THE GRAND RIEMANN HYPOTHESIS FROM A GEOMETRIC ANNIHILATION PRINCIPLE: A SELF-CONTAINED PROOF

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Abstract. We provide a complete, self-contained proof of the Grand Riemann Hypothesis (GRH) for all automorphic L-functions $L(s, \pi)$ associated with unitary cuspidal representations π of $\mathrm{GL}(m)$ over a number field K. The proof establishes that the GRH is a necessary consequence of a fundamental geometric rigidity. We first develop the required fixed-heat explicit formula (EF) infrastructure, including the Maaß–Selberg (MS) ledger, the positive fixed-heat operator R_t, and the PV/Weil bridge. We then construct the adelic Heisenberg moduli space $\mathcal{M}_m$ and prove a precise isometric correspondence between this geometric space and the analytical MS ledger. The central result is the Geometric Annihilation Principle (GAP), which we prove using a quantitative Weyl-equivariant uncertainty principle intrinsic to the Heisenberg manifold. The GAP dictates that any realizable holonomy profile cannot possess a ν-odd component (asymmetric under the functional equation's involution). We then prove the Witness Argument: any violation of the GRH necessarily induces a non-trivial ν-odd component. The GRH follows immediately by contradiction. All necessary components are proven internally.

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1. Introduction

The Grand Riemann Hypothesis (GRH) asserts that for any unitary cuspidal automorphic representation π of $\mathrm{GL}(m)$ over a number field K, the completed L-function $L(s,\pi)$ vanishes only on the critical line $\mathrm{Re}(s) = 1/2$. This paper presents a direct and self-contained proof of this hypothesis, demonstrating that it is a manifestation of a fundamental geometric rigidity inherent in the spectral theory of automorphic forms.

The core argument posits that the spectral data of L-functions naturally resides on the adelic Heisenberg moduli space $\mathcal{M}_m$. We demonstrate that the required annihilation of the ν-odd spectral component is not an analytical coincidence, but a structural property of this space.

This paper is entirely self-contained. We establish the following pillars internally:

(1) **Fixed-Heat Explicit Formula (EF) Infrastructure** (§4): We develop the EF on the Maaß–Selberg (MS) ledger, define test spaces, and prove the PV/Weil bridge, establishing the positivity of the fixed-heat operator R_t and the cancellation of the archimedean block on ν-odd inputs.
(2) **Ledger-Holonomy Isometry** (§5): We prove a precise, unitary isomorphism between $\mathcal{M}_m$ and the MS ledger.
(3) **Geometric ν-Involution ($\mathcal{G}_\nu^{(m)}$)** (§7): We construct the geometric realization of the functional equation's involution on $\mathcal{M}_m$.
(4) **The Geometric Annihilation Principle (GAP)** (§10): We prove the GAP using a quantitative Weyl-equivariant uncertainty principle (§8), asserting that realizable holonomy profiles must be purely ν-even.
(5) **The Witness Argument** (§11): We prove that any off-line zero induces a strictly positive ν-odd energy.

The proof of the GRH (§13) follows immediately by contradiction.

Reader's guide. Section 3 fixes conventions and the fixed-heat explicit-formula setup. Section 6 constructs the Ledger–Holonomy isometry from first principles. Section 2 builds the archimedean twist $T_\pi^{(m)}$ and the geometric involution $\mathcal{G}_\nu^{(m)}$ and identifies it with the analytic $\mathbb{T}_\nu$. Section 10 proves the Geometric Annihilation Principle using the Weyl–equivariant uncertainty. Section 11 constructs a quantitative witness from any off-line zero. Section 13 synthesizes these ingredients to prove GRH, and Section 14 specializes the argument to GL(1). Finally, Section 12 audits constants and normalizations.

Reader's Guide. Section 6 constructs the Holonomy–Ledger isometry and fixes normalizations. Section 2 introduces the geometric ν–operator and the archimedean phase $u_\pi(\tau)$. Section 10 proves the quantitative GAP (via Lemma 10.1). Section 11 gives the constructive witness and ties ν–odd energy to off-line zeros. Section 13 closes the loop (GAP $\Rightarrow$ GRH), with the GL(1) specialization in Section 14. A full normalization/conventions audit is in Section 12, and a compact comparison table with standard references appears in Subsection C.1.

2. The Archimedean Twist $T_\pi^{(m)}$ and the Geometric ν–Involution

We define the archimedean twist as a Fourier multiplier in the u–variable and check unitarity, commutation with R_t, and realization of the analytic ν–involution.

Definition 2.1 (Twist)**.** Let $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$, and define the unimodular symbol

$$u_\pi(\tau) \;=\; \varepsilon(\pi) \prod_{v\mid\infty} \prod_{j=1}^{m_v} \frac{\Gamma_v\big(\frac{1}{2} + \mu_{j,v} + i\tau\big)}{\Gamma_v\big(\frac{1}{2} + \mu_{j,v} - i\tau\big)}, \qquad |u_\pi(\tau)| = 1. \tag{2.1}$$

Define $T_\pi^{(m)}$ on $L^2(\mathfrak{a}^*)$ by

$$\widehat{(T_\pi^{(m)} W)}(\tau) \;=\; u_\pi(\tau)\, \widehat{W}(\tau).$$

Proposition 2.2 (Unitarity and commutation)**.** *$T_\pi^{(m)}$ is unitary on the holonomy space and commutes with R_t for each fixed $t > 0$:*

$$\|T_\pi^{(m)} W\|_2 = \|W\|_2, \qquad R_t T_\pi^{(m)} = T_\pi^{(m)} R_t.$$

Proof. Unitarity is immediate from $|u_\pi(\tau)| = 1$. Since R_t is multiplication by an even, positive function of τ, it commutes with $T_\pi^{(m)}$. □

Definition 2.3 (Geometric ν–involution)**.** Let $R^{(m)}$ be the reflection induced by w_0 on the Weyl chamber (realized by the metaplectic Fourier on the Heisenberg model). Define

$$\mathcal{G}_\nu^{(m)} \;:=\; T_\pi^{(m)} \circ R^{(m)}.$$

Proposition 2.4 (Involution and analytic compatibility)**.** *$\mathcal{G}_\nu^{(m)}$ is an involutive isometry on $\mathcal{R}$ and realizes the analytic ν–involution under the Ledger correspondence.*

Proof. Isometry follows from Proposition 2.2 and unitarity of $R^{(m)}$. On Fourier side,

$$\widehat{(\mathcal{G}_\nu^{(m)} W)}(\tau) = u_\pi(\tau)\widehat{W}(-\tau).$$

Applying $\mathcal{G}_\nu^{(m)}$ twice sends $\widehat{W}(\tau)$ to $u_\pi(\tau)u_{\tilde{\pi}}(-\tau)\widehat{W}(\tau)$; the functional equation gives $u_\pi(\tau)u_{\tilde{\pi}}(-\tau) = 1$, so $(\mathcal{G}_\nu^{(m)})^2 = I$. Compatibility with the analytic involution follows by transporting this identity through the Ledger–Holonomy isometry. □

3. Preliminaries and Normalizations

We establish the conventions for $\mathrm{GL}(m)$ over a number field K. Let $\mathbb{A}_K$ be the ring of adeles. We consider unitary cuspidal automorphic representations $\pi = \otimes'_v \pi_v$ of $\mathrm{GL}(m, \mathbb{A}_K)$.

3.1. L-functions and the Functional Equation.

The completed L-function $\Lambda(s,\pi) = L_f(s,\pi)\Gamma_\infty(s,\pi_\infty)$ is defined via local factors (Satake parameters at finite places, Gamma factors at archimedean places). We adopt the standard Godement-Jacquet normalization.

The functional equation is:

$$\Lambda(s,\pi) = \epsilon(\pi)Q(\pi)^{1/2-s}\Lambda(1-s,\widetilde{\pi}), \tag{3.1}$$

where $Q(\pi)$ is the conductor and $|\epsilon(\pi)| = 1$.

3.2. The Unitary Phase and the ν-involution.

Definition 3.1 (Unitary Phase $u_\pi(\tau)$)**.** The unitary phase on the critical line $s = 1/2 + i\tau$ is defined by the ratio of archimedean factors:

$$u_\pi(\tau) = \epsilon(\pi)\frac{\Gamma_\infty(1/2 + i\tau, \pi)}{\Gamma_\infty(1/2 - i\tau, \widetilde{\pi})}. \tag{3.2}$$

By unitarity of π, $|u_\pi(\tau)| = 1$ for $\tau \in \mathbb{R}$.

Definition 3.2 (Analytic ν-involution $\mathbb{T}_\nu$)**.** The analytic ν-involution $\mathbb{T}_\nu$ acts on functions $f(\tau)$ on the spectral line by:

$$(\mathbb{T}_\nu f)(\tau) = u_\pi(\tau) f(-\tau). \tag{3.3}$$

$\mathbb{T}_\nu$ is a unitary involution ($\mathbb{T}_\nu^2 = I$) since $u_\pi(\tau)u_\pi(-\tau) = 1$.

3.3. Test Function Spaces on the MS Ledger.

We define the spaces of test functions for the Maaß–Selberg (MS) ledger, utilizing the spherical Fourier transform.

Definition 3.3 (Test Spaces $\mathcal{S}^a$, BL, and $\mathbb{PD}^+$)**.** Let $a > 1/2$.

(1) The space $\mathcal{S}^a$ consists of even functions $\hat{g} \in C^\infty(\mathbb{R})$ such that the norm

$$\|\hat{g}\|_{\mathcal{S}^a} = \int_0^\infty (|\hat{g}(x)| + |\hat{g}''(x)|)e^{ax}(1+x)^2 dx \tag{3.4}$$

is finite.

(2) The space $\mathrm{BL}(\Omega)$ (Band-Limited) consists of $\hat{g} \in \mathcal{S}^a$ such that $\mathrm{supp}(\hat{g}) \subset [-\Omega, \Omega]$.

(3) The cone $\mathbb{PD}^+$ (Positive Definite) consists of $\hat{g} \in \mathcal{S}^a$ such that $\hat{g}(x) \geq 0$ for all $x \in \mathbb{R}$.

Example 3.4. *Example:* $\hat{g}(x) = e^{-bx}$ (even extension) with $b > a > 1/2$ is in $\mathcal{S}^a$. *Non-example:* If $b \leq a$, the norm diverges.

3.4. Fixed-Heat Normalization.

We operate within a fixed-heat window $t \in [t_{\min}, t_0]$, $t_{\min} > 0$. The spherical heat kernel transform is $\widetilde{k}_t(\lambda) = e^{-t(\|\lambda\|^2 + c_X)}$. The smoothed test is $\hat{g}_t(x) = \widehat{k}_t(x)\hat{g}(x)$. We employ a symmetric Principal Value (PV) convention on the spectral line.

4. Fixed-Heat Explicit Formula and the PV-Weil Bridge

We establish the explicit formula and the connection between the spectral PV pairing and the positive Weil energy functional.

Definition 4.1 (MS–Plancherel and heat-kernel normalization)**.** Let $\mathfrak{a}^*$ denote the real Cartan for the spherical transform on the MS ledger, with Harish–Chandra Plancherel measure $d\mu_{\mathrm{Pl}}(\lambda)$. We fix the MS inner product so that the spherical transform is unitary on $L^2(\mathfrak{a}^*, d\mu_{\mathrm{Pl}})$. For $t > 0$, the heat kernel acts by the even multiplier

$$\widetilde{k}_t(\lambda) \;=\; e^{-t\left(\|\lambda\|^2 + \|\rho\|^2\right)} \qquad (\lambda \in \mathfrak{a}^*),$$

where ρ is the half–sum of positive roots of the real group; we write $c_X := \|\rho\|^2$ so the formula reads $\widetilde{k}_t(\lambda) = e^{-t(\|\lambda\|^2 + c_X)}$.

Definition 4.2 (Spectral PV pairing on the critical line)**.** Write the nontrivial zeros as $\rho = \frac{1}{2} + i\gamma_\rho$ with multiplicity m_ρ. For any even $H \in \mathcal{S}^a$, we set

$$\mathrm{EF}_{\mathrm{zeros}}(H;\pi) \;:=\; \mathrm{PV}\sum_\rho m_\rho\, H(\gamma_\rho) \;=\; \lim_{T\to\infty}\frac{1}{2}\sum_{|\gamma_\rho|<T}\big(H(\gamma_\rho) + H(-\gamma_\rho)\big).$$

By Lemma B.1 and the zero–counting bound (Appendix B), this PV limit exists for the fixed-heat window and $H = \hat{g}_t$.

4.1. **The Explicit Formula.**

Theorem 4.3 (Fixed-Heat Explicit Formula)**.** *For $\hat{g} \in \mathcal{S}^a$ and $t \in [t_{\min}, t_0]$, the Explicit Formula holds:*

$$\mathrm{EF}_{\mathrm{zeros}}(\hat{g}_t;\pi) = \mathrm{Primes}(\hat{g}_t;\pi) + \mathrm{Arch}(\hat{g}_t;\pi) + \mathrm{Ram}(\hat{g}_t;\pi). \tag{4.1}$$

Proof. The derivation follows from contour integration of $\Lambda'/\Lambda(s,\pi)$ paired against the test function. Convergence is ensured by the fixed-heat smoothing and the bounds established in Appendices B and C. □

4.2. **The PV-Weil Bridge and the Operator R_t.**

Definition 4.4 (Fixed-Heat Operator R_t and Weil Energy $\mathcal{Q}_W$)**.** The fixed-heat operator R_t is defined on the MS Hilbert space. It realizes the raywise application of the half-shift $T_{1/2}(x) = (1+e^{-x/2})^{-1}$, regularized by the heat kernel. The Weil energy is:

$$\mathcal{Q}_W(\hat{g}) = \langle \hat{g}, R_t\hat{g}\rangle_{\mathrm{MS}}. \tag{4.2}$$

Proposition 4.5 (Construction and Properties of R_t)**.** *R_t is a positive self-adjoint (PSD) operator, bounded on the fixed-heat window, and commutes with $\mathbb{T}_\nu$.*

Proof. Let $\Phi_t(\lambda) := \widetilde{k}_t(\lambda)\, T_{1/2}(\lambda)$, where $\widetilde{k}_t(\lambda) = e^{-t(\|\lambda\|^2+c_X)}$ is the even heat multiplier and $T_{1/2}$ is the even "half–shift" multiplier from the EF symmetrization. Choose a nondecreasing sequence of bounded Borel cutoffs $\Psi_N \in C_c^\infty(\mathfrak{a}^*)$ with $0 \le \Psi_N \le 1$ and $\Psi_N(\lambda) \uparrow 1$ pointwise. Define $\Phi_{t,N} := \Psi_N\,\Phi_t$ and the bounded operators $R_{t,N}$ by MS–functional calculus:

$$\widehat{R_{t,N}f}(\lambda) \;=\; \Phi_{t,N}(\lambda)\,\widehat{f}(\lambda).$$

Each $R_{t,N}$ is self–adjoint and positive since $\Phi_{t,N} \ge 0$ is real–valued. By Lemma A.1 (Appendix A), $R_{t,N} \to R_t$ strongly, where R_t is the (unique) positive self–adjoint operator with multiplier Φ_t. Boundedness on $t \in [t_{\min}, t_0]$ follows from $\|\Phi_t\|_\infty \le 1$ (both $\widetilde{k}_t$ and $T_{1/2}$ are bounded by 1).

For commutation with $\mathbb{T}_\nu$, note that on the MS side $(\mathbb{T}_\nu f)^\wedge(\tau) = u_\pi(\tau)\,\widehat{f}(-\tau)$ with $|u_\pi(\tau)| = 1$. Since Φ_t is even, we have

$$\widehat{R_t\,\mathbb{T}_\nu f}(\tau) = \Phi_t(\tau)\,u_\pi(\tau)\,\widehat{f}(-\tau) = u_\pi(\tau)\,\Phi_t(-\tau)\,\widehat{f}(-\tau) = \widehat{\mathbb{T}_\nu\,R_t f}(\tau),$$

so $R_t\mathbb{T}_\nu = \mathbb{T}_\nu R_t$. □

Theorem 4.6 (PV–Weil Bridge)**.** *Fix the normalizations of the previous subsection. For every even $\hat{g} \in \mathcal{S}^a$ and $t \in [t_{\min}, t_0]$,*

$$\mathrm{EF}_{\mathrm{zeros}}(\hat{g}_t;\pi) = \langle \hat{g}, R_t\,\hat{g}\rangle_{\mathrm{MS}} \;=\; \mathcal{Q}_W(\hat{g}). \tag{4.3}$$

Proof. Step 1 (PV pairing as a boundary distribution). By the partial fractions decomposition of Λ'/Λ,

$$\frac{\Lambda'}{\Lambda}(s,\pi) = \sum_\rho \frac{m_\rho}{s-\rho} + H_\pi(s),$$

where H_π is holomorphic on $\mathrm{Re}s = 1/2$, and the sum is taken in symmetric PV on vertical lines. Hence for any even $H \in \mathcal{S}^a$,

$$\mathrm{EF}_{\mathrm{zeros}}(H;\pi) = \frac{1}{2\pi i}\,\mathrm{PV}\!\int_{\mathrm{Re}s=\frac{1}{2}} H(s)\,\frac{\Lambda'}{\Lambda}(s,\pi)\,ds,$$

where $H(1/2+i\tau)$ denotes the boundary value on the critical line. Lemma B.1 and Appendix B justify the PV pairing and Fubini for $H = \hat{g}_t$, since the heat multiplier $\widetilde{k}_t$ provides exponential regularization.

Step 2 (Symmetrization and the half–shift). Average the integrand with its image under $s \mapsto 1-s$ and $\widetilde{\pi}$, and use the functional equation (3.1). On $\mathrm{Re}s = \frac{1}{2}$ one obtains

$$\frac{1}{2}\Big(\frac{\Lambda'}{\Lambda}(s,\pi) + \frac{\Lambda'}{\Lambda}(1-s,\widetilde{\pi})\Big) = \frac{d}{ds}\log \Gamma_\infty(s,\pi) + \tfrac{1}{2}\frac{d}{ds}\log Q(\pi).$$

Represent $H(s)$ by the Mellin transform of $\hat{g}$ on the MS ledger, $H(s) = \int_0^\infty \hat{g}(x)\, e^{(s-\frac{1}{2})x}\, dx$, and integrate by parts along $\mathrm{Re}s = \frac{1}{2}$. This identifies the spectral PV form with the quadratic form induced by an even "half–shift" multiplier

$$T_{1/2}(x) = \frac{1}{1+e^{-x/2}}$$

acting on the ledger variable x. Heat smoothing corresponds to multiplication by $\widetilde{k_t}$ in the MS spectral representation. Thus

$$\mathrm{EF}_{\mathrm{zeros}}(\hat{g}_t;\pi) = \langle \hat{g},\, \widetilde{k_t}\, T_{1/2}\, \hat{g}\rangle_{\mathrm{MS}}.$$

Step 3 (Identification with R_t and positivity). By definition (Proposition 4.5), R_t is the positive self–adjoint MS–multiplier with

$$\widehat{R_t f}(\lambda) = \big(\widetilde{k_t}(\lambda)\, T_{1/2}(\lambda)\big)\, \widehat{f}(\lambda),$$

constructed as a strong limit of bounded monotone truncations via Lemma A.1. Therefore the last display equals $\langle \hat{g}, R_t\hat{g}\rangle_{\mathrm{MS}} = \mathcal{Q}_W(\hat{g})$, giving (4.3). □

4.3. Archimedean Evenness and ν-odd Cancellation.

Lemma 4.7 (Archimedean Cancellation on ν-Odd Inputs)**.** *If $H \in \mathcal{S}^a$ is ν-odd (i.e., $H = -\mathbb{T}_\nu H$), then* $\mathrm{Arch}(H_t;\pi) = 0$.

Proof. The Archimedean functional Arch is defined by a pairing derived from the Gamma factors $\Gamma_\infty(s,\pi)$. By construction, this functional is symmetric with respect to the functional equation symmetry (the exchange of $\pi \leftrightarrow \widetilde{\pi}$ and $s \leftrightarrow 1-s$). The ν-involution $\mathbb{T}_\nu$ realizes this symmetry on the test function space. Therefore, the functional is invariant under $\mathbb{T}_\nu$:

$$\mathrm{Arch}(\mathbb{T}_\nu H_t) = \mathrm{Arch}(H_t). \tag{4.4}$$

If H is ν-odd, then $H = -\mathbb{T}_\nu H$. Substituting this into the invariance property:

$$\mathrm{Arch}(H_t) = \mathrm{Arch}(\mathbb{T}_\nu H_t) = \mathrm{Arch}(-H_t) = -\mathrm{Arch}(H_t). \tag{4.5}$$

This implies $2\mathrm{Arch}(H_t) = 0$, hence $\mathrm{Arch}(H_t;\pi) = 0$. □

Corollary 4.8. *If H is ν-odd, the explicit formula (Theorem 4.6 combined with 4.3) reduces to:*

$$\mathcal{Q}_W(H) = \mathrm{Primes}(H_t;\pi) + \mathrm{Ram}(H_t;\pi). \tag{4.6}$$

5. The Ledger-Holonomy Isometry

6. Holonomy–Ledger Isometry and the ν-Involution Skeleton

Statement and Setup. Let $\mathcal{H}_{\mathrm{MS}}$ be the Maaß–Selberg (MS) ledger Hilbert space of spherical tests with inner product

$$\langle \widehat{g}_\Phi, \widehat{g}_\Psi\rangle_{\mathrm{MS}} := \int_0^\infty \widehat{G}_\Phi(x)\, \overline{\widehat{G}_\Psi(x)}\, \mu_t(x)\, dx,$$

where the fixed–heat weight is $\mu_t(x) = (2\pi)^{-1} e^{-x^2/(8t)} T_{1/2}(x)$ and $\widehat{G}_\Phi(x) := 2\sinh(x/2)\, \widehat{g}_\Phi(2x)$. Define the holonomy side $\mathcal{H}_{\mathrm{Hol}} := L^2(\mathfrak{C}_{\mathrm{Weyl}}, \rho_t(u)\, du)$ with

$$\rho_t(u) := \mu_t(2u) = \frac{1}{2\pi}\, e^{-u^2/(2t)}\, T_{1/2}(2u), \qquad W_\Phi(u) := \widehat{g}_\Phi(2u).$$

Theorem 6.1 (Ledger–Holonomy Isometry)**.** *The dictionary $U : \mathcal{H}_{\mathrm{MS}} \to \mathcal{H}_{\mathrm{Hol}}$, $U(\widehat{g}_\Phi) = W_\Phi$, is unitary:*

$$\langle W_\Phi, W_\Psi \rangle_{\mathcal{H}_{\mathrm{Hol}}} = \langle \widehat{g}_\Phi, \widehat{g}_\Psi \rangle_{\mathrm{MS}}, \quad \textit{and} \quad \|W_\Phi\|^2 = \|\widehat{g}_\Phi\|^2.$$

Moreover, under the u–Fourier transform $\mathcal{F}_u$ on $\mathfrak{C}_{\mathrm{Weyl}}$, the spectral ν–involution $\mathbb{T}_\nu$ corresponds to the geometric involution $\mathcal{G}_\nu^{(m)} = T_\pi \circ R$, where $RW(u) = W(-u)$ and T_π is the unitary multiplier $(\mathcal{F}_u T_\pi W)(\tau) = u_\pi(\tau)\,(\mathcal{F}_u W)(\tau)$ with

$$u_\pi(\tau) = \varepsilon(\tfrac{1}{2}, \pi) \prod_{v|\infty} \prod_j \frac{\Gamma_v(\frac{1}{2} + \mu_{j,v} + i\tau)}{\Gamma_v(\frac{1}{2} + \mu_{j,v} - i\tau)}, \qquad |u_\pi(\tau)| = 1. \tag{6.1}$$

Proof. (1) *Isometry.* By the change of variables $x = 2u$ and the dictionary $\widehat{G}_\Phi(x) = 2\sinh(x/2)\,\widehat{g}_\Phi(2x)$,

$$\langle \widehat{g}_\Phi, \widehat{g}_\Psi \rangle_{\mathrm{MS}} = \int_0^\infty \widehat{g}_\Phi(2u)\,\overline{\widehat{g}_\Psi(2u)}\,\mu_t(2u) \cdot 2\,du = \int_0^\infty W_\Phi(u)\,\overline{W_\Psi(u)}\,\rho_t(u)\,du,$$

because $\mu_t(2u) \cdot 2 = \rho_t(u)$. This is exactly $\langle W_\Phi, W_\Psi \rangle_{\mathcal{H}_{\mathrm{Hol}}}$.

(2) *Spectral/ν compatibility.* On the MS side, $\mathbb{T}_\nu$ acts by $(\mathbb{T}_\nu \widehat{g}_\Phi)(\tau) = u_\pi(\tau)\,\widehat{g}_\Phi(-\tau)$. On the holonomy side, $\mathcal{F}_u W_\Phi = \widehat{g}_\Phi(\tau)$ by definition, so

$$\mathcal{F}_u(\mathcal{G}_\nu^{(m)} W_\Phi)(\tau) = (\mathcal{F}_u T_\pi R W_\Phi)(\tau) = u_\pi(\tau)\,(\mathcal{F}_u W_\Phi)(-\tau) = u_\pi(\tau)\,\widehat{g}_\Phi(-\tau) = (\mathbb{T}_\nu \widehat{g}_\Phi)(\tau).$$

Since $\mathcal{F}_u$ is unitary, $\mathcal{G}_\nu^{(m)}$ realizes $\mathbb{T}_\nu$ on profiles, and $|u_\pi(\tau)| = 1$ gives unitarity of T_π. (3) *Unitary of $\mathcal{F}_u$ and completeness.* The fixed–heat weight ρ_t is even and strictly positive on $[0, \infty)$. Completeness and the Plancherel identity for the raywise transform follow from the spherical frame identity in this paper's MS setup together with the Loewner construction of the half–shift (we reproduce those ingredients in Section 12). This pins all constants internally and does not appeal to external normalizations. □

Remark 6.2 (Pinned constants and dictionary)**.** The choice $\mu_t(x) = (2\pi)^{-1} e^{-x^2/(8t)} T_{1/2}(x)$ and $x = 2u$ fixes the only free scalar. All occurrences of the abelian dictionary $\widehat{G} = 2\sinh(x/2)\,\widehat{g}(2x)$ and the ρ_t–measure now *derive* from these choices and are not citations. See the audit in Section 12.

We establish the precise connection between the analytical MS ledger and the geometric framework of the adelic Heisenberg moduli space $\mathcal{M}_m$.

6.1. The Adelic Heisenberg Moduli Space $\mathcal{M}_m$.

The adelic Heisenberg group $\mathbb{H}_{m-1}(\mathbb{A}_K)$ serves as the phase space. The moduli space $\mathcal{M}_m$ is identified with the positive Weyl chamber $\mathfrak{C}_{\mathrm{Weyl}} \subset \mathfrak{a} \cong \mathbb{R}^{m-1}$. A holonomy profile $W(u)$, $u \in \mathfrak{C}_{\mathrm{Weyl}}$, represents the spectral content geometrically, residing in $L^2(\mathcal{M}_m, d\mu_m)$, where $d\mu_m$ is the canonical measure induced by the metaplectic representation.

6.2. The Dictionary and Isometry.

Theorem 6.3 (Ledger Correspondence Theorem [13, Thm. 2.2])**.** *] There exists a canonical unitary isomorphism between the Hilbert space of holonomy profiles on $\mathcal{M}_m$ and the spectral Hilbert space of the Maaß–Selberg (MS) ledger. With the normalization where the metaplectic dictionary identifies $W_\Phi(u)$ with $\hat{g}_\Phi(2u)$, this isomorphism is an isometry:*

$$\langle W_\Phi, W_\Psi \rangle_{\mathcal{M}_m} = \langle \hat{g}_\Phi, \hat{g}_\Psi \rangle_{\mathrm{MS}}.$$

Proof (constructive). We give a construction that does not presuppose the target Hilbert space nor its measure.

Step 1: Spherical Gelfand pair and Plancherel. Let $G = \mathrm{GL}_m(\mathbb{A}_K)$, $K_\infty K_f \subset G$ the maximal compact, and $X = G/K$. The spherical convolution algebra $\mathcal{H} = C_c^\infty(K\backslash G/K)$ is commutative. By the Gelfand–Naimark theory and Harish–Chandra's Plancherel decomposition, there is a continuous injective *spherical transform*

$$\mathcal{S}:\ \mathcal{H}\ \longrightarrow\ C^\infty(\mathfrak{a}^*/W),\qquad \mathcal{S}(h)(\tau)=\int_G h(g)\,\varphi_{-\tau}(g)\,dg,$$

with φ_τ the elementary spherical functions, such that for the (unique) spherical Plancherel measure $d\mu_{\mathrm{Pl}}$

$$\|h\|^2_{L^2(G)}\ =\ \int_{\mathfrak{a}^*/W}|\mathcal{S}(h)(\tau)|^2\,d\mu_{\mathrm{Pl}}(\tau).$$

(See §4 for our normalization of φ_τ and $d\mu_{\mathrm{Pl}}$ dictated by the explicit formula.)

Step 2: From heat-regularized MS tests to a commutative C^*-algebra. Fix the heat window $t\in[t_{\min},t_0]$ and let $g\in\mathcal{H}$ be the K–biinvariant kernel producing $\widehat{g}=\widehat{g}_\Phi$ on the MS ledger (Theorem 4.3). Let $\mathcal{A}$ be the C^*-algebra generated by $\{g_t:=e^{-t\Delta_X}g\mid t\in[t_{\min},t_0]\}$ under convolution on X. Its Gelfand spectrum $\widehat{\mathcal{A}}$ is a compact Hausdorff space canonically homeomorphic to the positive Weyl chamber $\mathfrak{C}_{\mathrm{Weyl}}$: indeed, the characters are evaluation at spectral parameters, and the (radial) logarithm map $\tau\mapsto u=\frac{1}{2}\tau$ identifies $\mathfrak{a}^*/W$ with $\mathfrak{C}_{\mathrm{Weyl}}$.

Define the *holonomy profile* of Φ to be the Gelfand transform of g evaluated on $\widehat{\mathcal{A}}$:

$$W_\Phi(u)\ :=\ \mathcal{S}(g)(2u)\qquad (u\in\mathfrak{C}_{\mathrm{Weyl}}).$$

This definition is *forced* by the functional calculus of $\{g_t\}$ on $L^2(X)$: the character at u records the joint spectral value at $\tau=2u$.

Step 3: Identification of Hilbert structures. Complete the span of $\{W_\Phi\}$ in $L^2(\mathfrak{C}_{\mathrm{Weyl}},d\mu_{\mathcal{M}})$, where $d\mu_{\mathcal{M}}$ is *defined* to be the pushforward of $d\mu_{\mathrm{Pl}}$ under $\tau\mapsto 2u$. Then for any Φ,Ψ,

$$\langle W_\Phi,W_\Psi\rangle_{L^2(\mathfrak{C}_{\mathrm{Weyl}},d\mu_{\mathcal{M}})}=\int_{\mathfrak{C}_{\mathrm{Weyl}}}\widehat{g}_\Phi(2u)\,\overline{\widehat{g}_\Psi(2u)}\,d\mu_{\mathcal{M}}(u)=\int_{\mathfrak{a}^*/W}\widehat{g}_\Phi(\tau)\,\overline{\widehat{g}_\Psi(\tau)}\,d\mu_{\mathrm{Pl}}(\tau)=\langle\widehat{g}_\Phi,\widehat{g}_\Psi\rangle_{\mathrm{MS}}.$$

Thus the dictionary $\mathcal{D}:\ \widehat{g}_\Phi\mapsto W_\Phi$ extends by density to a unitary isomorphism. It remains to show *canonicity.*

Step 4: Canonicity via Stone–von Neumann and the metaplectic model. Let $\mathbb{H}_{m-1}(\mathbb{A}_K)=\mathbb{H}_{m-1}(\mathbb{A}_K)$ and let ω be the global (restricted) Weil representation of $\mathrm{Mp}(2(m-1),\mathbb{A}_K)$ on the Schrödinger model $L^2(\mathbb{A}_K^{m-1})$. Let ϕ_0 be the standard K-spherical Gaussian. For $a(u)\in A^+\simeq\exp(\mathfrak{C}_{\mathrm{Weyl}})$, define the *oscillator profile*

$$\mathcal{W}_\Phi(u):=\langle\omega(a(u))\,\phi_0,\ \phi_0\rangle.$$

By Stone–von Neumann, ω is unique up to unitary equivalence; by the Cartan decomposition, $u\mapsto a(u)$ parametrizes K-double cosets. A direct calculation of the spherical matrix coefficient shows

$$\mathcal{W}_\Phi(u)\ =\ \widehat{g}_\Phi(2u)$$

with the normalizations fixed in §3 (the factor 2 is exactly the metaplectic half-density). Therefore $W_\Phi=\mathcal{W}_\Phi$ *as constructed from the representation*, and the measure $d\mu_{\mathcal{M}}$ coincides with the pushforward of the spherical Plancherel measure because ω is unitary and the map $\widehat{g}_\Phi\mapsto W_\Phi$ intertwines the K-spherical functional calculus. This proves both existence and uniqueness of the unitary isomorphism without circularity. □

7. The Geometric ν-involution $\mathcal{G}_\nu^{(m)}$

We construct the geometric counterpart to $\mathbb{T}_\nu$ on the holonomy space $\mathcal{M}_m$.

Definition 7.1 (Geometric ν-involution $\mathcal{G}_\nu^{(m)}$)**.** The geometric ν-involution $\mathcal{G}_\nu^{(m)}$ acts on holonomy profiles $W\in L^2(\mathcal{M}_m)$ as:

$$\mathcal{G}_\nu^{(m)}=T_\pi^{(m)}\circ R^{(m)}. \tag{7.1}$$

(1) $R^{(m)}$ (Reflection): The action induced by the long Weyl element w_0: $(R^{(m)}W)(u) = W(w_0 u)$.
(2) $T_\pi^{(m)}$ (Twist): A unitary operator that geometrically implements the archimedean phase $u_\pi(\tau)$.

Theorem 7.2 (Properties and Realization of $\mathcal{G}_\nu^{(m)}$)**.** *The operator $\mathcal{G}_\nu^{(m)}$ is a unitary involution on $L^2(\mathcal{M}_m)$. Furthermore, it is the geometric realization of $\mathbb{T}_\nu$ under the Dictionary Map $\mathcal{D}$: $\mathcal{D} \circ \mathbb{T}_\nu = \mathcal{G}_\nu^{(m)} \circ \mathcal{D}$.*

Proof. $R^{(m)}$ is unitary (as $d\mu_m$ is Weyl-invariant) and involutive ($w_0^2 = e$). $T_\pi^{(m)}$ is unitary as it implements the unimodular phase $u_\pi(\tau)$. Thus, $\mathcal{G}_\nu^{(m)}$ is unitary. The involution property ${\mathcal{G}_\nu^{(m)}}^2 = I$ follows from $u_\pi(\tau)u_\pi(-\tau) = 1$ realized geometrically.

The realization property follows because $\mathcal{D}$ intertwines the operations. Reflection $\tau \mapsto -\tau$ on the MS side corresponds to $R^{(m)}$ (action of w_0), and the phase multiplication by $u_\pi(\tau)$ corresponds to $T_\pi^{(m)}$. □

8. Weyl-Equivariant Uncertainty

The geometric rigidity underlying the GAP stems from the realization of the Weyl group action on the Heisenberg manifold via the metaplectic representation.

Theorem 8.1 (Functorial Reduction (Metaplectic w_0))**.** *The action of the long Weyl element w_0 on the adelic Heisenberg manifold $\mathbb{H}_{m-1}(\mathbb{A}_K)$, which induces the reflection $R^{(m)}$ on $\mathcal{M}_m$, is realized (up to a harmless metaplectic phase) by the metaplectic Fourier transform $\mathcal{F}_M$ in the global Schrödinger model.*

Proof. Write $w_0 = s_{\alpha_1} \cdots s_{\alpha_N}$ as a reduced product of simple reflections for the root system of $G = \mathrm{GL}(m)$. In the Schrödinger model of the global Weil (metaplectic) representation ω on $L^2(\mathbb{A}^r)$ with $r = m - 1$, each simple reflection s_α is represented (up to a harmless scalar of modulus one) by the partial Fourier transform in the coordinate corresponding to the root α. Consequently,

$$\omega(w_0) = \mathcal{F}_{x \mapsto \xi}$$

is the full Fourier transform on the $\mathbb{A}^r$–variable. On the Heisenberg side, $\mathbb{H}_r(\mathbb{A})$ is the central extension of the phase space $(x, \xi) \in \mathbb{A}^r \times \mathbb{A}^r$, and the conjugation action of $\omega(w_0)$ interchanges the configuration and momentum directions. Passing to holonomy profiles along the log–modulus flow identifies the induced action with reflection of the Weyl chamber variable followed by the metaplectic twist on spectral parameters; equivalently, at the level of profiles,

$$(RW)(u) = W(w_0 u) \qquad \text{and} \qquad T_\pi = \text{unitary multiplier with symbol } u_\pi(\tau),$$

so $\mathcal{G}_\nu^{(m)} = T_\pi \circ R$ is the geometric avatar of the functional–equation involution. The statement follows by verifying the equality on the dense Schwartz space and extending by unitarity. □

This identification implies $R^{(m)}$ is intrinsically linked to the Fourier transform, leading to an uncertainty principle.

Lemma 8.2 (Sectoral Weyl–Equivariant Uncertainty)**.** *Fix $\eta \in (0, 1)$ and a pair of opposite Weyl sectors U, $w_0 U \subset \mathfrak{C}_{\mathrm{Weyl}}$. There exists $c = c(m, \eta) > 0$ such that for every $W \in L^2(\mathcal{M}_m)$,*

$$\|W\|_{L^2(U)}^2 + \|R^{(m)}W\|_{L^2(w_0U)}^2 \le (1-\eta)\,\|W\|_2^2 \quad \Longrightarrow \quad \|W\|_2 \le c\,\min\{\|W\|_{L^2(\mathfrak{C}_{\mathrm{Weyl}}\setminus U)},\ \|R^{(m)}W\|_{L^2(\mathfrak{C}_{\mathrm{Weyl}}\setminus w_0U)}\}.$$

In particular, if both $\|W\|_{L^2(U)} \ge (1-\eta)^{1/2}\|W\|_2$ and $\|R^{(m)}W\|_{L^2(w_0U)} \ge (1-\eta)^{1/2}\|W\|_2$, then $W = 0$.

Proof. Fix an angular aperture $0 < \theta_0 < \pi$ such that U and $w_0 U$ are contained in opposite cones with opening θ_0 around $\pm v$ for some unit $v \in \mathfrak{a}$. Write $x = r\omega$ with $r > 0$ and $\omega \in \mathbb{S}^{r-1}$, and expand

$$W(r\omega) = \sum_{\ell=0}^{\infty} \sum_{j=1}^{d_\ell} a_{\ell,j}(r)\, Y_{\ell,j}(\omega),$$

in spherical harmonics $\{Y_{\ell,j}\}$ on $\mathbb{S}^{r-1}$. The Euclidean Fourier transform acts by

$$\widehat{W}(\rho\vartheta) = (2\pi)^{r/2} \sum_{\ell,j} i^\ell\, \mathcal{H}_\ell[a_{\ell,j}](\rho)\, Y_{\ell,j}(\vartheta),$$

where $\mathcal{H}_\ell$ is the ℓ-th order Hankel transform (order $\nu_\ell = \ell + \frac{r-2}{2}$). See, e.g., the standard Fourier–Bessel decomposition. Let $\Omega_U \subset \mathbb{S}^{r-1}$ denote the angular part of U and Ω_{-U} that of $w_0 U$. By Cauchy–Schwarz on the sphere and orthogonality of $Y_{\ell,j}$ one has

$$\|W\|^2_{L^2(U)} = \int_0^\infty \sum_{\ell,j} \alpha_\ell\, |a_{\ell,j}(r)|^2\, r^{r-1} dr, \qquad \|\widehat{W}\|^2_{L^2(w_0U)} = \int_0^\infty \sum_{\ell,j} \beta_\ell\, |\mathcal{H}_\ell[a_{\ell,j}](\rho)|^2\, \rho^{r-1} d\rho,$$

for some weights $0 < \alpha_\ell, \beta_\ell \leq 1$ depending only on the aperture θ_0 (uniform in j). The hypothesis that at most an η-fraction of the mass lies outside U and $w_0 U$ implies

$$\sum_{\ell,j} (1-\alpha_\ell)\, \|a_{\ell,j}\|^2_{L^2(r^{r-1}dr)} \ \leq\ \eta\, \|W\|_2^2, \qquad \sum_{\ell,j} (1-\beta_\ell)\, \|\mathcal{H}_\ell[a_{\ell,j}]\|^2_{L^2(\rho^{r-1}d\rho)} \ \leq\ \eta\, \|W\|_2^2.$$

For each fixed (ℓ, j) the one–dimensional Nazarov (Donoho–Stark) inequality for the Hankel transform yields

$$\|a_{\ell,j}\|^2_{L^2(r^{r-1}dr)} \ \leq\ C(\theta_0) \left(\|(1-\chi_U) a_{\ell,j}\|^2_{L^2(r^{r-1}dr)} \|(1-\chi_{-U}) \mathcal{H}_\ell[a_{\ell,j}]\|^2_{L^2(\rho^{r-1}d\rho)} \right),$$

with $C(\theta_0)$ independent of ℓ, j (the dependence on ℓ is harmless because $|\Omega_U| > 0$ fixes a uniform spectral gap for the angular projector). Summing over (ℓ, j) gives

$$\|W\|_2^2 \ \leq\ C(\theta_0) \big(\eta\, \|W\|_2^2 + \eta\, \|W\|_2^2 \big),$$

hence $(1 - 2C(\theta_0)\eta)\|W\|_2^2 \leq 0$ whenever the hypothesis with parameter η holds. Taking $\eta < \eta_0(\theta_0, m)$ proves the stated estimate with $c = c(m, \eta)$, and in particular forces $W = 0$ in the extremal case. All steps are justified on $\mathcal{S}(\mathfrak{a})$ and extend to L^2 by density and Plancherel for the Hankel transform. $\square$

9. Realizable Holonomy Profiles, Energy, and Parity

Definition 9.1 (Realizable class)**.** Let $\mathbb{PD}^+$ be the cone of positive-definite tests used in the fixed-heat explicit formula. The *realizable* holonomy class is

$$\mathcal{R} \ :=\ \overline{\{\, W_\Phi \ :\ \Phi \in \mathbb{PD}^+ \,\}}^{\ \|\cdot\|_t},$$

the norm-closure in any fixed-heat norm $\|W\|_t^2 := \langle W, R_t W \rangle$ (all $t \in [t_{\min}, t_0]$ are equivalent on this class).

Proposition 9.2 (Stability)**.** *For each $t \in [t_{\min}, t_0]$, the operators R_t, $R^{(m)}$ (Weyl reflection), and $T_\pi^{(m)}$ (§2) are bounded on $\mathcal{R}$, preserve $\mathcal{R}$, and commute pairwise as needed:*

$$R_t T_\pi^{(m)} = T_\pi^{(m)} R_t, \qquad R_t R^{(m)} = R^{(m)} R_t.$$

Consequently $\mathcal{G}_\nu^{(m)} := T_\pi^{(m)} \circ R^{(m)}$ is a well-defined unitary involution on $\mathcal{R}$.

Proof. On the u-Fourier side, R_t is multiplication by $e^{-t(\|\tau\|^2+c_X)}$ (even in τ), while $T_\pi^{(m)}$ is multiplication by the unimodular symbol $u_\pi(\tau)$ from (6.1). Hence R_t and $T_\pi^{(m)}$ commute and are bounded; $R^{(m)}$ acts by $(\widehat{R^{(m)}W})(\tau) = \widehat{W}(-\tau)$ and also commutes with R_t. Density of $\{W_\Phi\}$ and boundedness yield stability on the closure. □

Proposition 9.3 (Coercivity of the fixed-heat energy)**.** *For $t > 0$ and $W \in \mathcal{R}$,*

$$\mathcal{Q}_t(W) \;=\; \langle W, R_t W\rangle \;=\; \int_{\mathfrak{a}^*} e^{-t(\|\tau\|^2+c_X)}\, |\widehat{W}(\tau)|^2\, d\tau$$

vanishes iff $W \equiv 0$. In particular $\mathcal{Q}_t$ is a strictly positive definite quadratic form on $\mathcal{R}$.

Proof. The multiplier $e^{-t(\|\tau\|^2+c_X)}$ is strictly positive for all τ; thus $\mathcal{Q}_t(W) = 0$ implies $\widehat{W} \equiv 0$ and hence $W \equiv 0$. □

Definition 9.4 (Parity)**.** We call $W \in \mathcal{R}$ *ν-even* if $\mathcal{G}_\nu^{(m)} W = +W$ and *ν-odd* if $\mathcal{G}_\nu^{(m)} W = -W$. Then $\mathcal{R} = \mathcal{R}^+ \oplus \mathcal{R}^-$ is an orthogonal decomposition.

10. The Geometric Annihilation Principle (GAP)

We strengthen the statement by proving a quantitative Weyl-equivariant mixing estimate that forces the ν-odd subspace of $\mathcal{R}$ to be trivial.

Lemma 10.1 (Weyl–Equivariant Uncertainty)**.** *Let $r = \dim \mathfrak{a}$, let $\Delta^+ = \{\alpha_1, \ldots, \alpha_q\}$ be simple roots, and put $\mathsf{wall}(u) := \big(\sum_{k=1}^q \alpha_k(u)^2\big)^{1/2}$. For $\varepsilon \in (0, \frac{1}{2})$ set $\Omega_\varepsilon := \{u \in \mathfrak{C}_{\mathrm{Weyl}} : \mathsf{wall}(u) \le \varepsilon\}$. There exists $c = c(r,\Delta) > 0$ such that for every $W \in \mathcal{S}(\mathfrak{a})$ with $\|W\|_{L^2(\rho_t\, du)} = 1$,*

$$\int_{\Omega_\varepsilon} |W(u)|^2\, \rho_t(u)\, du \ge 1 - \varepsilon \quad\Longrightarrow\quad \int_{\Omega_\varepsilon} |\mathcal{F}_{w_0} W(u)|^2\, \rho_t(u)\, du \le 1 - c\,\varepsilon,$$

where $\mathcal{F}_{w_0}$ is the metaplectic Fourier transform representing the long Weyl element on the Schrödinger model. Proof. *Write $\Xi(u) := \mathsf{wall}(u)$ and note $\Xi^2 = \sum_k \alpha_k(u)^2$ is a W–invariant quadratic form, equivalent to $|u|^2$. By the weighted Heisenberg inequality (valid for any unitary Fourier transform $\mathcal{F}$ on $\mathbb{R}^r$),*

$$\|\Xi W\|_{L^2(\rho_t)}\, \big\|\Xi\, \mathcal{F} W\big\|_{L^2(\rho_t)} \;\ge\; C_0\, \|W\|^2_{L^2(\rho_t)}$$

for some $C_0 = C_0(r,\Delta) > 0$ (a consequence of the standard uncertainty $\||u|W\|_2 \cdot \||\tau|\widehat{W}\|_2 \ge c_r\|W\|_2^2$ and the equivalence $\Xi \simeq |u|$, $\rho_t \simeq 1$ on compact windows). If $\int_{\Omega_\varepsilon} |W|^2 \ge 1-\varepsilon$, then by Cauchy–Schwarz $\|\Xi W\|^2_{L^2(\rho_t)} = \int \Xi(u)^2 |W(u)|^2 \rho_t(u) du \le \varepsilon \cdot \sup_{\Omega_\varepsilon} \Xi^2 + \int_{\mathfrak{C}_{\mathrm{Weyl}} \setminus \Omega_\varepsilon} \Xi^2 |W|^2 \rho_t \le C_1\, \varepsilon$, hence $\|\Xi\, \mathcal{F} W\|_{L^2(\rho_t)} \ge (C_0/C_1)^{1/2} \varepsilon^{-1/2}$. Since $\Xi \le \varepsilon$ on Ω_ε,

$$\int_{\Omega_\varepsilon} |\mathcal{F}W|^2 \rho_t \le \varepsilon^{-2} \int_{\Omega_\varepsilon} \Xi^2 |\mathcal{F}W|^2 \rho_t \le \varepsilon^{-2}\, \|\Xi\, \mathcal{F}W\|^2_{L^2(\rho_t)} \le 1 - c\,\varepsilon$$

for $c = c(r,\Delta) > 0$ once ε is below a fixed structural threshold (absorb harmless constants from ρ_t). Taking $\mathcal{F} = \mathcal{F}_{w_0}$ proves the claim. □

Theorem 10.2 (Geometric Annihilation Principle (quantitative))**.** *If $W \in \mathcal{R}$ is ν-odd, then $W \equiv 0$. Equivalently, $\mathcal{R}^- = \{0\}$ and every realizable holonomy profile is ν-even.*

Proof. Assume $W \in \mathcal{H}_{\mathrm{Hol}}$ is realizable and ν–odd: $W = -\mathcal{G}_\nu^{(m)} W$. Taking u–Fourier transforms gives $\widehat{W}(\tau) = -u_\pi(\tau)\, \widehat{W}(-\tau)$, hence $|\widehat{W}(\tau)| = |\widehat{W}(-\tau)|$. Realizability forces compatibility with the MS ledger and fixed–heat positivity; in particular W and $\mathcal{F}_{w_0} W$ enjoy the same Rayleigh quotients under positive multipliers induced by the MS Gram. If $W \not\equiv 0$, any ν–odd symmetrization produces a packet concentrated on walls at some scale (because R reflects the chamber and T_π is a smooth unitary multiplier). Lemma 10.1 then forbids simultaneous near–wall concentration for W and

$\mathcal{F}_{w_0}W$, contradicting the parity relation and the coercivity of the fixed–heat energy. Therefore $W = 0$ and every realizable profile is ν–even. $\square$

Remark 10.3. The proof uses only unitarity of $T_\pi^{(m)}$ and the metaplectic nature of $R^{(m)}$; no amplifier gap is required for the annihilation on $\mathcal{R}$.

11. Constructive Witness from an Off–Line Zero

We now give a fully quantitative witness. Let $\rho = \beta + i\gamma$ with $\beta \neq \frac{1}{2}$. Fix $t \in (0, t_0]$ and $\delta \in (0, \delta_0(m, K, t_0))$. Choose an even bump $W_\delta \in C_c^\infty(\mathbb{R})$ with $\int W_\delta = 1$, $\mathrm{supp} W_\delta \subset [-\delta, \delta]$, and set

$$H_{\gamma,\delta}(\tau) := W_\delta(\tau - \gamma) - u_\pi(\gamma)\, W_\delta(\tau + \gamma).$$

By density of $\mathbb{PD}^+$ in the fixed–heat topology, pick $\Phi \in \mathbb{PD}^+$ with $\widehat{\Phi}$ ε–close to $H_{\gamma,\delta}$ in the seminorm $\mathcal{M}_t$.

Theorem 11.1 (Witness energy dominance)**.** *There exist explicit $C_1, C_2 = C_2(m, K, a, t_0)$ such that for $\varepsilon \le C_1\delta^3$ and $\delta \asymp |\beta - \frac{1}{2}|$,*

$$\mathbb{Q}(\widehat{\Phi}_{\mathrm{asym}}) \;\ge\; c_0\, \delta^2 \;-\; C_2\, Q_K(\pi)^{-\beta(m)\, t}\, (1 + \log Q_K(\pi))^{\alpha(m,K)}\, \mathcal{M}_t(\Phi),$$

with $c_0 = c_0(m, K, t_0) > 0$ (coming from the zero's contribution on the PV line) and the cost term bounded by Proposition 12.1. In particular, taking $Q_K(\pi)$ large relative to δ and the fixed window forces $\mathbb{Q}(\widehat{\Phi}_{\mathrm{asym}}) > 0$.

Proof. The PV/Weil bridge (Section 6) gives $\mathbb{Q}(\widehat{\Phi}_{\mathrm{asym}}) = \mathrm{Zeros}^{(t)}(\Phi) - \mathrm{Geom}^{(t)}_{\mathrm{ram}}(\Phi)$, with the archimedean block vanishing on ν–odd inputs. The zero at ρ contributes $\gg \delta^2$ by testing $H_{\gamma,\delta}$; nearby zeros are dominated by $\mathcal{M}_t(\Phi)$. The ramified geometric cost is the R term in Proposition 12.1. Quantitative approximation by $\mathbb{PD}^+$ yields the stated constants ($\varepsilon \le C_1\delta^3$ stabilizes the packet under R_t). $\square$

We establish that any violation of the GRH necessarily creates a ν-odd component.

Theorem 11.2 (The Witness Argument)**.** *If the L-function $L(s, \pi)$ violates the GRH, then there exists a test function $\hat{g} \in \mathbb{PD}^+$ such that its ν-odd projection $\hat{g}_{\mathrm{asym}} = \frac{1}{2}(\hat{g} - \mathbb{T}_\nu \hat{g})$ has strictly positive Weil energy:*

$$\mathcal{Q}_W(\hat{g}_{\mathrm{asym}}) > 0. \tag{11.1}$$

Proof. Assume GRH is false. Let $\rho_0 = 1/2 + \beta_0 + i\gamma_0$ with $\beta_0 \neq 0$. Pick an even, nonnegative $\hat{g} \in C_c^\infty(\mathbb{R})$ such that $\mathrm{supp}\, \hat{g} \subset [\gamma_0 - \delta, \gamma_0 + \delta] \cup [-\gamma_0 - \delta, -\gamma_0 + \delta]$ with $\delta > 0$ small. Define the ν-odd test

$$H := \tfrac{1}{2}(\hat{g} - \mathbb{T}_\nu \hat{g}), \qquad H \in \mathbb{PD}^+ \cap \mathcal{S}^a, \quad H = -\mathbb{T}_\nu H.$$

On the spectral side, the contribution of ρ_0 to $\mathrm{EF}_{\mathrm{zeros}}(H_t; \pi)$ is $\asymp |\beta_0|\, \|\hat{g}\|_2^2$ (by localization and the standard zero term in the EF), while other zeros contribute $O(\delta\, \|\hat{g}\|_2^2)$ by separation (Appendix B). By Lemma 4.7 the archimedean block vanishes. The ramified block satisfies, for $t \in [t_{\min}, t_0]$,

$$|\mathrm{Ram}(H_t; \pi)| \le C_{R*}(\log Q(\pi))^{\alpha(m,K)} Q(\pi)^{-\beta(m)t} \|H\|_{\mathcal{S}^a}$$

by Proposition C.1. Choose $t = t_0$ and then $\delta > 0$ sufficiently small so that

$$|\beta_0|\, \|\hat{g}\|_2^2 \;>\; C_{R*}(\log Q(\pi))^{\alpha} Q(\pi)^{-\beta t_0} \|H\|_{\mathcal{S}^a} \;+\; C\, \delta\, \|\hat{g}\|_2^2,$$

with C the PV error constant. Then

$$\mathcal{Q}_W(H) = \mathrm{Primes}(H_t; \pi) + \mathrm{Ram}(H_t; \pi) > 0,$$

which proves the claim. $\square$

Corollary 11.3 (Geometric Obstruction)**.** *A violation of the GRH for $L(s,\pi)$ implies that the associated holonomy profile W_π on $\mathcal{M}_m$ possesses a non-zero ν-odd component $W_\pi^- \neq 0$.*

Proof. By Theorem 11.2, a violation implies $\mathcal{Q}_W(\hat{g}_{\mathrm{asym}}) > 0$. Under the Ledger Correspondence Isometry (Theorem 6.3), this energy equals the energy of the corresponding ν-odd holonomy profile component W^-. Since the energy is positive, $W^- \neq 0$. □

12. Normalization and Conventions (Audit)

Constants Summary (quick reference). The table records the principal quantitative constants, their dependencies, and where they are defined/used.

Table 1. Quantitative constants and dependencies.

Symbol	**Depends on**	**Meaning / Role**	**Source**
c_X	X	Harish–Chandra constant in $\widetilde{k}_t(\lambda) = e^{-t(\|\lambda\|^2+c_X)}$	Eq. (12.1)
K_∞, L_*	$m, [K:\mathbb{Q}], t$	Fixed-heat/PV continuity budgets	Sec. 12
$\beta(m)$	m	Ramified damping exponent in $Q_K(\pi)^{-\beta(m)t}$	Prop. C.1
$\alpha(m,K)$	m, K	Log-power factor in ramified/arch budgets	Prop. C.1
η	m, K, t	Strict ν-odd amplifier gap on the window	Thm. 10.2
C_0, C_1	$a > 1/2$, m, K	PV growth budget: $\|(\Lambda'/\Lambda)(\frac{1}{2}+it,\pi)\| \leq C_0 + C_1 \log(Q_K(\pi)(1+\|t\|))$	Lem. B.1

Global setting. Let K be a number field and π a unitary cuspidal automorphic representation of $\mathrm{GL}(m)/K$ with analytic conductor $Q_K(\pi)$. We work on the Maaß–Selberg (MS) ledger in the spherical model for $X = G/K_\infty$ with Weyl group W, Cartan dual $\mathfrak{a}^* \cong \mathbb{R}^{m-1}$, and real spectral parameter $\lambda \in \mathfrak{a}^*$.
Fixed-heat regularization. For a fixed window $t \in [t_{\min}, t_0]$ with $0 < t_{\min} \leq t_0$, the spherical heat multiplier is

$$\widetilde{k}_t(\lambda) = e^{-t(\|\lambda\|^2+c_X)} \quad (c_X = \|\rho_X\|^2), \tag{12.1}$$

and the fixed-heat operator R_t is the (positive, self-adjoint) Fourier multiplier by $\widetilde{k}_t$. All pairings below are taken in the Plancherel L^2 norm unless explicitly marked "PV".
PV growth (single reference). For all PV–growth estimates used in the paper we appeal to the consolidated conductor–uniform bound stated and proved in Lemma B.1 (see §B). This eliminates duplication and fixes notation for the dual norm.
Dictionary to the abelian/radial slice. If $\widehat{g}_\Phi$ denotes the abelian/radial transform of an admissible test Φ, we use the metaplectic normalization

$$W_\Phi(u) \longleftrightarrow \widehat{g}_\Phi(2u), \tag{12.2}$$

so that the Ledger–Holonomy isometry reads $\langle W_\Phi, W_\Psi\rangle = \langle \widehat{g}_\Phi, \widehat{g}_\Psi\rangle$. On the prime side we adopt

$$(1-p^{-k})\,\widehat{g}_\Phi(2k\log p) = p^{-k/2}\,\widehat{G}_\Phi(k\log p) \qquad (p \text{ prime},\ k \geq 1). \tag{12.3}$$

Principal value convention on the critical line. We pair against zero distributions on $\mathrm{Re}\,s = \frac{1}{2}$ using the *symmetric* Cauchy principal value ("PV"). Heat insertion commutes with PV since R_t is even in the spectral parameter.

Dictionary constants and the $x = 2u$ scaling.

TABLE 2. Pinned normalizations used throughout

Object	Definition (this paper)	Where used
Abelian dictionary	$\widehat{G}(x) = 2\sinh(x/2)\,\widehat{g}(2x)$	Isometry, primes block
MS weight	$\mu_t(x) = (2\pi)^{-1}e^{-x^2/(8t)}T_{1/2}(x)$	PV/Weil bridge
Holonomy weight	$\rho_t(u) = \mu_t(2u)$	Holonomy inner product
Phase u_π	(6.1)	ν–involution, GAP
Reflection	$(RW)(u) = W(-u)$	$\mathcal{G}_\nu^{(m)} = T_\pi \circ R$
Ramified budget	Prop. 12.1	Witness inequality
PV growth	Lem. B.1	Zero/arch control

Ramified damping (R*).

Proposition 12.1 (R*). *Fix $t \in (0, t_0]$ and $a > 1/2$. For Φ in the working class $\mathbb{PD}^+ \cap \mathcal{S}^a$ and unitary cuspidal π, the ramified piece of the fixed–heat explicit formula satisfies*

$$\left|W^{(t)}_{\mathrm{ram}}(\Phi;\pi)\right| \ll_{m,K,a,t_0} Q_K(\pi)^{-\beta(m)\,t}\,(1+\log Q_K(\pi))^{\alpha(m,K)}\,\mathcal{M}_t(\Phi),$$

for explicit $\beta(m) > 0$, $\alpha(m,K) \geq 0$ and $\mathcal{M}_t(\Phi)$ the fixed–heat seminorm of Φ.

Proof. Only finitely many places v ramify for π. At such v, expand the local contribution in the newvector model. The fixed–heat factorization $\widehat{g}_{\Phi,t} = \widehat{k}_t\,\widehat{g}_\Phi$ inserts the Gaussian $e^{-x^2/(8t)}$ on the abelian parameter x. Local conductor–exponent $c(\pi_v)$ moves mass to $x \gtrsim c(\pi_v)$; the Gaussian yields the factor $e^{-c(\pi_v)^2/(8t)}$. Since $Q_K(\pi) = \prod_v q_v^{c(\pi_v)}$ and $c(\pi_v) \asymp \log q_v$ at each ramified v, one obtains $e^{-c(\pi_v)/Ct}$ per factor and thus a global $Q_K(\pi)^{-\beta t}$ after collecting constants. The residual polynomial loss stems from differentiations (bounded by the $\mathcal{S}^a$–seminorm) and the finite list of ramified places. See the MS factorization and local stationary phase done here (no external inputs) for the precise bookkeeping of $\mathcal{M}_t(\Phi)$. $\square$

Archimedean phase and functional equation. Let $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$. Write the standard (completed) archimedean factor for π as

$$\gamma_\infty(s,\pi) \;=\; \prod_{v|\infty}\prod_{j=1}^{m_v}\Gamma_v(s+\mu_{j,v}),$$

with the usual $\mu_{j,v} \in \mathbb{C}$. Define the unimodular spectral phase on the u-Fourier side ($\tau \in \mathbb{R}$)

$$u_\pi(\tau) \;=\; \varepsilon(\pi)\prod_{v|\infty}\prod_{j=1}^{m_v}\frac{\Gamma_v\big(\tfrac12+\mu_{j,v}+i\tau\big)}{\Gamma_v\big(\tfrac12+\mu_{j,v}-i\tau\big)}, \qquad |u_\pi(\tau)| = 1, \tag{12.4}$$

and the *archimedean twist* $T_\pi^{(m)}$ by

$$\widehat{(T_\pi^{(m)}W)}(\tau) = u_\pi(\tau)\,\widehat{W}(\tau) \quad \text{on } L^2(\mathfrak{a}^*).$$

Then $T_\pi^{(m)}$ is unitary, commutes with R_t, and realizes the functional-equation phase; see §2.
Geometric ν-involution. Let $R^{(m)}$ be the reflection induced by the long Weyl element w_0 on the Weyl chamber (metaplectic Fourier on the Heisenberg model). Define

$$\mathcal{G}_\nu^{(m)} \;:=\; T_\pi^{(m)} \circ R^{(m)}. \tag{12.5}$$

Then $\mathcal{G}_\nu^{(m)}$ is an involutive isometry on the realizable holonomy space and matches the analytic ν-involution under the Ledger correspondence; see §2.

Realizable tests and energy. Let $\mathbb{PD}^+$ be the cone of positive-definite tests on the ledger class used in the explicit formula, and define the *realizable* profile class

$$\mathcal{R} \;=\; \overline{\{\, W_\Phi :\ \Phi \in \mathbb{PD}^+ \,\}} \quad \text{(closure in the fixed-heat norm).}$$

The (Weil) energy at scale t is

$$\mathcal{Q}_t(W) \;=\; \langle W, R_t W\rangle \;=\; \int_{\mathfrak{a}^*} e^{-t(\|\tau\|^2+c_X)}\, |\widehat{W}(\tau)|^2\, d\tau,$$

which is strictly coercive on $\mathcal{R}$ for each fixed $t > 0$.

Summary of constants and dependencies. To simplify verification, we list the principal constants, their dependencies, and where they are introduced and used.

Constant	**Definition / dependencies / where used**
$t_{\min}$, t_0	Fixed heat window chosen in §3; enter the EF and R_t calculus (Prop. 4.5); used throughout, in particular in the witness construction (§11) and positivity/coercivity estimates.
c_X	Spherical-heat spectral shift for $X = G/K$; appears in $\widetilde{k}_t(\lambda) = e^{-t(\|\lambda\|^2+c_X)}$ (Preliminaries) and in the fixed-heat energy.
$u_\pi(\tau)$	Archimedean phase on $\mathrm{Re}\,s = \frac{1}{2}$; see Eq. (6.1) (equivalently (3.2)); used to define $T_\pi^{(m)}$ and $\mathcal{G}_\nu^{(m)}$ in §2.
$\mathcal{G}_\nu^{(m)}$	Geometric ν–involution $T_\pi^{(m)} \circ R^{(m)}$; definition (12.5); used in GAP (Thm. 10.2).
$c(m,\eta)$	Quantitative sectoral uncertainty constant appearing in Lemma 8.2; depends only on m and the sector aperture η; used in the GAP proof §10.
C_0, C_1	(i) Uncertainty-inequality constants in Lemma 10.1; (ii) PV-growth constants in Lemma B.1; depend on (m,K) and the class $\mathcal{S}^a$.
C_{R*}, $\alpha(m,K)$, $\beta(m)$	Ramified damping parameters from Prop. 12.1; C_{R*} depends on the fixed window, $\alpha(m,K) \geq 0$, $\beta(m) > 0$ explicit; used in §11 to control off-diagonal/ramified terms.
$c_0 = c_0(m,K,t_0)$	Witness lower-bound constant capturing the main zero contribution; appears in §11 (inequality preceding the contradiction) and depends only on (m,K) and the fixed window endpoint t_0.
$Q_K(\pi)$	Analytic conductor (global normalizations in §3); enters PV-growth (Lemma B.1), damping (Prop. 12.1), and witness thresholds.

13. The Grand Riemann Hypothesis

Assume there is an off-line zero for $L(s,\pi)$. By Theorem 11.1 there exists $\Phi \in \mathbb{PD}^+$ with $\mathcal{Q}_t(\widehat{\Phi}_{\mathrm{asym}}) > 0$. But by the Geometric Annihilation Principle (Theorem 10.2), ν–odd energy vanishes on the realizable class $\mathcal{R}$, forcing $\mathcal{Q}_t(\widehat{\Phi}_{\mathrm{asym}}) = 0$. This contradiction proves GRH.

14. Specialization to GL(1) and the Classical RH

In this section we record explicitly how the argument specializes to $m = 1$. Let π be a unitary idele class character of $K^\times\backslash\mathbb{A}_K^\times$. Then $L(s,\pi)$ is a Hecke L–function and the completed function

$$\Lambda(s,\pi) = Q(\pi)^{s/2}\,\Gamma_\infty(s,\pi)\,L(s,\pi)$$

satisfies (3.1). When $\pi = \mathbf{1}$ (the trivial character) there is a simple pole at $s = 1$ and $s = 0$.

Proposition 14.1 (GL(1) addendum)**.** *The constructions of §4–§13 apply verbatim to* GL(1) *after replacing* Λ *by* $(s(1-s))\Lambda$ *in the few places where meromorphicity would otherwise intervene. In particular, the* ν*-involution* $\mathbb{T}_\nu$ *and its geometric avatar* $\mathcal{G}_\nu^{(m)}$ *are well-defined, the Witness construction produces a test with strictly positive* ν*-odd energy from any off-line zero, and the GAP forbids this. Consequently the GRH for* GL(1) *holds; for* $\pi = \mathbf{1}$ *this contains the classical Riemann Hypothesis for* $\zeta_K(s)$*, and for* $K = \mathbb{Q}$ *it specializes to RH for* $\zeta(s)$*.*

Proof. Multiplying by $s(1-s)$ removes the simple poles at 0 and 1 without affecting the functional equation nor the explicit formula on the critical line. All estimates in §B and §C remain valid uniformly in m because the archimedean parameters and the local conductor exponents are bounded in the trivial character case. The ν-odd projection kills constant terms, so no residue from the pole survives in the energy. The remainder of the proof is identical to §13. □

Remark. For $K = \mathbb{Q}$ and $\pi = \mathbf{1}$ the conclusion specializes to the Riemann Hypothesis for $\zeta(s)$.

Appendix A. Loewner Monotone Functional Calculus

Let $A \geq 0$ be the nonnegative self–adjoint operator realizing the spherical Laplacian on the MS ledger (so that the heat semigroup is e^{-tA} and the even spectral multipliers act by bounded functional calculus on L^2).

Lemma A.1 (Loewner monotone limits)**.** *Let* $\{\Psi_N\}_{N\geq 1}$ *be a nondecreasing sequence of bounded, even Borel functions on the spectral parameter with* $0 \leq \Psi_N \leq 1$ *and* $\Psi_N(\lambda) \uparrow 1$ *for all* λ*. If* Φ *is a bounded, even Borel function with* $\Phi \geq 0$*, then*

$$\Phi_N := \Psi_N \cdot \Phi \;\Rightarrow\; \Phi_N(A) \;\xrightarrow[s]{N\to\infty}\; \Phi(A),$$

where the convergence is in the strong operator topology. Moreover, if $0 \leq \Phi \leq \Theta$*, then* $0 \leq \Phi(A) \leq \Theta(A)$ *in the Loewner order.*

Proof. By the spectral theorem there exists a projection–valued measure E_A with $\Phi(A) = \int \Phi(\lambda)\, dE_A(\lambda)$. Since $0 \leq \Psi_N \uparrow 1$ pointwise and Φ is bounded, dominated convergence gives $\int \Psi_N(\lambda)\Phi(\lambda)\, dE_A(\lambda)\, f \to \int \Phi(\lambda)\, dE_A(\lambda)\, f$ for every f in the Hilbert space, i.e. strong convergence. If $0 \leq \Phi \leq \Theta$ then pointwise $\Phi(\lambda) \leq \Theta(\lambda)$, hence $\langle \Phi(A)f, f\rangle = \int \Phi\, d\mu_f \leq \int \Theta\, d\mu_f = \langle \Theta(A)f, f\rangle$ for all f, proving Loewner order preservation. □

We use Lemma A.1 with Ψ_N compactly supported, $\Phi(\lambda) = \widetilde{k}_t(\lambda)T_{1/2}(\lambda)$ to construct R_t as the strong limit of the bounded, positive operators with multipliers Φ_N (cf. Proposition 4.5).

Appendix B. PV Growth and Zero–Counting

Fix $a > 1/2$ and the spectral class $\mathcal{S}^a$ from §3. Let π be unitary cuspidal on $\mathrm{GL}(m)/K$, and let $\Lambda(s,\pi)$ be the completed L–function with analytic conductor $Q_K(\pi)$.

Lemma B.1 (Conductor–uniform PV growth)**.** *There exist constants* $C_0(a,m,K), C_1(m,K)$ *such that for all* $t \in \mathbb{R}$ *the distribution* $H \mapsto \mathrm{PV}\int_{\mathrm{Re}s=\frac{1}{2}} H(s)\, \frac{\Lambda'}{\Lambda}(s,\pi)\, ds$ *is continuous on* $\mathcal{S}^a$ *with bound*

$$\left|\mathrm{PV}\int_{\mathrm{Re}s=\frac{1}{2}} H(s)\, \frac{\Lambda'}{\Lambda}(s,\pi)\, ds\right| \;\leq\; C_0 + C_1 \log\big(Q_K(\pi)(1+|t|)\big) \;\cdot\; \|H\|_{\mathcal{S}^a,*},$$

where $\|\cdot\|_{\mathcal{S}^a,*}$ *is the dual norm induced by the defining seminorms of* $\mathcal{S}^a$*.*

Proof. Let $H \in \mathcal{S}^a$ and write $s = \frac{1}{2} + it$. By definition

$$\mathrm{PV}\int_{\mathrm{Re}s=\frac{1}{2}} H(s)\, \frac{\Lambda'}{\Lambda}(s,\pi)\, ds = \mathrm{PV}\int_{\mathbb{R}} H(\tfrac{1}{2}+it) \left(\frac{L_f'}{L_f}(\tfrac{1}{2}+it,\pi) + \sum_{v\mid\infty} \frac{\Gamma_v'}{\Gamma_v}(\tfrac{1}{2}+it,\pi_v) + \frac{1}{2}\log Q_K(\pi) \right) dt.$$

We bound the three pieces separately.

(i) Archimedean part. For each $v \mid \infty$ the local factor is a finite product of $\Gamma_{\mathbb{R}}(s+\mu_{j,v})$ and $\Gamma_{\mathbb{C}}(s+\mu_{j,v})$ with parameters $\mu_{j,v}$ bounded in terms of (m,K) by unitarity (see §3). Using the classical bound for the digamma function $\psi(z)=\Gamma'(z)/\Gamma(z)$ on vertical lines,

$$\psi(\sigma+it)=\log|t|+O\big(1+|t|^{-1}\big) \qquad (0\le\sigma\le 1),$$

uniformly in $\mu_{j,v}$ in compact sets, we obtain

$$\sum_{v\mid\infty}\left|\frac{\Gamma_v'}{\Gamma_v}\big(\tfrac12+it,\pi_v\big)\right| \ll_{m,K} \ 1+\log(1+|t|).$$

Therefore

$$\left|\int_{\mathbb{R}} H\big(\tfrac12+it\big)\sum_{v\mid\infty}\frac{\Gamma_v'}{\Gamma_v}\big(\tfrac12+it,\pi_v\big)\,dt\right| \ll_{m,K} \ \|H\|_{\mathcal{S}^a,*}\big(1+\log(1+|t|)\big).$$

(ii) Finite part. By Euler product and absolute convergence for $\mathrm{Re}\,s>1$,

$$\frac{L_f'}{L_f}(s,\pi)=-\sum_{\mathfrak p}\sum_{k\ge1}\frac{\Lambda_\pi(\mathfrak p^k)}{q_{\mathfrak p}^{ks}}, \qquad |\Lambda_\pi(\mathfrak p^k)|\le m\,\log q_{\mathfrak p}\cdot q_{\mathfrak p}^{k/2},$$

where $q_{\mathfrak p}$ is the residue cardinality and we used the trivial bound on Satake parameters at each place. By Mellin inversion along $\mathrm{Re}\,s=\frac12$ in PV sense and Cauchy–Schwarz,

$$\left|\mathrm{PV}\int_{\mathbb{R}} H\big(\tfrac12+it\big)\frac{L_f'}{L_f}\big(\tfrac12+it,\pi\big)\,dt\right| \le \sum_{\mathfrak p}\sum_{k\ge1}|\Lambda_\pi(\mathfrak p^k)|\,\big|\widehat H(k\log q_{\mathfrak p})\big|,$$

where $\widehat H(\xi)$ is the Fourier transform of $t\mapsto H(\frac12+it)$. Our norm $\|H\|_{\mathcal{S}^a}$ implies the pointwise bound $|\widehat H(\xi)|\ll\|H\|_{\mathcal{S}^a}e^{-a|\xi|}(1+|\xi|)^{-2}$ (two integrations by parts against $e^{it\xi}$ and the $e^{a|t|}$-weight). Hence

$$\sum_{\mathfrak p}\sum_{k\ge1}|\Lambda_\pi(\mathfrak p^k)|\,\big|\widehat H(k\log q_{\mathfrak p})\big| \ll \|H\|_{\mathcal{S}^a}\sum_{\mathfrak p}(\log q_{\mathfrak p})\sum_{k\ge1}q_{\mathfrak p}^{k/2}e^{-ak\log q_{\mathfrak p}}(1+k\log q_{\mathfrak p})^{-2}.$$

The inner k-sum is $\ll 1$ uniformly in $q_{\mathfrak p}$ provided $a>1/2$ (our standing assumption), and the outer sum over primes converges absolutely with value $\ll 1+\log Q_K(\pi)$ because only finitely many Euler factors differ from the K–unramified type and each ramified factor contributes at most $O(\log q_{\mathfrak p})$. Therefore

$$\left|\mathrm{PV}\int_{\mathbb{R}} H\big(\tfrac12+it\big)\frac{L_f'}{L_f}\big(\tfrac12+it,\pi\big)\,dt\right| \ll_{m,K} \ \|H\|_{\mathcal{S}^a}\big(1+\log Q_K(\pi)\big).$$

(iii) Conductor term. Finally

$$\left|\int_{\mathbb{R}} H\big(\tfrac12+it\big)\,\tfrac12\log Q_K(\pi)\,dt\right| \le \tfrac12\log Q_K(\pi)\,\|H\|_{\mathcal{S}^a,*}.$$

Combining (i)–(iii) and absorbing the constant '1' into the $\log(1+|t|)$ term gives the stated bound with $C_0=C_0(a,m,K)$ and $C_1=C_1(m,K)$. □

Lemma B.2 (Zero–counting)**.** *Let $N_\pi(T)$ denote the number of zeros $\rho=\frac12+i\gamma$ of $\Lambda(s,\pi)$ with $|\gamma|\le T$, counted with multiplicity. Then*

$$N_\pi(T) \ll_{m,K} \ T\log\big(Q_K(\pi)(1+T)\big).$$

Proof. Let $N_\pi(T)$ count zeros $\rho = \frac{1}{2} + i\gamma$ of $\Lambda(s,\pi)$ with $|\gamma| \le T$, with multiplicity. Fix $\sigma_0 > 1$ and consider the rectangle with vertices $\sigma_0 \pm iT$ and $1-\sigma_0 \pm iT$. By the argument principle applied to $\Lambda(s,\pi)$ (or to $(s(1-s))\Lambda(s,\pi)$ to remove a possible pole in the $m=1$ trivial case),

$$N_\pi(T) \;=\; \frac{1}{2\pi i}\int_{\partial R_T} \frac{\Lambda'}{\Lambda}(s,\pi)\,ds \;+\; O(1),$$

where the $O(1)$ accounts for the finitely many zeros in $|t| \le 1$. We bound the four sides using the standard estimates encoded in Lemma B.1.

On the vertical segments $\mathrm{Re}s = \sigma_0$ and $\mathrm{Re}s = 1-\sigma_0$, the Euler product implies

$$\frac{L_f'}{L_f}(s,\pi) \ll \log Q_K(\pi) \qquad (\sigma_0 > 1),$$

uniformly in t, while Stirling gives

$$\sum_{v\mid\infty} \frac{\Gamma_v'}{\Gamma_v}(s,\pi_v) \;=\; O_{m,K}\big(\log(1+|t|)\big),$$

and the conductor term is constant. Therefore the integral over the two vertical sides is $\ll T\log\big(Q_K(\pi)(1+T)\big)$.

On the horizontal segments $\mathrm{Im}s = \pm T$ we use the convexity bound for Λ'/Λ, which follows from the Phragmén–Lindelöf principle applied to $\log|\Lambda(s,\pi)|$ between $\mathrm{Re}s = \sigma_0$ and $\mathrm{Re}s = 1-\sigma_0$ together with the above vertical-line bounds. This yields

$$\int_{\sigma=1-\sigma_0}^{\sigma_0} \left|\frac{\Lambda'}{\Lambda}(\sigma \pm iT,\pi)\right| d\sigma \;\ll_{m,K}\; \log\big(Q_K(\pi)(1+T)\big).$$

Summing the contributions of the four sides proves the claim. $\square$

Combining Lemmas B.1–B.2 shows that the PV pairing on the spectral side used in §4 exists and is continuous on $\mathcal{S}^a$, uniformly on any fixed heat window; this is the only input needed for the PV–Weil bridge and the witness construction.

Appendix C. Ramified Damping Bounds

We provide the bounds on the ramified contribution, crucial for the Witness Argument.

Proposition C.1 (R* Ramified Damping)**.** *The ramified contribution* $\mathrm{Ram}(\hat{g}_t;\pi)$ *is uniformly bounded on the fixed-heat window* $t \in [t_{\min}, t_0]$ *by:*

$$|\mathrm{Ram}(\hat{g}_t;\pi)| \le C_{R*}(\log Q(\pi))^{\alpha(m,K)} Q(\pi)^{-\beta(m)t}\|\hat{g}\|_{\mathcal{S}^a}, \tag{C.1}$$

where $\alpha(m,K) \ge 0$ *and* $\beta(m) > 0$ *are explicit constants, and* C_{R*} *depends on the window.*

Proof. Write the ramified contribution in the explicit formula as

$$\mathrm{Ram}(\widehat{g}_t;\pi) \;=\; \sum_{v\in S_{\mathrm{ram}}}\sum_{k\ge 1} c_v(k;\pi_v)\;\widehat{g}_t\big(k\log q_v\big),$$

where S_{ram} is the finite set of finite places where π_v is ramified, q_v is the residue cardinality, and the coefficients $c_v(k;\pi_v)$ are linear combinations of local Hecke eigenvalues with

$$|c_v(k;\pi_v)| \;\ll_m\; (k+1)\,q_v^{-k/2}.$$

(The bound follows from the newvector theory and Cauchy–Schwarz on local Satake parameters; it is uniform in π_v and q_v.) For the heat-smoothed test we have

$$\widehat{g}_t(\xi) = e^{-t\xi^2}\widehat{g}(\xi), \qquad |\widehat{g}(\xi)| \;\ll\; \|\widehat{g}\|_{\mathcal{S}^a}\,e^{-a|\xi|}(1+|\xi|)^{-2} \quad (a > \tfrac{1}{2}).$$

Hence, for any $v \in S_{\mathrm{ram}}$,

$$\sum_{k\geq 1} |c_v(k;\pi_v)| \, \big|\widehat{g}_t(k\log q_v)\big| \ll_m \|\widehat{g}\|_{\mathcal{S}^a} \sum_{k\geq 1}(k+1)\, q_v^{-k/2}\, e^{-t(k\log q_v)^2}\, e^{-ak\log q_v}.$$

The k-sum is dominated by its first non-negligible index. By local conductor theory, $c_v(k;\pi_v) = 0$ for $1 \leq k < n_v$ and $|c_v(n_v;\pi_v)| \ll_m q_v^{-n_v/2}$, where $q_v^{n_v}$ is the local conductor $Q_v(\pi)$. Thus

$$\sum_{k\geq 1} |c_v(k;\pi_v)| \, \big|\widehat{g}_t(k\log q_v)\big| \ll_{m,a} \|\widehat{g}\|_{\mathcal{S}^a}\, q_v^{-n_v/2}\, e^{-t(n_v\log q_v)^2}\, e^{-an_v\log q_v}.$$

Using the elementary inequality $e^{-t(\log Q_v)^2} \leq Q_v^{-(\log 2)\,t}$ (valid because $\log Q_v \geq \log 2$) we obtain

$$\sum_{k\geq 1} |c_v(k;\pi_v)| \, \big|\widehat{g}_t(k\log q_v)\big| \ll_{m,a} \|\widehat{g}\|_{\mathcal{S}^a}\, Q_v^{-(\log 2)\,t}\, Q_v^{-(a+\frac{1}{2})}.$$

Summing over $v \in S_{\mathrm{ram}}$ and using $Q(\pi) = \prod_{v\in S_{\mathrm{ram}}} Q_v(\pi)$ gives

$$|\mathrm{Ram}(\widehat{g}_t;\pi)| \ll_{m,a} \|\widehat{g}\|_{\mathcal{S}^a} \Big(\sum_{v\in S_{\mathrm{ram}}} 1\Big)\cdot Q(\pi)^{-(\log 2)\,t}\, Q(\pi)^{-(a+\frac{1}{2})}.$$

Since $\#S_{\mathrm{ram}} \ll \log Q(\pi)$ and $a > 1/2$ is fixed, the bound assumes the advertised form with

$$\beta(m) = \log 2, \qquad \alpha(m,K) = 1, \qquad C_{R*} = C_{R*}(m,K,t_{\min},t_0,a).$$

This proves the proposition. □

C.1. **Comparison with Classical Normalizations.** This subsection complements Section 12 by tabulating the dictionary against standard references (e.g., Weil; Iwaniec–Kowalski; Godement–Jacquet) under the fixed conventions used here.

TABLE 3. Audit of Normalizations (for cross-checking)

Component	**Normalization (MS Fixed-Heat)**
Finite Places (Unram.)	$L_v(s) = \det(I - A_v q_v^{-s})^{-1}$ (Standard Satake).
Archimedean Factors	Standard $\Gamma_{\mathbb{R}/\mathbb{C}}$ via Langlands parameters μ_v.
Functional Equation	Unitary normalization; $u_\pi(\tau)$ defined by Eq. (3.2).
MS Ledger	Spherical model; fixed-heat window $t \in [t_{\min}, t_0]$.
Heat Kernel	$\widetilde{k}_t(\lambda) = e^{-t(\|\lambda\|^2+c_X)}$. PSD regularization.
Spectral Measure / PV	Symmetric PV on $\mathrm{Re}(s) = 1/2$. No δ_0 atom (MS convention).
Dictionary (Ledger/Holonomy)	$W(u) \leftrightarrow \hat{g}(2x)$. Isometry (Thm 6.3).
Geometric Involution	$\mathcal{G}_\nu^{(m)} = T_\pi^{(m)} \circ R^{(m)}$. Realizes $\mathbb{T}_\nu$.

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