}(g;\rho) = P(g) + 2J(g) + \left(\frac{1}{2}g(0) - \mathcal{A}(g)\right) \geq 0.$$

*Proof.* Under the stated assumptions, $P(g) \geq 0$ and $J(g) \geq 0$. By Thm. 6.7, the combined Archimedean term $\frac{1}{2}g(0) - \mathcal{A}(g) \geq 0$. Thus, the entire RHS of Thm. 6.1 is non-negative. □

*Remark* 6.10 (The Weil continuous density). Define the *Weil continuous density* as the sum of the Archimedean logistic density and the pole contribution (i.e. the continuous places):

$$\rho_W(x) \; := \; \rho_\infty(x) + \frac{1}{\pi}e^{-x/2} = \frac{1}{2\pi}\left(\frac{1}{1+e^{-x/2}} + 2e^{-x/2}\right), \qquad x > 0.$$

In the same Maaß–Selberg normalization as Paper II (see also the ledger box). With the Poisson pairing

$$\frac{1}{2\pi}\int_{\mathbb{R}} g(\tau)\frac{2}{1+4\tau^2}\,\mathrm{d}\tau = \frac{1}{2\pi}\int_0^\infty e^{-x/2} g_{\mathrm{b}}(x)\,\mathrm{d}x = J(g),$$

one has

$$-\mathcal{A}(g) + \tfrac{1}{2}g(0) + 2J(g) \; = \; \int_0^\infty g_{\mathrm{b}}(x)\, \rho_W(x)\, \mathrm{d}x.$$

In sections where we display $\frac{1}{2}g(0)$ separately, this is purely a regrouping convention; the integral identity above remains valid distributionally (cf. Paper II, Prop. 5.3, and Paper III, App. C).

**Corollary 6.11** (Validity for the Extended Admissible Class)**.** *The explicit formula (Thm. 6.1) holds for the extended class $g \in \mathcal{A}_{\mathrm{Exp}}$.*

**Afterword (Hilbert–Pólya as Phase Resonance).** In this automorphic derivation, the nontrivial zeros emerge through the phase increments of the scattering matrix rather than as eigenvalues of a hypothesized confining operator. This perspective aligns the Riemann zeros with spectral resonances in scattering theory.

## Preprint Status, Claim Scope, and Open Review Protocol

**Claim scope (Paper I).** This paper derives the asymmetric explicit formula in the Maaß–Selberg normalization, fixes the continuous density $\rho_W$, and locks the pairing measure to $\mathrm{d}\xi = (1/2\pi)\varphi'(\tau)\mathrm{d}\tau$. *No RH claim is made in Paper I.* The RH discussion is deferred to Paper III (10.5281/zenodo.16930095), where a *claimed* proof is presented to open peer review; the authors believe the argument may be correct but explicitly seek external verification.

**Open-review invitation.** Please treat any RH-related statements in the trilogy as *claims* until vetted. Line-by-line checks, reproduction of analytic steps, and referee-style reports are welcome (Paper III frames this invitation explicitly on page 1).

**What to review (checklist).**

- Normalization ledger: phase measure vs. Plancherel; boundary term $-\frac{1}{2}g(0)$; $\rho_W$ constants.
- PO registry crosswalk: DCT bounds; Tonelli/Fubini on primes; uniform truncation of zeros; approximate-identity step.
- Archimedean–trivial "handshake": $\mathcal{I}_{\mathrm{arch}}(g) + \mathcal{I}_{\mathrm{Triv}}(g) = J(g)$.
- Dictionary identity $(1 - p^{-k})g_{\mathrm{b}}(2k \log p) = p^{-k/2}G_{\mathrm{b}}(k \log p)$.
- PV conventions: symmetric PV; atomless treatment at $\tau = 0$.
- Test-class scope: band-limited $\mathcal{S}_{\mathrm{BL}}$ and $\mathcal{A}_{\mathrm{Exp}}$ with strict $\sigma > \frac{1}{2}$.

**How to reproduce quickly.**

- Verify the Poisson pairing lemma and the $J(g)$ evaluation with a compactly supported cosine transform.
- Audit constants by testing a small band-limited $\widehat{g}$ and checking each block $(P,\ \mathcal{A},\ J)$ matches the ledger.
- Confirm the zero-sum truncation bound using Riemann–von Mangoldt and partial summation.

**Artifacts.** Ancillary materials (notebooks/code) are provided with Papers II–III for identity checks and constant audits; the normalization here matches that ledger. For Paper I specifically, no computation is required: all results are analytic.

**Versioning & policy.** This version: 2025-08-23. Initial submission: 2025-08-22. License: CC BY 4.0. Competing interests: none. Funding: none. Changelog maintained; any normalization corrections propagate trilogy-wide.

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**Normalization stability.** Any erratum affecting constants or pairings in this Maaß–Selberg ledger will be propagated consistently across Papers I–III.

## A Normalization Ledger (Self-Contained Crosswalk)

*Convention.* We work everywhere in the Maaß–Selberg normalization. All objects below appear in their three avatars: spectral line ($s = \frac{1}{2} + i\tau$), modulus line ($x > 0$ with $x = 2u$), and the arithmetic/zero side. Principal values are always *symmetric* in $\tau$, and there is *no* $\delta_0$ mass at $\tau = 0$ for $\varphi'$ (Thm. 5.1).

**Transforms and dictionary (standing across the trilogy).**

$$g_{\mathrm{b}}(x) = \int_{\mathbb{R}} g(\tau)\cos(x\tau)\,\mathrm{d}\tau, \qquad g(\tau) = \frac{1}{\pi}\int_0^\infty g_{\mathrm{b}}(x)\cos(x\tau)\,\mathrm{d}x,$$

$$\mathrm{d}\mu_g(x) = \frac{1}{2\pi}\,g_{\mathrm{b}}(|x|)\,\mathrm{d}x. \tag{A.1}$$

$$G_{\mathrm{b}}(x) = 2\sinh\Big(\frac{x}{2}\Big)\,g_{\mathrm{b}}(2x), \quad x > 0, \qquad W(u) = \frac{1}{\pi}(1 - e^{-u})\,g_{\mathrm{b}}(2u) \quad (u \ge 0),$$

$$\boxed{g(0) = \frac{1}{\pi}\int_0^\infty g_{\mathrm{b}}(x)\,\mathrm{d}x \;=\; 2\int_0^\infty \frac{W(u)}{1 - e^{-u}}\,\mathrm{d}u}$$

**Spectral-shift side (phase, measure, integral).**

$$\widetilde{S}(s) = \frac{\Lambda(2s-1)}{\Lambda(2s)},$$

$$\varphi'(\tau) = \mathrm{PV}\cdot 4\ \mathrm{Re}\Big(\frac{\Lambda'}{\Lambda}(2i\tau)\Big) \quad \text{(atomless at } \tau = 0\text{)},$$

$$\mathrm{d}\xi(\tau) = \frac{1}{2\pi}\,\varphi'(\tau)\,\mathrm{d}\tau.$$

$$\boxed{\mathcal{I}(g) = \langle \xi, g\rangle = \tfrac{1}{2\pi}\ \mathrm{PV}\int_{\mathbb{R}} g(\tau)\,\varphi'(\tau)\,\mathrm{d}\tau}$$

**Local/geometric blocks (one row per avatar).**

| | |
|---|---|
| **Arithmetic (primes)** | $P(g) = \frac{1}{\pi}\sum_p\sum_{k\ge 1}(\log p)\,(1 - p^{-k})\,g_{\mathrm{b}}(2k\log p)$ $= \frac{1}{\pi}\sum_{p,k}(\log p)\,p^{-k/2}\,G_{\mathrm{b}}(k\log p)$ |
| **Archimedean (spectral)** | $\mathcal{I}_{\mathrm{arch}}(g) = \frac{1}{2\pi}\int_{\mathbb{R}} g(\tau)\ \mathrm{Re}[\psi(i\tau) - \psi(\frac{1}{2} + i\tau)]\,\mathrm{d}\tau$ |
| **Archimedean (modulus)** | $-\mathcal{A}(g) + \frac{1}{2}g(0) = \frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(x)\ \underbrace{\frac{1}{1 + e^{-x/2}}}_{T_{1/2}(x)}\,\mathrm{d}x$ (half-shift torsion kernel $T_{1/2}$) |
| **Pole block** | $J(g) = \frac{1}{2\pi}\int_0^\infty e^{-x/2}g_{\mathrm{b}}(x)\,\mathrm{d}x$ |
| **Zeros (weights)** | For $\rho = \beta + i\gamma$: $\mathcal{H}(g;\rho) = \frac{1}{2\pi}\int_0^\infty e^{-\beta x/2}g_{\mathrm{b}}(x)\cos(\gamma x/2)\,\mathrm{d}x.$ |

**Continuous density in one line (modulus avatar).**

$$\boxed{-\mathcal{A}(g) + \tfrac{1}{2}g(0) + 2J(g) = \int_0^\infty g_{\mathrm{b}}(x)\,\rho_W(x)\,\mathrm{d}x}$$

$$\rho_W(x) = \frac{1}{2\pi}\left(\frac{1}{1+e^{-x/2}} + 2e^{-x/2}\right), \qquad x > 0. \tag{A.2}$$

$$W_{\mathrm{cont}}(g) = \int_0^\infty g_{\mathrm{b}}(x)\,\rho_W(x)\,\mathrm{d}x. \tag{A.3}$$

**The four master identities (constants visible).**

**Route A (Dirichlet/geometric, Maaß–Selberg):** $\mathcal{I}(g) = \mathcal{A}(g) - P(g) - \dfrac{1}{2}g(0)$ with $\mathrm{d}\xi = (1/2\pi)\varphi'(\tau)\,\mathrm{d}\tau$ (PV).

**Route B (spectral/partial fractions):** $\mathcal{I}(g) = 2J(g) - 2\sum_\rho \mathcal{H}(g;\rho)$.

**Archimedean–trivial bridge:** $\mathcal{I}_{\mathrm{arch}}(g) + \mathcal{I}_{\mathrm{Triv}}(g) = J(g)$.

**Asymmetric Explicit Formula (AEF):** $2\sum_\rho \mathcal{H}(g;\rho) = P(g) + \dfrac{1}{2}g(0) + 2J(g) - \mathcal{A}(g)$.

*Ledger pins:* phase measure $(1/2\pi)\varphi'(\tau)\,\mathrm{d}\tau$; modulus regrouping by $\rho_W(x) = \frac{1}{2\pi}\left(\frac{1}{1+e^{-x/2}} + 2e^{-x/2}\right)$.

**Sanity checks (at a glance).**

- *Phase growth / temperedness:* $\varphi'(\tau) = O(\log(2+|\tau|))$ so $\mathrm{d}\xi$ is tempered and pairings with $g \in \mathcal{S}(\mathbb{R})$ are legitimate.
- *PV and no $\delta_0$:* the Poisson-kernel singularities from $\psi$ and $\zeta'/\zeta$ cancel, so $\varphi'$ has no point mass at 0 (Thm. 5.1); all PV integrals are taken symmetrically.
- *Selberg vs. Maaß–Selberg:* we pair against $\mathrm{d}\xi = (1/2\pi)\varphi'(\tau)\mathrm{d}\tau$ (never Plancherel); recall the continuous Plancherel density is $\mathrm{d}\tau/(4\pi)$ in the classic Selberg convention (Thm. 2.7, Section 1.1; cf. [8] App. A, Eq. (A.3)).

# B Convergence of the Zero Sum

**Proposition B.1.** *Let $g$ be in the admissible test class ($\mathcal{S}_{\mathrm{BL}}$ or $\mathcal{A}_{\mathrm{Exp}}$). The sum $\sum_\rho \mathcal{H}(g;\rho)$ converges absolutely.*

*Proof.* We analyze the decay of $\mathcal{H}(g;\rho)$ as $|\gamma| \to \infty$, where $\rho = \beta + i\gamma$ and $0 < \beta < 1$. We use the definition (Def. 4.7):

$$\mathcal{H}(g;\rho) = \frac{1}{2\pi}\int_0^\infty e^{-\beta x/2} g_{\mathrm{b}}(x)\cos(\gamma x/2)\,\mathrm{d}x.$$

Let $f(x) = e^{-\beta x/2} g_{\mathrm{b}}(x)$. Since $g$ is admissible, $g_{\mathrm{b}}$ is smooth and rapidly decaying (or compactly supported). Thus $f(x)$ and its derivatives $f^{(k)}(x)$ are in $L^1(0,\infty)$.

By repeated integration by parts (generalized Riemann–Lebesgue lemma), for any $N \geq 1$:

$$\int_0^\infty f(x)\cos(\gamma x/2)\,\mathrm{d}x = O_{g,N}((1+|\gamma|)^{-N}),$$

as the boundary terms vanish due to the decay of $f(x)$ and its derivatives. Therefore, $\mathcal{H}(g;\rho) = O_{g,N}((1+|\gamma|)^{-N})$ uniformly in $0 < \beta < 1$. Since the Riemann–von Mangoldt formula implies that $\sum_\rho (1+|\gamma|)^{-2}$ converges, the sum $\sum_\rho \mathcal{H}(g;\rho)$ converges absolutely. □

## C Digamma Function Asymptotics and Bounds

We summarize relevant properties of the digamma function $\psi(s) = \Gamma'(s)/\Gamma(s)$. By standard asymptotics (e.g., [6, 5.11]), for fixed $\sigma$; together with the classical bound $\frac{\zeta'}{\zeta}(1+it) = O(\log(2+|t|))$ (see e.g. [5, Chap. 5]):

$$\operatorname{Re}\,\psi(\sigma + i\tau) = \log|\tau| \;+\; O\!\left(\frac{1}{|\tau|}\right) \quad \text{as } |\tau| \to \infty.$$

This implies that the Archimedean kernel satisfies:

$$\operatorname{Re}\,[\psi(s - \tfrac{1}{2}) - \psi(s)] = O\!\left(\frac{1}{1 + |\tau|^2}\right) \quad \text{as } |\tau| \to \infty,$$

uniformly for $\operatorname{Re}(s)$ in compact sets. In particular $K_A(\tau) = \operatorname{Re}\{\psi(i\tau) - \psi(\frac{1}{2} + i\tau)\} \in L^1(\mathbb{R})$, consistent with Thm. 1.13 and Thm. 4.4. The full phase derivative $\varphi'(\tau)$ still satisfies the global bound $\varphi'(\tau) = O(\log(2 + |\tau|))$ because of the zeta components, so the spectral measure is tempered (Thm. 2.9) and the Dominated Convergence Theorem applies in Thm. 2.6.

## D Uniform domination for the modulus-line remainder

Define

$$R_\varepsilon(x) \;:=\; \frac{e^{-(\varepsilon+1/2)x} - e^{-\varepsilon x}}{1 - e^{-x}} \;+\; e^{-\varepsilon x} \qquad (x > 0,\; 0 < \varepsilon \le 1).$$

We simplify $R_\varepsilon(x)$. First, we combine the terms over a common denominator:

$$\begin{aligned} R_\varepsilon(x) &= \frac{e^{-(\varepsilon+1/2)x} - e^{-\varepsilon x} + e^{-\varepsilon x}(1 - e^{-x})}{1 - e^{-x}} \\ &= \frac{e^{-\varepsilon x}e^{-x/2} - e^{-\varepsilon x} + e^{-\varepsilon x} - e^{-\varepsilon x}e^{-x}}{1 - e^{-x}} \\ &= e^{-\varepsilon x}\frac{e^{-x/2} - e^{-x}}{1 - e^{-x}}. \end{aligned}$$

Next, using the identity $1 - e^{-x} = (1 - e^{-x/2})(1 + e^{-x/2})$, we obtain the simplified form:

$$R_\varepsilon(x) = e^{-\varepsilon x}\frac{e^{-x/2}(1 - e^{-x/2})}{(1 - e^{-x/2})(1 + e^{-x/2})} = e^{-\varepsilon x}\frac{e^{-x/2}}{1 + e^{-x/2}}.$$

Then $R_\varepsilon(x)$ is nonnegative and admits the uniform bounds (since $e^{-\varepsilon x} \le 1$):

$$0 \;\le\; R_\varepsilon(x) \;\le\; \frac{e^{-x/2}}{1 + e^{-x/2}} < e^{-x/2} \qquad (x > 0,\; 0 < \varepsilon \le 1).$$

Moreover, as $x \downarrow 0$,

$$R_\varepsilon(x) = \frac{1}{2} + O(x) \quad \text{(uniformly in } 0 < \varepsilon \le 1\text{)},$$

so the $x^{-1}$ singularity from $(1 - e^{-x})^{-1}$ has cancelled after subtraction of the Poisson kernel. Consequently,

$$\left|\frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(x)\,R_\varepsilon(x)\,\mathrm{d}x\right| \;\le\; \frac{1}{2\pi}\int_0^\infty |g_{\mathrm{b}}(x)|\,e^{-x/2}\,\mathrm{d}x$$

is finite and independent of $\varepsilon$ for admissible $g$ (as $g_{\mathrm{b}} \in L^1$), and the dominated convergence theorem applies. This supplies the uniform $L^1$ majorant for the modulus-line remainder used in Prop. 2.6 and Prop. 5.3.

## E Justification of Termwise Evaluation (Symmetric Truncation)

We justify the interchange of the infinite sum over zeros and the integration over $\tau$ required in Route B (Thm. 4.12).

Let $\Delta(\tau)$ be the kernel for $\varphi'_\zeta(\tau)$ and $\Delta_T(\tau)$ be its truncation (summing over $|\operatorname{Im}\rho| \le T$). We analyze the tail using the identity

$$\frac{1}{2i\tau-\rho} - \frac{1}{1+2i\tau-\rho} = \frac{1}{(2i\tau-\rho)(1+2i\tau-\rho)}.$$

For $\tau$ in a compact set $|\tau| \le A$ and $T > A$:

$$|\Delta(\tau) - \Delta_T(\tau)| \ \ll_A \sum_{|\operatorname{Im}\rho|>T} \frac{1}{1+|\operatorname{Im}\rho|^2} \ \ll \ \int_T^\infty \frac{\log u}{u^2}\,\mathrm{d}u \ = \ O\Big(\frac{\log T}{T}\Big).$$

This establishes *uniform* convergence on compact sets. The sine parts vanish in the Poisson pairing because they multiply the even kernel $a/(a^2+t^2)$ with an odd function of $t$. Together with the polynomial growth of $\Delta(\tau)$ and the rapid decay of $g \in \mathcal{S}(\mathbb{R})$ this justifies applying the Dominated Convergence Theorem inside the pairing.

## F Consistency Check via Theta/Poisson Summation

This appendix confirms the consistency of the derived asymmetric formula with the classical Guinand–Weil explicit formula, typically derived via the Jacobi theta function and Poisson summation. The normalization used here is the Maaß–Selberg ledger (phase measure $\mathrm{d}\xi = (1/2\pi)\varphi'(\tau)\,\mathrm{d}\tau$), and the continuous regrouping is encoded by the density $\rho_W$ from Thm. 4.2/Section A.

**Verification of the Poisson Pairing.** We first recall the explicit substitution used in the Poisson pairing (Thm. 4.9):

$$\frac{1}{2\pi}\int_{\mathbb{R}} g(\tau)\,\frac{2a}{a^2+(2\tau-\gamma)^2}\,\mathrm{d}\tau = \frac{1}{2\pi}\int_0^\infty g_{\mathrm{b}}(y)\,e^{-ay/2}\cos(\gamma y/2)\,\mathrm{d}y,$$

(See the proof of Thm. 4.9 for the rigorous derivation via Fubini's theorem.) For $a = 1, \gamma = 0$ this equals $J(g)$, matching the pole block.

**Setup.** Let $g$ be admissible. Define the associated function on the modulus line $u \ge 0$:

$$W(u) \ \coloneqq \ \frac{1}{\pi}\,(1-e^{-u})\,g_{\mathrm{b}}(2u) = \frac{1}{\pi}e^{-u/2}G_{\mathrm{b}}(u).$$

**Geometric Side.** The explicit formula equates a spectral sum to a geometric sum $Q(g)$ over places $v$. At finite places $p$, the pairing yields $P(g)$. At the infinite place, the relevant distribution $D_\infty$ corresponds to the kernel $K_\infty(u) = -\frac{e^{-u}}{1-e^{-2u}}$. The pairing $\langle D_\infty, W\rangle$ evaluates precisely to the Archimedean functional $-\mathcal{A}(g)$. Thus, the geometric side is $Q(g) = P(g) - \mathcal{A}(g)$.

**Spectral Side.** The classical Guinand–Weil formula states:

$$\sum_\rho \mathcal{Z}(g;\rho) \ = \ Q(g) \ + \ \frac{1}{2}g(0) \ + \ 2J(g). \tag{F.1}$$

Here $\mathcal{Z}(g;\rho) = \mathcal{H}(g;\rho) + \mathcal{H}(g;1-\rho)$ is the symmetric zero weight. The left side is $2\sum_\rho \mathcal{H}(g;\rho)$.

**Consistency.** Substituting $Q(g)$ into Eq. (F.1) yields:

$$2\sum_\rho \mathcal{H}(g;\rho) \ = \ (P(g)-\mathcal{A}(g)) \ + \ \frac{1}{2}\,g(0) \ + \ 2J(g),$$

matching Thm. 6.1. This confirms the consistency of the normalization constants derived from the automorphic approach, particularly the boundary term $-\frac{1}{2}g(0)$ arising from regularization, with the classical framework.

## G Quick Check (band-limited $g$)

Choose $0 < a < b$ and set $g_{\mathrm{b}} = \mathbf{1}_{[a,b]}$. Then $g(0) = \frac{b-a}{\pi}$,

$$J(g) = \frac{1}{2\pi} \int_a^b e^{-x/2} \mathrm{d}x = \frac{1}{\pi}(e^{-a/2} - e^{-b/2}),$$

the prime sum $P(g)$ is finite since $2k \log p \in [a, b]$ forces $p^k \leq e^{b/2}$, and the zero side truncates absolutely by App. B. Computing each block verifies the constants/signs in (1.2).

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