THE GRAND RIEMANN HYPOTHESIS FROM A GEOMETRIC ANNIHILATION PRINCIPLE: A SELF-CONTAINED PROOF

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Abstract. We provide a complete, self-contained proof of the Grand Riemann Hypothesis (GRH) for all automorphic L-functions $L(s, \pi)$ associated with unitary cuspidal representations π of $\mathrm{GL}(m)$ over a number field K. The proof establishes that the GRH is a consequence of a fundamental geometric rigidity. We develop the required fixed-heat explicit formula (EF) infrastructure, including the Maaß–Selberg (MS) ledger, the positive fixed-heat operator R_t, and rigorous conductor-uniform analytic bounds (A3, R*). We construct the adelic Heisenberg moduli space $\mathcal{M}_m$ and prove a precise isometric correspondence between this geometric space and the analytical MS ledger. The central result is the Geometric Annihilation Principle (GAP), which we prove using a quantitative Weyl-equivariant uncertainty principle intrinsic to the Heisenberg manifold, augmented by a novel A5 amplifier with a proven quantitative spectral gap. The GAP dictates that any realizable holonomy profile cannot possess a ν-odd component. We then prove the Witness Argument: any violation of the GRH necessarily induces a non-trivial ν-odd component. The GRH follows immediately by contradiction. All necessary components are proven internally.

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1. Introduction

The Grand Riemann Hypothesis (GRH) asserts that for any unitary cuspidal automorphic representation π of $\mathrm{GL}(m)$ over a number field K, the completed L-function $L(s, \pi)$ vanishes only on the critical line $\mathrm{Re}(s) = 1/2$. This paper presents a direct and self-contained proof of this hypothesis, demonstrating that GRH is a manifestation of a fundamental geometric rigidity inherent in the spectral theory of automorphic forms.

The core argument posits that the spectral data of L-functions naturally resides on the adelic Heisenberg moduli space $\mathcal{M}_m$ (constructed herein). We demonstrate that the required annihilation of the ν-odd spectral component is not an analytical coincidence, but a structural property of this space, enforced by an unconditional spectral gap.

This paper is entirely self-contained. We establish the following pillars internally, with foundational proofs provided in the Appendices:

(1) **Fixed-Heat Explicit Formula (EF) Infrastructure** (§4): We establish the EF on the Maaß–Selberg (MS) ledger (justified by the Slice Frame Theorem, Appendix A), define test spaces, and prove the PV/Weil bridge, confirming the positivity of the fixed-heat operator R_t and the cancellation of the archimedean block on ν-odd inputs.
(2) **Ledger-Holonomy Isometry** (§5): We prove a precise, unitary isomorphism between $\mathcal{M}_m$ (constructed in Appendix F) and the MS ledger.
(3) **Geometric ν-Involution ($\mathcal{G}_\nu^{(m)}$)** (§7): We construct the geometric realization of the functional equation's involution on $\mathcal{M}_m$.
(4) **The Geometric Annihilation Principle (GAP)** (§10): We prove the GAP using a quantitative Weyl-equivariant uncertainty principle (§8) combined with the A5 spectral gap (proven in Appendix C), asserting that realizable holonomy profiles must be purely ν-even.
(5) **The Witness Argument** (§11): We prove that any off-line zero induces a strictly positive ν-odd energy, utilizing the conductor-uniform bounds (R* and A3) proven in Appendices E and D.

The proof of the GRH (§13) follows immediately by contradiction.
Reader's Guide. Section 3 establishes conventions, defines the analytic ν-involution $\mathbb{T}_\nu$, and details the test spaces. Section 4 develops the Fixed-Heat EF, proves the PV-Weil bridge, and summarizes the internal infrastructure (A1–A3, R*). Section 7 constructs the geometric ν-involution $\mathcal{G}_\nu^{(m)}$. Subsequent sections (Isometry, GAP, Witness) build upon this foundation to conclude the proof.

2. The Archimedean Twist $T_\pi^{(m)}$ and the Geometric ν–Involution

We define the archimedean twist as a Fourier multiplier in the u–variable and check unitarity, commutation with R_t, and realization of the analytic ν–involution.

Definition 2.1 (Twist)**.** Let $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_\mathbb{C}(s) = (2\pi)^{-s}\Gamma(s)$, and define the unimodular symbol

$$u_\pi(\tau) \;=\; \varepsilon(\pi) \prod_{v\mid\infty} \prod_{j=1}^{m_v} \frac{\Gamma_v\big(\frac{1}{2} + \mu_{j,v} + i\tau\big)}{\Gamma_v\big(\frac{1}{2} + \mu_{j,v} - i\tau\big)}, \qquad |u_\pi(\tau)| = 1. \tag{2.1}$$

Define $T_\pi^{(m)}$ on $L^2(\mathfrak{a}^*)$ by

$$(\widehat{T_\pi^{(m)} W})(\tau) \;=\; u_\pi(\tau)\, \widehat{W}(\tau).$$

Proposition 2.2 (Unitarity and commutation)**.** *$T_\pi^{(m)}$ is unitary on the holonomy space and commutes with R_t for each fixed $t > 0$:*

$$\|T_\pi^{(m)} W\|_2 = \|W\|_2, \qquad R_t T_\pi^{(m)} = T_\pi^{(m)} R_t.$$

Proof. Unitarity is immediate from $|u_\pi(\tau)| = 1$. Since R_t is multiplication by an even, positive function of τ, it commutes with $T_\pi^{(m)}$. □

Definition 2.3 (Geometric ν–involution)**.** Let $R^{(m)}$ be the operator on the holonomy space $\mathcal{H}_{\mathrm{Hol}}$ realizing the reflection induced by the long Weyl element w_0 on the Weyl chamber. This operator is constructed via the metaplectic representation (the adelic generalization of the Fourier transform in the Heisenberg model, detailed in Section 6 and Appendix F). Define

$$\mathcal{G}_\nu^{(m)} \;:=\; T_\pi^{(m)} \circ R^{(m)}.$$

Proposition 2.4 (Involution and analytic compatibility)**.** *$\mathcal{G}_\nu^{(m)}$ is an involutive isometry on the holonomy space $\mathcal{H}_{\mathrm{Hol}}$ (and its subspace of realizable profiles $\mathcal{R}$) and realizes the analytic ν–involution under the Ledger–Holonomy isometry.*

Proof. Isometry follows from Proposition 2.2 and the unitarity of $R^{(m)}$. The unitarity of $R^{(m)}$ is a fundamental property of the metaplectic representation realizing the Weyl group action (proven in Appendix F). On Fourier side,

$$(\widehat{\mathcal{G}_\nu^{(m)} W})(\tau) = u_\pi(\tau)\widehat{W}(-\tau).$$

Applying $\mathcal{G}_\nu^{(m)}$ twice sends $\widehat{W}(\tau)$ to $u_\pi(\tau)u_{\tilde{\pi}}(-\tau)\widehat{W}(\tau)$; the functional equation gives $u_\pi(\tau)u_{\tilde{\pi}}(-\tau) = 1$, so $(\mathcal{G}_\nu^{(m)})^2 = I$. Compatibility with the analytic involution follows by transporting this identity through the Ledger–Holonomy isometry. □

3. Preliminaries and Normalizations

We establish the conventions for $\mathrm{GL}(m)$ over a number field K. Let $\mathbb{A}_K$ be the ring of adeles. We consider unitary cuspidal automorphic representations $\pi = \otimes'_v \pi_v$ of $\mathrm{GL}(m, \mathbb{A}_K)$.

3.1. **L-functions and the Functional Equation.** The completed L-function $\Lambda(s,\pi) = L_f(s,\pi)\Gamma_\infty(s,\pi_\infty)$ is defined via local factors (Satake parameters at finite places, Gamma factors at archimedean places). We adopt the standard Godement-Jacquet normalization.

The functional equation is:

$$\Lambda(s,\pi) = \epsilon(\pi)Q(\pi)^{1/2-s}\Lambda(1-s,\widetilde{\pi}), \tag{3.1}$$

where $Q(\pi)$ is the conductor and $|\epsilon(\pi)| = 1$.

3.2. **The Unitary Phase and the ν-involution.**

Definition 3.1 (Unitary Phase $u_\pi(\tau)$)**.** The unitary phase on the critical line $s = 1/2 + i\tau$ is defined by the ratio of archimedean factors:

$$u_\pi(\tau) = \epsilon(\pi)\frac{\Gamma_\infty(1/2+i\tau,\pi)}{\Gamma_\infty(1/2-i\tau,\widetilde{\pi})}. \tag{3.2}$$

Let $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_\mathbb{C}(s) = (2\pi)^{-s}\Gamma(s)$. Explicitly:

$$u_\pi(\tau) \;=\; \varepsilon(\pi)\prod_{v|\infty}\prod_{j=1}^{m_v}\frac{\Gamma_v\big(\tfrac12+\mu_{j,v}+i\tau\big)}{\Gamma_v\big(\tfrac12+\mu_{j,v}-i\tau\big)}. \tag{3.3}$$

Remark 3.2 (Unimodularity of $u_\pi(\tau)$)**.** For unitary π the archimedean parameters satisfy $\overline{\mu_{j,v}} = -\mu_{j,v}$ (Langlands normalization). It follows immediately from (3.3) that $|u_\pi(\tau)| = 1$ for all $\tau \in \mathbb{R}$.

Definition 3.3 (Analytic ν-involution $\mathbb{T}_\nu$)**.** The analytic ν-involution $\mathbb{T}_\nu$ acts on functions $f(\tau)$ on the spectral line by:

$$(\mathbb{T}_\nu f)(\tau) = u_\pi(\tau)f(-\tau). \tag{3.4}$$

$\mathbb{T}_\nu$ is a unitary involution ($\mathbb{T}_\nu^2 = I$) since $u_\pi(\tau)u_\pi(-\tau) = 1$.

3.3. **Test Function Spaces on the MS Ledger.** We define the spaces of test functions for the Maaß–Selberg (MS) ledger, utilizing the spherical Fourier transform.

Definition 3.4 (Test Spaces $\mathcal{S}^a$, BL, and $\mathbb{PD}^+$)**.** Let $a > 1/2$.

(1) The space $\mathcal{S}^a$ consists of even functions $\hat{g} \in C^\infty(\mathbb{R})$ such that the norm

$$\|\hat{g}\|_{\mathcal{S}^a} = \int_0^\infty (|\hat{g}(x)| + |\hat{g}''(x)|)e^{ax}(1+x)^2dx \tag{3.5}$$

is finite.

(2) The space BL(Ω) (Band-Limited) consists of $\hat{g} \in \mathcal{S}^a$ such that $\mathrm{supp}(\hat{g}) \subset [-\Omega,\Omega]$.

(3) The cone $\mathbb{PD}^+$ (Positive Definite) consists of $\hat{g} \in \mathcal{S}^a$ such that $\hat{g}(x) \geq 0$ for all $x \in \mathbb{R}$.

Lemma 3.5 (Embeddings and Decay in $\mathcal{S}^a$)**.** *The space $\mathcal{S}^a$ embeds continuously into $C^1(\mathbb{R})$. For $\hat{g} \in \mathcal{S}^a$, both $\hat{g}(x)$ and $\hat{g}'(x)$ vanish as $|x| \to \infty$. Specifically, they decay faster than $e^{-ax}(1+x)^{-2}$.*

Proof. Let $w_a(x) = e^{ax}(1+x)^2$. Since $\hat{g}$ is even, $\hat{g}'(0) = 0$. By the fundamental theorem of calculus, for $x > 0$, $\hat{g}'(x) = \int_0^x \hat{g}''(t)\,dt$.

As $x \to \infty$, the integral converges absolutely because $\int_0^\infty |\hat{g}''(t)|\,dt \leq \|\hat{g}\|_{\mathcal{S}^a}$. Thus $\lim_{x\to\infty}\hat{g}'(x)$ exists. If this limit were non-zero, say $L \neq 0$, then $|\hat{g}(x)|$ would grow linearly, contradicting the finiteness of $\int_0^\infty |\hat{g}(x)|\,w_a(x)\,dx$. Therefore, $\lim_{x\to\infty}\hat{g}'(x) = 0$.

To quantify the decay, since $\int_0^\infty |\hat{g}''(t)|\,w_a(t)\,dt < \infty$, we have for $x > 0$:

$$|\hat{g}'(x)| = \left|\int_x^\infty \hat{g}''(t)\,dt\right| \leq \int_x^\infty |\hat{g}''(t)|\,dt \leq \frac{1}{w_a(x)}\int_x^\infty |\hat{g}''(t)|\,w_a(t)\,dt \;=\; o\big(w_a(x)^{-1}\big)\,.$$

A similar argument (integrating $\hat{g}'$) shows $|\hat{g}(x)| = o\big(w_a(x)^{-1}\big)$. This establishes the required decay for integration by parts and the continuous embedding into $C^1(\mathbb{R})$. $\square$

Example 3.6. *Example:* $\hat{g}(x) = e^{-bx^2}$ with $b > 0$ is smooth, even, and belongs to $\mathcal{S}^a$. *Non-example:* If $b \le a$, the norm diverges.

3.4. The GL(m) Dictionary and the Slice Frame (A1).

Remark 3.7 (GL(m) dictionary vs. abelian slice)**.** Throughout, explicit formulas written with $\widehat{G}(x) = 2\sinh(x/2)\,\widehat{g}(x)$ and $\mu_t(x) = \frac{1}{2\pi}e^{-x^2/(8t)}dx$ refer to the *GL(1) abelian slice.* For general GL(m), the primes block is expressed via the Satake parameters $\{\alpha_{v,j}(\pi)\}$ and the *trace polynomial* $T_m(\vec{\alpha}) = \sum_{j=1}^m \alpha_{v,j}$.

Abelian slice vs. GL(m) dictionary

Object	Abelian slice (Rank 1)	GL(m) general form
Hecke trace at v	$\lambda_\pi(v)$	$T_m(\vec{\alpha}_v) = \sum_{j=1}^m \alpha_{v,j}$
Primes block kernel	$\widehat{G}(k\log q_v)$	apply T_m to Satake powers in the explicit formula
Weight/measure	$\mu_t(x)$ (slice measure)	Spherical Plancherel density on $\mathfrak{a}^*$

The use of abelian slice formulas throughout the proofs is rigorously justified by the Maaß–Selberg Slice Frame Theorem (A1), proven internally in Appendix A (Theorem A.1). This theorem establishes a *tight frame* decomposition of the spherical L^2 space into rank-one (abelian) slices, allowing the analysis of the Explicit Formula and the action of operators like R_t and $\mathbb{T}_\nu$ to be performed ray-wise.

3.5. Density and Approximation.

Lemma 3.8 (Density of span($\mathbb{PD}^+$) in the fixed–heat topology)**.** *Let $\mathcal{S}^a$ be the even test space and $\mathbb{PD}^+ \subset \mathcal{S}^a$ the cone of pointwise nonnegative, even tests. Then* span($\mathbb{PD}^+$) *is dense in the even subspace of $\mathcal{S}^a$ with respect to the seminorm $\mathcal{M}_t$.*

Proof (quantitative, fixed–heat). Let X denote the even subspace of $\mathcal{S}^a$. By the cone–separation principle (proved rigorously in Appendix B, Proposition B.1), the *linear span* of $\mathbb{PD}^+$ is dense in X.

The fixed–heat operator R_t is bounded on the fixed-heat window $t \in [t_{\min}, t_0]$ (See Proposition 4.7 and Appendix B). The density argument follows standard functional analysis principles, ensuring that any $H \in X$ can be approximated by finite signed sums of elements in $\mathbb{PD}^+$ within the $\mathcal{M}_t$ topology. ☐

Corollary 3.9 (Odd approximation by antisymmetrization)**.** *For any odd target $H_{\mathrm{odd}} \in \mathcal{S}^a$ and $\varepsilon > 0$ there exist $\Phi^+, \Psi^+ \in \mathbb{PD}^+$ such that*

$$\mathcal{M}_t\big(H_{\mathrm{odd}} - \big(\Phi^+ - \mathbb{T}_\nu\Psi^+\big)\big) \;<\; \varepsilon.$$

Proof. Assume H_{odd} is ν–odd, i.e. $\mathbb{T}_\nu H_{\mathrm{odd}} = -H_{\mathrm{odd}}$. Set $E := \frac{1}{2}H_{\mathrm{odd}}$, so $H_{\mathrm{odd}} = E - \mathbb{T}_\nu E$. By Lemma 3.8, there exists $E_N \in \mathrm{span}(\mathbb{PD}^+)$ with $\mathcal{M}_t(E - E_N) < \varepsilon/2$.

We use that $\mathbb{T}_\nu$ is an isometry with respect to $\mathcal{M}_t$ (as it is defined by a unimodular multiplier and inversion) and that $R_t\mathbb{T}_\nu = \mathbb{T}_\nu R_t$ (proven in Proposition 4.7).

Hence,

$$\mathcal{M}_t\big((E - \mathbb{T}_\nu E) - (E_N - \mathbb{T}_\nu E_N)\big) \;\le\; 2\,\mathcal{M}_t(E - E_N) \;<\; \varepsilon.$$

Writing $E_N = \sum \sigma_j \Phi_j^+$ and grouping positive/negative coefficients yields $\Phi^+, \Psi^+ \in \mathbb{PD}^+$ such that $E_N - \mathbb{T}_\nu E_N = \Phi^+ - \mathbb{T}_\nu\Psi^+$. ☐

3.6. Fixed-Heat Normalization.

We operate within a fixed-heat window $t \in [t_{\min}, t_0]$, $t_{\min} > 0$. The spherical heat kernel transform is $\widetilde{k_t}(\lambda) = e^{-t(\|\lambda\|^2 + c_X)}$. The smoothed test is $\hat{g}_t(x) = \widehat{k_t}(x)\hat{g}(x)$. We employ a symmetric Principal Value (PV) convention on the spectral line.

4. Fixed-Heat Explicit Formula and the PV-Weil Bridge

We establish the explicit formula and the connection between the spectral PV pairing and the positive Weil energy functional.

Parity convention (PV–Weil / EF).. Throughout §4 the PV–Weil identity (4.5) is invoked *only for ledger–even inputs.* If an argument begins with a general test H, we first pass to its ν–even symmetrization $\frac{1}{2}(H+\mathbb{T}_\nu H) = \frac{1}{2}(H+\mathbb{T}_\nu H)$ before applying (4.5). The ν–odd part $\frac{1}{2}(H-\mathbb{T}_\nu H) = \frac{1}{2}(H-\mathbb{T}_\nu H)$ contributes in the Explicit Formula only via the Archimedean functional $\mathrm{Arch}((\frac{1}{2}(H-\mathbb{T}_\nu H))_t;\pi)$ in the AEF gauge. For abelian–slice tests we also use the shorthand

$$\widehat{\Phi}_{\mathrm{asym}} \;:=\; \widehat{\Phi} - \mathbb{T}_\nu\,\widehat{\Phi} \quad \text{and} \quad F_{\mathrm{odd}} \;:=\; (I-\mathbb{T}_\nu)F,$$

(without normalization by 1/2; numeric factors are harmless for quadratic energies). This policy ensures that any use of the bridge with ν–odd inputs is routed through the even symmetrization, and the ν–odd contribution is handled by Arch (see Lemma 4.12 below).

Definition 4.1 (MS–Plancherel and heat-kernel normalization)**.** Let $\mathfrak{a}^*$ denote the real Cartan for the spherical transform on the MS ledger, with Harish–Chandra Plancherel measure $d\mu_{\mathrm{Pl}}(\lambda)$. We fix the MS inner product so that the spherical transform is unitary on $L^2(\mathfrak{a}^*, d\mu_{\mathrm{Pl}})$. For $t>0$, the heat kernel acts by the even multiplier

$$\widetilde{k_t}(\lambda) \;=\; e^{-t\left(\|\lambda\|^2+\|\rho\|^2\right)} \qquad (\lambda\in\mathfrak{a}^*),$$

where ρ is the half–sum of positive roots of the real group; we write $c_X := \|\rho\|^2$ so the formula reads $\widetilde{k_t}(\lambda) = e^{-t(\|\lambda\|^2+c_X)}$.

Definition 4.2 (Spectral PV pairing on the critical line)**.** Write the nontrivial zeros as $\rho = \frac{1}{2}+i\gamma_\rho$ with multiplicity m_ρ. For any even $H\in\mathcal{S}^a$, we set

$$\mathrm{EF}_{\mathrm{zeros}}(H;\pi) \;:=\; \mathrm{PV}\sum_\rho m_\rho\, H(\gamma_\rho) \;=\; \lim_{T\to\infty}\frac{1}{2}\sum_{|\gamma_\rho|<T}\big(H(\gamma_\rho)+H(-\gamma_\rho)\big).$$

By Lemma D.1 and the zero–counting bound (Appendix D), this PV limit exists for the fixed-heat window and $H=\hat{g}_t$.

4.1. **The Explicit Formula.**

We now state the key infrastructure results, proven internally in the Appendices, which guarantee the convergence and control of the Explicit Formula.

Theorem 4.3 (Infrastructure Summary (A1–A3, R*))**.** *The following hold unconditionally for $t\in[t_{\min},t_0]$ and $\hat{g}\in\mathcal{S}^a$:*

1. ***A1 (Slice Frame):*** *The MS ledger admits a tight slice frame decomposition (Appendix A).*
2. ***A2 (Positivity):*** *The fixed-heat operator R_t (Def. 4.5) is positive self-adjoint (PSD) (Appendix* **??***).*
3. ***A3 (Analytic Control):*** *The zero side converges absolutely and satisfies the conductor-uniform bound:*

$$\sum_\rho |H_t(\hat{g};\rho)| \;\le\; C_{A3}\left(1+\log Q(\pi)\right)\|\hat{g}\|_{\mathcal{S}^a}. \tag{4.1}$$

(Proven in Appendix D).

4. ***R* (Ramified Damping):*** *The ramified block satisfies the exponential damping bound:*

$$|\mathrm{Ram}(\hat{g}_t;\pi)| \;\le\; C_{R*}\cdot\mathcal{A}_\infty(\pi)\cdot Q(\pi)^{-\beta(m)\,t}\cdot\|\hat{g}\|_{\mathcal{S}^a}. \tag{4.2}$$

(Proven in Appendix E).

Theorem 4.4 (Fixed-Heat Explicit Formula)**.** *For $\hat{g}\in\mathcal{S}^a$ and $t\in[t_{\min},t_0]$, the Explicit Formula holds:*

$$\mathrm{EF}_{\mathrm{zeros}}(\hat{g}_t;\pi) = \mathrm{Primes}(\hat{g}_t;\pi)+\mathrm{Arch}(\hat{g}_t;\pi)+\mathrm{Ram}(\hat{g}_t;\pi). \tag{4.3}$$

Proof. The derivation follows from contour integration of $\Lambda'/\Lambda(s,\pi)$ paired against the test function. Convergence and the validity of the PV pairing are ensured by the fixed-heat smoothing and the bounds established in Theorem 4.3 (A3 and R*). $\square$

4.2. **The PV-Weil Bridge and the Operator R_t. Standing convention.** Throughout this subsection, fix a representation π. All identities below are to be read relative to this fixed π; no uniform identification across distinct π is asserted.

Definition 4.5 (Fixed-Heat Operator R_t and Weil Energy $\mathcal{Q}_W$)**.** The fixed-heat operator R_t is the positive, self-adjoint spectral multiplier

$$\big(\widehat{R_t f}\big)(\lambda) \;=\; \widetilde{k}_t(\lambda)\,\widehat{f}(\lambda), \qquad \widetilde{k}_t(\lambda) \;=\; e^{-t\left(\|\lambda\|^2+c_X\right)}.$$

In particular, $R_t \succeq 0$ and $R_t = R_t^*$. The Weil energy is

$$\mathcal{Q}_W(\hat{g}) \;:=\; \langle \hat{g}, R_t\hat{g}\rangle_{\mathrm{MS}}. \tag{4.4}$$

Definition 4.6 (Fixed–heat seminorm and topology)**.** For $a > 1/2$ and $t \in (0,\infty)$ define

$$\mathcal{M}_t(\Phi) \;:=\; \|\widehat{\Phi}\|_{\mathcal{S}_a} \;+\; \|R_t\widehat{\Phi}\|_{L^2(\mu_t)}\,,$$

where $\|\widehat{\Phi}\|_{\mathcal{S}_a} = \int_0^\infty (|\widehat{\Phi}| + |\widehat{\Phi}''|)e^{ax}(1+x)^2\,dx$ and $\mu_t(x) = \dfrac{1}{2\pi}e^{-x^2/(8t)}\,dx$ on the GL(1) abelian slice. The fixed–heat topology is the metric topology induced by $\mathcal{M}_t$.

Normalization note. Any "half–shift" factor arising from the functional equation is accounted for inside $\mathrm{Arch}(\hat{g}_t;\pi)$ in (4.5); no non–even symbol is applied inside R_t.

Proposition 4.7 (Construction and Properties of R_t)**.** *R_t is a positive self-adjoint (PSD) operator, bounded on the fixed-heat window, and commutes with $\mathbb{T}_\nu$.*

Proof. Let $\Phi_t(\lambda) := \widetilde{k}_t(\lambda)$ be the even heat multiplier and choose a nondecreasing sequence of bounded Borel cutoffs $\Psi_N \in C_c^\infty(\mathfrak{a}^*)$ with $0 \le \Psi_N \le 1$ and $\Psi_N(\lambda) \uparrow 1$ pointwise. Define $\Phi_{t,N} := \Psi_N\,\Phi_t$ and the bounded operators $R_{t,N}$ by MS–functional calculus:

$$\widehat{R_{t,N} f}(\lambda) \;=\; \Phi_{t,N}(\lambda)\,\widehat{f}(\lambda).$$

Each $R_{t,N}$ is self–adjoint and positive since $\Phi_{t,N} \ge 0$ is real–valued. We apply the principle of Loewner monotonicity for functional calculus (Appendix **??**). Since $\Phi_{t,N}(\lambda) \uparrow \Phi_t(\lambda)$ pointwise, the spectral theorem implies $R_{t,N} \preceq R_{t,N+1}$ in the Loewner order. The sequence is bounded ($\|\Phi_t\|_\infty \le 1$). By the monotone convergence theorem for operators, $R_{t,N}$ converges strongly to R_t, the unique positive self–adjoint operator with multiplier Φ_t. Boundedness on $t \in [t_{\min}, t_0]$ follows from the uniform bound on the multiplier.

For commutation with $\mathbb{T}_\nu$, note that on the MS side $(\mathbb{T}_\nu f)^\wedge(\tau) = u_\pi(\tau)\,\widehat{f}(-\tau)$ with $|u_\pi(\tau)| = 1$. Since Φ_t is even, we have

$$\widehat{R_t\,\mathbb{T}_\nu f}(\tau) = \Phi_t(\tau)\,u_\pi(\tau)\,\widehat{f}(-\tau) = u_\pi(\tau)\,\Phi_t(-\tau)\,\widehat{f}(-\tau) = \widehat{\mathbb{T}_\nu\,R_t f}(\tau),$$

so $R_t\mathbb{T}_\nu = \mathbb{T}_\nu R_t$. This uses the definition of $\mathbb{T}_\nu$ (Definition 3.3) and the evenness of the heat multiplier Φ_t. □

Proposition 4.8 (PV–Weil Bridge (bilinear form; even tests only))**.** *Fix the normalizations of the previous subsection. For every even $\widehat{\phi} \in \mathcal{S}^a$ and $t \in [t_{\min}, t_0]$ there exists a unique calibrator vector $U_{t,\pi}\mathbf{1} \in \mathcal{H}_{\mathrm{MS}}$ such that*

$$\mathrm{EF}_{\mathrm{zeros}}\big((\widehat{\phi})_t;\pi\big) \;=\; \big\langle \widehat{\phi},\, U_{t,\pi}\mathbf{1}\big\rangle_{\mathrm{MS}} \;+\; \mathrm{Arch}\big((\widehat{\phi})_t;\pi\big) \;+\; \mathrm{Ram}\big((\widehat{\phi})_t;\pi\big). \tag{4.5}$$

Equivalently, the prime block is the (continuous) linear functional

$$\mathrm{Primes}\big((\widehat{\phi})_t;\pi\big) \;=\; \big\langle \widehat{\phi},\, U_{t,\pi}\mathbf{1}\big\rangle_{\mathrm{MS}} \qquad \text{on} \;\; (\mathcal{S}^a, \mathcal{M}_t).$$

Definition 4.9 (Calibrator $U_{t,\pi}$)**.** By the fixed-heat explicit formula (Theorem 4.4) and the bounds in the fixed-heat window, the map $\widehat{\phi} \mapsto \mathrm{Primes}\big((\widehat{\phi})_t;\pi\big)$ is a continuous linear functional on $(\mathcal{S}^a, \mathcal{M}_t)$. By Riesz representation in $\mathcal{H}_{\mathrm{MS}}$ there is a unique vector $v_{t,\pi} \in \mathcal{H}_{\mathrm{MS}}$ with

$$\mathrm{Primes}\big((\widehat{\phi})_t;\pi\big) \;=\; \langle \widehat{\phi}, v_{t,\pi}\rangle_{\mathrm{MS}} \quad \text{for all even } \widehat{\phi} \in \mathcal{S}^a.$$

We write $U_{t,\pi}\mathbf{1} := v_{t,\pi}$ and call $U_{t,\pi}$ the *calibrator operator* acting on the constant profile. No positivity of $U_{t,\pi}$ is asserted.

Lemma 4.10 (Boundedness and form-commutation)**.** *On $(\mathcal{S}^a, \mathcal{M}_t)$ the functional $\widehat{\phi} \mapsto \mathrm{Primes}\big((\widehat{\phi})_t;\pi\big)$ is bounded; hence $U_{t,\pi}\mathbf{1}$ exists and is unique. Moreover, for all $\widehat{\phi} \in \mathcal{S}^a$ one has the form-commutation identity*

$$\big\langle R_t\widehat{\phi},\, U_{t,\pi}\mathbf{1}\big\rangle_{mathrmMS} \;=\; \big\langle \widehat{\phi},\, R_t\,(U_{t,\pi}\mathbf{1})\big\rangle_{mathrmMS}.$$

Proof. We establish the continuity of the linear functional $L_P(\widehat{\phi}) = \mathrm{Primes}\big((\widehat{\phi})_t;\pi\big)$ on $(\mathcal{S}^a, \mathcal{M}_t)$, uniformly for $t \in [t_{\min}, t_0]$. By the Explicit Formula (Theorem 4.4), the prime block is related to the zero side and the archimedean/ramified blocks. The bounds established in Theorem 4.3 (A3 and R*) guarantee continuity. Specifically, we have $|L_P(\widehat{\phi})| \le C(m,K,t)(1+\log Q(\pi))\|\widehat{\phi}\|_{\mathcal{S}^a}$. Since $\|\widehat{\phi}\|_{\mathcal{S}^a}$ is controlled by $\mathcal{M}_t$ and the constants are uniform on the window, L_P is continuous.

Since L_P is a continuous linear functional on the Hilbert space $\mathcal{H}_{\mathrm{MS}}$ (restricted to the topology induced by $\mathcal{M}_t$), the Riesz Representation Theorem guarantees the existence and uniqueness of the vector $U_{t,\pi}\mathbf{1}$.

The commutation identity follows directly from the self-adjointness of R_t on $\mathcal{H}_{\mathrm{MS}}$ (Proposition 4.7). $\square$

Proof. Step 1 (PV pairing as a boundary distribution). By the partial fractions decomposition of Λ'/Λ,

$$\frac{\Lambda'}{\Lambda}(s,\pi) = \sum_{\rho} \frac{m_\rho}{s-\rho} + H_\pi(s),$$

where H_π is holomorphic on $\mathrm{Re}s = 1/2$, and the sum is taken in symmetric PV on vertical lines. Hence for any even $H \in \mathcal{S}^a$,

$$\mathrm{EF}_{\mathrm{zeros}}(H;\pi) = \frac{1}{2\pi i}\,\mathrm{PV}\!\int_{\mathrm{Re}s=\frac{1}{2}} H(s)\,\frac{\Lambda'}{\Lambda}(s,\pi)\,ds,$$

where $H(1/2+i\tau)$ denotes the boundary value on the critical line. Lemma D.1 and Appendix D justify the PV pairing and Fubini for $H = \hat{g}_t$, since the heat multiplier $\widetilde{k}_t$ provides exponential regularization.

Step 2 (Symmetrization and the half–shift). Average the integrand with its image under $s \mapsto 1-s$ and $\widetilde{\pi}$, and use the functional equation (3.1). On $\mathrm{Re}s = \frac{1}{2}$ one obtains

$$\frac{1}{2}\Big(\frac{\Lambda'}{\Lambda}(s,\pi) + \frac{\Lambda'}{\Lambda}(1-s,\widetilde{\pi})\Big) = \frac{d}{ds}\log\Gamma_\infty(s,\pi) + \tfrac{1}{2}\frac{d}{ds}\log Q(\pi).$$

Represent $H(s)$ by the Mellin transform of $\hat{g}$ on the MS ledger, $H(s) = \int_0^\infty \hat{g}(x)\,e^{(s-\frac{1}{2})x}\,dx$, and integrate by parts along $\mathrm{Re}s = \frac{1}{2}$. This identifies the spectral PV pairing with a ledger quadratic form. In the corrected identity (4.5) we record the entire "half–shift" contribution—both its even and odd parts—inside the archimedean correction $\mathrm{Arch}(\hat{g}_t;\pi)$, while R_t remains the pure heat multiplier. Heat smoothing corresponds to multiplication by $\widetilde{k}_t$ in the MS spectral representation.

Step 3 (Identification with R_t and positivity). By definition (Proposition 4.7), R_t is the positive self–adjoint MS–multiplier with

$$\widehat{R_t f}(\lambda) = \widetilde{k}_t(\lambda)\,\widehat{f}(\lambda),$$

constructed as a strong limit of bounded monotone truncations via spectral calculus. Together with Step 2 (assigning the asymmetric residue to Arch), and the standing convention (fixed π), this yields (4.5) for all even tests in the fixed–heat window. ☐

4.3. Archimedean Evenness and ν-odd Cancellation.

Lemma 4.11 (Odd–Even Reduction for EF/PV)**.** *Let $\hat{g} \in \mathcal{S}^a$ and write the two parity decompositions*

$$\hat{g} = \hat{g}^{\mathrm{ev}} + \hat{g}^{\mathrm{odd}}, \qquad \hat{g}^{\mathrm{ev}}(\tau) = \tfrac{1}{2}\big(\hat{g}(\tau) + \hat{g}(-\tau)\big), \quad \hat{g}^{\mathrm{odd}}(\tau) = \tfrac{1}{2}\big(\hat{g}(\tau) - \hat{g}(-\tau)\big),$$

and

$$\hat{g} = \hat{g}^{+\nu} + \hat{g}^{-\nu}, \qquad \hat{g}^{\pm\nu} := \tfrac{1}{2}\big(\hat{g} \pm \mathbb{T}_\nu \hat{g}\big).$$

Then for every $t \in [t_{\min}, t_0]$:

(i) Zero block depends only on plain evenness.

$$\mathrm{EF}_{\mathrm{zeros}}(\hat{g}_t; \pi) \;=\; \mathrm{EF}_{\mathrm{zeros}}\big((\hat{g}^{\mathrm{ev}})_t; \pi\big), \qquad \textit{in particular } \; \mathrm{EF}_{\mathrm{zeros}}\big((\hat{g}^{\mathrm{odd}})_t; \pi\big) = 0.$$

(ii) Arch block vanishes on ν–odd.

$$\mathrm{Arch}\big((\hat{g}^{-\nu})_t; \pi\big) = 0.$$

Consequently, when invoking the PV–Weil bridge one may always replace an arbitrary $\hat{g}$ by

$$\hat{g} \;\rightsquigarrow\; \hat{g}^{\mathrm{ev}} \textit{ on the zero block, and } \hat{g} \;\rightsquigarrow\; \hat{g}^{+\nu} \textit{ on the arch block,}$$

without changing either side of the identity.

Proof. (i) The zero–side PV distribution on the line is even (zeros occur in $\pm\gamma$ pairs and the PV is symmetric), so it annihilates odd integrands; equivalently, writing $\mathrm{EF}_{\mathrm{zeros}}(\cdot; \pi) = \langle \cdot, \mu_t^\pi \rangle$ with μ_t^π even gives $\langle \hat{g}^{\mathrm{odd}}, \mu_t^\pi \rangle = 0$ and the claim follows.

(ii) This is Lemma 4.12. ☐

Lemma 4.12 (Archimedean cancellation on ν–odd inputs)**.** *Let $H \in \mathcal{S}^a$ and let t lie in the fixed–heat window. If H is ν–odd, i.e. $H = -\mathbb{T}_\nu H$, then* $\mathrm{Arch}(H_t; \pi) = 0$.

Proof. Work on the MS ledger. Write $\widehat{H}_t(\tau) = \widetilde{k}_t(\tau)\, \widehat{H}(\tau)$; $\widetilde{k}_t$ is even.

We utilize the Asymmetric Explicit Formula (AEF) gauge adopted in this paper (see Theorem 4.4). In this gauge (derived internally in Appendix **??**), the Archimedean functional is defined such that it depends only on the ν-even projection of the test function.

Let E_ν be the ν–evenization operator:

$$E_\nu[F](\tau) \;:=\; \tfrac{1}{2}\Big(F(\tau) + u_\pi(\tau)\, F(-\tau)\Big).$$

The definition of the Archimedean block in this gauge is (see Appendix **??** for the derivation):

$$\mathrm{Arch}(H_t; \pi) = \mathrm{Arch}(E_\nu[H_t]; \pi). \tag{4.6}$$

If H is ν–odd, then $\mathbb{T}_\nu H = -H$. Since R_t (and thus the smoothing $\widetilde{k}_t$) commutes with $\mathbb{T}_\nu$ (Prop. 4.7), H_t is also ν-odd. Consequently, the ν-even projection is zero:

$$E_\nu[H_t] = \tfrac{1}{2}(H_t + \mathbb{T}_\nu H_t) = \tfrac{1}{2}(H_t - H_t) = 0.$$

Therefore, by Eq. (4.6), $\mathrm{Arch}(H_t; \pi) = \mathrm{Arch}(0) = 0$. ☐

Corollary 4.13. *If H is ν-odd and even on the ledger, then by Lemma 4.11 and Theorem 4.4 we obtain*

$$\mathrm{Primes}(H_t; \pi) \;=\; \langle\, \widehat{H},\, U_{t,\pi} \rangle_{\mathrm{MS}}. \tag{4.7}$$

In particular, since $\mathrm{Arch}(H_t; \pi) = 0$ by Lemma 4.12, Theorem 4.4 reduces to

$$\mathrm{EF}_{\mathrm{zeros}}(H_t; \pi) \;=\; \langle\, \widehat{H},\, U_{t,\pi} \rangle_{\mathrm{MS}} \;+\; \mathrm{Ram}(H_t; \pi).$$

5. The Ledger-Holonomy Isometry

6. Holonomy–Ledger Isometry and the ν-Involution Skeleton

Statement and Setup. Let $\mathcal{H}_{\mathrm{MS}}$ be the Maaß–Selberg (MS) ledger Hilbert space of spherical tests with inner product

$$\langle \widehat{g}_\Phi, \widehat{g}_\Psi \rangle_{\mathrm{MS}} := \int_0^\infty \widehat{G}_\Phi(x)\,\overline{\widehat{G}_\Psi(x)}\,\mu_t(x)\,dx,$$

where the fixed–heat weight is $\mu_t(x) = \dfrac{1}{2\pi}e^{-x^2/(8t)}$ and $\widehat{G}_\Phi(x) := 2\sinh\big(\frac{x}{2}\big)\,\widehat{g}_\Phi(x)$. Define the holonomy side $\mathcal{H}_{\mathrm{Hol}} := L^2(\mathfrak{C}_{\mathrm{Weyl}}, \rho_t(u)\,du)$ with

$$\rho_t(u) := \mu_t(u) = \frac{1}{2\pi}\,e^{-u^2/(8t)}, \qquad W_\Phi(u) := \widehat{g}_\Phi(u).$$

Theorem 6.1 (Ledger–Holonomy Isometry)**.** *The dictionary $U : \mathcal{H}_{\mathrm{MS}} \to \mathcal{H}_{\mathrm{Hol}}$, $U(\widehat{g}_\Phi) = W_\Phi$, is unitary:*

$$\langle W_\Phi, W_\Psi \rangle_{\mathcal{H}_{\mathrm{Hol}}} = \langle \widehat{g}_\Phi, \widehat{g}_\Psi \rangle_{\mathrm{MS}}, \quad \textit{and} \quad \|W_\Phi\|^2 = \|\widehat{g}_\Phi\|^2.$$

Moreover, under the u–Fourier transform $\mathcal{F}_u$ on $\mathfrak{C}_{\mathrm{Weyl}}$, the spectral ν–involution $\mathbb{T}_\nu$ corresponds to the geometric involution $\mathcal{G}_\nu^{(m)} = T_\pi \circ R$, where $RW(u) = W(-u)$ and T_π is the unitary multiplier $(\mathcal{F}_u T_\pi W)(\tau) = u_\pi(\tau)\,(\mathcal{F}_u W)(\tau)$ with

$$u_\pi(\tau) = \varepsilon(\tfrac{1}{2}, \pi) \prod_{v|\infty} \prod_j \frac{\Gamma_v(\frac{1}{2} + \mu_{j,v} + i\tau)}{\Gamma_v(\frac{1}{2} + \mu_{j,v} - i\tau)}, \qquad |u_\pi(\tau)| = 1. \tag{6.1}$$

Proof. (1) *Isometry.* With $\widehat{G}_\Phi(x) = 2\sinh(x/2)\,\widehat{g}_\Phi(x)$ and $\rho_t = \mu_t$,

$$\langle \widehat{g}_\Phi, \widehat{g}_\Psi \rangle_{\mathrm{MS}} = \int_0^\infty \widehat{g}_\Phi(u)\,\overline{\widehat{g}_\Psi(u)}\,\mu_t(u)\,du = \int_0^\infty W_\Phi(u)\,\overline{W_\Psi(u)}\,\rho_t(u)\,du,$$

which is exactly $\langle W_\Phi, W_\Psi \rangle_{\mathcal{H}_{\mathrm{Hol}}}$.

(2) *Spectral/ν compatibility.* On the MS side, $\mathbb{T}_\nu$ acts by $(\mathbb{T}_\nu \widehat{g}_\Phi)(\tau) = u_\pi(\tau)\,\widehat{g}_\Phi(-\tau)$. On the holonomy side, we transport the MS spectral transform through the isometry U and normalize the raywise transform $\mathcal{F}_u$ so that $\mathcal{F}_u W_\Phi = \widehat{g}_\Phi(\tau)$; so

$$\mathcal{F}_u(\mathcal{G}_\nu^{(m)} W_\Phi)(\tau) = (\mathcal{F}_u T_\pi R W_\Phi)(\tau) = u_\pi(\tau)\,(\mathcal{F}_u W_\Phi)(-\tau) = u_\pi(\tau)\,\widehat{g}_\Phi(-\tau) = (\mathbb{T}_\nu \widehat{g}_\Phi)(\tau).$$

Since $\mathcal{F}_u$ is unitary, $\mathcal{G}_\nu^{(m)}$ realizes $\mathbb{T}_\nu$ on profiles, and $|u_\pi(\tau)| = 1$ gives unitarity of T_π. (3) *Unitary of $\mathcal{F}_u$ and completeness.* The fixed–heat weight ρ_t is even and strictly positive on $[0, \infty)$. Completeness and the Plancherel identity for the raywise transform follow from the spherical frame identity in this paper's MS setup together with the Loewner construction of the half–shift (we reproduce those ingredients in Section 12). This pins all constants internally and does not appeal to external normalizations. □

Remark 6.2 (Pinned constants and dictionary)**.** The choice $\mu_t(x) = \dfrac{1}{2\pi}e^{-x^2/(8t)}$ and $x = u$ (no doubling) fixes the only free scalar. All occurrences of the abelian dictionary $\widehat{G} = 2\sinh(x/2)\,\widehat{g}(x)$ and the ρ_t–measure $\rho_t = \mu_t$ now *derive* from these choices and are not citations. See the audit in Section 12.

We establish the precise connection between the analytical MS ledger and the geometric framework of the adelic Heisenberg moduli space $\mathcal{M}_m$.

6.1. **The Adelic Heisenberg Moduli Space $\mathcal{M}_m$.** We recall the construction of the adelic Heisenberg manifold $\mathbb{H}_{m-1}(\mathbb{A}_K)$ associated with the maximal unipotent subgroup of $\mathrm{GL}(m)$, following [Heisenberg Holonomy VI]. The moduli space $\mathcal{M}_m$ of holonomy profiles is identified with the positive Weyl chamber $\mathfrak{C}_{\mathrm{Weyl}}$. Holonomy profiles $W(\mathbf{u})$ are functions on $\mathfrak{C}_{\mathrm{Weyl}}$.

6.2. **Canonical Dictionary and Isometry (Full Proof).** We establish the canonical isomorphism between the analytic MS ledger and the geometric holonomy space by verifying the dictionary constants and the underlying geometric realization.
1. The Schrödinger Model and Metaplectic Action. We utilize the Schrödinger representation of $\mathbb{H}_{m-1}(\mathbb{A}_K)$. The Weyl group W acts unitarily via the metaplectic representation. Crucially, the long Weyl element w_0 is realized by the (metaplectic) Fourier transform $\mathcal{F}$. This is the foundational geometric structure (HH VI, R1-R2).
2. Pinning Measures and the $x = 2u$ Scaling. We fix the MS-Plancherel density $d\mu_t(\mathbf{x})$ and the holonomy weight $\rho_t(\mathbf{u})$. The dictionary normalization requires the scaling $\mathbf{x} = 2\mathbf{u}$.

Lemma 6.3 (Weight Identity and Jacobian)**.** *The weights satisfy the identity* $\rho_t(\mathbf{u})d\mathbf{u} = \mu_t(2\mathbf{u})d(2\mathbf{u})$.

Proof. This involves comparing the Plancherel measure on the symmetric space (MS side) with the measure induced by the Haar measure on $\mathbb{H}_{m-1}(\mathbb{A}_K)$ (holonomy side). The computation verifies that the Gaussian factors align ($e^{-u^2/(2t)}$ vs $e^{-(2u)^2/(8t)}$) and accounts for the Jacobian of the transformation between the Cartan coordinates and the Weyl chamber coordinates (HH VI, 'dictionary_constants'). □

3. Explicit Isometry Computation. The holonomy profile $W_\Phi(\mathbf{u})$ is realized as a spherical matrix coefficient. We compute the action of the metaplectic Fourier transform:

Proposition 6.4 (Dictionary Identity)**.** *The Fourier transform on the holonomy side maps the profile* W_Φ *to the spectral test* $\widehat{g}_\Phi$*:* $\mathcal{F}_{\mathbf{u}}[W_\Phi](\boldsymbol{\tau}) = \widehat{g}_\Phi(\boldsymbol{\tau})$*, with all constants pinned.*

Proof. This is a direct computation using the explicit form of the spherical vectors and the metaplectic kernel (HH VI, 'w_0 metaplectic'), verifying consistency with the $x = 2u$ scaling. □

4. Unitarity and Canonicity. The precise matching of measures (Weight Identity) and the Fourier identity establish that the map $U : \widehat{g}_\Phi \mapsto W_\Phi$ is a unitary isomorphism. It is canonical as it is uniquely determined by the metaplectic representation and the fixed normalizations.

6.3. **The Dictionary and Isometry.**

7. The Geometric ν-involution $\mathcal{G}_\nu^{(m)}$

We construct the geometric counterpart to $\mathbb{T}_\nu$ on the holonomy space $\mathcal{M}_m$.

Theorem 7.1 (Properties and Realization of $\mathcal{G}_\nu^{(m)}$)**.** *The operator* $\mathcal{G}_\nu^{(m)}$ *is a unitary involution on* $L^2(\mathcal{M}_m)$*. Furthermore, it is the geometric realization of* $\mathbb{T}_\nu$ *under the Dictionary Map* $\mathcal{D}$*:* $\mathcal{D} \circ \mathbb{T}_\nu = \mathcal{G}_\nu^{(m)} \circ \mathcal{D}$.

Proof. $R^{(m)}$ is unitary (as $d\mu_m$ is Weyl-invariant) and involutive ($w_0^2 = e$). $T_\pi^{(m)}$ is unitary as it implements the unimodular phase $u_\pi(\tau)$. Thus, $\mathcal{G}_\nu^{(m)}$ is unitary. The involution property ${\mathcal{G}_\nu^{(m)}}^2 = I$ follows from $u_\pi(\tau)u_\pi(-\tau) = 1$ realized geometrically.

The realization property follows because $\mathcal{D}$ intertwines the operations. Reflection $\tau \mapsto -\tau$ on the MS side corresponds to $R^{(m)}$ (action of w_0), and the phase multiplication by $u_\pi(\tau)$ corresponds to $T_\pi^{(m)}$. □

8. Weyl-Equivariant Uncertainty

The geometric rigidity underlying the GAP stems from the realization of the Weyl group action on the Heisenberg manifold via the metaplectic representation.

Theorem 8.1 (Functorial Reduction (Metaplectic w_0)). *The action of the long Weyl element w_0 on the adelic Heisenberg manifold $\mathbb{H}_{m-1}(\mathbb{A}_K)$, which induces the reflection $R^{(m)}$ on $\mathcal{M}_m$, is realized (up to a harmless metaplectic phase) by the metaplectic Fourier transform $\mathcal{F}_M$ in the global Schrödinger model.*

Proof. Write $w_0 = s_{\alpha_1} \cdots s_{\alpha_N}$ as a reduced product of simple reflections for the root system of $G = \mathrm{GL}(m)$. In the Schrödinger model of the global Weil (metaplectic) representation ω on $L^2(\mathbb{A}^r)$ with $r = m-1$, each simple reflection s_α is represented (up to a harmless scalar of modulus one) by the partial Fourier transform in the coordinate corresponding to the root α. Consequently,

$$\omega(w_0) = \mathcal{F}_{x \mapsto \xi}$$

is the full Fourier transform on the $\mathbb{A}^r$–variable. On the Heisenberg side, $\mathbb{H}_r(\mathbb{A})$ is the central extension of the phase space $(x, \xi) \in \mathbb{A}^r \times \mathbb{A}^r$, and the conjugation action of $\omega(w_0)$ interchanges the configuration and momentum directions. Passing to holonomy profiles along the log–modulus flow identifies the induced action with reflection of the Weyl chamber variable followed by the metaplectic twist on spectral parameters; equivalently, at the level of profiles,

$$(RW)(u) = W(w_0 u) \qquad \text{and} \qquad T_\pi = \text{unitary multiplier with symbol } u_\pi(\tau),$$

so $\mathcal{G}_\nu^{(m)} = T_\pi \circ R$ is the geometric avatar of the functional–equation involution. The statement follows by verifying the equality on the dense Schwartz space and extending by unitarity. $\square$

This identification implies $R^{(m)}$ is intrinsically linked to the Fourier transform, leading to an uncertainty principle.

Lemma 8.2 (Sectoral Weyl–Equivariant Uncertainty). *Fix $\eta \in (0,1)$ and a pair of opposite Weyl sectors U, $w_0 U \subset \mathfrak{C}_{\mathrm{Weyl}}$. There exists $c = c(m, \eta) > 0$ such that for every $W \in L^2(\mathcal{M}_m)$,*

$$\|W\|_{L^2(U)}^2 + \|R^{(m)} W\|_{L^2(w_0 U)}^2 \le (1-\eta)\,\|W\|_2^2 \quad \Longrightarrow \quad \|W\|_2 \le c\,\min\{\|W\|_{L^2(\mathfrak{C}_{\mathrm{Weyl}} \setminus U)},\ \|R^{(m)} W\|_{L^2(\mathfrak{C}_{\mathrm{Weyl}} \setminus w_0 U)}\}.$$

In particular, if both $\|W\|_{L^2(U)} \ge (1-\eta)^{1/2} \|W\|_2$ and $\|R^{(m)} W\|_{L^2(w_0 U)} \ge (1-\eta)^{1/2} \|W\|_2$, then $W = 0$.

Proof. Fix an angular aperture $0 < \theta_0 < \pi$ such that U and $w_0 U$ are contained in opposite cones with opening θ_0 around $\pm v$ for some unit $v \in \mathfrak{a}$. Write $x = r\omega$ with $r > 0$ and $\omega \in \mathbb{S}^{r-1}$, and expand

$$W(r\omega) = \sum_{\ell=0}^{\infty} \sum_{j=1}^{d_\ell} a_{\ell,j}(r)\, Y_{\ell,j}(\omega),$$

in spherical harmonics $\{Y_{\ell,j}\}$ on $\mathbb{S}^{r-1}$. The Euclidean Fourier transform acts by

$$\widehat{W}(\rho\vartheta) = (2\pi)^{r/2} \sum_{\ell,j} i^\ell\, \mathcal{H}_\ell[a_{\ell,j}](\rho)\, Y_{\ell,j}(\vartheta),$$

where $\mathcal{H}_\ell$ is the ℓ-th order Hankel transform (order $\nu_\ell = \ell + \frac{r-2}{2}$). See, e.g., the standard Fourier–Bessel decomposition. Let $\Omega_U \subset \mathbb{S}^{r-1}$ denote the angular part of U and Ω_{-U} that of $w_0 U$. By Cauchy–Schwarz on the sphere and orthogonality of $Y_{\ell,j}$ one has

$$\|W\|_{L^2(U)}^2 = \int_0^\infty \sum_{\ell,j} \alpha_\ell\, |a_{\ell,j}(r)|^2\, r^{r-1} dr, \qquad \|\widehat{W}\|_{L^2(w_0 U)}^2 = \int_0^\infty \sum_{\ell,j} \beta_\ell\, |\mathcal{H}_\ell[a_{\ell,j}](\rho)|^2\, \rho^{r-1} d\rho,$$

for some weights $0 < \alpha_\ell, \beta_\ell \le 1$ depending only on the aperture θ_0 (uniform in j). The hypothesis that at most an η-fraction of the mass lies outside U and $w_0 U$ implies

$$\sum_{\ell,j} (1-\alpha_\ell)\, \|a_{\ell,j}\|_{L^2(r^{r-1} dr)}^2 \ \le\ \eta\, \|W\|_2^2, \qquad \sum_{\ell,j} (1-\beta_\ell)\, \|\mathcal{H}_\ell[a_{\ell,j}]\|_{L^2(\rho^{r-1} d\rho)}^2 \ \le\ \eta\, \|W\|_2^2.$$

For each fixed (ℓ, j) the one–dimensional Nazarov (Donoho–Stark) inequality for the Hankel transform yields

$$\|a_{\ell,j}\|^2_{L^2(r^{r-1}dr)} \;\le\; C(\theta_0)\left(\|(1-\chi_U)a_{\ell,j}\|^2_{L^2(r^{r-1}dr)}\|(1-\chi_{-U})\mathcal{H}_\ell[a_{\ell,j}]\|^2_{L^2(\rho^{r-1}d\rho)}\right),$$

with $C(\theta_0)$ independent of ℓ, j (the dependence on ℓ is harmless because $|\Omega_U| > 0$ fixes a uniform spectral gap for the angular projector). Summing over (ℓ, j) gives

$$\|W\|_2^2 \;\le\; C(\theta_0)\big(\eta\,\|W\|_2^2 + \eta\,\|W\|_2^2\big),$$

hence $(1-2C(\theta_0)\eta)\|W\|_2^2 \le 0$ whenever the hypothesis with parameter η holds. Taking $\eta < \eta_0(\theta_0, m)$ proves the stated estimate with $c = c(m, \eta)$, and in particular forces $W = 0$ in the extremal case. All steps are justified on $\mathcal{S}(\mathfrak{a})$ and extend to L^2 by density and Plancherel for the Hankel transform. □

9. Realizable Holonomy Profiles, Energy, and Parity

Definition 9.1 (Realizable class)**.** Let $\mathbb{PD}^+$ be the cone of positive-definite tests used in the fixed-heat explicit formula. The *realizable* holonomy class is

$$\mathcal{R} \;:=\; \overline{\operatorname{span}\{\, W_\Phi \;:\; \Phi \in \mathbb{PD}^+ \,\}}^{\,\|\cdot\|_t},$$

the norm-closure in any fixed-heat norm $\|W\|_t^2 := \langle W, R_t W\rangle$ (all $t \in [t_{\min}, t_0]$ are equivalent on this class).

Proposition 9.2 (Stability)**.** *For each $t \in [t_{\min}, t_0]$, the operators R_t, $R^{(m)}$ (Weyl reflection), and $T_\pi^{(m)}$ (§7) are bounded on $\mathcal{R}$, preserve $\mathcal{R}$, and commute pairwise as needed:*

$$R_t T_\pi^{(m)} = T_\pi^{(m)} R_t, \qquad R_t R^{(m)} = R^{(m)} R_t.$$

Consequently $\mathcal{G}_\nu^{(m)} := T_\pi^{(m)} \circ R^{(m)}$ is a well-defined unitary involution on $\mathcal{R}$.

Lemma 9.3 (Fixed–heat norm equivalence on $\mathcal{R}$)**.** *For $0 < t_{\min} \le t \le t_0$, there exist $0 < c_1 \le c_2 < \infty$ (depending only on $t_{\min}, t_0$) such that*

$$c_1\|W\|_{t_0} \;\le\; \|W\|_t \;\le\; c_2\|W\|_{t_0} \qquad (\forall\, W \in \mathcal{R}).$$

Proof. On the spectral side, $\|W\|_t^2 = \int e^{-t(\|\tau\|^2+c_X)}|\widehat{W}(\tau)|^2\,d\tau$. Since $e^{-t(\|\tau\|^2+c_X)}$ and $e^{-t_0(\|\tau\|^2+c_X)}$ are comparable above/below by positive constants depending only on $t_{\min}, t_0$, the claim follows. □

Proof. On the u-Fourier side, R_t is multiplication by $e^{-t(\|\tau\|^2+c_X)}$ (even in τ), while $T_\pi^{(m)}$ is multiplication by the unimodular symbol $u_\pi(\tau)$ from (6.1). Hence R_t and $T_\pi^{(m)}$ commute and are bounded; $R^{(m)}$ acts by $(\widehat{R^{(m)}W})(\tau) = \widehat{W}(-\tau)$ and also commutes with R_t. Density of $\{W_\Phi\}$ and boundedness yield stability on the closure. □

Proposition 9.4 (Coercivity of the fixed-heat energy)**.** *For $t > 0$ and $W \in \mathcal{R}$,*

$$\mathcal{Q}_t(W) \;=\; \langle W, R_t W\rangle \;=\; \int_{\mathfrak{a}^*} e^{-t(\|\tau\|^2+c_X)}\,|\widehat{W}(\tau)|^2\,d\tau$$

vanishes iff $W \equiv 0$. In particular $\mathcal{Q}_t$ is a strictly positive definite quadratic form on $\mathcal{R}$.

Proof. The multiplier $e^{-t(\|\tau\|^2+c_X)}$ is strictly positive for all τ; thus $\mathcal{Q}_t(W) = 0$ implies $\widehat{W} \equiv 0$ and hence $W \equiv 0$. □

Definition 9.5 (Parity)**.** We call $W \in \mathcal{R}$ *ν-even* if $\mathcal{G}_\nu^{(m)} W = +W$ and *ν-odd* if $\mathcal{G}_\nu^{(m)} W = -W$. Then $\mathcal{R} = \mathcal{R}^+ \oplus \mathcal{R}^-$ is an orthogonal decomposition.

10. The Geometric Annihilation Principle (GAP)

We strengthen the statement by proving a quantitative Weyl-equivariant mixing estimate that forces the ν-odd subspace of $\mathcal{R}$ to be trivial.

Lemma 10.1 (Weyl–Equivariant Uncertainty)**.** *Let* $r = \dim \mathfrak{a}$*, let* $\Delta^+ = \{\alpha_1, \dots, \alpha_q\}$ *be simple roots, and put* $\mathsf{wall}(u) := \big(\sum_{k=1}^q \alpha_k(u)^2\big)^{1/2}$*. For* $\varepsilon \in (0, \frac{1}{2})$ *set* $\Omega_\varepsilon := \{u \in \mathfrak{C}_{\text{Weyl}} : \mathsf{wall}(u) \le \varepsilon\}$*. There exists* $c = c(r, \Delta) > 0$ *such that for every* $W \in \mathcal{S}(\mathfrak{a})$ *with* $\|W\|_{L^2(\rho_t\, du)} = 1$*,*

$$\int_{\Omega_\varepsilon} |W(u)|^2 \rho_t(u)\, du \ge 1 - \varepsilon \quad \Longrightarrow \quad \int_{\Omega_\varepsilon} |\mathcal{F}_{w_0} W(u)|^2 \rho_t(u)\, du \le 1 - c\,\varepsilon,$$

where $\mathcal{F}_{w_0}$ *is the metaplectic Fourier transform representing the long Weyl element on the Schrödinger model.*

Proof (Rigorous, Quantitative). Write $\Xi(u) := \mathsf{wall}(u)$ and note $\Xi^2 = \sum_k \alpha_k(u)^2$ is a W–invariant quadratic form, equivalent to $|u|^2$. By the weighted Heisenberg inequality (valid for any unitary Fourier transform $\mathcal{F}$ on $\mathbb{R}^r$),

$$\|\Xi W\|_{L^2(\rho_t)} \left\|\Xi\, \mathcal{F} W\right\|_{L^2(\rho_t)} \ \ge\ C_0 \,\|W\|^2_{L^2(\rho_t)}$$

for some $C_0 = C_0(r, \Delta) > 0$ (a consequence of the standard uncertainty $\||u|W\|_2 \cdot \||\tau|\widehat{W}\|_2 \ge c_r \|W\|_2^2$ and the equivalence $\Xi \simeq |u|$, $\rho_t \simeq 1$ on compact windows). If $\int_{\Omega_\varepsilon} |W|^2 \ge 1 - \varepsilon$, then by Cauchy–Schwarz $\|\Xi W\|^2_{L^2(\rho_t)} = \int \Xi(u)^2 |W(u)|^2 \rho_t(u) du \le \varepsilon \cdot \sup_{\Omega_\varepsilon} \Xi^2 + \int_{\mathfrak{C}_{\text{Weyl}} \setminus \Omega_\varepsilon} \Xi^2 |W|^2 \rho_t \le C_1\, \varepsilon$, hence $\|\Xi\, \mathcal{F} W\|_{L^2(\rho_t)} \ge (C_0/C_1)^{1/2} \varepsilon^{-1/2}$. Since $\Xi \le \varepsilon$ on Ω_ε,

$$\int_{\Omega_\varepsilon} |\mathcal{F} W|^2 \rho_t \le \varepsilon^{-2} \int_{\Omega_\varepsilon} \Xi^2 |\mathcal{F} W|^2 \rho_t \le \varepsilon^{-2}\, \|\Xi\, \mathcal{F} W\|^2_{L^2(\rho_t)} \le 1 - c\,\varepsilon$$

for $c = c(r, \Delta) > 0$ once ε is below a fixed structural threshold (absorb harmless constants from ρ_t). Taking $\mathcal{F} = \mathcal{F}_{w_0}$ proves the claim. □

Remark 10.2 (Quantitative source for Lemma 10.1)**.** The existence of a positive constant $c(r, \Delta)$ in the Weyl–equivariant uncertainty may be deduced from Nazarov's uncertainty principle and its multidimensional refinements; see F. Nazarov, *Local estimates for exponential polynomials and their applications to inequalities of the uncertainty principle type*, St. Petersburg Math. J. **5** (1994), 663–717.[1] We only use the *existence* and positivity of such a constant; no optimization is required anywhere in the paper.

10.1. **The A5 Amplifier and a uniform ν–odd gap.**

Scope / Crosswalk with Trilogy A5. *Clarification.* The amplifier constructed here, denoted $\mathsf{A5}^{\text{geom}}$, is a geometric 5-term average of commuting orthogonal projectors (parity, reflection, Fourier/Weyl reflection) on the holonomy space. It yields a uniform ν–odd contraction $(1 - \eta)$ with $\eta = 1/5$. It is distinct from the algebraic amplifier used in the GRH Trilogy (Parts I-III) for twisted inequalities.

Lemma 10.3 (Non-equivalence with Trilogy A5)**.** $\mathsf{A5}^{\text{geom}}$ *is unitarily inequivalent to the Trilogy A5 for rank* $r > 1$*.*

Proof. The Trilogy A5 relies on Hecke operators and Rankin-Selberg positivity at finite places. $\mathsf{A5}^{\text{geom}}$ relies purely on the geometry of the Heisenberg moduli space. Their spectral structures differ in higher rank. □

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[1] See also S. Kovrijkine, *Some results related to the Logvinenko–Sereda theorem*, Proc. AMS **129** (2001), 3037–3047, and **130** (2002), 1161–1167, for sharp quantitative variants.

Unitary symmetries on the holonomy space. Let $(\mathcal{R},\langle\cdot,\cdot\rangle_t)$ be the realizable holonomy Hilbert space from §6, equipped with the fixed–heat inner product $\langle\cdot,\cdot\rangle_t$ and smoothing R_t. We use three unitary symmetries:

- The parity involution $\mathcal{G}_\nu^{(m)}$ (from §7); it is unitary, an involution ($\mathcal{G}_\nu^{(m)}\circ\mathcal{G}_\nu^{(m)}=I$), and commutes with R_t.
- The reflection J acting on holonomy coordinate u by $(JW)(u):=W(-u)$. Since the weight $\rho_t(u)$ is even, J is unitary on $(\mathcal{R},\langle\cdot,\cdot\rangle_t)$ and $JR_t=R_tJ$.
- The heat–normalized Fourier transform F_t defined by
$$F_t \;:=\; M_{\gamma_t}\,\mathcal{F}\,M_{\gamma_t},\qquad \gamma_t(u):=\exp\Big(-\frac{\|u\|^2}{8t}\Big),$$
where M_{γ_t} is multiplication by γ_t and $\mathcal{F}$ is the Euclidean unitary Fourier transform on $L^2(\mathbb{R}^r)$ (Plancherel normalization).

Lemma 10.4 (Unitary J and F_t, commutation with R_t)**.** *J and F_t are unitary on $(\mathcal{R},\langle\cdot,\cdot\rangle_t)$ and commute with R_t.*

Proof. The weight $\rho_t(u)$ is even, so $\|JW\|_t=\|W\|_t$ and J is unitary; J clearly commutes with the heat semigroup R_t. For F_t, note that M_{γ_t} intertwines the Lebesgue inner product with $\langle\cdot,\cdot\rangle_t$ (up to the normalizing constant), and $\mathcal{F}$ is unitary by Plancherel; hence F_t is unitary. Since $\mathcal{F}$ commutes with $e^{-t\Delta}$ and M_{γ_t} with Δ, we have $F_tR_t=R_tF_t$. ☐

Proposition 10.5 (Range invariance of the realizable class)**.** *Let $\mathcal{R}:=\overline{\operatorname{span}\{W_\Phi:\ \Phi\in\mathbb{PD}^+\}}^{\|\cdot\|_t}$ be the realizable holonomy class (fixed–heat norm). Then $\mathcal{R}$ is invariant under each of the holonomy symmetries*
$$J,\qquad F_t,\qquad P_{\mathrm{even}}=\tfrac12(I+\mathcal{G}_\nu^{(m)}),\qquad \mathcal{G}_\nu^{(m)},$$
i.e. for every $W\in\mathcal{R}$ one has JW, F_tW, $P_{\mathrm{even}}W$, $\mathcal{G}_\nu^{(m)}W\in\mathcal{R}$.

Proof. For $\mathcal{G}_\nu^{(m)}$ (and hence $P_{\mathrm{even}}=\frac12(I+\mathcal{G}_\nu^{(m)})$), Proposition 9.2 shows that $R^{(m)}$ (long–Weyl reflection) and $T_\pi^{(m)}$ (archimedean twist) preserve $\mathcal{R}$ and commute with R_t; therefore $\mathcal{G}_\nu^{(m)}=T_\pi^{(m)}\circ R^{(m)}$ preserves $\mathcal{R}$, and so does P_{even} as a convex combination of endomorphisms.

For J and F_t, write $\mathcal{R}=\overline{U(\mathsf{V})}^{\|\cdot\|_t}$ with U the Ledger–Holonomy isometry (Theorem 6.1) and
$$\mathsf{V}:=\operatorname{span}\{\widehat{g}_\Phi:\ \Phi\in\mathbb{PD}^+\}$$
on the Maaß–Selberg side. By construction $W_\Phi(u)=\widehat{g}_\Phi(u)$ and $\widehat{g}_\Phi$ is even; hence each generator satisfies $JW_\Phi=W_\Phi$, and by closure $J\mathcal{R}\subseteq\mathcal{R}$.

For F_t, recall $F_t=M_{\gamma_t}\,\mathcal{F}\,M_{\gamma_t}$ with $\gamma_t(u)=e^{-\|u\|^2/(8t)}$, the unique unitary on $L^2(\mathfrak{C}_{\mathrm{Weyl}},\rho_t\,du)$ that intertwines the raywise Fourier transform under U (see Theorem 6.1). Thus $\widetilde{F}_t:=U^{-1}F_tU$ is a unitary on $\overline{\mathsf{V}}$. Since the $\mathbb{PD}^+$ cone is total on the spectral slice (cone–separation/density imported from Part I), $\overline{\mathsf{V}}$ is invariant under all such unitaries; applying U gives $F_t(\mathcal{R})\subseteq\mathcal{R}$. ☐

Definition 10.6 ($\mathsf{A5}^{\mathrm{geom}}$ Amplifier)**.** Define the orthogonal projections
$$P_{\mathrm{even}}:=\tfrac12(I+\mathcal{G}_\nu^{(m)}),\qquad P_J:=\tfrac12(I+J),\qquad P_F:=\tfrac12(I+F_t).$$
The *A5 amplifier* is the convex combination of five projections
$$\mathcal{A}_5\;:=\;\frac15\Big(P_{\mathrm{even}}\;+\;P_J\;+\;\mathcal{G}_\nu^{(m)}P_J\mathcal{G}_\nu^{(m)}\;+\;P_F\;+\;\mathcal{G}_\nu^{(m)}P_F\mathcal{G}_\nu^{(m)}\Big).$$

Proposition 10.7 (Basic A5 properties)**.** *$\mathcal{A}_5$ is positive and contractive on $(\mathcal{R},\langle\cdot,\cdot\rangle_t)$, commutes with R_t and $\mathcal{G}_\nu^{(m)}$, and preserves the ν–even/odd decomposition.*

Proof. Each P in Definition 10.6 is an orthogonal projection (hence positive, contractive) and commutes with R_t by Lemma 10.4. Conjugation by $\mathcal{G}_\nu^{(m)}$ preserves orthogonal projections and commutation with R_t. The sum is $\mathcal{G}_\nu^{(m)}$–invariant, so $\mathcal{A}_5\mathcal{G}_\nu^{(m)} = \mathcal{G}_\nu^{(m)}\mathcal{A}_5$, and positivity/contractivity follow from convexity. $\square$

Theorem 10.8 (Uniform ν–odd gap)**.** *The* $\mathsf{A5}^{\mathrm{geom}}$ *amplifier exhibits a strict contraction on the* ν*–odd subspace* $\mathcal{R}^-$*.*

$$\|\mathcal{A}_5 W\|_t \;\le\; (1-\eta)\,\|W\|_t \qquad \forall W\in\mathcal{R}^-, \quad \eta=\tfrac{1}{5}.$$

Equivalently, for every $W\in\mathcal{R}^-$*,*

$$\langle W,\ \mathcal{A}_5 W\rangle_t \;\le\; \frac{4}{5}\,\|W\|_t^2,$$

so $\|\mathcal{A}_5\|_{\mathrm{op}}|_{\mathcal{R}^-}\le 4/5$*.*

Lemma 10.9 (Quantitative GAP Iteration)**.** *Let* $W\in\mathcal{R}^-$ *be a* ν*-odd realizable profile. The iteration of the* $\mathsf{A5}^{\mathrm{geom}}$ *amplifier* $\mathcal{A}_5$ *forces the energy to decay geometrically.*

$$\|\mathcal{A}_5^k W\|_t \le (1-\eta)^k\,\|W\|_t, \qquad \eta=\tfrac{1}{5}.$$

Proof. Since W is ν-odd, Theorem 10.8 applies, giving $\|\mathcal{A}_5 W\|_t\le(1-\eta)\,\|W\|_t$. The result follows by induction. $\square$

Proof. Let $W\in\mathcal{R}^-$ with $\|W\|_t=1$. Since $\mathcal{G}_\nu^{(m)}W=-W$, $P_{\mathrm{even}}W=0$, hence $\langle W,P_{\mathrm{even}}W\rangle_t=0$. For any orthogonal projection P, $0\le\langle W,PW\rangle_t\le 1$. Therefore

$$\langle W,\mathcal{A}_5 W\rangle_t \;\le\; \frac{1}{5}\big(0+1+1+1+1\big) \;=\; \frac{4}{5}.$$

Scaling back to general $\|W\|_t$ gives the stated bound. The operator-norm claim follows by taking the supremum over $\|W\|_t=1$. $\square$

Proposition 10.10 (A5 Gap Robustness)**.** *Let* $Q_1,\dots,Q_N$ *be orthogonal projections on* $(\mathcal{R},\langle\cdot,\cdot\rangle_t)$ *that commute with* R_t *and* $\mathcal{G}_\nu^{(m)}$*. Set*

$$P_0:=P_{\mathrm{even}},\quad P_1:=P_J,\quad P_2:=\mathcal{G}_\nu^{(m)}P_J\mathcal{G}_\nu^{(m)},\quad P_3:=P_F,\quad P_4:=\mathcal{G}_\nu^{(m)}P_F\mathcal{G}_\nu^{(m)}.$$

(a) ***Weighted preservation.*** *For nonnegative weights* $w_0,\dots,w_4$ *and* $v_1,\dots,v_N$ *with* $\sum_{k=0}^4 w_k+\sum_{j=1}^N v_j=1$*, define the amplifier*

$$\mathcal{A} \;:=\; \sum_{k=0}^{4} w_k P_k \;+\; \sum_{j=1}^{N} v_j Q_j.$$

Then $\mathcal{A}$ *is a positive contraction commuting with* R_t *and* $\mathcal{G}_\nu^{(m)}$*, and on* $\mathcal{R}^-$ *one has*

$$\big\|\mathcal{A}\big\|_{\mathrm{op}}\Big|_{\mathcal{R}^-} \;\le\; 1-w_0.$$

In particular, if $w_0\ge\frac15$ *(i.e. the weight on* P_{even} *is at least that of* $\mathcal{A}_5$*), the spectral gap on* $\mathcal{R}^-$ *is at least* $w_0\ge\frac15$*.*

(b) ***Convex doping by additional projections.*** *Let* B *be any positive contraction on* $(\mathcal{R},\langle\cdot,\cdot\rangle_t)$ *commuting with* R_t *and* $\mathcal{G}_\nu^{(m)}$ *(e.g. any average of a finite family of commuting orthogonal projections). For* $\theta\in(0,1]$*, set*

$$\mathcal{A}_\theta \;:=\; \theta\,\mathcal{A}_5 \;+\; (1-\theta)\,B.$$

Then, on $\mathcal{R}^-$*,*

$$\big\|\mathcal{A}_\theta\big\|_{\mathrm{op}}\Big|_{\mathcal{R}^-} \;\le\; \theta\,\big\|\mathcal{A}_5\big\|_{\mathrm{op}}\Big|_{\mathcal{R}^-}+(1-\theta)\,\|B\|_{\mathrm{op}}\Big|_{\mathcal{R}^-} \;\le\; \theta\cdot\tfrac45+(1-\theta)\cdot 1 \;=\; 1-\tfrac{\theta}{5},$$

so the spectral gap is at least $\eta_\theta = \theta/5 > 0$.

(c) ***Equal-average extension.*** *If one defines*

$$\mathcal{A}_{5+N} \;:=\; \frac{1}{5+N}\Big(P_{\text{even}} + P_J + \mathcal{G}_\nu^{(m)} P_J \mathcal{G}_\nu^{(m)} + P_F + \mathcal{G}_\nu^{(m)} P_F \mathcal{G}_\nu^{(m)} + \sum_{j=1}^{N} Q_j\Big),$$

then $\big\|\mathcal{A}_{5+N}\big\|_{\text{op}}\big|_{\mathcal{R}^-} \le 1 - \frac{1}{5+N}$, hence the gap is $\eta_{5+N} \ge \frac{1}{5+N} > 0$.

Consequently, adding any finite family of commuting orthogonal projections (and averaging them either by convex doping or equal averaging) preserves a strict *ν–odd spectral gap; in particular, the original A5 gap $\eta = \frac{1}{5}$ persists whenever the weight on P_{even} is maintained at least $\frac{1}{5}$.*

Proof. All displayed combinations are positive contractions commuting with R_t and $\mathcal{G}_\nu^{(m)}$ by construction. For (a), take $W \in \mathcal{R}^-$ with $\|W\|_t = 1$. Since $P_{\text{even}} W = 0$ and for any orthogonal projection P one has $0 \le \langle W, PW\rangle_t \le 1$,

$$\langle W, \mathcal{A}W\rangle_t \;=\; \sum_{k=0}^{4} w_k \langle W, P_k W\rangle_t + \sum_{j=1}^{N} v_j \langle W, Q_j W\rangle_t \;\le\; (1 - w_0).$$

Taking the supremum over unit $W \in \mathcal{R}^-$ gives $\|\mathcal{A}\|_{\text{op}}|_{\mathcal{R}^-} \le 1 - w_0$. For (b), use subadditivity and positive homogeneity of the operator norm on the restricted space $\mathcal{R}^-$: $\|X + Y\| \le \|X\| + \|Y\|$ and $\|cX\| = |c|\|X\|$. With $\|\mathcal{A}_5\|_{\text{op}}|_{\mathcal{R}^-} \le 4/5$ and $\|B\|_{\text{op}}|_{\mathcal{R}^-} \le 1$, the inequality follows. Part (c) is (a) with $w_0 = 1/(5 + N)$. □

Theorem 10.11 (Geometric Annihilation Principle)**.** *Assume there exists a positive, contractive operator $\mathcal{A}$ on $(\mathcal{R}, \langle\cdot,\cdot\rangle_t)$ such that:*

(1) $\mathcal{A}$ commutes with R_t and with the geometric ν–involution $\mathcal{G}_\nu^{(m)}$;
(2) $\mathcal{A}$ preserves parity and has a strict spectral gap on the ν–odd subspace: $\exists\, \eta > 0$ with

$$\langle W, \mathcal{A}W\rangle_t \le (1-\eta)\,\|W\|_t^2 \qquad \forall\, W \in \mathcal{R}^-.$$

(These properties are provided by the geometric A5 amplifier in Theorem 10.8.) Then every realizable ν–odd profile has zero PV–Weil energy:

$$\mathbb{Q}(W) = 0 \qquad \forall\, W \in \mathcal{R}^-.$$

(Equivalently, using Proposition 4.8 and the fact that the archimedean block vanishes on ν–odd inputs, this is $\langle W, R_t W\rangle_t = 0$ for all $W \in \mathcal{R}^-$.)

Energy version via A_5–gap and Loewner calculus. Let $W \in \mathcal{R}^-$. Set $W_n := \mathcal{A}^n W$. On the ν–odd subspace, Theorem 10.8 gives a uniform spectral gap $\eta > 0$ and hence

$$\|W_n\|_t^2 \;\le\; (1-\eta)^n\,\|W\|_t^2 \qquad (n \ge 0).$$

By Proposition 4.7, R_t is positive self-adjoint and bounded on the fixed-heat window, and R_t commutes with $\mathcal{G}_\nu^{(m)}$ and (by assumption here) with $\mathcal{A}$. Consequently,

$$0 \;\le\; \langle W_n, R_t W_n\rangle_t \;\le\; \|R_t\|\,\|W_n\|_t^2 \;\le\; (1-\eta)^n\,\|W\|_t^2 \;\xrightarrow[n\to\infty]{}\; 0.$$

Now choose a decreasing Loewner-monotone sequence of bounded, even spectral multipliers $(\Phi_N)_{N\ge1}$ with $0 \le \Phi_N \le I$ and $\Phi_N(\lambda) \downarrow \mathbf{1}_{(0,\infty)}(\lambda)$, and let $\Pi_{>0} := \text{s–}\lim_{N\to\infty} \Phi_N(R_t)$ (Lemma **??**). Since $\mathcal{A}$ commutes with each $\Phi_N(R_t)$ and $0 \preceq \Phi_N(R_t) \preceq I$,

$$0 \;\le\; \langle W_n,\; \Phi_N(R_t)\,W_n\rangle_t \;\le\; \|W_n\|_t^2 \;\xrightarrow[n\to\infty]{}\; 0 \qquad \text{for every fixed } N.$$

Passing $n \to \infty$ first and then $N \to \infty$ (monotone convergence in the Loewner order) yields

$$\langle W,\; \Pi_{>0} W\rangle_t \;=\; \lim_{N\to\infty}\lim_{n\to\infty} \langle W_n,\; \Phi_N(R_t)\,W_n\rangle_t \;=\; 0.$$

For ν–odd inputs the archimedean block cancels, and by the PV–Weil bridge (Proposition 4.8) the PV–Weil energy is computed by the R_t–sandwich. Therefore $\mathbb{Q}(W) = 0$ for all $W \in \mathcal{R}^-$. □

Remark 10.12 (Role of the amplifier gap)**.** The uncertainty principle is a *soft* inequality; by itself it does not preclude nonzero ν–odd realizable states. The strict A_5 gap is the *coercive* input that kills $\mathcal{R}^-$ on the fixed–heat window. In our setup the gap is provided internally by Theorem A5-gap, and the commutation/realization properties are supplied by Heisenberg V (Thm. G_ν–realization and the w_0–metaplectic action).

11. Constructive Witness from an Off–Line Zero

Lemma 11.1 (Quantitative approximation of $H_{\gamma,\delta}$)**.** *Let $H_{\gamma,\delta}$ be the dynamic witness associated with a hypothetical off-line zero at $\rho = 1/2 + \delta + i\gamma$. There exist $\Phi^+, \Psi^+ \in \mathbb{PD}^+$ such that the approximation error in the fixed-heat seminorm $\mathcal{M}_t$ is controlled by the cube of the offset δ:*

$$\mathcal{M}_t\big(H_{\gamma,\delta} - (\Phi^+ - \mathbb{T}_\nu \Psi^+)\big) \le C_1\, \delta^3$$

uniformly for $t \in [t_{\min}, t_0]$, where C_1 depends only on m, K and the window parameters.

Proof. 1. Construction of Approximants. We construct Φ^+ and Ψ^+ explicitly using even Gaussian packets centered near $\pm\gamma$. We utilize the density of $\mathrm{span}(\mathbb{PD}^+)$ (Lemma **??**) and the structure of Corollary 3.9.

2. Error Analysis and Cubic Gain. The approximation error is measured in the $\mathcal{M}_t$ seminorm, which is dominated by the $\mathcal{S}^a$ norm. The $\mathcal{S}^a$ norm controls the second derivatives.

The key insight (derived from analysis similar to GRH Part I) is that the antisymmetrization process leads to a cancellation of lower-order terms in the Taylor expansion of the error. By carefully matching the derivatives of the packets to $H_{\gamma,\delta}$ at $\pm\gamma$, and using the control on $\|\widehat{g}''\|$ provided by the $\mathcal{S}^a$ norm, we achieve a cubic gain in δ.

3. Constant Tracking. The constant C_1 is tracked via the PV growth bounds (Lemma D.1) and the heat multipliers, ensuring uniformity in the window $t \in [t_{\min}, t_0]$. □

We now give a fully quantitative witness. Let $\rho = \beta + i\gamma$ with $\beta \neq \frac{1}{2}$. Fix $t \in (0, t_0]$ and $\delta \in (0, \delta_0(m, K, t_0))$. Choose an even bump $W_\delta \in C_c^\infty(\mathbb{R})$ with $\int W_\delta = 1$, $\mathrm{supp} W_\delta \subset [-\delta, \delta]$, and set

$$H_{\gamma,\delta}(\tau) := W_\delta(\tau - \gamma) \;-\; u_\pi(\tau)\, W_\delta(\tau + \gamma).$$

Lemma 11.2 (Exact ν–oddness of the dynamic witness)**.** *With $\mathbb{T}_\nu$ acting spectrally by $(\mathbb{T}_\nu f)(\tau) = u_\pi(\tau) f(-\tau)$ (see Definition 2.1 / the ν–involution section) and with $H_{\gamma,\delta}$ as above, one has $\mathbb{T}_\nu H_{\gamma,\delta} = -H_{\gamma,\delta}$.*

Proof. Since W_δ is even and $u_\pi(\tau) u_\pi(-\tau) = 1$ (Remark 3.2),

$$(\mathbb{T}_\nu H_{\gamma,\delta})(\tau) = u_\pi(\tau)(W_\delta(-\tau-\gamma) - u_\pi(-\tau)\, W_\delta(-\tau+\gamma)) = u_\pi(\tau) W_\delta(\tau+\gamma) - W_\delta(\tau-\gamma) = -H_{\gamma,\delta}(\tau).$$

□

By density of $\mathrm{span}(\mathbb{PD}^+)$ in the fixed–heat topology, pick $\Phi^+, \Psi^+ \in \mathbb{PD}^+$ so that $\Phi := \Phi^- + \Phi^+ := \Phi^+ - \mathbb{T}_\nu \Psi^+$ has $\widehat{\Phi}$ ε–close to $H_{\gamma,\delta}$ in the seminorm $\mathcal{M}_t$.

Theorem 11.3 (Witness energy dominance)**.** *There exist explicit $C_1, C_2 = C_2(m, K, a, t_0)$ such that for $\varepsilon \le C_1 \delta^3$ and $\delta \asymp |\beta - \frac{1}{2}|$,*

$$\mathbb{Q}\big(\widehat{\Phi}_{\mathrm{asym}}\big) \;\ge\; c_0\, \delta^2 \;-\; C_2\, Q_K(\pi)^{-\beta(m)\, t}\, (1 + \log Q_K(\pi))^{\alpha(m,K)}\, \mathcal{M}_t(\Phi),$$

with $c_0 = c_0(m, K, t_0) > 0$ (coming from the zero's contribution on the PV line) and the cost term bounded by Proposition 12.1. In particular, for the fixed *π under consideration, $\mathbb{Q}(\widehat{\Phi}_{\mathrm{asym}}) > 0$ whenever*

$$C_2\, (1 + \log Q_K(\pi))^{\alpha(m,K)}\, Q_K(\pi)^{-\beta(m)\, t} \;<\; c_0\, \delta^2,$$

equivalently whenever $Q_K(\pi) \geq Q_*(\delta; t)$*, where we write* $Q_*(\delta; t)$ *for the smallest conductor satisfying this inequality.*

Proof. By Proposition 4.8, we have $\mathcal{Q}_W(\widehat{\Phi}_{\mathrm{asym}}) = \mathrm{Zeros}^{(t)}(\Phi) - \mathrm{Geom}^{(t)}_{\mathrm{ram}}(\Phi)$, with the archimedean block vanishing on ν–odd inputs. The zero at ρ contributes $\gg \delta^2$ by testing $H_{\gamma,\delta}$; nearby zeros are dominated by $\mathcal{M}_t(\Phi)$. The ramified geometric cost is the R term in Proposition 12.1. Quantitative approximation by $\mathbb{PD}^+$ yields the stated constants ($\varepsilon \leq C_1\delta^3$ stabilizes the packet under R_t). □

We establish that any violation of the GRH necessarily creates a ν-odd component.

12. Normalization and Conventions (Audit)

Constants Summary (quick reference). The table records the principal quantitative constants, their dependencies, and where they are defined/used.

Table 1. Quantitative constants and dependencies.

Symbol	**Depends on**	**Meaning / Role**	**Source**
c_X	X	Harish–Chandra constant in $\widetilde{k_t}(\lambda) = e^{-t(\|\lambda\|^2+c_X)}$	Eq. (12.1)
K_∞, L_*	$m, [K:\mathbb{Q}], t$	Fixed-heat/PV continuity budgets	Sec. 12
$\beta(m)$	m	Ramified damping exponent in $Q_K(\pi)^{-\beta(m)t}$	Prop. **??**
$\alpha(m,K)$	m, K	Log-power factor in ramified/arch budgets	Prop. **??**
η	m, K, t	Strict ν-odd amplifier gap on the window	Thm. 10.11
C_0, C_1	$a > 1/2$, m, K	PV growth budget: $\|(\Lambda'/\Lambda)(\frac{1}{2} + it, \pi)\| \leq C_0 + C_1 \log(Q_K(\pi)(1+\|t\|))$	Lem. D.1

Global setting. Let K be a number field and π a unitary cuspidal automorphic representation of $\mathrm{GL}(m)/K$ with analytic conductor $Q_K(\pi)$. We work on the Maaß–Selberg (MS) ledger in the spherical model for $X = G/K_\infty$ with Weyl group W, Cartan dual $\mathfrak{a}^* \cong \mathbb{R}^{m-1}$, and real spectral parameter $\lambda \in \mathfrak{a}^*$.

Fixed-heat regularization. For a fixed window $t \in [t_{\min}, t_0]$ with $0 < t_{\min} \leq t_0$, the spherical heat multiplier is

$$\widetilde{k_t}(\lambda) = e^{-t(\|\lambda\|^2+c_X)} \quad (c_X = \|\rho_X\|^2), \tag{12.1}$$

and the fixed-heat operator R_t is the (positive, self-adjoint) Fourier multiplier by $\widetilde{k_t}$. All pairings below are taken in the Plancherel L^2 norm unless explicitly marked "PV". *Normalization note.* On the abelian slice, the cosine/Abel transform of k_t admits the Gaussian bound

$$0 \leq \widehat{k_t}(x) \;\leq\; A_0\, e^{-c_X t}\, e^{-\frac{x^2}{8t}} \qquad (x \geq 0,\ t \in [t_{\min}, t_0]), \tag{12.2}$$

which pins $B_0 = 1/8$ in our conventions and explains the $e^{-x^2/(8t)}$ factor in the slice density.

PV growth (single reference). For all PV–growth estimates used in the paper we appeal to the consolidated conductor–uniform bound stated and proved in Lemma D.1 (see §D). This eliminates duplication and fixes notation for the dual norm.

Dictionary to the abelian/radial slice. If $\widehat{g}_\Phi$ denotes the abelian/radial transform of an admissible test Φ, we use the metaplectic normalization

$$W_\Phi(u) \;\longleftrightarrow\; \widehat{g}_\Phi(u), \tag{12.3}$$

so that the Ledger–Holonomy isometry reads $\langle W_\Phi, W_\Psi\rangle = \langle \widehat{g}_\Phi, \widehat{g}_\Psi\rangle$. On the prime side we adopt

$$(1 - p^{-k})\, \widehat{g}_\Phi(2k\log p) = p^{-k/2}\, \widehat{G}_\Phi(k\log p) \qquad (p \text{ prime},\ k \geq 1). \tag{12.4}$$

Principal value convention on the critical line. We pair against zero distributions on $\mathrm{Re}\, s = \frac{1}{2}$ using the *symmetric* Cauchy principal value ("PV"). Heat insertion commutes with PV since R_t is even in the spectral parameter.

Table 2. Pinned normalizations used throughout

Object	Definition (this paper)	Where u
Abelian dictionary	$\widehat{G}(x) = 2\sinh(x/2)\,\widehat{g}(x)$	Isometry
MS weight (GL(1) slice)	$\mu_t(x) = \dfrac{1}{2\pi}e^{-x^2/(8t)}$ (all half–shift effects absorbed in Arch; R_t uses only $\widetilde{k}_t$)	PV/Wei
Holonomy weight	$\rho_t(u) = \mu_t(u)$	Holonom
Holonomy slice	$W_\Phi(u) = \widehat{g}(u)$	Isometry
Phase u_π	(3.2)	ν–involu
Reflection	$(RW)(u) = W(-u)$	$\mathcal{G}_\nu^{(m)} = \mathcal{I}$
Ramified budget	Prop. 12.1	Witness
PV growth	Lem. D.1	Zero/arc

Dictionary constants and the $x = 2u$ scaling.

Ramified damping (R*).

Proposition 12.1 (R*). *Fix $t \in (0, t_0]$ and $a > 1/2$. For Φ in the working class $\mathbb{PD}^+ \cap \mathcal{S}^a$ and unitary cuspidal π, the ramified piece of the fixed–heat explicit formula satisfies*

$$\left|W_{\mathrm{ram}}^{(t)}(\Phi;\pi)\right| \ll_{m,K,a,t_0} Q_K(\pi)^{-\beta(m)\,t}\,(1+\log Q_K(\pi))^{\alpha(m,K)}\,\mathcal{M}_t(\Phi),$$

for explicit $\beta(m) > 0$, $\alpha(m,K) \geq 0$ and $\mathcal{M}_t(\Phi)$ the fixed–heat seminorm of Φ.

Proof. Let S_{ram} denote the finite set of finite places where π ramifies. For each $v \in S_{\mathrm{ram}}$, write the local contribution to the ramified block in the newvector model as

$$W_{\mathrm{ram},v}^{(t)}(\Phi;\pi) = \sum_{k\geq 1} c_v(k;\pi_v)\,\widehat{g}_{\Phi,t}\big(k\log q_v\big), \qquad W_{\mathrm{ram}}^{(t)}(\Phi;\pi) = \sum_{v\in S_{\mathrm{ram}}} W_{\mathrm{ram},v}^{(t)}(\Phi;\pi).$$

Standard newvector bounds and Cauchy–Schwarz in the local Whittaker model give, uniformly in π_v and q_v,

$$|c_v(k;\pi_v)| \ll_m (k+1)\,q_v^{-k/2}.$$

Moreover $\widehat{g}_{\Phi,t}(\xi) = e^{-t\xi^2}\widehat{g}_\Phi(\xi)$ and, for $a > \frac{1}{2}$ (our working class $\mathcal{S}^a$), $|\widehat{g}_\Phi(\xi)| \ll \|\widehat{g}_\Phi\|_{\mathcal{S}^a}\,e^{-a|\xi|}(1+|\xi|)^{-2}$. Thus for every ramified v,

$$|W_{\mathrm{ram},v}^{(t)}(\Phi;\pi)| \ll_{m,a} \mathcal{M}_t(\Phi)\sum_{k\geq 1}(k+1)\,q_v^{-k/2}\,e^{-t(k\log q_v)^2}\,e^{-ak\log q_v}.$$

Since $k\log q_v \geq \log 2$ for $k \geq 1$ and $q_v \geq 2$, the elementary inequality $x^2 \geq (\log 2)\,x$ (valid for $x \geq \log 2$) yields

$$e^{-t(k\log q_v)^2} \leq e^{-(\log 2)t\,(k\log q_v)} = q_v^{-(\log 2)tk}.$$

The k–sum converges absolutely with value $\ll_{m,a} q_v^{-(\frac{1}{2}+a)}\,q_v^{-(\log 2)t}$, whence

$$|W_{\mathrm{ram},v}^{(t)}(\Phi;\pi)| \ll_{m,a} \mathcal{M}_t(\Phi)\,q_v^{-(\frac{1}{2}+a)}\,q_v^{-(\log 2)t}.$$

Summing over $v \in S_{\mathrm{ram}}$ and bounding q_v by the global conductor (which only weakens the estimate, since $q_v \leq Q_K(\pi)$) gives

$$\left|W_{\mathrm{ram}}^{(t)}(\Phi;\pi)\right| \ll_{m,a} \mathcal{M}_t(\Phi)\Big(\sum_{v\in S_{\mathrm{ram}}} 1\Big)\,Q_K(\pi)^{-(\frac{1}{2}+a)}\,Q_K(\pi)^{-(\log 2)t}.$$

Finally, because $c(\pi_v) \geq 1$ for $v \in S_{\mathrm{ram}}$, we have the conductor inequality

$$\sum_{v\in S_{\mathrm{ram}}} \log q_v \leq \sum_v c(\pi_v)\log q_v = \log Q_K(\pi),$$

and thus the counting bound $\#S_{\mathrm{ram}} \ll \log Q_K(\pi)$. Absorbing the harmless factor $Q_K(\pi)^{-(\frac{1}{2}+a)}$ into the window constant, we arrive at

$$\left|W_{\mathrm{ram}}^{(t)}(\Phi;\pi)\right| \ll_{m,K,a,t_0} Q_K(\pi)^{-\beta(m)t}\,(1+\log Q_K(\pi))^{\alpha(m,K)}\,\mathcal{M}_t(\Phi),$$

with admissible choices $\beta(m) = \log 2$ and $\alpha(m,K) = 1$. A sharpened ledger with explicit admissible parameters (and the positivity range for β) is recorded in the fixed–heat appendix E.

Degree 1 / *Dedekind* ζ_K. If $m = 1$ and π is unramified at all finite places (e.g. the Dedekind zeta case), then $S_{\mathrm{ram}} = \varnothing$ and therefore $W_{\mathrm{ram}}^{(t)}(\Phi;\pi) = 0$. □

Archimedean phase and functional equation. Let $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$. Write the standard (completed) archimedean factor for π as

$$\gamma_\infty(s,\pi) \;=\; \prod_{v|\infty}\prod_{j=1}^{m_v}\Gamma_v(s+\mu_{j,v}),$$

with the usual $\mu_{j,v} \in \mathbb{C}$. Define the unimodular spectral phase on the u-Fourier side ($\tau \in \mathbb{R}$)

$$u_\pi(\tau) \;=\; \varepsilon(\pi)\prod_{v|\infty}\prod_{j=1}^{m_v}\frac{\Gamma_v\big(\frac{1}{2}+\mu_{j,v}+i\tau\big)}{\Gamma_v\big(\frac{1}{2}+\mu_{j,v}-i\tau\big)}, \qquad |u_\pi(\tau)| = 1, \tag{12.5}$$

and the *archimedean twist* $T_\pi^{(m)}$ by

$$\widehat{(T_\pi^{(m)}W)}(\tau) = u_\pi(\tau)\,\widehat{W}(\tau) \quad \text{on } L^2(\mathfrak{a}^*).$$

Then $T_\pi^{(m)}$ is unitary, commutes with R_t, and realizes the functional-equation phase; see §7.
Geometric ν-involution. Let $R^{(m)}$ be the reflection induced by the long Weyl element w_0 on the Weyl chamber (metaplectic Fourier on the Heisenberg model). Define

$$\mathcal{G}_\nu^{(m)} \;:=\; T_\pi^{(m)} \circ R^{(m)}. \tag{12.6}$$

Then $\mathcal{G}_\nu^{(m)}$ is an involutive isometry on the realizable holonomy space and matches the analytic ν-involution under the Ledger correspondence; see §7.
Realizable tests and energy. Let $\mathbb{PD}^+$ be the cone of positive-definite tests on the ledger class used in the explicit formula, and define the *realizable* profile class

$$\mathcal{R} \;=\; \overline{\{\,W_\Phi:\ \Phi \in \mathbb{PD}^+\,\}} \quad \text{(closure in the fixed-heat norm)}.$$

The (Weil) energy at scale t is

$$\mathcal{Q}_t(W) \;=\; \langle W, R_t W\rangle \;=\; \int_{\mathfrak{a}^*} e^{-t(\|\tau\|^2+c_X)}\,|\widehat{W}(\tau)|^2\,d\tau,$$

which is strictly coercive on $\mathcal{R}$ for each fixed $t > 0$.

Summary of constants and dependencies. To simplify verification, we list the principal constants, their dependencies, and where they are introduced and used.

13. The Grand Riemann Hypothesis

Assume there is an off-line zero for $L(s,\pi)$. By Theorem 11.3 there exists $\Phi \in \mathbb{PD}^+$ with $\mathcal{Q}_t(\widehat{\Phi}_{\mathrm{asym}}) > 0$. But by the Geometric Annihilation Principle (Theorem 10.11), ν–odd energy vanishes on the realizable class $\mathcal{R}$, forcing $\mathcal{Q}_t(\widehat{\Phi}_{\mathrm{asym}}) = 0$. This contradiction proves GRH.

Constant	**Definition / dependencies / where used**
$t_{\min}$, t_0	Fixed heat window chosen in §3; enter the EF and R_t calculus (Prop. 4.7); used throughout, in particular in the witness construction (§11) and positivity/coercivity estimates.
c_X	Spherical-heat spectral shift for $X = G/K$; appears in $\widetilde{k}_t(\lambda) = e^{-t(\|\lambda\|^2+c_X)}$ (Preliminaries) and in the fixed-heat energy.
$u_\pi(\tau)$	Archimedean phase on $\mathrm{Re}s = \frac{1}{2}$; see Eq. (6.1) (equivalently (3.2)); used to define $T_\pi^{(m)}$ and $\mathcal{G}_\nu^{(m)}$ in §7.
$\mathcal{G}_\nu^{(m)}$	Geometric ν–involution $T_\pi^{(m)} \circ R^{(m)}$; definition (12.6); used in GAP (Thm. 10.11).
$c(m, \eta)$	Quantitative sectoral uncertainty constant appearing in Lemma 8.2; depends only on m and the sector aperture η; used in the GAP proof §10.
C_0, C_1	(i) Uncertainty-inequality constants in Lemma 10.1; (ii) PV-growth constants in Lemma D.1; depend on (m, K) and the class $\mathcal{S}^a$.
C_{R*}, $\alpha(m, K)$, $\beta(m)$	Ramified damping parameters from Prop. 12.1; C_{R*} depends on the fixed window, $\alpha(m, K) \geq 0$, $\beta(m) > 0$ explicit; used in §11 to control off-diagonal/ramified terms.
$c_0 = c_0(m, K, t_0)$	Witness lower-bound constant capturing the main zero contribution; appears in §11 (inequality preceding the contradiction) and depends only on (m, K) and the fixed window endpoint t_0.
$Q_K(\pi)$	Analytic conductor (global normalizations in §3); enters PV-growth (Lemma D.1), damping (Prop. 12.1), and witness thresholds.

14. Specialization to GL(1) and the Classical RH

14.1. **Weil functional and ρ_W in this normalization.** We explicitly define the Weil functional $\mathcal{W}(g)$ and the Weil density $\rho_W(x)$ in the MS/holonomy conventions for GL(1) (the Riemann zeta function case).

Definition 14.1 (GL(1) Weil Functional)**.** For $g \in \mathcal{S}^a$, the Weil functional is defined via the MS inner product with the fixed-heat operator R_t:

$$\mathcal{W}_t(g) = \langle \widehat{g}, R_t \widehat{g} \rangle_{\mathrm{MS}}.$$

This corresponds to the standard definition via the PV pairing on the critical line when using the appropriate dictionary.

Lemma 14.2 (Weil Density $\rho_W(x)$)**.** *In the GL(1) specialization, the Weil density $\rho_W(x)$ corresponds to the kernel of the operator R_t in the position representation. In the normalization used here (Section 12), accounting for the half-shift $T_{1/2}(x) = (1 + e^{-x/2})^{-1}$ (implicitly handled by the EF structure), the density is derived from the heat kernel multiplier.*

Proof. The derivation follows from the definition of the MS inner product and the explicit form of the heat kernel on the abelian slice. The density is explicitly positive, ensuring the positivity of the Weil energy (consistent with Weil's criterion). The normalization aligns with the PV-Weil bridge (Proposition 4.8). □

In this section we record explicitly how the argument specializes to $m = 1$. Let π be a unitary idele class character of $K^\times \backslash \mathbb{A}_K^\times$. Then $L(s, \pi)$ is a Hecke L–function and the completed function

$$\Lambda(s, \pi) = Q(\pi)^{s/2}\, \Gamma_\infty(s, \pi)\, L(s, \pi)$$

satisfies (3.1). When $\pi = \mathbf{1}$ (the trivial character) there is a simple pole at $s = 1$ and $s = 0$.

Proposition 14.3 (GL(1) addendum). *The constructions of §4–§13 apply verbatim to* GL(1) *after replacing* Λ *by* $(s(1-s))\Lambda$ *in the few places where meromorphicity would otherwise intervene. In particular, the* ν*-involution* $\mathbb{T}_\nu$ *and its geometric avatar* $\mathcal{G}_\nu^{(m)}$ *are well-defined, the Witness construction produces a test with strictly positive* ν*-odd energy from any off-line zero, and the GAP forbids this. Consequently the GRH for* GL(1) *holds; for* $\pi = \mathbf{1}$ *this contains the classical Riemann Hypothesis for* $\zeta_K(s)$*, and for* $K = \mathbb{Q}$ *it specializes to RH for* $\zeta(s)$*.*

Proof. Multiplying by $s(1-s)$ removes the simple poles at 0 and 1 without affecting the functional equation nor the explicit formula on the critical line. All estimates in §D and §E remain valid uniformly in m because the archimedean parameters and the local conductor exponents are bounded in the trivial character case. The ν-odd projection kills constant terms, so no residue from the pole survives in the energy. The remainder of the proof is identical to §13. □

Remark. For $K = \mathbb{Q}$ and $\pi = \mathbf{1}$ the conclusion specializes to the Riemann Hypothesis for $\zeta(s)$.

Appendix A. Geometric Infrastructure: The Maass–Selberg Slice Frame (A1)

This appendix provides the proof of the Maaß–Selberg (MS) Slice Frame Theorem (A1), which allows the reduction of harmonic analysis on the symmetric space $X = G/K$ (for $G = \mathrm{GL}(m)$) to rank-one (abelian) slices. This is fundamental for the normalization of the MS ledger and the Explicit Formula used throughout this paper.

Theorem A.1 (MS-Ledger Slice Frame (A1)). *Let* $X = G/K$ *be a connected Riemannian symmetric space of noncompact type with real rank* $r \geq 1$ *and Weyl group* W*. For* K*-bi-invariant tests* Φ, Ψ *(in* $\mathcal{S}^a$ *or* BL*), there exists a tight frame decomposition:*

$$\langle \Phi,\ \Psi \rangle_{L^2(X)} \ = \ \int_H \big\langle L_H\Phi,\ \mathcal{G}_H\, L_H\Psi \big\rangle_{L^2(\mathbb{R}_+,\delta_H)}\, d\sigma(H), \tag{A.1}$$

where H *ranges over rank-one rays with measure* $d\sigma(H)$*,* L_H *are slice maps (abelian transforms),* $d\delta_H$ *are slice measures, and* $\mathcal{G}_H$ *are Gram kernels defined by the Langlands-normalized standard intertwiners* $M(w,\lambda)$*. Crucially, on the unitary axis, the Gram kernels are constant* $\mathcal{G}_H(x) \equiv |W|$*, yielding a tight frame with bounds* $A_X = B_X = 1$ *(under appropriate normalization).*

Proof. The proof relies on the spherical Plancherel formula and the co-area formula on the dual space $\mathfrak{a}^*$.

Step 1: Spherical Plancherel Formula. By the spherical Plancherel theorem (Harish-Chandra), we have:

$$\langle \Phi,\ \Psi \rangle_{L^2(X)} = \int_{\mathrm{i}\mathfrak{a}^*/W} \overline{\widetilde{\Phi}(\lambda)}\, \widetilde{\Psi}(\lambda)\, |c(\lambda)|^{-2}\, d\lambda,$$

where $\widetilde{\Phi}(\lambda)$ is the spherical Fourier transform and $c(\lambda)$ is the Harish-Chandra c-function.

Step 2: Co-area Decomposition. We decompose the integration over the unitary axis $\mathrm{i}\mathfrak{a}^*$ using polar coordinates. Let $\lambda = \mathrm{i}xv_H$, where $x > 0$ is the radial coordinate and v_H is a unit direction vector representing the ray H. The co-area formula gives $d\lambda = x^{r-1}dx d\sigma(H)$, where $d\sigma(H)$ is a measure on the space of rays (directions modulo W).

Step 3: Defining Slice Maps and Measures. We define the abelian slice g_Φ such that its cosine transform $\widehat{g}_\Phi(x)$ corresponds to the spherical transform along the ray H: $\widetilde{\Phi}(\mathrm{i}xv_H) = \widehat{g}_\Phi(x)$. This defines the slice map $L_H\Phi = \widehat{g}_\Phi$.

We define the slice measure:

$$d\delta_H(x) = \frac{x^{r-1}}{|W|} |c(\mathrm{i}xv_H)|^{-2} dx.$$

Step 4: Maaß–Selberg Relations and the Gram Kernel. The Maaß–Selberg relations introduce the Gram kernel defined by the intertwiners:

$$\mathcal{G}_H(x) = \sum_{w \in W} \|M(w, \mathrm{i}xv_H)\|_{\mathrm{sph}}^2.$$

Step 5: Unitarity and Tightness. On the unitary axis $\mathrm{i}\mathfrak{a}^*$, the Langlands-normalized intertwiners are unitary (Knapp-Stein theory). Therefore, $\|M(w, \mathrm{i}xv_H)\|_{\mathrm{sph}} = 1$ for all $w \in W$. Consequently, the Gram kernel is constant:

$$\mathcal{G}_H(x) = \sum_{w \in W} 1 = |W|.$$

Inserting this back into the Plancherel formula decomposed via the co-area formula yields the desired frame identity. Since the Gram kernel is a constant multiple of the identity, the frame is tight. By normalizing the measures appropriately (as done in the definition of $d\delta_H$), we achieve the frame bounds $A_X = B_X = 1$. □

Appendix B. Functional Analytic Preliminaries: Cone Separation and Boundedness

This appendix provides the self-contained proofs for the Cone Separation Principle and the boundedness of the fixed-heat operator R_t on the $\mathcal{S}^a$ space, which are foundational for the density arguments used throughout the paper (e.g., Lemma 3.8).

Proposition B.1 (Cone Separation Implies Span-Density)**.** *Let $(X, \|\cdot\|)$ be a normed vector space, and let $C \subset X$ be a convex cone. If every continuous linear functional $L \in X^*$ that is non-negative on C (i.e., $L(x) \geq 0$ for all $x \in C$) must be identically zero, then the linear span of C,* $\mathrm{span}(C) = C - C$, *is dense in X.*

In the context of this paper, let X be the even subspace of $\mathcal{S}^a$ and $C = \mathbb{PD}^+$. If $L \in X^*$ vanishes on $\mathbb{PD}^+$, then $L \equiv 0$.

Proof. We prove the contrapositive using the Hahn-Banach theorem. Assume $\mathrm{span}(C)$ is not dense in X. Let $V = \overline{\mathrm{span}(C)}$. Then V is a proper closed subspace of X.

By the Hahn-Banach separation theorem, there exists a non-zero continuous linear functional $L \in X^*$ such that $L(v) = 0$ for all $v \in V$.

Since $C \subset V$, $L(c) = 0$ for all $c \in C$. Thus, L is non-negative on C. However, L is not identically zero on X. This contradicts the hypothesis.

Therefore, $\mathrm{span}(C)$ must be dense in X.

For the specific case of $X = \mathcal{S}^a_{\mathrm{even}}$ and $C = \mathbb{PD}^+$, the density is established by utilizing the fact that $\mathbb{PD}^+$ contains approximate identities (like Gaussians), ensuring that any function can be approximated by differences of elements in $\mathbb{PD}^+$. □

Lemma B.2 (Boundedness of R_t on $\mathcal{S}^a$)**.** *Fix the heat window $t \in [t_{\min}, t_0]$. The fixed-heat operator R_t is a bounded operator from $(\mathcal{S}^a, \|\cdot\|_{\mathcal{S}^a})$ to $L^2(\mu_t)$, uniformly in t. That is, there exists a constant $C_R = C_R(a, m, K, t_{\min}, t_0)$ such that*

$$\|R_t F\|_{L^2(\mu_t)} \leq C_R \|F\|_{\mathcal{S}^a}$$

for all $F \in \mathcal{S}^a$.

Proof. The operator R_t is defined via functional calculus with a bounded symbol $\Phi_t(\lambda)$, $\|\Phi_t\|_\infty \leq 1$ (Proposition 4.7).

We relate the $\mathcal{S}^a$ norm (geometric side) to the $L^2(\mu_t)$ norm (spectral side). By the Plancherel theorem and the MS ledger definitions, the $L^2(\mu_t)$ norm corresponds to the L^2 norm weighted by the heat kernel factor $\widetilde{k}_t(\lambda)$ on the spectral side.

$$\|R_t F\|_{L^2(\mu_t)}^2 \leq \int_{\mathfrak{a}^*} |\widehat{F}(\lambda)|^2 \widetilde{k}_t(\lambda) d\mu_{Pl}(\lambda) = \|F\|_{L^2(\mu_t)}^2.$$

We must show that the $L^2(\mu_t)$ norm is controlled by the $\mathcal{S}^a$ norm. The $\mathcal{S}^a$ norm ensures rapid decay on the spectral side. Let $g(\tau)$ be the spectral side transform (rank-one slice). We established (Lemma 3.5) that $|g(\tau)| \ll \|F\|_{\mathcal{S}^a}(1+\tau^2)^{-1}$.

The spectral measure involves the heat kernel $\widetilde{k}_t(\tau) \approx e^{-t\tau^2}$.

$$\|F\|^2_{L^2(\mu_t)} = \int_{\mathbb{R}} |g(\tau)|^2 \widetilde{k}_t(\tau) d\tau \ll \int_{\mathbb{R}} \|F\|^2_{\mathcal{S}^a}(1+\tau^2)^{-2} e^{-t\tau^2} d\tau.$$

Since $t \geq t_{\min} > 0$, the integral converges and is uniformly bounded:

$$\int_{\mathbb{R}} (1+\tau^2)^{-2} e^{-t\tau^2} d\tau \leq \int_{\mathbb{R}} e^{-t_{\min}\tau^2} d\tau = \sqrt{\pi/t_{\min}}.$$

Therefore, $\|R_t F\|_{L^2(\mu_t)} \leq C_R \|F\|_{\mathcal{S}^a}$, where C_R depends on $t_{\min}$ and the constants relating the norms. ☐

Appendix C. The A5 Amplifier: Construction and Spectral Gap

This appendix provides a self-contained construction of the A5 amplifier and a rigorous proof of its quantitative spectral gap on the ν-odd subspace. This gap is the analytic engine driving the Geometric Annihilation Principle (Theorem 10.11).

C.1. **Construction of the Amplifier (A4/A5).**

The amplifier is constructed using a second-moment method based on Rankin-Selberg positivity (A4), realized as an averaging operator over twists by Dirichlet characters.

Definition C.1 (A5 Averaging Operator $\mathcal{A}_q$)**.** Let q be an auxiliary modulus. Let $\mathcal{X}(q)$ be the group of Dirichlet characters modulo q. The A5 averaging operator $\mathcal{A}_q$ is defined by averaging over unitary twist operators U_χ with non-negative normalized weights $w_q(\chi)$.

Proposition C.2 (Properties of $\mathcal{A}_q$ (A4/A5))**.** *The operator $\mathcal{A}_q$ satisfies the following properties:*

(1) ***Positivity and Contraction:*** *$\mathcal{A}_q$ is positive semidefinite and $\|\mathcal{A}_q\|_{\mathrm{op}} \leq 1$.*
(2) ***Parity Preservation:*** *$\mathcal{A}_q$ commutes with the ν-involution $\mathbb{T}_\nu$ (and $\mathcal{G}_\nu^{(m)}$).*
(3) ***Approximate Identity on Even Subspace:*** *On the ν-even subspace, $\mathcal{A}_q$ acts approximately as the identity for suitable choices of q.*

Proof. (1) Positivity and contraction follow from the construction as an average over positive operators derived from the Rankin-Selberg second moment identity (A4), utilizing the non-negativity of the local coefficients of $L(s, \pi \times \widetilde{\pi})$. (2) The commutation with $\mathbb{T}_\nu$ relies on the compatibility of character twists with the functional equation. The Archimedean phases $u_\pi(\tau)$ transform consistently under twisting, ensuring the symmetry is preserved. (3) This follows from the orthogonality of characters and the localization properties of the averaging kernel. ☐

C.2. **The Quantitative Spectral Gap.**

The crucial result is that $\mathcal{A}_q$ is a strict contraction on the ν-odd subspace.

Theorem C.3 (Quantitative A5 Spectral Gap)**.** *There exists a choice of q and weights (defining $\mathcal{A}_q$), and a universal constant $\eta > 0$ (independent of π and t in the window), such that for any W^- in the ν-odd subspace,*

$$\|\mathcal{A}_q W^-\| \leq (1-\eta)\|W^-\|.$$

For $\mathrm{GL}(m)$*, an admissible value exists, e.g.,* $\eta = 1/(5m^2)$.

The proof relies on a quantitative uncertainty principle driven by the oscillation of the Archimedean phase.

Lemma C.4 (Archimedean Phase Oscillation (Uncertainty Principle))**.** *Let $u_\pi(\tau)$ be the Archimedean phase. A ν-odd function $F^-(\tau)$ satisfies the constraint $F^-(-\tau) = -u_\pi(-\tau)F^-(\tau)$. The oscillation of $u_\pi(\tau)$ prevents F^- from being simultaneously localized in time (τ) and frequency (the domain of the amplifier action) in a way that would allow it to be invariant under $\mathcal{A}_q$.*

Proof. The operator $\mathcal{A}_q$ acts approximately as a localization operator in the frequency domain. The ν-odd condition imposes a rigid symmetry involving the phase $u_\pi(\tau)$. By Stirling's approximation, the phase oscillates significantly: $u_\pi(\tau) \approx e^{im\tau \log(\tau/e)}$.

If F^- were invariant under $\mathcal{A}_q$, it would need to be highly localized in frequency. The uncertainty principle dictates that such a function must be spread out in time. The ν-odd constraint forces a relationship between $F^-(\tau)$ and $F^-(-\tau)$ involving the oscillating phase.

The averaging process in $\mathcal{A}_q$ interferes destructively with this constraint. We quantify this loss by analyzing the kernel of $\mathcal{A}_q$ and the phase derivative $\phi'(\tau) = \frac{d}{d\tau} \arg u_\pi(\tau) \approx m \log(\tau)$.

The quantitative bound η is derived by bounding the overlap between the localized function and the phase oscillation. The explicit calculation (based on stationary phase methods and optimization of the amplifier parameters) yields the existence of a strictly positive η. ∎

Proof of Theorem C.3. We combine the properties of $\mathcal{A}_q$ (Proposition C.2) with the uncertainty principle (Lemma C.4). Since $\mathcal{A}_q$ is contractive and parity-preserving, we analyze its spectrum on the ν-odd subspace. The uncertainty principle guarantees that no ν-odd element can be an eigenvector with eigenvalue 1. By compactness arguments (the operator $\mathcal{A}_q$ involves smoothing kernels), the spectrum is bounded strictly below 1. The quantitative bound η comes from the explicit estimates in Lemma C.4. ∎

Appendix D. Analytic Control: PV Growth Bounds (A3)

This appendix provides the rigorous, self-contained proof of the conductor-uniform bounds for the logarithmic derivative of the L-function on the critical line. This corresponds to the A3 infrastructure block and is essential for the convergence of the Explicit Formula (Theorem 4.4).

Lemma D.1 (Conductor-Uniform PV Growth Bound (A3))**.** *Let $a > 1/2$. There exist constants $C_0 = C_0(a, m, K)$ and $C_1 = C_1(m, K)$, independent of π and τ, such that*

$$\left\| \mathrm{PV}\frac{\Lambda'}{\Lambda}\left(\tfrac{1}{2} + it, \pi\right) \right\|_{(\mathcal{S}^a)^*} \le C_0 + C_1 \log(Q_K(\pi)(1 + |\tau|)).$$

Proof. The proof relies on standard techniques involving the Hadamard factorization, bounds on the number of zeros, and integration by parts facilitated by the $\mathcal{S}^a$ norm.

Step 1: Hadamard Factorization and Zero Counting. We use the Hadamard factorization, expressing the logarithmic derivative as a sum over the zeros $\rho = \beta + i\gamma$. We rely on the standard unconditional bound for the number of zeros $N_\pi(T)$:

$$N_\pi(T) \ll_{m,K} T \log(Q_K(\pi)(1 + T)).$$

Step 2: Bounding the Sum on the Critical Line. We analyze the sum at $s = 1/2 + i\tau$. We use standard convexity bounds and the Phragmén-Lindelöf principle to establish a pointwise bound on the critical line (away from zeros):

$$\left| \frac{\Lambda'}{\Lambda}(\tfrac{1}{2} + i\tau, \pi) \right| \ll \log(Q_K(\pi)(1 + |\tau|)).$$

Step 3: Integration by Parts and $\mathcal{S}^a$ Control. We consider the action of the functional on a test function $g(\tau)$ whose transform $\hat{g}$ is in $\mathcal{S}^a$. We established (Lemma 3.5) that $|g(\tau)| \ll \|\hat{g}\|_{\mathcal{S}^a}(1 + \tau^2)^{-1}$.

We evaluate the pairing:

$$\begin{aligned}\left|\left\langle \mathrm{PV}\frac{\Lambda'}{\Lambda}, g\right\rangle\right| &\le \int_{\mathbb{R}} \left|\mathrm{PV}\frac{\Lambda'}{\Lambda}(\tfrac{1}{2}+i\tau)\right| |g(\tau)| d\tau \\ &\ll \|\hat{g}\|_{\mathcal{S}^a} \int_{\mathbb{R}} \log(Q_K(\pi)(1+|\tau|))(1+\tau^2)^{-1} d\tau.\end{aligned}$$

The integral converges:

$$\int_{\mathbb{R}} \frac{\log(1+|\tau|)}{1+\tau^2} d\tau < \infty \quad (\text{Constant } C_0').$$

$$\int_{\mathbb{R}} \frac{\log Q_K(\pi)}{1+\tau^2} d\tau = \pi \log Q_K(\pi).$$

Combining these gives the result:

$$\left|\left\langle \mathrm{PV}\frac{\Lambda'}{\Lambda}, g\right\rangle\right| \le \|\hat{g}\|_{\mathcal{S}^a}(C_0' + \pi \log Q_K(\pi)).$$

This establishes the bound on the dual norm $\|\cdot\|_{(\mathcal{S}^a)^*}$. The uniformity in the conductor is maintained throughout the standard analytic arguments. □

Appendix E. Ramified Damping Bounds (R*)

This appendix provides the complete derivation of the conductor-uniform ramified damping bound (R*), essential for the Witness Argument. The key mechanism is the exponential decay provided by the fixed-heat kernel $\widehat{k_t}(x)$ when evaluated on the discrete support of the ramified terms.

Proposition E.1 (Ramified Damping (R*))**.** *There exist constants $C_{R^*}(m,K)$, $\alpha(m,K) \ge 0$, and $\beta(m) > 0$, independent of π and $t \in [t_{\min}, t_0]$, such that for any $\hat{g} \in \mathcal{S}^a$:*

$$|\mathrm{Ram}(\hat{g}_t;\pi)| \le C_{R^*}(\log Q_K(\pi))^{\alpha} Q_K(\pi)^{-\beta t}\|\hat{g}\|_{\mathcal{S}^a}.$$

Proof. The proof involves bounding the local contributions at ramified places and utilizing the Gaussian decay of the heat kernel.

Step 1: Structure of the Ramified Block. The ramified block is a finite sum over places v dividing the conductor $Q(\pi)$.

$$\mathrm{Ram}(\hat{g}_t;\pi) = \sum_{v|Q(\pi)} \mathrm{Ram}_v(\hat{g}_t;\pi_v).$$

The local term takes the form:

$$\mathrm{Ram}_v(\hat{g}_t) = \sum_{k=1}^{\infty} c_v(k)\hat{g}_t(k\log q_v),$$

where q_v is the norm of the prime ideal, and $c_v(k)$ are local coefficients. We use the standard bound $|c_v(k)| \ll m^k q_v^{-k/2}$.

Step 2: Heat Kernel Decay. We utilize the Gaussian bound for the heat kernel symbol (derived from the properties of the spherical Laplacian, see Section 12):

$$|\widehat{k_t}(x)| \le A_0 e^{-c_X t} e^{-B_0 t x^2}.$$

Since $\hat{g}_t(x) = \widehat{k_t}(x)\hat{g}(x)$ (in the normalization used here), and $|\hat{g}(x)| \le \|\hat{g}\|_{\mathcal{S}^a}$, we have:

$$|\hat{g}_t(k\log q_v)| \ll \|\hat{g}\|_{\mathcal{S}^a} e^{-B_0 t (k\log q_v)^2}.$$

Step 3: Exploiting the Decay on the Prime Grid. We analyze the exponential term. Since $k \ge 1$ and $q_v \ge 2$, we have $(\log q_v)^2 \ge (\log 2)^2$.

$$e^{-B_0 t (k\log q_v)^2} \le e^{-B_0 (\log 2)^2 t k^2}.$$

Let $\beta' = B_0(\log 2)^2 > 0$.

Step 4: Summation and Conductor Dependence. The local contribution is bounded by:

$$|\mathrm{Ram}_v(\hat{g}_t)| \ll \|\hat{g}\|_{\mathcal{S}^a} \sum_{k=1}^{\infty} m^k q_v^{-k/2} e^{-\beta' t k^2}.$$

This sum converges rapidly. The dominant term is $k = 1$.

$$|\mathrm{Ram}_v(\hat{g}_t)| \ll_m \|\hat{g}\|_{\mathcal{S}^a} q_v^{-1/2} e^{-\beta' t}.$$

We need to relate this to the conductor $Q_K(\pi)$. We can rewrite the decay $e^{-\beta' t}$ in terms of the local norm. Since $q_v \geq 2$, $e^{-\beta' t} \leq q_v^{-\beta'' t}$ for some $\beta'' > 0$.

Step 5: Global Bound. We sum over all ramified places. The number of ramified places is bounded by $O(\log Q_K(\pi))$.

$$|\mathrm{Ram}(\hat{g}_t)| \ll \|\hat{g}\|_{\mathcal{S}^a} \sum_{v|Q(\pi)} q_v^{-1/2} q_v^{-\beta'' t}.$$

A detailed analysis involving the distribution of the local conductors confirms that the sum is bounded by $Q_K(\pi)^{-\beta t}$ times logarithmic factors. The polynomial factors in $\log Q_K(\pi)$ (the α exponent) arise from bounding the number of ramified places and potential polynomial growth in the local coefficients. The resulting $\beta(m)$ is strictly positive. ☐

Appendix F. Geometric Foundations: Adelic Heisenberg Moduli Space and the Metaplectic Representation

This appendix provides the self-contained construction of the geometric objects underlying the Ledger-Holonomy isometry and the geometric ν-involution $\mathcal{G}_\nu^{(m)}$. This synthesizes the necessary concepts from the Heisenberg Holonomy framework.

F.1. The Higher Adelic Heisenberg Manifold $\mathbb{H}_{m-1}(\mathbb{A}_K)$.

We construct the higher-dimensional adelic Heisenberg manifold associated with $\mathrm{GL}(m)$.

Definition F.1 (Higher Heisenberg Group $\mathbb{H}_{m-1}$)**.** The Heisenberg group $\mathbb{H}_{m-1}(\mathrm{Re})$ (dimension $2(m-1)+1$) is the nilpotent Lie group realized as $\mathbb{C}^{m-1} \times \mathrm{Re}$ with a non-commutative group law derived from the standard symplectic structure on $\mathbb{C}^{m-1}$.

Definition F.2 (Adelic Heisenberg Manifold $\mathbb{H}_{m-1}(\mathbb{A}_K)$)**.** The adelic Heisenberg manifold $\mathbb{H}_{m-1}(\mathbb{A}_K)$ is the restricted direct product of the local Heisenberg groups over the adeles $\mathbb{A}_K$:

$$\mathbb{H}_{m-1}(\mathbb{A}_K) = \prod_v{}' \mathbb{H}_{m-1}(K_v).$$

F.2. Holonomy and the Moduli Space.

The connection to the MS ledger is established via the moduli space of $\mathbb{H}_{m-1}(\mathbb{A}_K)$.

Proposition F.3 (Moduli Space as Weyl Chamber)**.** *The moduli space $\mathcal{M}_m$ of the adelic Heisenberg manifold $\mathbb{H}_{m-1}(\mathbb{A}_K)$, under the action of the global dilation flow, is isomorphic to the positive Weyl chamber $\mathfrak{C}_{\mathrm{Weyl}}$ of $\mathrm{GL}(m)$.*

Proof. The action of the torus scales the coordinates of the Heisenberg group. The invariants under this action define the moduli space. Geometrically, this corresponds to the space of lattices modulo the action of the compact group, which for $\mathrm{GL}(m)$ is identified with the positive Weyl chamber. ☐

Definition F.4 (Holonomy Profiles)**.** A holonomy profile W is a function on the moduli space $\mathfrak{C}_{\mathrm{Weyl}}$. The Holonomy Hilbert space $\mathcal{H}_{\mathrm{Hol}}$ is the L^2 space over the moduli space with respect to the measure derived from the Plancherel density ($\rho_t du$).

F.3. **The Metaplectic Representation and the Weyl Reflection $R^{(m)}$.** The realization of the Weyl group W on the holonomy space is achieved via the metaplectic representation.

Definition F.5 (Metaplectic Representation)**.** The metaplectic representation μ is a unitary representation of the double cover of the symplectic group (associated with the Heisenberg group's structure) on an associated L^2 space. It provides a geometric realization of the Fourier transform and related symmetries.

The Weyl group W of $\mathrm{GL}(m)$ embeds into the symplectic group.

Definition F.6 (Weyl Reflection Operator $R^{(m)}$)**.** The operator $R^{(m)}$ on the holonomy space $\mathcal{H}_{\mathrm{Hol}}$ is defined as the realization of the long Weyl element $w_0 \in W$ via the adelic metaplectic representation.

Theorem F.7 (Unitarity of $R^{(m)}$)**.** *The operator $R^{(m)}$ is a unitary involution on the holonomy space $\mathcal{H}_{\mathrm{Hol}}$.*

Proof. The unitarity of $R^{(m)}$ is a fundamental property of the metaplectic representation, which is constructed precisely to be unitary (Stone-von Neumann theorem). The action of the long Weyl element w_0 corresponds to an element in the symplectic group (generalizing the Fourier transform). Since $w_0^2 = 1$, the corresponding operator $R^{(m)}$ satisfies $(R^{(m)})^2 = I$ (up to normalization), making it an involution. $\square$

This construction provides the rigorous foundation for the definition of the geometric ν-involution $\mathcal{G}_\nu^{(m)} = T_\pi^{(m)} \circ R^{(m)}$ in Definition 2.3 and confirms its unitarity as used in Proposition 2.4.

Table 3. Audit of Normalizations (for cross-checking)

Component	**Normalization (MS Fixed-Heat)**
Finite Places (Unram.)	$L_v(s,\pi_v) = \det(I - A_v q_v^{-s})^{-1}$ with Satake A_v $(\mathrm{Std}\circ\pi_v)$; Haar self-dual w.r.t. ψ; $\mathrm{vol}(K_v) = 1$.
Archimedean Γ-factors	$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$; $\gamma(s,\pi_v)$ via Langlands parameters $\mu_{j,v}$; $h_{\pi_v}(\tau) = \mathrm{Re}\frac{d}{ds}\log\gamma(s,\pi_v)\big\rvert_{s=1/2+i\tau}$ (even in τ).
Functional Equation	Unitary normalization; $u_\pi(\tau)$ given by Eq. (3.2).
MS Ledger	Spherical model; fixed-heat window $t \in [t_{\min}, t_0]$.
Heat Kernel (spectral)	$\widetilde{k}_t(\lambda) = e^{-t(\lVert\lambda\rVert^2 + c_X)}$ with $c_X = \lVert\rho_X\rVert^2$; PSD regularizer on the ledger.
Spectral PV	Symmetric PV on $\mathrm{Re}(s) = \frac{1}{2}$; no δ_0 atom (MS convention).
Abelian Slice Density (GL(1))	$\rho_W(x) = \frac{1}{2\pi}e^{-x^2/(8t)}$ on $x \geq 0$ (even slice weight; asymmetry allocated to Arch).
Dictionary (Ledger/Holonomy)	$W(u) \leftrightarrow \widehat{g}(u)$; Prime grid $x = 2k\log N\mathfrak{p}$; $\widehat{G}(x) = 2\sinh(x/2)\,\widehat{g}(x)$.
Geometric Involution	$\mathcal{G}_\nu^{(m)} = T_\pi^{(m)} \circ R^{(m)}$ (realizes $\mathbb{T}_\nu$ on the ledger).

F.4. **Label Concordance with Series.** To facilitate cross-referencing, we provide a concordance table mapping key labels in this document to their canonical labels in the series map (combined.json) and antecedent papers.

Concept	This Paper Label	Series Canonical Label
PV-Weil Bridge	`thm:pv_weil_bridge` (aliased `lem:PVWeil`)	`lem:PVWeil`
Riesz Rep (Primes)	`lem:Utp`	`lem:Utp`
Fixed-Heat Operator	`prop:Rt_properties`	`prop:Rt_properties`
Weyl Uncertainty	`lem:weyl-uncertainty`	`lem:weyl-uncertainty`
GAP	`thm:GAP`	`thm:GAP`
Witness Argument	`thm:witness`	`thm:witness`
Ramified Damping (R*)	`prop:Rstar` / `prop:ramified_damping`	`prop:Rast` (GRH I)

Finite-place MS measure and Satake normalization (pinned). We explicitly fix the conventions at finite places.

- **Satake Parameterization:** At an unramified place v, we use the unitary normalization where the Satake parameters satisfy $|\alpha_{v,j}| = 1$. The local L-factor is $L(s, \pi_v) = \det(I - A_v q_v^{-s})^{-1}$.
- **Measure on the Spherical Line:** The Haar measure on G_v is normalized such that the maximal compact subgroup K_v has volume 1. The local Plancherel measure $d\mu_{\mathrm{Pl},v}$ is normalized consistently with the global measure (Section 6).
- **Induced Weights:** These normalizations induce the weights $p^{-k/2}$ and the scaling $2k \log p$ used in the prime dictionary (Eq. (12.4)), consistent with the standard Explicit Formula conventions (e.g., Iwaniec-Kowalski).

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