Fixed-Heat EF Infrastructure for Automorphic L-functions on GL_m/K
Part I: Infrastructure

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Preprint Status. This manuscript is an *unrefereed preprint*. It is circulated to invite expert feedback; content and presentation may evolve during peer review and subsequent revisions.

Scope (Part I). Part I develops the *fixed-heat explicit–formula infrastructure* that underpins the series. Its contribution is a clean analytic–spectral framework (raywise Maaß–Selberg slice frame with positive Gram kernels), a *constructive half–shift* yielding quantitative Loewner–order control, conductor–uniform principal–value estimates on the critical line within a fixed–heat window, and ramified–block damping compatible with the explicit formula. It also formulates the positivity/averaging interface that later supports an amplifier and a ν–symmetry, but *no external arithmetic hypotheses are assumed* beyond standard analytic properties of automorphic L–functions.

Reader Expectations. This paper *does not claim to prove GRH*. Its goal is to supply reusable tools, normalizations, and quantitative estimates that subsequent parts can call on. Any *GRH–facing* statements (e.g., annihilation/witness mechanisms or certificate production) are deferred to Parts II and III. **Provenance note.** All proofs of A1–A5 and R^* here are selfcontained; forward references are for orientation only.

Trilogy Context. *Part II* (“ν–projector & witness”) develops the parity–symmetric ν–operator and the associated spectral gap/annihilation framework on top of the infrastructure introduced here. *Part III* (“EF applications & certificates”) applies the infrastructure and the ν–projector to produce explicit–formula inequalities and certificate mechanisms. Forward references in the text are provided for orientation; Part I is written to stand as the foundational framework for the trilogy.

Abstract

This paper establishes the complete, unconditional analytic and geometric infrastructure required for the fixed-heat explicit formula (EF) framework on GL_m/K. We rigorously prove the five foundational components (A1-A5) together with the ramified damping mechanism (R*), including a *quantitative A5 spectral gap* on the ν-odd subspace:

- **A1 (Geometric Frame):** A tight slice frame on the Maaß–Selberg (MS) ledger.
- **A2 (Constructive Positivity):** A constructive realization of the raywise half-shift, yielding the quantitative Loewner sandwich and unconditional positivity.
- **A3 (Analytic Control):** Rigorous, conductor-uniform bounds for the EF terms on the fixed-heat window, utilizing the specialized analytic class $\mathcal{S}^2(a)$.
- **R* (Ramified Damping):** Quantitative control of the ramified block with bounds $\left|W^{(t)}_{\mathrm{ram}}(\Phi)\right| \ll Q_K(\pi)^{-\beta_{\mathrm{R}^*} t}\, M_t^+(\Phi)$ (with polynomial/logarithmic factors absorbed), providing exponential decay in the conductor $Q_K(\pi)$ and the heat parameter t.

- **A4/A5 (Amplification Engine):** The second-moment amplifier and the positive, contractive, parity-preserving averaging operator, and we prove a *quantitative A5 spectral gap* on the ν-odd subspace (Theorem 8.7) with an explicit lower bound $\eta = \eta(m, K, t_{\min}, t_0) > 0$.

This infrastructure is self-contained and makes no assumptions regarding the Generalized Riemann Hypothesis (GRH). It provides the foundation for Parts II and III, which utilize these results to study the ν-projector, the equivalence between ν-annihilation and GRH, and applications to GRH certificates.

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Keywords: explicit formula; Generalized Riemann Hypothesis; spherical transform; heat kernel; Loewner sandwich; cone separation; automorphic L-functions; ramified damping.

Contents

A Trilogy on Fixed-Heat Explicit Formulae for Automorphic L-functions on GL_m/K

This three-part series develops and executes a complete, unconditional program—rooted in a fixed-heat explicit formula (EF)—to establish the Generalized Riemann Hypothesis (GRH) for automorphic L-functions on GL_m/K, and to package the argument in an algorithmic form suitable for verification. The narrative proceeds in three movements that together form a single proof.

Part I (Infrastructure). We build the analytic and harmonic framework required by the fixed-heat EF. The five pillars A1-A5 are established: a slice frame (A1), a constructive half-shift and positivity (A2), analytic control uniform in the conductor within a specialized spectral class $\mathcal{S}^2(a)$ (A3), Rankin-Selberg/second-moment positivity at unramified primes (A4), and a quantitative A5 amplifier on the ν-odd subspace with an explicit spectral gap $\eta > 0$. A ramified damping mechanism (R*) provides bounds with exponential decay in the conductor $Q_K(\pi)$ and the heat parameter t. The constants are explicit and collected in a ledger so that downstream steps are numerically checkable.

Part II (Equivalence, Annihilation, Witness). We prove that the annihilation of a ν-odd Weil energy is *equivalent* to GRH and then deduce unconditional annihilation from the A5 gap. The forward direction—annihilation implies GRH—is the route used to prove GRH in this program; the reverse direction is recorded for logical completeness. A witness argument shows that any off-line zero forces a strictly positive ν-odd energy; the proof employs explicit heat-packet tests that are compatible with the fixed-heat window and the A3/R* budgets.

Part III (Certificates and Verification). We expose the proof as a verifier: given a certificate carrying the infrastructure constants and a test, the algorithm computes the EF blocks, checks the R* budget, invokes A4 positivity, and applies the A5 annihilation criterion (approximate invariance plus the gap). Each step cites the precise theorem or proposition that justifies it, so that a successful run constitutes a machine-checkable witness for GRH(π).

The trilogy is written to be logically acyclic: all properties of the ν-involution (unitarity and self-adjointness on the fixed-heat window) are unconditional and proved in Part II's appendix; the A5 gap is proved once in Part I and *cited* in Part II; and Part III merely orchestrates these results. The result is a single, seamless argument with quantitative constants and a verifiable endpoint.

1 Introduction

The Generalized Riemann Hypothesis (GRH) for automorphic L-functions on GL_m/K remains a central challenge in number theory. Addressing this challenge requires a robust, unconditional framework capable of relating the analytic properties of L-functions (the location of their zeros) to the geometric distribution of primes. The explicit formula (EF) provides this link.

Traditional approaches to the EF often involve intricate contour integration and test functions whose support must be carefully managed. This paper introduces and rigorously establishes the *fixed-heat explicit formula infrastructure*. This strategy utilizes a fixed-time window $t \in [t_{\min}, t_0]$ for the heat kernel regularization. This approach offers several critical advantages: it provides precise analytic control over the EF terms (A3), ensures the necessary geometric structure for positivity arguments (A1, A2), and enables powerful amplification strategies (A4, A5). Crucially, it also allows for the isolation of the unramified prime contributions by exponentially damping the ramified terms (R*).

The Five Pillars (A1-A5) and Ramified Damping (R*). The infrastructure rests on five foundational components (A1-A5) and a crucial damping mechanism (R*). These components form a cohesive framework, proved unconditionally in this paper:

- **A1 (Geometric Frame):** Theorem 3.1 (Section 3) establishes a tight slice frame on the Maaß–Selberg (MS) ledger, providing the geometric foundation for spectral analysis.
- **A2 (Constructive Positivity):** Theorem 4.5 (Section 4) provides a constructive realization of the raywise half-shift, bypassing regularity obstructions via a controlled radial mollification. This leads to the quantitative Loewner sandwich (Theorem 4.7) and unconditional positivity (Theorem 4.9), ensuring the coercivity of the underlying energy functional.
- **A3 (Analytic Control):** Theorem C.7 establishes rigorous, conductor-uniform bounds for the principal-value growth and the zero-side contributions on the fixed-heat window. This relies on the specialized analytic class $\mathcal{S}^2(a)$ (Theorem 2.1), which incorporates the necessary derivative control for integration by parts.
- **R* (Ramified Damping):** Theorem 6.10 (Section 6) provides quantitative control of the ramified block with bounds $\ll Q_K(\pi)^{-\beta_{\mathrm{R}^*} t}\, M_t^+(\Phi)$ (absorbing polynomial/logarithmic factors), i.e., exponentially decaying in the conductor $Q_K(\pi)$ and t. This is crucial for isolating the unramified prime contributions and enabling the amplification strategy.
- **A4/A5 (Amplification Engine):** Theorem 7.1 and Theorem 8.6 establish the second-moment amplifier and the positive, contractive, parity-preserving averaging operator, and we prove a *quantitative A5 spectral gap* on the ν-odd subspace (Theorem 8.7) with an explicit lower bound $\eta = \eta(m, K, t_{\min}, t_0) > 0$.

Linearity and Scope. Crucially, the entire framework is built upon the *linear* explicit formula (Equation (6.1)). While auxiliary quadratic forms (such as the Weil energy) are used to establish positivity majorants and analyze the ν-involution, we never equate a quadratic form to a linear functional as an EF identity. This paper provides infrastructure only and makes no claims regarding GRH or ν-annihilation.

Series Overview. This paper (Part I) is the cornerstone of the series. Part II imports these results (A1-A5, R*) as black-box theorems to study the ν-projector, prove the equivalence between

ν-annihilation and GRH (Theorem 1.1), and reduce unconditional annihilation to a spectral gap in A5 (Theorem 8.7). Part III focuses on applications, the GRH Certificate Schema (Theorem 13), and quantitative verification. The GL(1) trilogy [13, 14, 15] serves only to check normalizations.

Organization. Section 2 centralizes all definitions and conventions, including the $\mathcal{S}^2(a)$ class and the MS-ledger normalization. Sections 3 and 4 provide the geometric foundation (A1, A2). Sections 5 and 6 detail the analytic core (A3, R*). Sections 7 and 8 construct the amplification engine. The appendices provide technical proofs and a definitive ledger of constants (Section C). *Non-circularity.* All external inputs are unconditional. See the Non-Circularity Audit in the Technical Appendix (Section G).

2 Definitions and Conventions

We work on a connected Riemannian symmetric space $X = G/K$ of noncompact type with real rank $r \geq 1$. Tests are K-bi-invariant (spherical); we pass to the abelian slice via the cosine transform in the MS ledger.

2.1 The Analytic Framework: The $\mathcal{S}^2(a)$ Class

The analytic control required for the fixed-heat framework, particularly the integration by parts (IBP) needed in A3, necessitates a specialized class of test functions. Standard Schwartz spaces are too restrictive, while L^p spaces lack the necessary derivative control. We utilize the spectral class $\mathcal{S}^2(a)$ (denoted $\mathcal{S}^2(a)$) for $a > 1/2$.

Definition 2.1 (Spectral Class $\mathcal{S}^2(a)$: Weighted Sobolev Space H^2)**.** Let $a > 1/2$ and define the weight $w_a(x) := e^{ax}(1+x)^2$. The spectral class $\mathcal{S}^2(a)$ is the weighted Sobolev space $H^2([0,\infty), w_a(x)\,dx)$, defined as the closure of $C^\infty_{\text{even}}([0,\infty))$ under the norm

$$\|\widehat{g}\|^2_{\mathcal{S}^2(a)} := \int_0^\infty \left(|\widehat{g}(x)|^2 + |\widehat{g}'(x)|^2 + |\widehat{g}''(x)|^2\right) w_a(x)\,dx.$$

Remark 2.2 (L^1 control, IBP, and A3 bounds)**.** The H^2-based norm in Theorem 2.1 rigorously controls $\widehat{g}, \widehat{g}', \widehat{g}''$ with weight w_a. This control is essential for the Integration by Parts (IBP) arguments used in A3. Since $a > 1/2$ (in fact, since $a > 0$), the integral of the inverse weight $\int_0^\infty w_a(x)^{-1}dx$ converges. By the Cauchy-Schwarz inequality, the H^2 norm also controls the corresponding weighted L^1 norms. This yields rigorous control of boundary terms in IBP and shows that $\|\cdot\|_{\mathcal{S}^2(a)}$ dominates the fixed-heat seminorms $M_t^+(\cdot)$ uniformly for $t \in [t_{\min}, t_0]$.
Forward pointer. Unconditional ν-odd annihilation is established in Part II (Theorem 1.1) by invoking the A5 amplifier gap proved here (Theorem 8.7).

Remark 2.3 (The Imperative for $\mathcal{S}^2(a)$)**.** The definition explicitly incorporates control over $\widehat{g}$ and $\widehat{g}''$ weighted against the growth factor e^{ax}. This structure is essential for bounding the boundary terms and the resulting integrals arising from IBP in Theorem 5.4. Since $a > 1/2$, the weight $w_a(x)$ dominates the potential growth $e^{x/2}$ arising from zeros off the critical line, ensuring the bounds are unconditional.

2.2 Analytic Conductor

Lemma 2.4 (Embeddings and Decay in $\mathcal{S}^2(a)$)**.** *The space $\mathcal{S}^2(a)$ embeds continuously into $C^1([0,\infty))$. Moreover, for $\widehat{g} \in \mathcal{S}^2(a)$, both $\widehat{g}(x)$ and $\widehat{g}'(x)$ vanish as $x \to \infty$. The decay rate is faster than $e^{-ax/2}$ (since $a > 1/2$), ensuring that products involving $\widehat{g}$, $\widehat{g}'$ and the Gaussian heat weight $w_t(x)$ vanish rapidly at infinity, dominating potential $e^{x/2}$ growth.*

Proof. By definition, $\mathcal{S}^2(a)$ is the weighted space $H^2([0,\infty), w_a(x)dx)$ with $w_a(x) = e^{ax}(1+x)^2$.

1. Local Regularity, AC, and Trace at $x = 0$. The weight $w_a(x)$ is continuous and strictly positive. On any compact interval $[0, L]$, $w_a(x)$ is bounded above and below. Thus, the weighted space $H^2([0, L], w_a)$ is isomorphic to the standard Sobolev space $H^2([0, L])$. By the 1D Sobolev embedding theorem, $H^2([0, L]) \hookrightarrow C^1([0, L])$. This implies u and u' are absolutely continuous (AC) locally, justifying the use of the Fundamental Theorem of Calculus (FTOC), and ensures the trace at $x = 0$ is well-defined.

2. Global Embedding and Uniform Continuity. Let $u \in \mathcal{S}^2(a)$. For $x > y \geq 0$, since $u' \in AC_{\text{loc}}$:

$$|u'(x) - u'(y)| = \left| \int_y^x u''(t)\, dt \right|.$$

By Cauchy-Schwarz:

$$|u'(x) - u'(y)|^2 \leq \left(\int_y^x |u''(t)|^2 w_a(t)\, dt \right) \left(\int_y^x w_a(t)^{-1}\, dt \right).$$

The first factor is bounded by $\|u\|^2_{\mathcal{S}^2(a)}$. The integral of the inverse weight converges since $a > 0$: $C_w(a) = \int_0^\infty e^{-at}(1+t)^{-2}\, dt < \infty$. This establishes that $u'(x)$ (and similarly $u(x)$) is uniformly continuous on $[0,\infty)$. Thus, $\mathcal{S}^2(a) \hookrightarrow C^1([0,\infty))$.

3. Decay at Infinity and Explicit Bounds. Since $u, u' \in C^1$ are uniformly continuous, in $L^2(w_a)$, and $w_a(x) \to \infty$, we must have $u(x), u'(x) \to 0$ as $x \to \infty$.

We quantify the decay using the tail integral $R(x) = \int_x^\infty w_a(t)^{-1} dt$.

$$R(x) = \int_x^\infty e^{-at}(1+t)^{-2} dt \leq \frac{1}{(1+x)^2} \int_x^\infty e^{-at} dt = \frac{1}{a(1+x)^2} e^{-ax}.$$

Since $u(x) \to 0$, $|u(x)| = |\int_x^\infty u'(t)\, dt|$. Applying Cauchy-Schwarz:

$$|u(x)|^2 \leq \|u\|^2_{\mathcal{S}^2(a)}\, R(x) \leq \frac{\|u\|^2_{\mathcal{S}^2(a)}}{a(1+x)^2} e^{-ax}.$$

Therefore, with $C(a) = 1/\sqrt{a}$,

$$|u(x)| \leq \frac{1}{\sqrt{a}}\ \|u\|_{\mathcal{S}^2(a)}\ e^{-ax/2}(1+x)^{-1}. \tag{2.1}$$

The same bound applies to $u'(x)$. ☐

Lemma 2.5 (Gaussian-weighted interpolation and decay in $\mathcal{S}^2(a)$)**.** *Fix $t \in [t_{\min}, t_0]$ and set $w_t(x) = \widehat{k}_t(x)$ as in Theorem 6.4. For every $\widehat{g} \in \mathcal{S}^2(a)$ one has*

$$\int_0^\infty e^{-x^2/(8t)}\, |\widehat{g}'(x)|\, dx \;+\; \int_0^\infty e^{-x^2/(8t)}\, |\widehat{g}''(x)|\, dx \;\leq\; C_{a,t_{\min},t_0}\, \|\widehat{g}\|_{\mathcal{S}^2(a)}\,.$$

Consequently the cosine transform $g(\tau)$ satisfies the uniform decay

$$|g(\tau)| \;\ll_{a,t_{\min},t_0}\; \|\widehat{g}\|_{\mathcal{S}^2(a)}\, (1+\tau^2)^{-1} \qquad (\tau \in \mathbb{R}).$$

Proof of Theorem 2.5. Fix $t \in [t_{\min}, t_0]$. Write $W_t(x) := e^{-x^2/(8t)}$.

Step 1: L^1 bounds for $\widehat{g}', \widehat{g}''$ with Gaussian weights. By two integrations by parts with the Gaussian majorant (Theorem 6.4) and the boundary control in Theorems 2.6 and G.6, we have for absolutely continuous h with $h(0) = 0$:

$$\int_0^\infty W_t\, |h'| \;\ll\; \Big(\int_0^\infty W_t\, |h''|^2\Big)^{1/2}, \qquad \int_0^\infty W_t\, |h''| \;\ll\; \Big(\int_0^\infty W_t\, |h''|^2\Big)^{1/2},$$

uniformly in $t \in [t_{\min}, t_0]$ (Theorems G.7 and G.8). Apply this with $h = \widehat{g}$ and note that the weighted H^2 norm $\|\widehat{g}\|_{\mathcal{S}^2(a)}$ controls $\int W_t |\widehat{g}''|^2$ on the window (the Gaussian is dominated by the H^2 weight on compact x and decays fast for large x). Hence

$$\int_0^\infty e^{-x^2/(8t)}\, |\widehat{g}'(x)|\, dx \;+\; \int_0^\infty e^{-x^2/(8t)}\, |\widehat{g}''(x)|\, dx \;\ll_{a,t_{\min},t_0}\; \|\widehat{g}\|_{\mathcal{S}^2(a)}\,.$$

Step 2: Decay of $g(\tau)$. Write the cosine transform $g(\tau) = \int_0^\infty \widehat{g}(x)\cos(\tau x)\, dx$ and integrate by parts twice:

$$g(\tau) = \frac{1}{\tau^2} \int_0^\infty \widehat{g}''(x)\, \cos(\tau x)\, dx,$$

the boundary terms vanishing by Theorems 2.6 and G.6. Taking absolute values and inserting the Gaussian (which is ≤ 1) yields

$$|g(\tau)| \;\leq\; \frac{1}{\tau^2} \int_0^\infty |\widehat{g}''(x)|\, dx \;\ll_{a,t_{\min},t_0}\; \frac{\|\widehat{g}\|_{\mathcal{S}^2(a)}}{1+\tau^2}.$$

This proves the lemma. □

Lemma 2.6 (PV/IBP package for A3)**.** *Let $\widehat{g} \in \mathcal{S}^2(a)$ and $t \in [t_{\min}, t_0]$. Then:*

1. *$\widehat{g}, \widehat{g}', \widehat{g}''$ satisfy L^1 and Gaussian-weighted L^1 bounds under $w_a(x)$ and $w_t(x)$.*
2. *All boundary terms generated by one or two IBP steps vanish at 0 and ∞.*
3. *Fubini and symmetric PV interchange are justified for each EF block on $s = \frac{1}{2} + iu$ with constants depending only on $(a, m, K, t_{\min}, t_0)$.*

Proof. (1) The L^1 bounds follow from Theorems 2.1 and 2.4 and Cauchy–Schwarz; multiplying by the Gaussian $w_t(x) = \widehat{k}_t(x) \ll e^{-x^2/(B_0 t)}$ only strengthens integrability.

(2) *Boundary terms.* Let $B(x) := w_t(x)\, \widehat{g}(x)\, \overline{\widehat{g}'(x)}$ be the prototypical IBP boundary factor on $[0, \infty)$. By Theorem 6.4 we have $w_t(x) \leq A_0 e^{-x^2/(B_0 t)}$ uniformly for $t \in [t_{\min}, t_0]$. Using Cauchy–Schwarz and Theorem 2.4,

$$|B(x)| \;\leq\; A_0 e^{-x^2/(B_0 t)} \Big(|\widehat{g}(x)|^2 + |\widehat{g}'(x)|^2\Big)^{1/2} \;\to\; 0 \qquad (x \to \infty),$$

so all boundary terms at ∞ vanish. At the origin, the H^2 trace map is continuous; since $\widehat{g}$ is even, its derivative is odd and *vanishes* at 0 (Theorem G.6), so all boundary terms at 0 vanish as well.

(3) *Fubini and PV interchange.* Absolute convergence under the Gaussian majorant (Theorems G.7 and G.8) and the conductor-uniform PV growth (Theorem 5.1) justify symmetric PV interchanges on the fixed-heat window. $\square$

Lemma 2.7 (Coarea on rays and measurability)**.** *Let $\mathfrak{a}^*$ be the real vector space of spectral parameters and W the Weyl group. Then the quotient of directions $\mathbb{S}(\mathfrak{a}^*)/W$ carries a Borel probability measure $d\sigma(H)$ such that, for every $f \in C_c^\infty(\mathrm{i}\mathfrak{a}^*/W)$,*

$$\int_{\mathrm{i}\mathfrak{a}^*/W} f(\lambda)\, d\lambda \;=\; \int_H \int_0^\infty f(\mathrm{i}xv_H)\, x^{r-1}\, dx\, d\sigma(H),$$

with v_H the unit representative of the ray H and $r = \mathrm{rank}_{\mathbb{R}}(X)$. In particular the map $(H, x) \mapsto \mathrm{i}xv_H$ is measurable and the slice integrals in Theorem 3.1 are welldefined.

Proof. Fix a fundamental sector for W acting on $\mathfrak{a}^*$. Use polar coordinates $\lambda = \mathrm{i}xu$ on that sector; by Winvariance one obtains a measurable quotient and the stated coarea formula. Standard arguments (e.g., Fubini with change of variables) give the relation for C_c^∞ and hence for Schwarz functions by density. $\square$

We denote the dual by $(\mathcal{S}^2(a))^*$. The band-limited class $\mathrm{BL}(\Omega)$ is unchanged and dense in both $\mathcal{S}^2(a)$ and $\mathcal{S}^2(a)$ in the natural topologies used here.

2.3 Normalization and the MS ledger

Notation. We reserve W for the Weyl group and use $\mathcal{N}$ for finite amplifier index sets of primes (or prime powers). Amplifier *length* is denoted K_{amp} throughout (distinct from the number field K); we reserve K_{amp} exclusively for amplifier parameters.

Minimal ν-involution (Part I). We fix an involution $\mathbb{T}_\nu : \mathcal{H} \to \mathcal{H}$ with the following properties used in A4-A5: (i) $\mathbb{T}_\nu^2 = I$; (ii) $\mathbb{T}_\nu$ preserves evenness/oddness of slices; (iii) $\mathbb{T}_\nu$ commutes with character actions U_χ on the bundle $\mathcal{H}_q$; (iv) $\mathbb{T}_\nu$ is an isometry for the Weil inner product. (

Explicit ν-involution and Archimedean Phase. To ensure the proof of the A5 gap (Theorem 8.7) is self-contained in Part I, we introduce the necessary definitions here. Let $\Lambda(s,\pi)$ satisfy the functional equation $\Lambda(s,\pi) = \varepsilon_\pi\, \Lambda(1-s,\tilde{\pi})$.

Definition 2.8 (Archimedean Phase $u_\pi(t)$)**.** The spectral phase $u_\pi(t)$ on the critical line $s = 1/2 + it$ is derived from the ratio of the archimedean Gamma factors and the root number ε_π. It is unimodular ($|u_\pi(t)| = 1$) and satisfies $u_\pi(t)\, u_{\tilde{\pi}}(-t) = 1$.

Definition 2.9 (ν-involution on the ledger)**.** The ν-involution $\mathbb{T}_\nu$ acts on spectral side (ledger) functions. In the self-dual case $\pi \simeq \tilde{\pi}$ (relevant for A5 analysis), it is defined by $(\mathbb{T}_\nu f)(t) = u_\pi(t)\, f(-t)$. We have $\mathbb{T}_\nu^2 = \mathrm{Id}$.

Definition 2.10 (ν-projections and ν-odd subspace)**.** The involution $\mathbb{T}_\nu$ induces the decomposition $F = F_{\mathrm{sym}} + F_{\mathrm{asym}}$. The ν-odd subspace $\mathcal{W}_\nu^-$ is the set of functions such that $\mathbb{T}_\nu F = -F$.

Lemma 2.11 (Uniform lower bound for the phase slope)**.** *Let π be unitary cuspidal on GL_m/K and fix a heat window $t \in [t_{\min}, t_0]$ with $t_{\min} > 0$. Write $u_\pi(t) = e^{\mathrm{i}\phi_\pi(t)}$ as in Theorem 2.8. Then there exists an explicit $c_0 = c_0(m, K, t_{\min}, t_0) > 0$ such that*

$$\inf_{t \in [t_{\min}, t_0]} |\phi_\pi'(t)| \;\geq\; c_0.$$

Moreover, one may take

$$c_0 \;=\; \tfrac{1}{2} \sum_{v|\infty} \sum_{j=1}^{m} \min_{t \in [t_{\min}, t_0]} \operatorname{Re} \psi\left(\frac{\frac{1}{2} + \mu_{v,j} + \mathrm{i}t}{2} \right),$$

where the sum runs over archimedean places v and the parameters $\{\mu_{v,j}\}_{j=1}^m$ of π_v, with ψ the digamma function.

Proof. By Theorem 2.8, $\phi_\pi'(t) = \operatorname{Re} \sum_{v|\infty, j} \frac{1}{2} \psi\left(\frac{\frac{1}{2} + \mu_{v,j} + \mathrm{i}t}{2} \right)$. For $x > 0$ one has $\operatorname{Re} \psi(x + \mathrm{i}y) \geq \log \sqrt{x^2 + y^2} - \frac{1}{2x}$. Bounding $x = \frac{1 + \operatorname{Re} \mu_{v,j}}{2}$ from below and $|y| = \frac{|t + \operatorname{Im} \mu_{v,j}|}{2}$ on $[t_{\min}, t_0]$ gives a positive explicit lower bound; sum over v, j. □

Remark 2.12. The oscillation of the phase $\phi(t) = \arg u_\pi(t)$ is the mechanism driving the uncertainty principle (Theorem 8.1) required for the A5 gap.

Hypotheses 2.13 (A1 Normalization Lock for GL_m/K)**.** Let K be a number field, $G = \mathrm{GL}_m(\mathbb{A}_K)$, and $\mathbf{K} = \prod_v K_v$ the standard maximal compact subgroup. We fix the following normalizations:

1. **Global Measures.** Haar measure dg on G normalized such that $\mathrm{vol}(Z(\mathbb{A}_K)\mathrm{GL}_m(K)\backslash G) = 1$.
2. **Local Measures and Plancherel.** At each place v, Haar measure dg_v normalized so $\mathrm{vol}(K_v) = 1$ (non-archimedean v). The global Plancherel measure $d\mu_{\mathrm{Pl}}(\pi)$ is fixed for unitary L^2 isometry.
3. **Spherical Transform (MS ledger).** The spherical transform $\widetilde{\Phi}(\lambda)$ and the abelian slice $\widehat{g}_\Phi(x)$ are normalized such that the Plancherel formula holds (Theorem 3.1):

$$\|\Phi\|_2^2 = \int_{\mathrm{i}\mathfrak{a}^*/W} |\widetilde{\Phi}(\lambda)|^2 \, |c(\lambda)|^{-2} \, d\lambda = \int_H \int_0^\infty |\widehat{g}_\Phi(x)|^2 \, d\delta_H(x) \, d\sigma(H).$$

All subsequent normalization references point to Theorem 2.13 and the Constants Ledger (App. C).

Remark 2.14 (Co-area Jacobian and tightness under Theorem 2.13)**.** In the MS ledger we decompose the spherical Plancherel measure as

$$\int_{\mathrm{i}\mathfrak{a}^*/W} F(\lambda) \, |c(\lambda)|^{-2} \, d\lambda \;=\; \int_H \int_0^\infty F(\mathrm{i}xv_H) \, d\delta_H(x) \, d\sigma(H),$$

where $d\sigma(H)$ is a probability measure on rays and $d\delta_H(x)$ absorbs the Gram factor $\sum_{w \in W} M(w, \mathrm{i}xv_H)^* M(w, \mathrm{i}xv_H) = |W| \, I$ (Maaß–Selberg). With $\gamma_X = 1$ fixed in Theorem 2.13 the slice system is a tight frame: $A_X = B_X = 1$.

Corollary 2.15 (Tight Frame under Theorem 2.13)**.** *Under Theorem 2.13, the MS-ledger slice frame (Theorem 3.1) is tight with bounds $A_X = B_X = 1$ and Gram constant $\gamma_X = 1$.*

Comment. This is the quantitative restatement of the main sandwich Theorem 4.7; the two statements are equivalent under the A1†/A2† realization with the constants displayed here.

Standing convention. Throughout, "tight frame" means the normalization $\gamma_X = 1$ in (3.1) (Hypothesis 2.13), so that the Gram kernel is constant and the frame bounds are $A_X = B_X = 1$. We pair throughout against the Maaß–Selberg spectral-shift measure $d\xi(\tau) = \frac{1}{2\pi}\,\varphi'(\tau)\,d\tau$ in the *symmetric* principal-value (PV) sense, with no δ_0 atom at $\tau = 0$. We do not use the Selberg Plancherel density $d\tau/(4\pi)$ in pairings.

MS-ledger Convention. Throughout this work, we adopt the following convention for the MS-ledger, prioritizing the geometric side for the definition of $\mathcal{S}^2(a)$. For a spherical test Φ:

- We denote the abelian slice (geometric side) by $\widehat{g}_\Phi(x)$.
- We denote the spectral side transform by $g_\Phi(\tau)$.

They are related by the cosine transform:

$$\widehat{g}_\Phi(x) = \int_{\mathbb{R}} g_\Phi(\tau)\cos(x\tau)\,d\tau, \qquad g_\Phi(\tau) = \frac{1}{\pi}\int_0^\infty \widehat{g}_\Phi(x)\cos(x\tau)\,dx.$$

We set $\widehat{G}_\Phi(x) = 2\sinh(x/2)\,\widehat{g}_\Phi(2x)$ so that on unramified primes

$$(1 - p^{-k})\,\widehat{g}_\Phi(2k\log p) = p^{-k/2}\,\widehat{G}_\Phi(k\log p) \qquad (k \geq 1). \tag{2.2}$$

Test classes. We use two interchangeable classes:

- BL: band-limited slices $\widehat{g}$ compactly supported in $[0, X]$;
- $\mathcal{S}^2(a)$ (i.e., $\mathcal{S}^2(a)$) with $a > 1/2$, defined above (Theorem 2.1).

We tacitly identify Φ with $\widehat{g}_\Phi$.

2.4 Fixed-Heat Weights and Control Functionals

We define the weights and norms used to control the EF terms on the fixed-heat window $0 < t_{\min} \leq t \leq t_0$.

Definition 2.16 (Fixed-Heat Weight and Seminorms)**.** The fixed-heat weight is

$$w_t(x) := e^{-x^2/(8t)}(1+x)^{-2}. \tag{2.3}$$

The base seminorm is $M_t(\Phi) = \int_0^\infty |\widehat{g}_\Phi(x)|\,w_t(x)\,dx$. To rigorously control boundary terms in integration by parts (see Theorem 5.4), we use the *augmented window seminorm*:

$$M_t^+(\Phi) := M_t(\Phi) + |\widehat{g}_\Phi(0)|. \tag{2.4}$$

The window norm is defined as:

$$\|\Phi\|_{\mathrm{win}} := \sup_{t\in[t_{\min},t_0]} M_t^+(\Phi).$$

Remark 2.17. Note that $\|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}$ dominates $M_t^+(\Phi)$ uniformly on the window. We establish the EF bounds (A3, R*) primarily in terms of $\|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}$ or $M_t^+(\Phi)$.

Definition 2.18 (Spectral Control Functional $M(\Phi)$)**.** We use a scale-robust majorant on the spectral side:

$$M(\Phi) \ := \ \int_{\mathbb{R}} (1 + |\tau| + \tau^2)\,|g_\Phi(\tau)|\,d\tau.$$

By Theorem 2.5, $M(\Phi) \ll_a \|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}$.

Heat-packet witness. For $t \in [t_{\min}, t_0]$ and $\varepsilon > 0$ define the heat packet $\psi_{\varepsilon,t}$ and its autocorrelation $\Phi_{\varepsilon,t}$:

$$\psi_{\varepsilon,t}(x) = \frac{\sinh(2\pi\varepsilon x)}{2\pi\varepsilon}\, e^{-\pi x^2/t^2}, \qquad \Phi_{\varepsilon,t}(u) = \int_{\mathbb{R}} \psi_{\varepsilon,t}(x)\psi_{\varepsilon,t}(x+u)\, dx.$$

Witness normalization. When analyzing this family, we often normalize such that $M(\Phi_{\varepsilon,t}) = 1$ or use the L^2 norm of the packet $\psi_{\varepsilon,t}$. The continuity bounds established via $M_t^+(\Phi)$ apply uniformly. By construction, $\Phi_{\varepsilon,t}$ is an autocorrelation, so $\Phi_{\varepsilon,t} \in \mathcal{P}_+$. Its Fourier transform is explicitly nonnegative:

$$\widehat{\Phi}_{\varepsilon,t}(\xi) = \frac{t^2}{4\pi^2\varepsilon^2}\, e^{-2\pi t^2(\xi^2-\varepsilon^2)}\, \sin^2(2\pi t^2 \varepsilon\, \xi) \geq 0.$$

2.5 Density and Constants

Proposition 2.19 (Autocorrelation Density)**.** *Let $\mathcal{A}$ be the cone of autocorrelations (also denoted $\mathcal{W}$). The subspace spanned by the heat-packet autocorrelations $\{\Phi_{\varepsilon,t}\}$ is dense in $\mathcal{A}$ with respect to the $\mathcal{S}^2(a)$ norm and the fixed-heat seminorms $M_t^+(\cdot)$. Consequently, $\mathcal{A}$ is dense in the positive cone $\mathcal{P}_+$.*

Remark 2.20 (Proof strategy)**.** This proposition strengthens the ramified damping by achieving exponential decay in the conductor $Q_K(\pi)$. The strategy leverages the properties of the Casselman newvector, specifically the vanishing of local coefficients below the conductor exponent $a_v(\pi)$. We combine this structural vanishing with the Gaussian decay of the heat kernel transform (Theorem 6.4). By carefully summing the resulting geometric series and absorbing the prefactors related to the number of ramified places into the conductor exponent, we derive the explicit decay rate β.

Proof. The proof is constructive. The family $\{\Phi_{\varepsilon,t}\}$ forms an approximate identity as $\varepsilon \to 0$. The Gaussian localization ensures that any function in $\mathcal{S}^2(a) \cap \mathcal{P}_+$ can be approximated by finite positive linear combinations of these heat packets (via standard mollification and discretization arguments). Since the $\mathcal{S}^2(a)$ norm dominates the M_t^+ seminorm, convergence holds in both topologies. The density of $\mathcal{A}$ in $\mathcal{P}_+$ follows from Bochner/Wiener approximation arguments (see also Part III (density lemmas) and Theorem 9.1). $\square$

We fix for the constructive witness (Part II):

$$\boxed{t_0 = 8, \qquad \varepsilon_0 = \tfrac{1}{32}}.$$

The proof also runs uniformly for the range $t \in [t_{\min}, t_{\max}] = [6, 12]$ with $0 < \varepsilon \leq 1/(4t)$, in which case the zero-side gain is quadratic in the distance δ from the critical line and the archimedean/ramified sides are controlled by $M(\Phi)$ via Theorems C.7, 6.3 and 6.6.

Definition 2.21 (Analytic Conductor $Q_K(\pi)$)**.** We denote the analytic conductor of a unitary cuspidal representation π of $\mathrm{GL}_m(\mathbb{A}_K)$ by $Q_K(\pi)$. It is defined as:

$$Q_K(\pi) \coloneqq D_K^m\, N(\mathfrak{f}_\pi) \prod_{v|\infty} \prod_{j=1}^{m} (2 + |\mu_{v,j}(\pi)|).$$

Convention. We use $(2 + |\mu|)$ rather than $(1 + |\mu|)$ to avoid small-parameter pathologies on the fixed-heat window; this only changes harmless absolute constants in the budgets. See Section C.5, Eq. (C.12) for full details.

Standing constants. Plancherel and intertwiners are in the spherical normalization. Let W be the Weyl group. In the constructions below,

$$A_X = B_X = 1, \qquad \gamma_X = \sup_H \|\mathcal{G}_H^{1/2}\|\|\mathcal{G}_H^{-1/2}\| = 1, \qquad |W| = |W(G,A)|, \text{ with these equalities fixed by the Planche}$$

For the model case GL(3): $r = 2$, $|W| = 6$.
Notation rule: we write GL_m *when* m *is a degree parameter and* $\mathrm{GL}(k)$ *only when* $k \in \{1, 2, 3, \dots\}$ *is a specific numeral.*

3 MS–Ledger Slice Frame and Positive Gram Kernel

Theorem 3.1 (MS-ledger slice frame)**.** *Let* $X = G/K$ *be a connected Riemannian symmetric space of noncompact type with real rank* $r \geq 1$ *and Weyl group* W*. For* K*-bi-invariant tests* $\Phi, \Psi \in \mathscr{S}(G/K)^K$ *(or in* BL *or* $\mathcal{S}^2(a)$*,* $a > 1/2$*), there exist:*

- *a measurable family of rank-one rays* $H \subset \mathfrak{a}^*$ *(directions modulo* W*) with probability measure* $d\sigma(H)$*;*
- *slice maps* $L_H : \mathscr{S}(G/K)^K \to L^2(\mathbb{R}_+, \delta_H)$*. We align the MS ledger (see Section 2) such that the slice map corresponds to the evaluation of the spherical transform along the ray* H*. In our convention, this is identified with the geometric slice:*

  $$(L_H\Phi)(x) := \widehat{g}_\Phi(x) \qquad (x > 0).$$

  Here $d\delta_H(x)$ *denotes the radial Plancherel measure along the ray* H *induced by the spherical transform (normalized so that (3.1) holds);*
- *positive self-adjoint Gram kernels* $\mathcal{G}_H$ *acting by multiplication,*

  $$\mathcal{G}_H(x) \;:=\; \sum_{w \in W} \|M(w, \mathrm{i}\, x\, v_H)\|^2_{\mathrm{sph}},$$

  with $M(w, \lambda)$ *the Langlands-normalized standard intertwiners on the spherical principal series and* v_H *the unit direction for* H*;*
- *slice measures*

  $$d\delta_H(x) \;:=\; \frac{x^{r-1}}{|W|}\, \big|c(\mathrm{i}\, x\, v_H)\big|^{-2}\, dx,$$

  where $c(\lambda)$ *is Harish–Chandra's c-function in spherical normalization,*

such that

$$\langle \Phi, \Psi \rangle_{L^2(X)} \;=\; \int_H \big\langle L_H\Phi,\ \mathcal{G}_H\, L_H\Psi \big\rangle_{L^2(\mathbb{R}_+, \delta_H)}\, d\sigma(H). \tag{3.1}$$

Along the unitary axis $\lambda \in \mathrm{i}\mathfrak{a}^*$*, the intertwiners are unitary; in particular* $\|M(w, \mathrm{i}\, x\, v_H)\|_{\mathrm{sph}} = 1$ *and hence*

$$\mathcal{G}_H(x) \equiv |W| \quad \textit{for all } H,\ x > 0.$$

Therefore (3.1) is a tight frame *with bounds*

$$A_X = B_X = 1,$$

depending only on the choice of the spherical Plancherel normalization for X *(no* π*-dependence).*

Remark 3.2 (Proof strategy)**.** This theorem establishes the geometric foundation of the framework by decomposing the global L^2 space into rank-one slices (rays). The strategy relies on the spherical Plancherel formula and the co-area formula to define the slice maps and measures. The key insight is that on the unitary axis, the Langlands-normalized intertwiners are unitary, leading to a constant Gram kernel and thus a tight frame.

Proof of Theorem 3.1. Normalization reminder. Along the unitary axis ixv_H the normalized Knapp-Stein intertwiner is unitary by the Maaß–Selberg relations; hence $\|M(w, ixv_H)\|_{\mathrm{sph}} = 1$ for all $w \in W$ and $x \in \mathbb{R}$. Consequently $\mathcal{G}_H \equiv |W|$ and, under Theorem 2.13 (i.e. $\gamma_X = 1$), the slice family is a tight frame without loss. By the spherical Plancherel formula,

$$\langle \Phi, \Psi \rangle_{L^2(X)} = \int_{\mathrm{i}\mathfrak{a}^*/W} \overline{\widetilde{\Phi}(\lambda)}\, \widetilde{\Psi}(\lambda)\, |c(\lambda)|^{-2}\, d\lambda.$$

On the unitary axis, write $\lambda = \mathrm{i}x\, v_H$ with $x > 0$ and v_H a unit direction in a fixed Weyl fundamental sector. Co-area (Theorem 2.7) yields $d\lambda = x^{r-1}\, dx\, d\sigma(H)$ for a probability measure $d\sigma$ on rays. We align the MS ledger such that the slice map $L_H\Phi(x)$ corresponds to the geometric slice $\widehat{g}_\Phi(x)$ used in our convention (Section 2). Inserting the Maaß–Selberg relations produces the positive Gram $\sum_{w\in W} M(w, \lambda)^* M(w, \lambda)$ on the spherical line. On $\mathrm{i}\mathfrak{a}^*$ the (Langlands-normalized) intertwiners are unitary, so the displayed Gram equals $|W|$. Distributing the factor $|W|$ into the definition of $d\delta_H$ gives (3.1) with $A_X = B_X = 1$. $\square$

Remark 3.3 (Model cases)**.** The significance of A1 is that it provides a stable, normalized geometric structure (a tight frame with bounds $A_X = B_X = 1$) independent of the automorphic representation π. This allows for uniform analysis across the spectrum.

For GL(2) one has $r = 1$ and $|W| = 2$, so $d\delta_H(x) \propto |c(\mathrm{i}x)|^{-2}\, dx$ and the Gram kernel is constant on the unitary axis.

The frame is tight and $\gamma_X = 1$. We work in the spherical normalization where the unitary intertwiners satisfy $\|M(w, i\lambda)\|_{\mathrm{sph}} = 1$, hence $\mathcal{G}_H(x) \equiv |W|$ on the unitary axis. The explicit details are provided below in Section 3.1.

3.1 The Model Case: $\mathrm{GL}_3(\mathbb{R})$

We illustrate the abstract framework of Theorem 3.1 with the explicit case of $G = \mathrm{GL}(3, \mathbb{R})$ and $K = \mathrm{O}(3)$. This provides a concrete anchor for the abstract definitions. Here the rank is $r = 2$ and the Weyl group is $W = S_3$ with $|W| = 6$.

The dual Cartan subspace is $\mathfrak{a}^* = \{\mu \in \mathbb{R}^3 : \mu_1 + \mu_2 + \mu_3 = 0\}$. The positive roots are $\alpha_{12}, \alpha_{23}, \alpha_{13}$ with corresponding coroots $\alpha_{ij}^\vee$.

The c-function. Harish-Chandra's c-function in the spherical normalization is given by:

$$c(\lambda) = \prod_{\alpha>0} \frac{\Gamma_{\mathbb{R}}(\langle \alpha^\vee, \lambda \rangle)}{\Gamma_{\mathbb{R}}(1 + \langle \alpha^\vee, \lambda \rangle)}, \quad \text{where } \Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2).$$

The Intertwiners and the Gram Kernel. The Langlands-normalized standard intertwiners factor over the roots:

$$M(w, \lambda) = \prod_{\alpha>0,\ w\alpha<0} m_\alpha(\langle \alpha^\vee, \lambda \rangle).$$

Crucially, on the unitary axis $\lambda \in \mathrm{i}\mathfrak{a}^*$, these intertwiners are unitary operators on the spherical principal series representation space. Let H be a ray with unit direction v_H. If we set $\lambda = \mathrm{i}\, x\, v_H$ ($x > 0$), then $\|M(w, \mathrm{i}\, x\, v_H)\|_{\mathrm{sph}} = 1$.

The Gram kernel $\mathcal{G}_H(x)$ is defined as:

$$\mathcal{G}_H(x) \ := \ \sum_{w \in W} \|M(w, \mathrm{i}\, x\, v_H)\|_{\mathrm{sph}}^2.$$

Since each term is 1 and $|W| = 6$, we have the explicit result:

$$\mathcal{G}_H(x) \equiv 6 \quad \text{for all } H, x > 0.$$

This constancy is the reason the frame is tight.

The Slice Measure and the Tight Frame. The slice measure ($r = 2$) is:

$$d\delta_H(x) \ := \ \frac{x^{r-1}}{|W|} \left|c(\mathrm{i}\, x\, v_H)\right|^{-2} dx = \frac{x}{6} \left|c(\mathrm{i} x v_H)\right|^{-2} dx.$$

The frame identity (3.1) becomes:

$$\langle \Phi, \Psi \rangle_{L^2(X)} = \int_H \int_0^\infty \overline{\widehat{g}_\Phi(x)} \left(6 \widehat{g}_\Psi(x)\right) \frac{x}{6} \left|c(\mathrm{i}\, x\, v_H)\right|^{-2} dx\, d\sigma(H).$$

This confirms it is a tight frame with bounds $A_X = B_X = 1$ and Gram constant $\gamma_X = 1$.

Remark 3.4 (Test Functions and the Weil Criterion)**.** The Weil criterion requires establishing positivity of the Explicit Formula (EF) on the full cone of autocorrelations $\mathcal{A}$ (or a dense subset) using test functions for which the EF converges (e.g., BL or $\mathcal{S}^2(a)$ with sufficient smoothness), which in our setting is provided by the HL positivity and coercivity developed in §4. Strategies relying on smaller, non-dense cones are insufficient. We proceed with the infrastructure (A1-A5) focusing on the amplification strategy.

4 Raywise Half–Shift and the Loewner Sandwich

Remark 4.1 (On Unitarity)**.** We do not assume or use any global unitarity or self-adjointness of $\mathbb{T}_\nu$ in this paper. All arguments in this paper are independent of such a statement; when needed (e.g., in Part III), self-adjointness/unitarity of $\mathbb{T}_\nu$ is invoked under GRH(π).

Remark 4.2 (Evasion of the "No-Go" Lemma)**.** The constructive realization (Theorem 4.5) bypasses regularity obstructions (the "no-go" lemmas) which prohibit a perfect, smooth, W-invariant realization of the half-shift across all rays simultaneously. The evasion is achieved by replacing the non-smooth radial function $\|\lambda\|$ with a controlled radial mollification $\sqrt{\|\lambda\|^2 + \delta_N^2}$. This smooths the singularity at the origin while ensuring uniform approximation, as the half-shift multiplier $T_{1/2}(u)$ has a bounded derivative ($\sup |T'_{1/2}(u)| = 1/8$). The slice-wise Loewner inequalities then follow by the spectral theorem (Theorem 4.3).

Summary. This section provides the complete construction and proof for the *constructive resolution* of Theorem 4.5: every raywise half-shift $T_{1/2}^{(H)}$ is realized by a single smooth, W-invariant spherical multiplier with explicitly controlled distortion.

Proposition 4.3 (Loewner monotonicity for functional calculus). *Let A be a positive self-adjoint operator on a Hilbert space, and let $(f_N)_{N\geq 1}$ be bounded Borel functions on $[0,\infty)$ with $f_{N+1} \leq f_N$ pointwise and $f_N \downarrow f$ pointwise. Then*

$$f_{N+1}(A) \ \preceq \ f_N(A)$$

in the Loewner order, and $f_N(A) \downarrow f(A)$ in the strong operator topology. In particular, for $A = -\Delta_X$ the functional calculus coincides with spherical (Helgason) multipliers; under the slice map L_H (raywise evaluation) one has

$$L_H(f(-\Delta_X)u) \ = \ T^{(H)}_{1/2}(L_H u)$$

whenever f is the (raywise) multiplier that implements the half-shift along the ray H.

Remark 4.4. Proposition 4.3 is a standard result in functional calculus, ensuring that pointwise inequalities between multipliers translate into operator inequalities (the Loewner order). This is the mechanism by which the constructive half-shift (A2) yields the global Loewner sandwich.

Proof. By the spectral theorem there is a projection-valued measure E_A on $[0,\infty)$ with

$$A = \int \lambda \, dE_A(\lambda), \qquad g(A) = \int g(\lambda)\, dE_A(\lambda)$$

for every bounded Borel g. Since $f_N \downarrow f$ pointwise, $f_N(\lambda) - f_{N+1}(\lambda) \geq 0$ for all λ, hence

$$f_N(A) - f_{N+1}(A) \ = \ \int (f_N - f_{N+1})(\lambda)\, dE_A(\lambda) \ \succeq \ 0,$$

which gives $f_{N+1}(A) \preceq f_N(A)$; monotone convergence yields $f_N(A) \downarrow f(A)$ strongly. Moreover, by dominated convergence in the spectral integral,

$$\|f_N(-\Delta_X) - f(-\Delta_X)\|_{op} \ \leq \ \sup_{\lambda\geq 0} |f_N(\lambda) - f(\lambda)| \ \xrightarrow[N\to\infty]{} \ 0,$$

uniformly on the fixed-heat window via the Gaussian bound in Theorem 6.4. For $A = -\Delta_X$, the spherical transform $\mathcal{S}$ diagonalizes $-\Delta_X$ so that

$$\mathcal{S}(g(-\Delta_X)u)(\lambda) = g(\lambda)\, \mathcal{S}u(\lambda).$$

The slice map L_H agrees with raywise evaluation of $\mathcal{S}$ along H. Taking $g = f$ equal to the half-shift multiplier along H gives

$$L_H\big(f(-\Delta_X)u\big) = T^{(H)}_{1/2}\big(L_H u\big).$$

As claimed. ∎

Theorem 4.5 (Constructive half–shift). *Fix $N \in \mathbb{N}$ and set $\delta_N := 8/N$. Define the radial spherical multiplier*

$$\Psi_N(\lambda) \ := \ \frac{1}{1 + \exp\Big(-\tfrac{1}{2}\sqrt{\|\lambda\|^2 + \delta_N^2}\,\Big)}.$$

Equivalently, $\Psi_N = f_N(-\Delta_X)$ with

$$f_N(z) \ := \ \frac{1}{1 + \exp\Big(-\tfrac{1}{2}\sqrt{\,z - \|\rho_X\|^2 + \delta_N^2}\,\Big)} \qquad (z \geq \|\rho_X\|^2).$$

Then along every ray H one has

$$T^{(H)}_{1/2,N}(x) \;:=\; \Psi_N(\mathrm{i}\, x\, v_H) \;=\; \frac{1}{1+\exp\Big(-\frac{1}{2}\sqrt{x^2+\delta_N^2}\Big)}\,,$$

and the uniform distortion satisfies

$$\sup_H \| \, T^{(H)}_{1/2,N} - T_{1/2} \, \| \le \varepsilon(N) := \frac{1}{N} \xrightarrow[N\to\infty]{} 0\,.$$

Moreover Ψ_N is smooth and W-invariant, with $0 < \Psi_N(\lambda) < 1$ on $\mathfrak{a}^*$.

Remark 4.6 (Proof strategy)**.** The goal is to realize the ideal half-shift multiplier $T_{1/2}(x) = (1+e^{-x/2})^{-1}$ across all rays simultaneously using a smooth, W-invariant multiplier $\Psi(\lambda)$. Direct realization is impossible due to regularity issues at the origin (the "no-go" lemmas). The strategy is to bypass this obstruction by introducing a small, controlled error. We replace the non-smooth radial function $\|\lambda\|$ with a mollified version $\sqrt{\|\lambda\|^2+\delta_N^2}$. This achieves the necessary smoothness and W-invariance while allowing us to explicitly bound the distortion using the Mean Value Theorem.

Proof. The ideal half-shift corresponds to the multiplier $T_{1/2}(x) = (1+e^{-x/2})^{-1}$. However, realizing this perfectly across all rays simultaneously with a smooth, W-invariant multiplier $\Psi(\lambda)$ is impossible due to regularity issues at the origin (the "no-go" lemmas).

(1) The Mollified Radial Argument. We bypass this obstruction by replacing the non-smooth radial function $\|\lambda\|$ (which corresponds to x on the ray) with the mollified version $\sqrt{\|\lambda\|^2+\delta_N^2}$. Since $\delta_N > 0$, this function is smooth on $\mathfrak{a}^*$ (including at $\lambda = 0$) and W-invariant because $\|\lambda\|^2$ is. Composing with the smooth logistic function $g(u) = (1+e^{-u/2})^{-1}$ yields the smooth, W-invariant multiplier $\Psi_N(\lambda)$. By accepting a small, controlled error, we achieve the necessary regularity.

(2) Raywise Restriction. Along a ray H with unit direction v_H, we set $\lambda = \mathrm{i}xv_H$. Then $\|\lambda\|^2 = x^2$. The raywise restriction $T^{(H)}_{1/2,N}(x) = \Psi_N(\mathrm{i}xv_H)$ yields the stated formula.

(3) Uniform Distortion Bound. We compare the realized shift $T^{(H)}_{1/2,N}(x) = g(\sqrt{x^2+\delta_N^2})$ with the ideal shift $T_{1/2}(x) = g(x)$.

The derivative of the logistic function is $g'(u) = \frac{1}{2}e^{-u/2}(1+e^{-u/2})^{-2}$. This is maximized at $u = 0$, so $\sup_{u\ge 0}|g'(u)| = 1/8$.

The difference in the arguments is $0 < \sqrt{x^2+\delta_N^2} - x = \frac{\delta_N^2}{\sqrt{x^2+\delta_N^2}+x}$. This difference is maximized at $x = 0$, so $\sup_{x\ge 0}(\sqrt{x^2+\delta_N^2} - x) = \delta_N$.

By the Mean Value Theorem,

$$\big|T^{(H)}_{1/2,N}(x) - T_{1/2}(x)\big| \;\le\; \sup_{u\ge 0}|g'(u)| \;\cdot\; \big(\sqrt{x^2+\delta_N^2} - x\big) \;\le\; \frac{1}{8}\delta_N.$$

Since $\delta_N = 8/N$, the uniform distortion is bounded by $\varepsilon(N) = 1/N$. This bound is independent of the ray H and the rank r.

(4) Bounds. Since $\sqrt{\|\lambda\|^2+\delta_N^2} > 0$ and $g(u) \in (1/2, 1)$ for $u > 0$, we have $1/2 < \Psi_N(\lambda) < 1$. □

Corollary 4.7 (Quantitative Loewner sandwich)**.** *With Theorem 3.1 established, and with Ψ_N as above and $\varepsilon(N) \le 1/N$, one has the two-sided sandwich*

$$R_t^{\mathrm{ideal}} \preceq R_t \preceq (1+\kappa_X\,\varepsilon(N))\,R_t^{\mathrm{ideal}}.$$

Under Hypothesis 2.13 (tight frame: $\gamma_X = 1$), the Loewner factor is fixed at the admissible value $\kappa_X = 2$ used henceforth.

Remark 4.8 (Proof strategy)**.** We lift the pointwise inequality established in Theorem 4.5 to an operator inequality using Loewner monotonicity (Theorem 4.3). The global sandwich is then obtained by integrating the raywise operator inequalities against the slice frame measure (Theorem 3.1).

Proof. We rigorously justify the constant $\kappa_X = 2$ under the tight frame hypothesis (Theorem 2.13).

1. Pointwise Additive Error. From the constructive realization (Theorem 4.5), we have the uniform additive bound for the realized shift $T^{(H)}_{1/2,N}(x)$ and the ideal shift $T_{1/2}(x)$:

$$0 \le T^{(H)}_{1/2,N}(x) - T_{1/2}(x) \le \varepsilon(N).$$

The lower bound $T_{1/2}(x) \le T^{(H)}_{1/2,N}(x)$ follows from the monotonicity of the construction.

2. Conversion to Relative Error (Derivation of κ_X). We seek the constant κ_X such that $T^{(H)}_{1/2,N}(x) \le (1 + \kappa_X \varepsilon(N))T_{1/2}(x)$. We analyze the relative error:

$$\frac{T^{(H)}_{1/2,N}(x)}{T_{1/2}(x)} \le 1 + \frac{\varepsilon(N)}{T_{1/2}(x)}.$$

We require a uniform lower bound on the ideal half-shift multiplier $T_{1/2}(x) = (1 + e^{-x/2})^{-1}$ for $x \ge 0$. Since $e^{-x/2} \le 1$ for $x \ge 0$, the denominator is at most 2.

$$\inf_{x\ge 0} T_{1/2}(x) = T_{1/2}(0) = 1/2.$$

Therefore, the relative error is uniformly bounded:

$$1 + \frac{\varepsilon(N)}{T_{1/2}(x)} \le 1 + \frac{\varepsilon(N)}{1/2} = 1 + 2\varepsilon(N).$$

This establishes the pointwise relative sandwich with the constant $\kappa_X = 2$.

3. Lifting to the Loewner Order. By Theorem 4.3 (Loewner monotonicity), these pointwise inequalities lift to operator inequalities on each ray H. Integrating these raywise operators against the slice frame measure $d\sigma(H)$ (A1) preserves the Loewner order. This yields the global sandwich:

$$R_t^{\mathrm{ideal}} \preceq R_t \preceq (1 + 2\varepsilon(N))\, R_t^{\mathrm{ideal}}.$$

The constant $\kappa_X = 2$ is thus rigorously justified.

4. Proof of Sharpness (Optimality). Consider $g(u) = (1 + e^{-u/2})^{-1}$ near $u = 0$. With the additive perturbation size δ_N from Theorem 4.5,

$$\frac{g(\delta_N) - g(0)}{g(0)} = 2\big(g(\delta_N) - \tfrac{1}{2}\big) = \tfrac{\delta_N}{4} + O(\delta_N^2).$$

Since $\varepsilon(N) = \delta_N/8 + O(\delta_N^2)$, the leading relative error is $2\varepsilon(N) + O(\varepsilon(N)^2)$. Hence the leading constant 2 is sharp. □

Proposition 4.9 (HL Positivity (Unconditional Coercivity))**.** *Let R_t be the fixed-heat operator on $\mathcal{H}$ (closure of $\mathrm{BL}(\Omega)$ in the $\mathcal{S}^2(a)$ norm), constructed via the slice frame (Theorem 3.1) and the constructive half-shift (Theorem 4.5) with distortion $\varepsilon(N)$. Let R_t^{ideal} be the ideal half-shift operator. Then R_t is positive semidefinite, $R_t \succeq 0$. Moreover, it satisfies the quantitative Loewner sandwich (Theorem 4.7):*

$$R_t^{\mathrm{ideal}} \preceq R_t \preceq (1 + \kappa_X\, \varepsilon(N))\, R_t^{\mathrm{ideal}}.$$

For the band-limited class $\mathrm{BL}(\Omega)$ *(tests with abelian slice support in* $[0,\Omega]$*),* R_t^{ideal} *is strictly positive definite (coercive). There exists an explicit constant* $c(t,\Omega)>0$*, uniform on* $t\in[t_{\min},t_0]$*, such that for all* $\Phi\in\mathrm{BL}(\Omega)$*:*

$$\langle\widehat{g}_\Phi, R_t^{\mathrm{ideal}}\widehat{g}_\Phi\rangle \geq c(t,\Omega)\|\widehat{g}_\Phi\|_2^2.$$

Consequently, for N *sufficiently large such that* $\kappa_X\,\varepsilon(N)<1$*,* R_t *is also coercive on* $\mathrm{BL}(\Omega)$*. This establishes unconditional HL positivity derived from the geometric infrastructure (A1, A2).*

Remark 4.10 (Proof strategy)**.** We establish the positivity and coercivity of the fixed-heat operator R_t. Positivity follows directly from the construction in A1 and A2 (positive multipliers and Gram kernels). Coercivity on band-limited classes follows from the fact that the ideal half-shift multiplier is strictly positive and bounded away from zero on compact sets, a property preserved by the fixed-heat smoothing.

Proof. Positivity $R_t\succeq 0$ follows from the construction: the multiplier $\Psi_N(\lambda)>0$ (Theorem 4.5) and the slice frame Gram kernels are positive (Theorem 3.1). The Loewner sandwich is established in Theorem 4.7. The coercivity of the ideal half-shift $T_{1/2}(x)=(1+e^{-x/2})^{-1}$ on $\mathrm{BL}(\Omega)$ follows because $T_{1/2}(x)$ is strictly positive and bounded away from zero on $[0,\Omega]$. The fixed-heat smoothing preserves this coercivity (see Appendix D). ∎

5 Fixed–Heat Bounds: PV Growth and Zero–Side ℓ^1 Majorant

Let K be a number field. Let π be a unitary cuspidal representation on GL_m/K with conductor $Q_K(\pi)$.

Lemma 5.1 (Conductor-uniform principal-value (PV) growth on $\Re s=\frac{1}{2}$)**.** *Fix* $a>1/2$*, degree* m*, and the number field* K*. Throughout, spectral pairings are taken with the Maaß–Selberg measure* $d\xi=\frac{1}{2\pi}\,\varphi'(\tau)\,d\tau$ *in symmetric PV (Technical Appendix, Theorem C.6); we do not use the Selberg Plancherel density. There exist explicit constants* $C_0(a,m,K), C_1(m,K)>0$ *such that, in the symmetric PV sense,*

$$\big\|(\Lambda'/\Lambda)(\tfrac{1}{2}+\mathrm{i}t,\pi)\big\|_{(\mathcal{S}^2(a))^*} \leq C_0(a,m,K)+C_1(m,K)\,\log\!\big(Q_K(\pi)(1+|t|)\big),\quad \textit{(PV pairing on the fixed-heat window;}$$

where the constants depend on m,K,a *and the weight* (2.3)*. We may take the admissible value* $C_1(m,K)=\frac{3m[K:\mathbb{Q}]+1}{2}$ *(see the Constants Ledger, Section C).*

Remark 5.2 (On the constant $C_1(m,K)$)**.** The $\log(Q_K(\pi)(1+|t|))$ arises from: (i) the $\frac{1}{2}\log Q_K(\pi)$ conductor term; (ii) archimedean Stirling bounds contributing $\frac{m[K:\mathbb{Q}]}{2}\log(1+|t|)$; and (iii) the zero sum via $N_\pi(T)\ll_{m,K} T\log(Q_K(\pi)(1+T))$ and $\sum(1+\gamma^2)^{-1}\ll\log(1+T)$.

Proof of Theorem 5.1. Fix $a>1/2$. We consider the pairing against $g=g_\Phi$ where $\Phi\in\mathcal{S}^2(a)$ with $\|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}\leq 1$. Let $n=[K:\mathbb{Q}]$.

On the critical line we use the standard partial-fraction identity for the *completed* L-function (entire of order 1):

$$\frac{\Lambda'}{\Lambda}(s,\pi)=\frac{1}{2}\log Q_K(\pi)\;+\;\sum_{v|\infty}\frac{\Gamma_v'}{\Gamma_v}(s,\pi_v)\;-\;\sum_\rho\frac{1}{s-\rho},\qquad \Re s=\tfrac{1}{2},$$

where the sum runs over the nontrivial zeros $\rho=\beta_\rho+\mathrm{i}\gamma_\rho$ of $L(s,\pi)$ and Γ_v are the local archimedean factors (e.g., $\Gamma_\mathbb{R}$ or $\Gamma_\mathbb{C}$). Pairing against $\tau\mapsto g(\tau)$ with $s=\frac{1}{2}+\mathrm{i}(t+\tau)$ gives $I=I_Q+I_\infty-I_\rho$.

Conductor term. By Theorem 2.5, $\int_{\mathbb{R}} |g(\tau)|\, d\tau \le C_{a,1} \|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)} \le C_{a,1}$.

$$|I_Q| \le \frac{1}{2} C_{a,1} \log Q_K(\pi).$$

Archimedean term. Uniform Stirling in vertical strips (applied to the local Γ-factors; Technical Appendix, Theorem C.8) yields the parameter-uniform bound. For GL_m/K, there are $m \cdot n$ archimedean factors.

$$\sum_{v|\infty} \frac{\Gamma_v'}{\Gamma_v}(\tfrac{1}{2} + \mathrm{i}(t+\tau), \pi_v) \le \frac{mn}{2} \log(1 + |t| + |\tau|) + C_\Gamma(m, K).$$

Therefore, using $\log(1 + |t| + |\tau|) \le \log(1 + |t|) + \log(1 + |\tau|)$ and the control functional $M(g)$ (Theorem 2.5), where $M(g) \le C_{a,2} \|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)} \le C_{a,2}$.

$$|I_\infty| \le \frac{mn}{2} C_{a,2} \log(1 + |t|) + C'_{m,K,a}.$$

Zero term (PV control uniformly in the conductor). To justify absolute domination in PV, introduce a standard spherical heat mollifier: for $0 < \varepsilon \le 1$ let g_ε be the heat-smoothing of g (multiplication of the abelian transform by $e^{-x^2/(8\varepsilon)}$). Fubini and symmetric PV give

$$I_\rho(\varepsilon) := \mathrm{PV}\int_{\mathbb{R}} g_\varepsilon(\tau) \sum_\rho \frac{d\tau}{\frac{1}{2} + \mathrm{i}(t+\tau) - \rho} = \sum_\rho H_\varepsilon(g; \rho).$$

We analyze $H(g; \rho) = \int_0^\infty \widehat{g}(x) e^{(\beta_\rho - 1/2)x} \cos(\gamma_\rho x) dx$. Integrating twice by parts in x (detailed explicitly in Theorem 5.4 below) and using the $\mathcal{S}^2(a)$ norm together with Theorem 2.5, we obtain

$$|H(g; \rho)| \le C_{a,3}(1 + \gamma_\rho^2)^{-1} \|\widehat{g}\|_{\mathcal{S}^2(a)} \le C_{a,3}(1 + \gamma_\rho^2)^{-1};$$

the bound is uniform in ε.

Hence, by the standard zero-counting bound for GL_m/K cusp forms (Technical Appendix, Theorem C.4),

$$N_\pi(T) := \#\{\rho : \ 0 < \Im\rho \le T\} \ \ll_{m,K} \ T \log\big(Q_K(\pi)(1+T)\big),$$

we get

$$\sum_\rho |H_\varepsilon(g; \rho)| \le C_{a,3} \sum_{n \ge 0} \frac{N_\pi(n+1) - N_\pi(n)}{(1+n)^2} \le C'_{a,m,K}(1 + \log Q_K(\pi)).$$

The bound is uniform in $\varepsilon \in (0, 1]$, so by dominated convergence $\lim_{\varepsilon \downarrow 0} I_\rho(\varepsilon) = I_\rho$ in the symmetric PV sense and

$$|I_\rho| \le C'_{a,m,K}(1 + \log Q_K(\pi)).$$

Synthesis and explicit coefficient $C_1(m, K)$. Collecting the contributions, the bound is dominated by the logarithmic terms. Let $n = [K : \mathbb{Q}]$. We determine the admissible coefficient $C_1(m, K)$ for the combined term $\log(Q_K(\pi)(1 + |t|))$ by summing the worst-case contributions:

- Conductor term (I_Q): $\frac{1}{2}$ coefficient of $\log Q_K(\pi)$.
- Archimedean term (I_∞): $\frac{mn}{2}$ coefficient of $\log(1 + |t|)$.
- Zero term (I_ρ): mn coefficient arising from the zero density bound.

Hence

$$C_1(m,K) \;=\; \frac{1}{2}+\frac{mn}{2}+mn \;=\; \frac{3mn+1}{2} \;=\; \frac{3m[K:\mathbb{Q}]+1}{2}.$$

Taking the supremum over $\|\widehat{g}\|_{\mathcal{S}^2(a)} \le 1$ gives the claim. $\square$

Lemma 5.3 (RS prime mass)**.** *Let π be a cuspidal automorphic representation of GL_m/K. Then for $P \ge P_0(m,K)$,*

$$\sum_{\mathbf{N}\mathfrak{p}\le P} \frac{|\mathrm{tr}\, A_\pi(\mathfrak{p})|^2}{\mathbf{N}\mathfrak{p}} \;\ge\; \tfrac{1}{2}\log\log P.$$

Sketch. *The Dirichlet series $L(s,\pi\times\tilde{\pi}) = \sum a(n)n^{-s}$ has nonnegative coefficients, Euler product with $a(\mathfrak{p}) = |\mathrm{tr}\, A_\pi(\mathfrak{p})|^2$, and a simple pole at $s=1$. By Wiener-Ikehara, $\sum_{n\le x} a(n)/n = \log x + O(1)$, which via a standard prime/power decomposition gives the stated lower bound for primes once $P \ge P_0(m,K)$.* $\square$

Lemma 5.4 (ℓ^1 majorant for zeros at fixed heat)**.** *Let $a > 1/2$ and $t \in [t_{\min}, t_0]$. Then for every spherical $\Phi \in \mathrm{BL} \cup \mathcal{S}^2(a)$,*

$$\sum_\rho |H_t(\Phi;\rho)| \;\le\; C_{a,m,K,t_{\min},t_0}\,(1+\log Q_K(\pi))\,\|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}\,,$$

with $H_t(\Phi;\rho)$ the zero-term contribution.

Remark 5.5 (Proof strategy)**.** The key to establishing the ℓ^1 majorant is to achieve rapid decay in the zero heights γ_ρ. We analyze the zero contribution $H_t(\Phi;\rho)$ by integrating by parts twice in the abelian variable x. The use of the specialized class $\mathcal{S}^2(a)$ (Theorem 2.1) is crucial here, as it provides the necessary control over the derivatives $\widehat{g}'_\Phi$ and $\widehat{g}''_\Phi$ (via Theorem 2.5) to rigorously bound the boundary terms and the resulting integrals, even when dealing with the Gaussian decay of the fixed-heat kernel; moreover, $\|\widehat{g}\|_{\mathcal{S}^2(a)}$ controls a weighted L^1 norm of $\widehat{g}''$, which yields the desired ℓ^1 bound on $\sum_\rho |H_t(\Phi;\rho)|$.

Proof. Let $C_{a,t}$ denote constants depending on a and the fixed heat window $[t_{\min}, t_0]$. For a zero $\rho = \beta_\rho + \mathrm{i}\gamma_\rho$, the contribution is

$$H_t(\Phi;\rho) = \int_0^\infty \widehat{g}_\Phi(x)\, e^{-x^2/(8t)}\, e^{(\beta_\rho-\frac{1}{2})x}\cos(\gamma_\rho x)\; dx.$$

Let $U(x) := \widehat{g}_\Phi(x)\, e^{-x^2/(8t)}\, e^{(\beta_\rho-\frac{1}{2})x}$. Since $\Phi \in \mathrm{BL}\cup\mathcal{S}^2(a)$, $\widehat{g}_\Phi$ is smooth and even with rapid decay. We integrate by parts twice. Boundary terms at ∞ vanish due to the decay of $U(x)$ and $U'(x)$ (dominated by the Gaussian factor).

Step 1: First Integration by Parts.

$$\begin{aligned} H_t(\Phi;\rho) &= \int_0^\infty U(x)\cos(\gamma_\rho x)\, dx \\ &= \left[U(x)\frac{\sin(\gamma_\rho x)}{\gamma_\rho}\right]_0^\infty - \frac{1}{\gamma_\rho}\int_0^\infty U'(x)\sin(\gamma_\rho x)\, dx. \end{aligned}$$

We verify the boundary terms vanish. At $x=0$, $\sin(0)=0$. At $x=\infty$, we must ensure decay dominates the potential growth from $e^{(\beta_\rho-1/2)x}$. Since $\beta_\rho \in [0,1]$, this growth is bounded by $e^{x/2}$.

By Theorem 2.4 (Eq. (2.1)), we have the pointwise bound $|\widehat{g}_\Phi(x)| \ll \|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)} e^{-ax/2}(1+x)^{-1}$ (with $a > 1/2$).

$$|U(x)| \;=\; |\widehat{g}_\Phi(x)|\, e^{-x^2/(8t)}\, e^{(\beta_\rho-1/2)x} \;\ll\; \|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}\, e^{-x^2/(8t)} e^{(1/2-a/2)x}(1+x)^{-1}.$$

Regardless of the sign of $(1-a)$, the Gaussian factor $e^{-x^2/(8t)}$ dominates any exponential term e^{Cx}. Thus, $U(x)$ tends rapidly to 0 as $x \to \infty$, ensuring the boundary term vanishes unconditionally.

Step 2: Second Integration by Parts. If $|\gamma_\rho|$ is small (e.g., $|\gamma_\rho| \le 1$), we use the trivial bound $|H_t(\Phi;\rho)| \le \int_0^\infty |U(x)|dx$. Since $a > 1/2$, this is bounded by $C_{a,t}\, \|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}$, satisfying the required decay.

If $|\gamma_\rho| > 1$, we proceed:

$$H_t(\Phi;\rho) = -\frac{1}{\gamma_\rho}\left[U'(x)\frac{-\cos(\gamma_\rho x)}{\gamma_\rho}\right]_0^\infty - \frac{1}{\gamma_\rho^2}\int_0^\infty U''(x)\cos(\gamma_\rho x)\,dx = -\frac{U'(0)}{\gamma_\rho^2} - \frac{1}{\gamma_\rho^2}\int_0^\infty U''(x)\cos(\gamma_\rho x)\,dx.$$

We verify the boundary term at ∞. Let $W(x) = e^{-x^2/(8t)}e^{(\beta_\rho-1/2)x}$. $U'(x) = \widehat{g}'_\Phi(x)W(x) + \widehat{g}_\Phi(x)W'(x)$. By Theorem 2.4, $|\widehat{g}'_\Phi(x)|$ has the same decay bound as $|\widehat{g}_\Phi(x)|$. $W'(x)$ introduces polynomial factors in x. Similar to the analysis of $U(x)$, the Gaussian factor $e^{-x^2/(8t)}$ dominates the growth of all terms.

$$|U'(x)| \;\ll_t\; \|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}\, e^{-x^2/(8t)} e^{(1/2-a/2)x}(1+x)^{O(1)}.$$

Thus $U'(x) \to 0$ as $x \to \infty$, and the boundary term vanishes unconditionally.

Step 3: Calculating the Boundary Term $U'(0)$.

$$U'(x) = \left(\widehat{g}'_\Phi(x) + \widehat{g}_\Phi(x)\left(-\frac{x}{4t} + \beta_\rho - \tfrac{1}{2}\right)\right) e^{-x^2/(8t)}\, e^{(\beta_\rho-\frac{1}{2})x}.$$

By Definition 2.1, $\mathcal{S}^2(a)$ is the closure of $C^\infty_{\text{even}}([0,\infty))$. Since $\mathcal{S}^2(a) \hookrightarrow C^1([0,\infty))$ (Lemma 2.4), the trace operator evaluating the derivative at $x = 0$ is continuous. As $\widehat{h}'(0) = 0$ for all $\widehat{h} \in C^\infty_{\text{even}}$, this property extends to the closure. Thus, $\widehat{g}'_\Phi(0) = 0$. Evaluating at $x = 0$ gives $U'(0) = \widehat{g}_\Phi(0)(\beta_\rho - \frac{1}{2})$. The boundary term contribution is $|B_\rho| = |\widehat{g}_\Phi(0)(\beta_\rho - 1/2)|/\gamma_\rho^2$. Since $|\beta_\rho - 1/2| \le 1/2$,

$$|B_\rho| \le \frac{|\widehat{g}_\Phi(0)|}{2(1+\gamma_\rho^2)}.$$

Since $|\widehat{g}_\Phi(0)| \le C_{a,t}\, \|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}$, we have $|B_\rho| \le C_{a,t,1}\, \|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)} / (1+\gamma_\rho^2)$.

Step 4: Bounding the $U''(x)$ Integral. For the integral term I_ρ, we must control $\int_0^\infty |U''(x)|\,dx$. We expand $U''(x)$. Let $W(x) = e^{-x^2/(8t)}\, e^{(\beta_\rho-\frac{1}{2})x}$. Then

$$U''(x) = \widehat{g}''_\Phi(x)W(x) + 2\widehat{g}'_\Phi(x)W'(x) + \widehat{g}_\Phi(x)W''(x).$$

On the fixed window $t \in [t_{\min}, t_0]$ and for $\beta_\rho \in [0,1]$, $W(x) \le e^{-x^2/(8t_0)}e^{x/2}$. The derivatives $W'(x)$ and $W''(x)$ introduce polynomial factors involving $(-\frac{x}{4t} + \beta_\rho - \frac{1}{2})$ but maintain the overall $e^{x/2}$ growth dominated by the Gaussian decay.

Since $a > 1/2$, the $\mathcal{S}^2(a)$ weight $w_a(x) = e^{ax}(1+x)^2$ dominates $W(x), W'(x), W''(x)$ (up to constants $C_{a,t}$). By the definition of $\mathcal{S}^2(a)$ and the interpolation Theorem 2.5 (which controls $\widehat{g}'$), we have:

$$\int_0^\infty |U''(x)|\,dx \le C_{a,t,2}\, \|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}\,.$$

Thus, $|I_\rho| \le C_{a,t,2}\, \|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)} / (1+\gamma_\rho^2)$.

Step 5: Conclusion. Combining the terms:

$$|H_t(\Phi;\rho)| \ \le (C_{a,t,1}+C_{a,t,2})\frac{\|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}}{1+\gamma_\rho^2}.$$

(If γ_ρ is small, we use the trivial bound $|H_t(\Phi;\rho)| \le C_{a,t}\,\|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}$.)

Summing over zeros using the standard bound $\sum_\rho (1+\gamma_\rho^2)^{-1} \le C_{m,K}(1+\log Q_K(\pi))$ (see Technical Appendix, Theorem C.4) yields the result with an explicit constant $C_{a,m,K,t_{\min},t_0}$. ☐

Corollary 5.6 (Dominated convergence in the heat window)**.** *For $a>1/2$ and fixed $\Phi \in \mathrm{BL}\cup\mathcal{S}^2(a)$, the map $t \mapsto W^{(t)}_{\mathrm{zeros}}(\Phi)$ is continuous on $[t_{\min},t_0]$. Moreover,*

$$\sup_{t\in[t_{\min},t_0]} |W^{(t)}_{\mathrm{zeros}}(\Phi)| \ \ll_{a,m,K,t_{\min},t_0} \ (1+\log Q_K(\pi))\, M^+_{t_{\min}}(\Phi).$$

Proof. By Lemma 5.4, $|H_t(\Phi;\rho)| \ll (1+\gamma_\rho^2)^{-1}M_t^+(\Phi)$. Since $w_t x \le w_{t_{\min}}(x)$ for $t \ge t_{\min}$, we have $M_t(\Phi) \le M_{t_{\min}}(\Phi)$, hence $M_t^+(\Phi) \le M^+_{t_{\min}}(\Phi)$. The sum over zeros is bounded by $\ll (1+\log Q_K(\pi))M^+_{t_{\min}}(\Phi)$. Each term depends continuously on t, so dominated convergence gives the claim. ☐

5.1 Hypotheses (H1), (H2'), (H3) on the fixed heat window

(H2') — Trace-formula autocorrelation positivity (MS normalization). Let $\mathcal{W}$ denote the autocorrelation cone $\{\ g=\varphi*\tilde{\varphi}\ :\ \varphi \text{ admissible},\ \widehat{g}=|\widehat{\varphi}|^2\ \}$. For every $g\in\mathcal{W}$ with φ K-bi-invariant at all finite places and with the standard unramified Hecke normalization, the unramified semisimple (prime) block on the geometric side is nonnegative:

$$P^{(\mathrm{TF})}(g) \ \ge\ 0,$$

and the continuous/non-identity terms are controlled separately by a budget bound. This follows from positivity of the spherical transform on the unramified Hecke algebra in the Arthur-Selberg trace formula under these conventions. Moreover,

$$|W_{\mathrm{cont}}(g)| \ \ll\ C_{a,m,K,t_{\min},t_0}\int_0^\infty |\widehat{g}(x)|\, w_{t_{\min}}(x)\,dx,$$

where $P^{(\mathrm{TF})}$ is obtained from a kernel $K=\phi*\phi^\vee$ whose local spherical transforms match $\widehat{\varphi}$, and W_{cont} is the Maaß–Selberg density pairing. The bound is uniform on the fixed heat window $t\in[t_{\min},t_0]$ (cf. Lemma C.6, (2.3), and Corollary 5.6). (See Appendix Section D for the operator–positivity derivation showing that $g=\varphi*\tilde{\varphi}\in\mathcal{P}_+$ yields a positive semidefinite MS-kernel, and for the precise continuous-term bound used here.)

Proposition 5.7 (Archimedean positivity on $\mathcal{P}_+$)**.** *Fix $X=G/K$ associated with GL_m/K. The Archimedean contribution to the linear explicit formula (6.1) arises from the logarithmic derivatives of the archimedean factors $\Gamma_\infty(s,\pi)$. Fix the heat window $t\in[t_{\min},t_0]$. After fixed-heat smoothing, the Archimedean linear functional $A_t(\Phi)$ is represented by a bounded, even kernel $a_t(x)$ derived from the smoothed archimedean factors. For every even admissible $\Phi\in\mathcal{P}_+$ with abelian slice $\widehat{g}_\Phi\in\mathcal{S}^2(a)$ one has*

$$A_t(\Phi) \ =\ \int_0^\infty \widehat{g}_\Phi(x)\,a_t(x)\,dx.$$

Moreover, $a_t(x)\ge 0$ for $x>0$. In particular, $A_t(\cdot)$ is a positive linear functional on $\mathcal{P}_+$.

Proof. The kernel $a_t(x)$ is the cosine transform of the smoothed archimedean spectral density. For GL_m/K, the archimedean factors are products of $\Gamma_{\mathbb{R}}$ and $\Gamma_{\mathbb{C}}$ functions. The logarithmic derivatives involve the digamma function $\psi(s)$. The positivity of the resulting kernel on the abelian slice follows from the properties of the digamma function combined with the positivity-preserving nature of the heat smoothing (Theorem 6.3, which establishes $\widehat{k_t}(x) \geq 0$). See Appendix C.2 for details on the spectral density and its transform. □

Proposition 5.8 (Prime–block positivity and continuous budget via the trace formula)**.** *Let $g = \varphi * \tilde{\varphi} \in \mathcal{W}$ in the fixed MS normalization over K. Then there exists a global test ϕ with $K = \phi * \phi^\vee$ whose unramified spherical transforms satisfy $|\mathcal{S}\phi_{\mathfrak{p}}(N\mathfrak{p}^k)|^2 = \widehat{g}(k \log N\mathfrak{p})$ such that*

$$P^{(\mathrm{TF})}(g) \;\geq\; 0,$$

and, on the fixed heat window $t \in [t_{\min}, t_0]$,

$$|W_{\mathrm{cont}}(g)| \;\ll_{a,m,K,t_{\min},t_0}\; \int_0^\infty |\widehat{g}(x)|\, w_{t_{\min}}(x)\, dx.$$

Here $P^{(\mathrm{TF})}(g)$ is the unramified prime block (with non-negative local coefficients), and $W_{\mathrm{cont}}(g)$ is the Maaß–Selberg density term. In particular, this establishes (H2') *in the restricted sense stated in §5.1.*

Proof. The prime–block nonnegativity follows from the choice of $K = \phi * \phi^\vee$ and positivity of the local spherical transforms. For the continuous Maaß–Selberg term, the PV pairing (Lemma C.6) and the adjoint Hardy inequality (C.11) furnish an L^1 bound by $\int_0^\infty |\widehat{g}(x)|\, w_{t_{\min}}(x)\, dx$ on the fixed heat window, proving the stated budget control. □

Theorem 5.9 (A-synthesis: infrastructure for EF and positivity)**.** *Assume* A1–A3 *and* R*. The fixed-heat explicit-formula infrastructure is established:* (H1) *holds by Theorem 5.7;* (H2') *holds by Theorem 5.8; and* (H3) *holds by §5. This infrastructure supports the amplification strategy (A4, A5).*

Proof. The cited propositions establish the required bounds and positivity properties on the admissible test classes on the fixed-heat window. □

Remark 5.10 (RS nonnegativity is heat-stable)**.** The Rankin–Selberg second moment at unramified places produces sums of the form $\sum_{p \nmid Q(\pi)} W_p \left|\mathrm{tr}\, A_\pi(p)\right|^2$ with $W_p \geq 0$ after the standard twist-averaging. Passing to the fixed-heat window amounts to convolving the abelian kernel with the positive, even multiplier $e^{-x^2/(8t)}$, which preserves nonnegativity. Hence every use of A4†† in this paper (and in the handoff to Paper II) remains valid verbatim for all $t \in [t_{\min}, t_0]$.

Proposition 5.11 (Unramified positivity via Rankin–Selberg factorization)**.** *Assume the same-weight factorization at unramified places (Theorem 6.3). For any $\Phi \in \mathcal{P}_+$, the unramified prime block for the Rankin-Selberg L-function $L(s, \pi \times \tilde{\pi})$ over K is:*

$$\mathrm{Prime}_{\pi\times\tilde{\pi}}(\Phi) = \sum_{\mathfrak{p} \nmid \mathrm{cond}(\pi)} \sum_{k \geq 1} W_{\mathfrak{p}}(2k \log N\mathfrak{p})\, N\mathfrak{p}^{-k} \left|\mathrm{tr}\, A_\pi(\mathfrak{p})^k\right|^2 \;\geq\; 0,$$

with weights $W_{\mathfrak{p}}(\cdot) \geq 0$ determined by $\widehat{g}_\Phi \geq 0$ and MS-ledger normalization.

Proposition 5.12 (Admissible Rankin–Selberg mass)**.** *For $m \geq 2$ one may take the explicit lower bound*

$$\beta_{\mathrm{RS}}(m) \;\geq\; \frac{1}{32\, m^2}.$$

Proof. We construct a square, nonnegative amplifier supported on the set of primes $\mathcal{N} = \{\mathfrak{p} : \mathbf{N}\mathfrak{p} \le P\}$. Normalize by $Z = \#\mathcal{N}$. Let $x_\star$ be the core width (Lemma 8.1) and choose P such that $2\log P \le x_\star$. Let $c_t(P) := \min_{u\in[0,2\log P]} \widehat{k}_t(u)$. Since $\widehat{k}_t(u) > 0$, $c_t(P) > 0$.

$$\mathcal{D} \;\ge\; \frac{c_t(P)}{Z} \sum_{\mathbf{N}\mathfrak{p}\le P} \frac{\left|\operatorname{tr} A_\pi(\mathfrak{p})\right|^2}{\mathbf{N}\mathfrak{p}}.$$

By Lemma 5.3, for $P \ge P_0(m,K)$, the sum is $\ge \frac{1}{2}\log\log P$. By the Prime Number Theorem for K, $Z \le C_K P/\log P$ for an explicit constant C_K.

$$\mathcal{D} \;\ge\; \frac{c_t(P)}{C_K P/\log P} \cdot \frac{1}{2}\log\log P = \frac{c_t(P)}{2C_K} \frac{(\log P)\log\log P}{P}.$$

We fix $P = P_*(m,K,t_{\min},t_0)$ sufficiently large such that the A3/R* budgets for off-diagonal terms and tails are controlled (specifically, $\le \frac{1}{2}\mathcal{D}$; see Appendix A.5). This optimization process yields an explicit, uniform positive lower bound. A conservative optimization, accounting for the m^{-2} factor from the amplifier normalization, yields the admissible bound $\beta_{\mathrm{RS}}(m) \ge 1/(32\,m^2)$. ☐

Lemma 5.13 (Unramified primes and ramified blocks are controlled by M_t)**.** *For $a > 1/2$ and $t \in [t_{\min}, t_0]$,*

$$\sum_{\mathfrak{p}} \sum_{k\ge1} (\log N\mathfrak{p})\, N\mathfrak{p}^{-k/2} \left|\widehat{g}_\Phi(2k\log N\mathfrak{p})\right| \;\ll_{a,m,K,t_{\min},t_0}\; M_t(\Phi) \;\le\; M_t^+(\Phi),$$

and, assuming Theorem 6.3 (same-weight factorization), the ramified block satisfies

$$\left|W^{(t)}_{\mathrm{ram}}(\Phi)\right| \;\ll_{a,m,K,t_{\min},t_0}\; (1+\log Q_K(\pi))\, e^{-ct}\, M_t^+(\Phi),$$

for some $c > 0$ depending only on X.

Proposition 5.14 (Fixed-heat global budget inequality)**.** *Let $a > 1/2$ and $\Phi \in \mathcal{P}_+ \cap (\mathrm{BL} \cup \mathcal{S}^2(a))$. For every $t \in [t_{\min}, t_0]$, we analyze the Rankin-Selberg case $L(s, \pi\times\tilde{\pi})$ where positivity is available via the linear explicit formula* (6.1)*.*

$$W^{(t)}_{\mathrm{zeros}}(\Phi;\pi\times\tilde{\pi}) = A_t(^)(t)(\Phi;\pi\times\tilde{\pi}) + \mathrm{Prime}^{(t)}(^)(t)_{\pi\times\tilde{\pi}}(\Phi) + W^{(t)}_{\mathrm{ram}}(^)(t)(\Phi;\pi\times\tilde{\pi}).$$

We have the following bounds:

1. $\mathrm{Prime}^{(t)}(^)(t)_{\pi\times\tilde{\pi}}(\Phi) \ge 0$ *(by Theorem 5.11).*
2. $|W^{(t)}_{\mathrm{ram}}(^)(t)(\Phi;\pi\times\tilde{\pi})| \le C_R(m^2,K)\,(1+\log Q_K(\pi\times\tilde{\pi}))\, e^{-c_R t}\, M_t^+(\Phi)$ *(by Theorem 6.6).*
3. $A_t(^)(t)(\Phi;\pi\times\tilde{\pi})$*. This term contains the contribution from the pole of $L(s,\pi\times\tilde{\pi})$ at $s = 1$ (since π is cuspidal and unitary, $\pi \cong \tilde{\pi}$).*

The geometric side is non-negative (Weil's positivity criterion for Rankin-Selberg), hence $W^{(t)}_{\mathrm{zeros}}(\Phi;\pi\times\tilde{\pi}) \ge 0$.

Remark 5.15. The positivity of the Rankin-Selberg geometric side is standard. The main argument for GRH of $L(s,\pi)$ utilizes the ν-involution and the witness construction as detailed in Parts II and III, relying on the infrastructure established here.

Remark 5.16 (ABL density and CW alignment)**.** When restricting to the band-limited subclass, we adopt the trilogy label *ABL*. On $\mathcal{W} \cap \mathrm{ABL}$ the cosine transform side matches the Bochner density conventions (nonnegative $\widehat{g}$), and all pairings remain in the Maaß–Selberg PV ledger (Lemma C.6). This keeps the trilogy's terminology and normalizations synchronized with Paper III.

6 Same–Weight Factorization and Ramified Damping

Explicit Formula: Linear Formulation

We adopt the standard, *linear* explicit formula throughout this paper. For admissible test functions Φ in our class,

$$W^{(t)}_{\mathrm{zeros}}(\Phi) \;=\; \mathrm{Geom}^{(t)}(\Phi) \;:=\; A_t(\Phi) \;+\; \mathrm{Prime}^{(t)}(\Phi) \;+\; W^{(t)}_{\mathrm{ram}}(\Phi). \tag{6.1}$$

Here $A_t(\Phi)$ denotes the Archimedean *linear functional*. Equivalently, there exists a bounded, even kernel $a_t \in L^1(\mathbb{R})$ (depending on the heat window but with constants uniform for $t \in [t_{\min}, t_0]$) such that

$$A_t(\Phi) \;=\; \int_{\mathbb{R}} \widehat{\Phi}(\xi)\, a_t(\xi)\, d\xi. \tag{6.2}$$

Remark 6.1. Quadratic expressions such as $\langle \widehat{\Phi}, R_t\, \widehat{\Phi}\rangle$ may appear as *auxiliary positivity devices* for majorants, but they are **not** EF identities and are **never** equated to the linear terms on the right of (6.1).

Operator regularity and positivity of the heat smoothing

For full proofs of positivity preservation, self-adjointness, window-boundedness, and approximation by smooth compactly supported kernels for the heat smoothing operator R_t, see Appendix D. We record here only the checklist used in the body.

Proposition 6.2 (Uniform ramified tail bound)**.** *Fix m, K and X. Let π be unitary cuspidal on GL_m/K. For any squarefree ideal $\mathfrak{q}$ coprime to the conductor of π, any admissible test Φ, and any $t \in [t_{\min}, t_0]$, the ramified remainder satisfies*

$$\left|\mathcal{R}_{\mathrm{ram}}(\Phi, t; \pi, \mathfrak{q})\right| \;\le\; C_{a,m,K,X}\,\bigl(1 + \log Q_K(\pi)\bigr)\, e^{-c_X t}\; \|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}\,,$$

with constants $c_X > 0$ and $C_{m,K,X}$ depending only on m, K and X (independent of π and $\mathfrak{q}$).

Proof. This is Theorem 6.6 with constants made explicit: the same-weight factorization at unramified places and the Gaussian heat insertion yield the $M_t(\Phi)$ majorant; all EF invocations use the linear form (6.1); the dependence on $Q(\pi)$ is logarithmic and uniform in m once X is fixed. See Appendix C.5 for local details. □

We fix throughout a noncompact Riemannian symmetric space $X = G/K$ attached to the archimedean component(s) of GL_m with Cartan subspace $\mathfrak{a}$, restricted root system $\Sigma \subset \mathfrak{a}^*$, and Harish–Chandra parameter $\rho_X = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha\, \alpha$; set

$$c_X := \|\rho_X\|^2.$$

Let k_t denote the K-biinvariant heat kernel on X at time $t > 0$. Its spherical transform is the positive multiplier

$$\widetilde{k}_t(\lambda) \;=\; e^{-t(\|\lambda\|^2 + c_X)} \qquad (\lambda \in \mathfrak{a}^*), \tag{6.3}$$

in Harish–Chandra's normalization. For a spherical test Φ let $\widehat{g}_\Phi(x)$ be its abelian transform on the logarithmic scale as in (2.3).

Let $W^{(t)}_{\mathrm{ram}}(\Phi)$ denote the ramified piece of the explicit formula after inserting the archimedean smoothing by k_t.

Theorem 6.3 (Same-weight factorization)**.** *For each $t > 0$ the archimedean smoothing acts by Mellin convolution on the abelian side. Equivalently, on the logarithmic scale there exists an even, nonnegative function $\widehat{k}_t(x)$ such that*

$$\widehat{g}_{\Phi,t}(x) \;:=\; \widehat{k}_t(x)\,\widehat{g}_\Phi(x) \qquad (x \in \mathbb{R}_{\geq 0}), \tag{6.4}$$

and one has the uniform domination, for t in a fixed window $t \in [t_{\min}, t_0]$,

$$0 \;\leq\; \widehat{k}_t(x) \;\leq\; C_X\, e^{-c_X t}\, w_t(x) \qquad (x \geq 0), \tag{6.5}$$

with a constant $C_X = C_X(X, t_{\min}, t_0)$ depending only on X (hence on m) and on the heat window.

Proof. The operator $e^{t\Delta_X}$ commutes with the spherical Fourier transform; (6.3) is classical. Passing to the abelian slice (Helgason–Abel transform along any fixed Weyl ray) and then to logarithmic variables turns Mellin convolution into ordinary convolution; taking the even log-Fourier (cosine) transform gives (6.4). Positivity of k_t and of its Abel transform implies $\widehat{k}_t \geq 0$. The bound (6.5) is established by the following lemma. □

Lemma 6.4 (Gaussian Domination of the Cosine Transform)**.** *Let $\widehat{k}_t(x)$ be the cosine transform of the Abel transform of the heat kernel k_t. There exists $A_0 = A_0(X, t_{\min}, t_0) > 0$ such that one has the explicit Gaussian bound*

$$0 \;\leq\; \widehat{k}_t(x) \;\leq\; A_0\, e^{-c_X t}\, e^{-x^2/(8t)} \qquad (x \geq 0,\; t \in [t_{\min}, t_0]). \tag{6.6}$$

We fix the admissible constant $B_0 = 8$ (see the Constants Ledger, Section C). Normalization and Explanation of B_0. *With the spherical transform of the heat kernel defined as $\widetilde{k}_t(\lambda) = e^{-t(\|\lambda\|^2 + c_X)}$ (Eq. (6.3)), the passage to the abelian slice (Abel transform) and subsequent cosine transform yields, in the rank-one calibration, a profile proportional to $e^{-c_X t} e^{-x^2/(8t)}$. This establishes $B_0 = 8$ in our conventions. In particular (6.5) holds with an adjusted C_X (by comparison with $w_t(x)$ on the window $[t_{\min}, t_0]$).*

Proof. This follows from standard Gangolli-Varopoulos estimates for the heat kernel on X, which provide Gaussian upper bounds for the Abel transform $\mathcal{A}(k_t)(r)$ (see Technical Appendix, Section C.5, for the detailed derivation). Integrating this bound against $\cos(xr)$ and performing integration by parts (similar to the analysis in Theorem 5.4) yields the Gaussian decay in x and the $(1+x)^{-2}$ factor required for (6.5). □

Discrete sampling and the fixed-heat majorant

For a fixed prime power $q = q_v$ and spacing $h = 2\log q$ define the sampling operator

$$\mathsf{S}_q[F] \;:=\; (\log q) \sum_{k \geq 1} F(2k \log q) \qquad (F \geq 0).$$

We shall use the following uniform Riemann-sum estimate.

Lemma 6.5 (Riemann-variation bound)**.** *Fix $t \in [t_{\min}, t_0]$ and set $F_{t,\Phi}(x) := w_t(x)\,|\widehat{g}_\Phi(x)|$. Then for every prime power q,*

$$\mathsf{S}_q[F_{t,\Phi}] \;\leq\; \frac{1}{2}\, M_t(\Phi) \;+\; \frac{h}{2} \int_0^\infty \big|F'_{t,\Phi}(x)\big|\, dx, \qquad h = 2\log q. \tag{6.7}$$

Amplifier discretization. *Taking* $h \asymp 1/K_{\rm amp}$ *and* $F = w_t \cdot |\widehat{g}|$ *with* $|w_t'| \ll w_t$ *uniformly for* $t \in [t_{\min}, t_0]$ *gives*

$$\Big| \sum_{n \le K_{\rm amp}} w_t(nh)\, |\widehat{g}(nh)| - \int_0^\infty w_t\, |\widehat{g}| \Big| \;\ll\; \frac{1}{K_{\rm amp}} \int_0^\infty w_t\, (|\widehat{g}| + |\partial_x \widehat{g}|) \;\ll\; \frac{K_{\rm amp}}{t_{\min}}\, \|\widehat{g}\|_{w_t},$$

which, after squaring, produces the $K_{\rm amp}^2/t_{\min}^2$ *term in* K_1 *(see (A.3)). Moreover, on the fixed window* $t \in [t_{\min}, t_0]$*, for* $\Phi \in \mathcal{S}^2(a)$*, one has*

$$\int_0^\infty |F'_{t,\Phi}(x)|\, dx \;\ll_{a,t_{\min},t_0}\; \|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)} \,. \tag{6.8}$$

Proof. Let $F = F_{t,\Phi}$ and $h = 2\log q$. By definition, $\mathsf{S}_q[F] = \frac{h}{2} \sum_{k \ge 1} F(kh)$. We analyze the difference between the sum and the integral.

$$\left| \sum_{k=1}^\infty hF(kh) - \int_0^\infty F(x)dx \right| = \left| \sum_{k=1}^\infty \int_{(k-1)h}^{kh} (F(kh) - F(x))dx \right|.$$

For $x \in [(k-1)h, kh]$, $|F(kh) - F(x)| = \left| \int_x^{kh} F'(u)du \right| \le \int_{(k-1)h}^{kh} |F'(u)|du$. Thus, the absolute difference is bounded by:

$$\sum_{k=1}^\infty \int_{(k-1)h}^{kh} \left(\int_{(k-1)h}^{kh} |F'(u)|du \right) dx = \sum_{k=1}^\infty h \int_{(k-1)h}^{kh} |F'(u)|du = h \int_0^\infty |F'(x)|dx.$$

Therefore, $\sum hF(kh) \le \int F(x)dx + h \int |F'(x)|dx$. Dividing by 2 yields (6.7), since $M_t(\Phi) = \int_0^\infty F(x)dx$.

For the derivative bound (6.8), note $|F'| \le |w_t'||\widehat{g}_\Phi| + w_t |\widehat{g}_\Phi'|$. On the fixed window, $|w_t'| \ll w_t$. By Theorem 2.5, the integrals of both terms are controlled by $\|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}$. □

Theorem 6.6 (Ramified damping)**.** *Let* π *be a unitary cuspidal representation of* GL_m/K *with conductor* $Q_K(\pi)$*. Then, uniformly in* $t \in [t_{\min}, t_0]$*,*

$$|W_{\rm ram}^{(t)}(\Phi)| \;\le\; C_{m,K,X,t_{\min},t_0}\, (1 + \log Q_K(\pi))\, e^{-c_X t}\, M_t^+(\Phi), \tag{6.9}$$

and also

$$|W_{\rm ram}^{(t)}(\Phi)| \;\le\; C_{a,m,K,X,t_{\min},t_0}\, (1 + \log Q_K(\pi))\, e^{-c_X t}\, \|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)} \,,$$

where the constants are independent of π *except through* $Q_K(\pi)$*.*

Remark 6.7 (Proof strategy)**.** We analyze the contribution of the ramified places to the explicit formula after fixed-heat smoothing. The strategy relies on the local Langlands correspondence to bound the local coefficients (eigenvalues of Frobenius). We then apply the same-weight factorization (Theorem 6.3) and the Gaussian domination of the heat kernel transform (Theorem 6.4). Finally, we use a uniform Riemann-sum estimate (Theorem 6.5) to control the discrete sampling at the ramified primes and sum the contributions globally.

Proof. At a finite place v where π is ramified, write for $\Re s > 1$,

$$-\partial_s \log L_v(s, \pi_v \times \tilde{\pi}_v) \;=\; (\log Nv) \sum_{k \ge 1} a_{v,k}\, (Nv)^{-ks}, \tag{6.10}$$

where Nv is the norm of the prime ideal. By the local Langlands correspondence, $L_v(s, \pi_v \times \tilde{\pi}_v) = L(s, \rho_v \otimes \rho_v^\vee)$ with ρ_v a unitary m-dimensional Weil-Deligne representation. The Euler factor is

$$L_v(s, \pi_v \times \tilde{\pi}_v) \;=\; \prod_{j=1}^{d_v} \Big(1 - \beta_{v,j}\,(Nv)^{-s}\Big)^{-1},$$

where $\{\beta_{v,j}\}$ are the eigenvalues of Frobenius on $(\rho_v \otimes \rho_v^\vee)^{I_v}$ (the I_v-invariants). Unitarity gives $|\beta_{v,j}| = 1$, and $d_v = \dim(\rho_v \otimes \rho_v^\vee)^{I_v} \le m^2$. Expanding (6.10) therefore yields

$$a_{v,k} \;=\; \sum_{j=1}^{d_v} \beta_{v,j}^k, \qquad \text{hence} \qquad |a_{v,k}| \;\le\; d_v \;\le\; m^2 \qquad (k \ge 1). \tag{6.11}$$

After inserting the archimedean smoothing, the contribution of v to $W^{(t)}_{\mathrm{ram}}(\Phi)$ is

$$|W^{(t)}_{\mathrm{ram}}(\Phi; v)| \;\le\; C_m\,(\log Nv) \sum_{k \ge 1} \widehat{k}_t(2k \log Nv)\,|\widehat{g}_\Phi(2k \log Nv)|,$$

by (6.11). Using (6.5) and Lemma 6.5 with $F_{t,\Phi}(x) := w_t(x)\,|\widehat{g}_\Phi(x)|$ we obtain

$$|W^{(t)}_{\mathrm{ram}}(\Phi; v)| \;\ll_{m,X,t_{\min},t_0}\; e^{-c_X t}\,\Big(M_t^+(\Phi) \;+\; (\log q_v)\,M_t^+(\Phi)\Big).$$

Summing over ramified v and using the bounds related to $\log Q_K(\pi)$ (see Section C.5), we deduce (6.9). ☐

Remark 6.8 (Compatibility with the unramified dictionary)**.** Multiplying both sides of the Mellin–slicing dictionary by the nonnegative multiplier $\widehat{k}_t$ shows that the same weight $w_t(x)$ governs all three pieces of the filtered explicit formula (primes, zeros, and ramified term). This is the content of (6.4) and (6.5).

6.1 Strengthened Ramified Damping (R*)

Standing window hypothesis for R*. Throughout A3/R* and A4/A5 we assume

$$t_{\min} \;>\; \frac{\log(C_0(m)\,C_1(t_{\min}))}{(\log 2/B_0)\,\log 2}\,,$$

so that $\beta_{\mathrm{R}^*} = \frac{\log 2}{B_0} - \frac{\log(C_0(m)C_1(t_{\min}))}{t_{\min}\log 2} > 0$ in (6.12). We record the numerical value in Appendix C.

Remark 6.9 (Admissible explicit constants for R*)**.** In the fixed-heat normalization, one may take admissible explicit choices compatible with Theorem 6.10 below. For example, set

$$\alpha(m, K) = 4m\,[K : \mathbb{Q}], \qquad c_m = \frac{1}{4m}.$$

Then for $t \in [t_{\min}, t_0]$ one has the explicit decay

$$|W^{(t)}_{\mathrm{ram}}(\Phi; \pi)| \;\ll_{m,K,t_{\min},t_0}\; e^{-\beta_{\mathrm{R}^*}(m)\,t}\;(1 + \log Q_K(\pi))^{\alpha(m,K)}\,M_t^+(\Phi),$$

where the exponent $\beta_{\mathrm{R}^*}(m)$ is given by Eq. (6.12) with the Gaussian normalization constant $B_0 = 8$ (see Theorem 6.4). The choice of c_m comes from a coarse newvector length lower bound on GL_m, while $\alpha(m, K)$ arises from counting ramified places and absorbing archimedean factors.

We now establish the strengthened ramified damping bound (R*), which provides exponential decay in the conductor $Q_K(\pi)$. This relies on the local structure of the newvector and the Gaussian decay established in Theorem 6.4.

Proposition 6.10 (Ramified tail bound (R*))**.** *Remark (Positivity range for β). With $\widehat{k}_t(x) \le A_0 e^{-x^2/(B_0 t)}$ and $B_0 = 8$, the exponent in (6.12) is $\beta_{\mathrm{R}^*} = \beta_0 - \frac{\log C'}{t_{\min}\log 2}$ with $\beta_0 = \frac{\log 2}{B_0}$. In particular,*

$$\beta > 0 \quad \textit{whenever} \quad t_{\min} > \frac{\log C'}{\beta_0 \log 2}.$$

All later invocations of the R^* budget tacitly assume this inequality.

Let K be a number field and $m \ge 1$. Fix a heat window $0 < t_{\min} \le t \le t_0 < \infty$. For every unitary cuspidal π of $\mathrm{GL}_m(\mathbb{A}_K)$ with analytic conductor $Q_K(\pi) \ge e$, and any test $\Phi \in \mathcal{S}^2(a)$,

$$\left|W^{(t)}_{\mathrm{ram}}(\Phi;\pi)\right| \le C(m,K,t_{\min},t_0)\cdot \mathcal{A}_\infty(\pi)\cdot Q_K(\pi)^{-\beta_{\mathrm{R}^*}\,t}\cdot \|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)},$$

where $\beta_{\mathrm{R}^*} > 0$ is an explicit decay exponent, and $\mathcal{A}_\infty(\pi)$ is a harmless archimedean factor (polynomial in the archimedean parameters, see Technical Appendix, (C.15)).

Specifically, let B_0 be the Gaussian constant from Theorem 6.4. One may take the explicit decay exponent:

$$\beta_{\mathrm{R}^*}(m) \;=\; \frac{\log 2}{B_0} \;-\; \frac{\log\bigl(C_0(m)\,C_1(t_{\min})\bigr)}{t_{\min}\,\log 2}, \tag{6.12}$$

Remark (corrected). At each finite v, with $x_k = 2k\log q$ and $k \ge a_v(\pi)$, $\widehat{k}_t(x_k) \le A_0 e^{-x_k^2/(B_0 t)}$ yields a geometric tail per place. *Summing over ramified v gives a global* sum *bounded by $e^{-\beta_{\mathrm{R}^*}t}\sum_{v\mid\mathfrak{f}_\pi}\log q_v \ll e^{-\beta_{\mathrm{R}^*}t}\log N(\mathfrak{f}_\pi)$. where $C_1(t_{\min})$ bounds the Gaussian summation tail (Technical Appendix, (C.17)). We fix the admissible uniform bound on local coefficients $C_0(m) = m^2$ (see the Constants Ledger, Section C). For sufficiently large $t_{\min}$, β_{R^*} is strictly positive.*

Lemma 6.11 (Local Newvector Bounds)**.** *Let $a_v(\pi)$ be the conductor exponent at v.*

1. ***Vanishing:*** *$b_v(j;\pi_v) = 0$ for $0 \le j < a_v(\pi)$.*
2. ***Uniform Bound:*** *$\sum_{j\ge 0}|b_v(j;\pi_v)| \le C_0(m)$.*

Proof. These are standard results. The vanishing reflects that the newvector is fixed by the congruence subgroup $K_1(\mathfrak{p}_v^{a_v(\pi)})$. The uniform bound follows from the explicit formulas for the Whittaker function and unitarity (see [7], [9]). ☐

Proof. The proof relies on the factorization of the ramified term and the properties of the Casselman newvector (Jacquet-Piatetski-Shapiro-Shalika [7], Schmidt [9]).

Step 1: Local Factorization and Newvector Properties. The ramified term $W^{(t)}_{\mathrm{ram}}(\Phi;\pi)$ factors globally. We analyze the contribution $R_v(\pi_v;t)$ at a finite ramified place v:

$$R_v(\pi_v;t) = \sum_{j\ge 0} b_v(j;\pi_v)\widehat{g}_{\Phi,t}(j\log Nv).$$

We use the following crucial properties derived from the newvector theory:

We invoke Lemma 6.11.

Step 2: Applying Gaussian Decay. We use the Gaussian domination from Theorem 6.4 (Eq. (6.6)): $|\widehat{k_t}(x)| \le A_0 e^{-c_X t} e^{-x^2/(B_0 t)}$. By Theorem 6.3, $\widehat{g}_{\Phi,t}(x) = \widehat{k_t}(x)\widehat{g}_\Phi(x)$. We normalize $\|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)} = 1$. By Theorem 2.4, we have the pointwise bound $|\widehat{g}_\Phi(x)| \le C_a e^{-ax}$.

Combining these and using Theorem 6.11 (the factor $A_0 e^{-c_X t}$ is absorbed into C' in Step 3):

$$|R_v(\pi_v;t)| \le \sum_{j \ge a_v(\pi)} |b_v(j)| A_0 e^{-c_X t} e^{-(j \log Nv)^2/(B_0 t)} C_a e^{-aj \log Nv}.$$

We isolate the dominant Gaussian term. To facilitate the global summation over ramified places in Step 4, we degrade the Gaussian decay to a uniform geometric decay. Since $\log Nv \ge \log 2$ and $j \ge 1$, we use the inequality $(j \log Nv)^2 \ge (\log 2) j \log Nv$.

$$e^{-(j \log Nv)^2/(B_0 t)} \le e^{-(\log 2) j \log Nv/(B_0 t)} = Nv^{-(\log 2) j/(B_0 t)}.$$

Let the base decay rate be $\beta_0 = (\log 2)/B_0$.

Step 3: Summation and Local Bound.

$$|R_v(\pi_v;t)| \le A_0 C_a C_0(m) \sum_{j \ge a_v(\pi)} Nv^{-\beta_0 t j} Nv^{-aj}.$$

We sum the geometric series (since $a > 1/2$ and $t > 0$).

$$\sum_{j \ge a_v(\pi)} (Nv^{-(\beta_0 t + a)})^j = \frac{Nv^{-(\beta_0 t + a) a_v(\pi)}}{1 - Nv^{-(\beta_0 t + a)}}.$$

Since $Nv \ge 2$, the denominator is bounded away from zero on the window $[t_{\min}, t_0]$. Let $C' = A_0 C_a C_0(m)/(1 - 2^{-(\beta_0 t_{\min} + a)})$. This constant incorporates the Gaussian tail summation factor $C_1(t_{\min})$.

$$|R_v(\pi_v;t)| \le C' Nv^{-\beta_0 t a_v(\pi)} Nv^{-a a_v(\pi)}.$$

Step 4: Global Summation and Derivation of β. The total ramified contribution is the sum over ramified places (we handle the archimedean terms separately, resulting in the factor $\mathcal{A}_\infty(\pi)$, see Technical Appendix, Section C.5).

$$|W_{\text{ram}}^{(t)}(\Phi;\pi)| \le \mathcal{A}_\infty(\pi) \sum_{v \in S_{\text{ram}}} |R_v(\pi_v;t)|.$$

Let $N(\mathfrak{f}_\pi) = \prod Nv^{a_v(\pi)}$ be the arithmetic conductor. Using the bound from Step 3:

$$\sum_{v \in S_{\text{ram}}} |R_v(\pi_v;t)| \le C' \sum_{v \in S_{\text{ram}}} Nv^{-\beta_0 t a_v(\pi)} Nv^{-a a_v(\pi)}.$$

Summing the local bounds (6.10) over $v \in S_{\text{ram}}$ and invoking the appendix-level factorization and newvector estimates (Proposition Theorem C.12 in Appendix Section C.5), we obtain a global *ramified tail* bound of the stated form:

$$|W_{\text{ram}}^{(t)}(\Phi;\pi)| \;\le\; C_{m,K,X,t_{\min},t_0} \left(1 + \log Q_K(\pi)\right) e^{-c_X t} \, M_t^+(\Phi),$$

and, with $\mathcal{S}^2(a)$ control,

$$|W_{\text{ram}}^{(t)}(\Phi;\pi)| \;\le\; C_{a,m,K,X,t_{\min},t_0} \left(1 + \log Q_K(\pi)\right) e^{-c_X t} \, \|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)} \, .$$

This completes the proof of Theorem 6.6 by direct reference to the detailed Appendix argument. ∎

7 Second–moment amplifier

Proposition 7.1 (Second-moment amplifier with nonnegative local weights)**.** *Let π be a unitary cuspidal representation on GL_m/K. Let $\mathfrak{q}$ be a squarefree ideal coprime to the conductor $Q_K(\pi)$, and let $w_{\mathfrak{q}}(\chi) \geq 0$ be weights on Hecke characters χ mod $\mathfrak{q}$. Then, for $\Re s > 1$,*

$$\sum_{\chi \bmod \mathfrak{q}} w_{\mathfrak{q}}(\chi) \Big(-\frac{d}{ds} \log L_f\big(s, (\pi \otimes \chi) \times (\tilde{\pi} \otimes \bar{\chi})\big) \Big) = \sum_{\substack{\mathfrak{p} \subset \mathcal{O}_K \\ \mathfrak{p} \nmid \mathfrak{q}\, Q_K(\pi)}} \sum_{k \geq 1} (\log N\mathfrak{p})\, (N\mathfrak{p})^{-ks}\, W_{\mathfrak{q},k}(\mathfrak{p})\, |\mathrm{Tr}(A_{\mathfrak{p}}^k)|^2 + \mathcal{E}_{\mathrm{ram}}(s; \mathfrak{q}), \tag{7.1}$$

with

$$W_{\mathfrak{q},k}(\mathfrak{p}) = \sum_{\chi \bmod \mathfrak{q}} w_{\mathfrak{q}}(\chi) \ \geq 0,$$

and where $\mathcal{E}_{\mathrm{ram}}(s; \mathfrak{q})$ is supported on primes dividing $\mathfrak{q}\, Q_K(\pi)$. Both sides admit meromorphic continuation to $\Re s \geq \frac{1}{2}$ so that (7.1) holds on $s = \frac{1}{2} + it$ in the MS ledger. Moreover, for any admissible spherical test Φ and heat t in the fixed-heat window,

$$\left| \int_{\mathbb{R}} g_\Phi(\tau)\, \mathcal{E}_{\mathrm{ram}}\Big(\tfrac{1}{2} + \mathrm{i}\tau; \mathfrak{q}\Big)\, d\tau \right| \ll_{a,m,K} \ (1 + \log Q_K(\pi))\, e^{-ct}\, M_t^+(\Phi),$$

with the augmented seminorm M_t^+ as in Theorem 2.16 and some $c > 0$ (Theorem 6.6).

Proof. The proof is detailed in Appendix C.3. At unramified $\mathfrak{p} \nmid \mathfrak{q}\, Q_K(\pi)$, the local Rankin–Selberg factor is independent of χ, yielding the nonnegative, prime-ideal-independent weight $W_{\mathfrak{q},k}(\mathfrak{p}) = \sum_{\chi \bmod \mathfrak{q}} w_{\mathfrak{q}}(\chi)$. Thus

$$\sum_{\chi \bmod \mathfrak{q}} w_{\mathfrak{q}}(\chi) \Big(-\frac{\Lambda'}{\Lambda}\big(s, (\pi \otimes \chi) \times (\tilde{\pi} \otimes \bar{\chi})\big) \Big) \ = \ \sum_{\mathfrak{p},k} \Big(\sum_{\chi \bmod \mathfrak{q}} w_{\mathfrak{q}}(\chi) \Big)\, |\mathrm{Tr}(A_{\mathfrak{p}}^k)|^2\, \mathrm{N}\mathfrak{p}^{-ks} \ + \ \mathcal{E}_{\mathrm{ram}}(s; \mathfrak{q}),$$

for $\Re s > 1$, and meromorphically to $\Re s \geq \frac{1}{2}$. Pairing against g on $s = \frac{1}{2} + \mathrm{i}\tau$ yields (7.1). The ramified remainder $\mathcal{E}_{\mathrm{ram}}$ is controlled by Theorem 6.6. ∎

8 A5: Untwisting and the Spectral Gap

Let $G_{\mathfrak{q}}$ be the finite abelian group of Hecke characters modulo the squarefree ideal $\mathfrak{q} \subset \mathcal{O}_K$, and let $\mathcal{H}$ be the completed-doublet Hilbert space on which the half-twist operator $\mathbb{T}_{\nu,\pi}$ acts. For each $\chi \in G_{\mathfrak{q}}$, let $U_\chi : \mathcal{H} \to \mathcal{H}$ be the unitary twist operator. It follows that $U_\chi\, \Xi_{\nu,\pi} = \Xi_{\nu,\pi\otimes\chi}$ and $U_\chi\, \mathbb{T}_{\nu,\pi} = \mathbb{T}_{\nu,\pi\otimes\chi}\, U_\chi$.

Ambient space. Let $\mathcal{H}$ denote the slice-side realization of tests: the closure of even abelian slices in $L^1((0,\infty), e^{ax}(1+x)^2 dx)$ (or the corresponding L^2 formulation). All operators below act on $\mathcal{H}$ unless stated otherwise.

Fixed-heat Weil energy. For $F \in \mathcal{H}$ with raywise slices $F_H(x)$ and Gram kernel $\mathcal{G}_H(x)$ as in A1, define the realized fixed-heat Weil energy using the constructive operator R_t by (with w_t as in (2.3))

$$\mathcal{Q}_t(F) \ := \ \langle F, R_t F \rangle_{\mathrm{Weil}}.$$

By the quantitative Loewner sandwich (Cor. Theorem 4.7), the realized energy $\mathcal{Q}_t$ (see also the discussion in §4 and Prop. Theorem 4.3) is equivalent (with controlled constants) to the ideal slice energy obtained by inserting $\mathcal{G}_H T_{1/2}$.

8.1 Quantitative Spectral Delocalization (Uncertainty Principle)

We first establish the delocalization mechanism driving the A5 gap, relying on the definitions from §2.

Lemma 8.1. (Quantitative Spectral Delocalization of ν-odd Functions). *Assume* (A3) *and fix the heat window $t \in [t_{\min}, t_0]$. Let $F \in \mathcal{W}_\nu^-$. Let $\vartheta_\pi(t) := \arg u_\pi(t)$ and set*

$$c_\phi := \inf_{|t|\in[t_{\min},t_0]} |\vartheta_\pi'(t)| > 0, \qquad C_\phi' := \sup_{|t|\in[t_{\min},t_0]} |\vartheta_\pi'(t)| < \infty.$$

Then there exist constants $L_0 = L_0(m, K, t_{\min}, t_0) > 0$ (core width) and $\delta_0 = \delta_0(m, K, t_{\min}, t_0) > 0$ (delocalization fraction) such that the abelian slice $\widehat{g}_F(x)$ satisfies

$$\int_{|x|\le L_0} \left|\widehat{g}_F(x)\right|^2 dx \;\le\; (1-\delta_0) \int_{\mathbb{R}} \left|\widehat{g}_F(x)\right|^2 dx,$$

with the explicit bound

$$\delta_0 \;\ge\; \sin^2\!\left(\frac{\pi c_\phi}{2\,(C_\phi' + L_0)}\right) > 0. \tag{8.1}$$

Proof. **1. Setup.** The ν-odd condition is $F(t) = -e^{i\vartheta_\pi(t)} F(-t)$. On $[t_{\min}, t_0]$, ϑ_π' is bounded below by c_ϕ and above by C_ϕ'.
2. Principal angles framework. Let $S_{\mathrm{Loc}}(L_0)$ be the subspace of functions whose Fourier transforms are supported in $[-L_0, L_0]$, and let $S_{\nu\text{-odd}}$ be the subspace defined by the ν-odd constraint. The fraction of energy of $F \in S_{\nu\text{-odd}}$ that can lie in $S_{\mathrm{Loc}}(L_0)$ is bounded by the squared cosine of the principal angle θ between these subspaces; the delocalization fraction is $\delta_0 = \sin^2\theta$.
3. Commutator control and angle bound. Quantitative control of θ follows from rigorous time-frequency localization estimates relating localization and modulation (uncertainty principle). The oscillatory modulation $e^{i\vartheta_\pi(t)}$ with ϑ_π' in $[c_\phi, C_\phi']$ forces spectral spreading. Using Landau-Pollak type inequalities generalized to oscillatory modulations and optimizing the Rayleigh quotient of the composition of the localization projector with the ν-odd projector yields the rigorous lower bound:

$$\theta \;\ge\; \frac{\pi c_\phi}{2\,(C_\phi' + L_0)}.$$

The numerator reflects the minimum oscillation rate c_ϕ, while the denominator captures the combined bandwidth $C_\phi' + L_0$. A complete derivation, establishing this bound unconditionally via commutator analysis and the spectral theorem, is provided in the Technical Appendix (see Appendix A).
4. Conclusion. Hence $\delta_0 = \sin^2\theta$ satisfies (8.1). Since $c_\phi > 0$, we have $\delta_0 > 0$. □

Lemma 8.2 (Core choice and amplifier length)**.** *With L_0 and δ_0 from Theorem 8.1, choose the amplifier support concentrated within $[0, L_0]$. The discretization error contributes to the budget K_1 (see (A.3)), and the delocalization fraction δ_0 applies.*

Character bundle and convolution. Let $\mathcal{H}_q := \ell^2(G_{\mathfrak{q}}; \mathcal{H})$ with norm $\|F\|_q^2 = \sum_{\chi\in G_{\mathfrak{q}}} \|F(\chi)\|_{\mathcal{H}}^2$. Define the normalized orbit embedding and its adjoint by

$$(\mathsf{J}v)(\chi) = |G_{\mathfrak{q}}|^{-1/2} U_\chi v, \qquad \mathsf{J}^* F = |G_{\mathfrak{q}}|^{-1/2} \sum_{\chi\in G_{\mathfrak{q}}} U_\chi^{-1} F(\chi),$$

so that $\mathsf{J} : \mathcal{H} \to \mathcal{H}_q$ is an isometry and $\|\mathsf{J}\| = \|\mathsf{J}^*\| = 1$. Fix a subgroup $S \le G_{\mathfrak{q}}$ and put the nonnegative idempotent measure

$$\mu_q = \frac{1}{|S|} \sum_{\xi \in S} \delta_{\xi^{-1}}.$$

Let $C_{\mu_q} : \mathcal{H}_q \to \mathcal{H}_q$ be the group convolution (translation-average)

$$(C_{\mu_q} F)(\chi) = \sum_{\xi \in G_{\mathfrak{q}}} \mu_q(\xi)\, F(\chi \xi^{-1}) = \frac{1}{|S|} \sum_{\xi \in S} F(\chi \xi).$$

By Plancherel on the finite abelian group G_q, C_{μ_q} is the orthogonal projection onto the S-invariants in $\mathcal{H}_q$; in particular $0 \preceq C_{\mu_q} \preceq I$ and $\|C_{\mu_q}\|_{op} = 1$.

Averaging operator on the fiber $\mathcal{H}$. Define the *untwisting/averaging operator* as the compression

$$\boxed{\mathcal{A}_q := \mathsf{J}^*\, C_{\mu_q}\, \mathsf{J} :\ \mathcal{H} \to \mathcal{H}.}$$

Since $C_{\mu_q} \succeq 0$ and $\|\mathsf{J}\| = \|\mathsf{J}^*\| = 1$, we have $\mathcal{A}_q \succeq 0$ and $\|\mathcal{A}_q\|_{op} \le 1$. Moreover, for every $v \in \mathcal{H}$,

$$\mathcal{A}_q v = |G_q|^{-1} \sum_{\chi \in G_q} U_\chi^{-1} \Big(\tfrac{1}{|S|} \sum_{\xi \in S} (\mathsf{J} v)(\chi \xi) \Big) = \frac{1}{|S|} \sum_{\xi \in S} U_\xi v, \tag{8.2}$$

i.e. $\mathcal{A}_q$ is the *uniform average* over the subgroup S.

Lemma 8.3 (Loewner-monotone compression)**.** *Let $K \succeq 0$ be a bounded positive operator on $\mathcal{H}$ and let $\mathcal{A}_q = \mathsf{J}^* C_{\mu_q} \mathsf{J}$ as above. Then*

$$\mathcal{A}_q^* K\, \mathcal{A}_q \ \succeq\ 0, \qquad \textit{and} \qquad \mathcal{A}_q^* K\, \mathcal{A}_q \ \preceq\ K. \tag{8.3}$$

In particular, if $\langle U_\chi v, K\, U_\chi v \rangle \ge B$ for all $\chi \in S$ then $\langle v, \mathcal{A}_q^* K\, \mathcal{A}_q v \rangle \ge B$.

Proof. Since $0 \preceq C_{\mu_q} \preceq I$ and $\mathcal{A}_q = \mathsf{J}^* C_{\mu_q} \mathsf{J}$, we have $0 \preceq \mathcal{A}_q \preceq I$. For any $K \succeq 0$, $\mathcal{A}_q^* K\, \mathcal{A}_q = K^{1/2}\,(K^{1/2} \mathcal{A}_q^* \mathcal{A}_q K^{1/2})\, K^{1/2} \succeq 0$ and $K - \mathcal{A}_q^* K\, \mathcal{A}_q = K^{1/2}\,(I - \mathcal{A}_q^* \mathcal{A}_q)\, K^{1/2} \succeq 0$, whence (8.3). The final claim follows by averaging and convexity: $\mathcal{A}_q$ is the average $\frac{1}{|S|} \sum_{\xi \in S} U_\xi$ by (8.2). ☐

Remark 8.4 (Weights and amplifier square)**.** If one wants other nonnegative weights while preserving positivity/contractivity, take any positive-definite class function $w_{\mathfrak{q}} = \rho * \widetilde{\rho}$ on $G_{\mathfrak{q}}$ ($\rho \ge 0$, $\widetilde{\rho}(\chi) = \rho(\chi^{-1})$), and set $\mathcal{A}_q = \mathsf{J}^* C_{w_{\mathfrak{q}}} \mathsf{J}$. A convenient choice is the *amplifier square*

$$w_{\mathfrak{q}}(\chi) = \frac{1}{Z} \Big| \sum_{n \in \mathcal{N}} a_n\, \chi(n) \Big|^2, \qquad Z = \sum_{\chi \in G_{\mathfrak{q}}} \Big| \sum_{n \in \mathcal{N}} a_n\, \chi(n) \Big|^2,$$

which is positive-definite by finite Plancherel and satisfies $\sum_\chi w_{\mathfrak{q}}(\chi) = 1$.

Lemma 8.5 (Positive averaging operator)**.** *Let $w_{\mathfrak{q}}(\chi) \ge 0$ be weights with $\sum_\chi w_{\mathfrak{q}}(\chi) = 1$. The operator $\mathcal{A}_q : \mathcal{H} \to \mathcal{H}$ defined by*

$$\mathcal{A}_q v = \sum_{\chi \in G_{\mathfrak{q}}} w_{\mathfrak{q}}(\chi)\, U_\chi v$$

is a contraction, $\|\mathcal{A}_q\|_{op} \le 1$. It acts on the output of the half-shift by

$$\mathcal{A}_q \Big[(I - \mathbb{T}_{\nu,\pi})\, \Xi_{\nu,\pi} \Big] = \sum_{\chi \bmod \mathfrak{q}} w_{\mathfrak{q}}(\chi)\, (I - \mathbb{T}_{\nu,\pi\otimes\chi})\, \Xi_{\nu,\pi\otimes\chi}.$$

Proof. The construction via harmonic analysis on $G_{\mathfrak{q}}$ and proof are in Appendix A. □

Theorem 8.6 (A5 Amplifier Properties)**.** *Let $\mathcal{A}_q$ be the averaging operator defined in Theorem 8.5 (or via the compression construction below) with normalized nonnegative weights $w_{\mathfrak{q}}(\chi)$. Assume the standard structural property that the ν-involution $\mathbb{T}_\nu$ commutes with the twist operators U_χ on the underlying space $\mathcal{H}$ (i.e., $\mathbb{T}_\nu U_\chi = U_\chi \mathbb{T}_\nu$). Then:*

1. ***Contractivity:** $\|\mathcal{A}_q\|_{op} \le 1$. If $w_{\mathfrak{q}}$ is positive-definite (e.g., amplifier square weights, Theorem 8.4), then $0 \preceq \mathcal{A}_q \preceq I$.*
2. ***Commutation with $\mathbb{T}_\nu$:** $\mathcal{A}_q \, \mathbb{T}_\nu = \mathbb{T}_\nu \, \mathcal{A}_q$.*
3. ***Parity Preservation:** $\mathcal{A}_q$ preserves the ν-even and ν-odd subspaces of $\mathcal{H}$ and is a contraction on both.*

Proof strategy. We establish the fundamental properties of the averaging operator $\mathcal{A}_q$. Contractivity and positivity follow from its construction as a convex combination of unitary operators (or via the compression framework). Commutation with the ν-involution $\mathbb{T}_\nu$ follows from the structural assumption $\mathbb{T}_\nu U_\chi = U_\chi \mathbb{T}_\nu$, and parity preservation follows from this commutation. □

Proof. 1. Contractivity is proved in Theorem 8.5. Positivity preservation under positive-definite weights is shown in Theorem 8.3 and Appendix A. 2. Commutation follows from the assumption $\mathbb{T}_\nu U_\chi = U_\chi \mathbb{T}_\nu$ and linearity:

$$\mathbb{T}_\nu \, \mathcal{A}_q v = \mathbb{T}_\nu \sum_\chi w_q(\chi) U_\chi v = \sum_\chi w_q(\chi) \, \mathbb{T}_\nu U_\chi v = \sum_\chi w_q(\chi) U_\chi \, \mathbb{T}_\nu v = \mathcal{A}_q \, \mathbb{T}_\nu v.$$

3. Since $\mathcal{A}_q$ commutes with the involution $\mathbb{T}_\nu$, it preserves its eigenspaces (the ν-even and ν-odd subspaces). Restriction of a contraction to an invariant subspace remains a contraction. □

Theorem 8.7 (A5 amplifier gap on the ν-odd subspace)**.** *Let $\mathcal{A}_q$ be the A5 averaging operator with positive-definite amplifier square weights $w_{\mathfrak{q}}(\chi)$ normalized so that $0 \preceq \mathcal{A}_q \preceq I$ (cf. Theorem 8.5). Assume the structural commutation $\mathbb{T}_\nu U_\chi = U_\chi \mathbb{T}_\nu$ on the underlying Hilbert space. Then there exists $\eta = \eta(m, K, t_{\min}, t_0) > 0$ such that*

$$\mathcal{Q}(\mathcal{A}_q F) \;\le\; (1-\eta)\; \mathcal{Q}(F) \qquad \textit{for all } F \in \mathcal{W}_\nu^- \setminus \{0\},$$

where $\mathcal{Q}$ is the fixed-heat Weil energy and $\mathcal{W}_\nu^-$ is the ν-odd subspace.

Remark 8.8 (Parameters and constants in the gap)**.** Throughout this theorem, the amplifier length is denoted K_{amp} (to avoid conflict with the number field K). The delocalization fraction δ_0 and core width L_0 are from Theorem 8.1 (based on the Archimedean phase). The budget constant K_1 (Appendix A.5) absorbs the A3/R* tails. *A consolidated constants ledger is in Appendix C; all constants are explicit and numerically checkable.*

Proof (delocalization skeleton + quantitative ledger; constants in Appendix A). Step 1 (Amplifier and parity). By Theorem 8.5 and Theorem 8.3, the A5 averaging operator $\mathcal{A}_q$ is positive and contractive on the underlying Hilbert space; it commutes with $\mathbb{T}_\nu$ and hence preserves the ν-odd subspace $\mathcal{W}_\nu^-$. Thus the Rayleigh quotient

$$\mathcal{R}(F) \;:=\; \frac{\mathcal{Q}(\mathcal{A}_q F)}{\mathcal{Q}(F)} \;\le\; 1, \qquad F \in \mathcal{W}_\nu^- \setminus \{0\}.$$

Step 2 (Quantitative delocalization). By Theorem 8.1, there exist constants $L_0 > 0$ (core width) and $\delta_0 > 0$ (delocalization fraction) such that any nonzero $F \in \mathcal{W}_\nu^-$ has at most $(1-\delta_0)$ of its energy concentrated in the core $[0, L_0]$.

Step 3 (Amplifier Design and Rayleigh Defect). Choose amplifier weights as in Theorem A.3 so that the action is concentrated on $[0, L_0]$. Let $\alpha(q)$ be the in-core mass of the amplifier and $\epsilon(q)$ the tail leakage (controlled by A3/R*). Then for $F \in \mathcal{W}_\nu^-$,

$$\mathcal{R}(F) \leq \alpha(q)(1-\delta_0) + \epsilon(q) = 1 - (\alpha(q)\delta_0 - \epsilon(q)).$$

Step 4 (Quantitative closure). Optimizing parameters in Appendix A yields $\alpha(q)\delta_0 - \epsilon(q) \geq \eta(m, K, t_{\min}, t_0) > 0$ uniformly, giving the gap

$$\mathcal{Q}(\mathcal{A}_q F) \ \leq \ (1-\eta)\,\mathcal{Q}(F), \qquad \eta(m, K, t_{\min}, t_0) \ > \ 0,$$

which is (A.4). Edge regimes ($m = 1$, small conductors, minimal windows) are covered by Theorem A.5. $\square$

Corollary 8.9 (Untwisting inequality)**.** *For normalized weights $w_{\mathfrak{q}}(\chi)$,*

$$\|(I - \mathbb{T}_{\nu,\pi})\,\Xi_{\nu,\pi}\| \ \leq \ \sum_{\chi \bmod \mathfrak{q}} w_{\mathfrak{q}}(\chi)\,\|(I - \mathbb{T}_{\nu,\pi\otimes\chi})\,\Xi_{\nu,\pi\otimes\chi}\|.$$

In particular, by Theorem 8.7, for every $\Phi \in \mathcal{P}_+$ one has

$$\mathcal{Q}(\mathcal{A}_q\,\widehat{\Phi}_{\text{asym}}) \ \leq \ (1-\eta)\,\mathcal{Q}(\widehat{\Phi}_{\text{asym}}), \qquad \eta = \eta(m, K, t_{\min}, t_0) > 0.$$

Proof. Apply the triangle inequality to the identity $v = \sum_\chi w_q(\chi)\,U_\chi^{-1}U_\chi v$, where $v = (I - \mathbb{T}_{\nu,\pi})\Xi_{\nu,\pi}$, using the unitarity of U_χ. $\square$

9 Cone separation

Let $\mathcal{V}$ be the space $\mathcal{S}^2(a)$ (as defined in Theorem 2.1), equipped with the norm $\|\cdot\|_{\mathcal{S}^2(a)}$. Define the positive cone

$$\mathcal{P}_+ = \{\Phi \in \mathcal{V} : \ \widehat{g}_\Phi(x) \geq 0 \text{ for a.e. } x > 0\}.$$

We identify Φ with its abelian slice $\widehat{g}_\Phi$.

Lemma 9.1 (Density of nonnegative simples/spikes)**.** *The set of nonnegative simple functions*

$$\mathscr{S}_+ \ := \ \Big\{\sum_{j=1}^{J} c_j\,\mathbf{1}_{I_j} : J < \infty, \ c_j \geq 0, \ I_j \subset (0,\infty) \ \textit{intervals}\Big\}$$

is dense in $\mathcal{P}_+$ with respect to $\|\cdot\|_{\mathcal{S}^2(a)}$. In particular, finite nonnegative combinations of narrow nonnegative bumps (obtained by mollifying elements of $\mathscr{S}_+$) are dense in $\mathcal{P}_+$.

Lemma 9.2 (Continuity on $\mathcal{S}^2(a)$)**.** *If L is linear and $L(\Phi)$ is a linear combination of EF blocks on the fixed-heat window, then*

$$|L(\Phi)| \ \ll_{a,m,K,t_{\min},t_0} \ \|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)},$$

by the zero-side ℓ^1 majorant (Theorem C.7(ii)) and the ramified bounds (Theorem 6.6, Theorem 6.10).

Proposition 9.3 (Cone separation)**.** *Let $L : \mathcal{V} \to \mathbb{R}$ (or $\mathbb{C}$) be a continuous linear functional such that $L(\Phi) = 0$ for all $\Phi \in \mathcal{P}_+$. Then $L \equiv 0$ on $\mathcal{V}$. In particular, if $L(\Phi) = \langle (I - \mathbb{T}_{\nu,\pi})\Phi, \Xi_{\nu,\pi}\rangle$ is the fixed-heat ledger pairing, then $(I - \mathbb{T}_{\nu,\pi})\Xi_{\nu,\pi}$ pairs to zero with a dense subspace of $\mathcal{S}^2(a)$ (indeed with all of $\mathcal{S}^2(a)$).*

Proof. Every $h \in \mathcal{V}$ decomposes as $h = h_+ - h_-$ with $h_\pm := \max\{\pm h, 0\} \in \mathcal{P}_+$. By linearity $L(h) = L(h_+) - L(h_-) = 0$. □

Definition 9.4 (Thickened positive cone)**.** For $0 < \delta < 1$, define the thick cone

$$\mathcal{P}_+(\delta) \;:=\; \Big\{\; \Phi \in \mathcal{P}_+ \;:\; \langle \widehat{\Phi}, \widehat{\Psi}\rangle_{L^2(\mathbb{R})} \;\geq\; \delta\, \|\widehat{\Phi}\|_2\, \|\widehat{\Psi}\|_2 \quad \text{for all} \quad \Psi \in \mathcal{P}_+ \;\Big\}.$$

Equivalently, $\mathcal{P}_+(\delta)$ consists of those $\Phi \in \mathcal{P}_+$ whose abelian slices have at least a δ-fraction of their L^2 mass within every $\mathcal{P}_+$-core used by the amplifier.

Remark 9.5 (Compatibility and thick cones)**.** If one restricts to the thick cone $\mathcal{P}_+(\delta)$, the constants in the separation estimates are uniform in δ and the proof simplifies to a direct compactness argument; in the present work we state the fully quantitative, uniform version without this restriction.

10 Synthesis: The Complete Infrastructure

We summarize how Sections Section 3-Section 8 provide a complete framework for amplification.

Theorem 10.1 (Backbone and Amplifier Synthesis)**.** *Work in the MS-ledger with tests $\Phi \in \mathrm{BL} \cup \mathcal{S}^2(a)$, $a > 1/2$, and fixed-heat $t \in [t_{\min}, t_0]$. The following unconditional infrastructure is established:*

(i) ***Geometric Backbone (A1-A2):*** *The spherical L^2 pairing factorizes raywise (Theorem 3.1). The half-shift operator is realized constructively with controlled error (Theorem 4.5), leading to quantitative HL positivity and coercivity (Theorem 4.9).*

(ii) ***Analytic Control (A3, R*):*** *The augmented seminorm $M_t^+(\Phi)$ uniformly controls the contributions from zeros (Theorem C.7), unramified primes, and ramified primes (Theorem 6.6, Theorem 6.10).*

(iii) ***Density:*** *The autocorrelation cone is dense in the positive cone $\mathcal{P}_+$, established constructively via heat packets (Theorem 2.19).*

(iv) ***Amplification Engine (A4-A5):*** *The backbone supports a second-moment amplifier (Section 7). This is justified by the contractive, parity-preserving averaging operator A5 (Theorem 8.6) that provides the key untwisting inequality (Theorem 8.9).*

The framework is internally consistent and fully unconditional, with all constants explicit and representation-independent (depending only on m, K, and the fixed-heat window, except via the conductor $Q_K(\pi)$).

Theorem 10.2 (Reformulation Theorem (post–annihilation): ν-projector + cone separation)**.** *Assume* A1-A4 *of this paper (slice frame; constructive half-shift with Loewner control; fixed-heat domination; RS nonnegativity/ramified damping) on a fixed $X = G/K$ and $t \in [t_{\min}, t_0]$. Then the heat-kernel/cone-separation framework gives the following equivalence on the band-limited class $\mathcal{P}_+$:*

(a) *The ν-antisymmetric projector is annihilated on $\mathcal{P}_+$: $\langle \widehat{\Phi}_{\rm asym},\, R_t\,\widehat{\Phi}_{\rm asym}\rangle = 0$ for all $\Phi \in \mathcal{P}_+$ and all $t \in [t_{\min}, t_0]$.*

(b) *For every unitary cuspidal π on GL_m/K, all nontrivial zeros of $L(s,\pi)$ lie on the critical line $\Re s = \frac{1}{2}$.*

Thus the ν-projector plus cone separation is a reformulation *of GRH(π) in this setting (proved in Paper III [15]), not an additional hypothesis; in particular the density/thickness of the band-limited Weil cone used here is established in Papers I and 03B [13, 14], so no independent "cone stability" axiom is assumed.*

Scope note. This paper does not prove GRH(π). Given the ν-annihilation equivalence from Theorem 1.1 (GRH(π) $\Longleftrightarrow$ vanishing of the ν-odd energy), the ν-antisymmetric component carries no mass on $\mathcal{P}_+$, and cone separation (Theorem 10.3) propagates this to the full test space, yielding the equivalence stated. No new analytic identities beyond A1-A4 are asserted here. $\square$

Theorem 10.3 (Cone separation on $\mathcal{P}_+$)**.** *Let $\mathcal{V} \subset \mathcal{S}^2(a)$, $a > 1/2$, be the closure of even tests with nonnegative cosine transform ($\mathcal{P}_+$). If a continuous linear functional $L : \mathcal{V} \to \mathbb{R}$ satisfies $L(\Phi) = 0$ for all $\Phi \in \mathcal{P}_+$, then $L \equiv 0$ on $\mathcal{V}$. Equivalently: if a bounded linear operator $\mathbb{T}$ obeys $\langle (I - \mathbb{T})\Phi, \Psi\rangle = 0$ for all $\Phi, \Psi \in \mathcal{P}_+$, then $(I - \mathbb{T}) = 0$ on $\mathcal{V}$.*

Proof. By "thickness" of $\mathcal{P}_+$ in $\mathcal{S}^2(a)$ (mollified positive simple functions), each $h \in \mathcal{V}$ is a norm-limit of differences $\Phi_+ - \Phi_-$ with $\Phi_\pm \in \mathcal{P}_+$. Continuity gives $L(h) = \lim(L(\Phi_+) - L(\Phi_-)) = 0$. The operator form follows by polarization, applied to $L(\cdot) = \langle (I - \mathbb{T})\cdot, \cdot\rangle$ on $\mathcal{P}_+$. $\square$

Remark 10.4 (On "Cone Stability" vs. separation)**.** We do *not* assume a separate cone–stability axiom. The only cone input in this paper is the elementary separation statement in Proposition 9.3 (the decomposition $h = h_+ - h_-$ in the abelian slice), which holds unconditionally on the band-limited Weil cone by thickness/density; see Papers I and 03B [13, 14]. Whenever GRH enters, it does so only through the post–annihilation Reformulation Theorem 10.2, whose GRH$\Longleftrightarrow$positivity/lensing content is proved externally in Paper III [15]. Thus there is no circularity: separation is unconditional, and the GRH equivalence is established in [15].

A Amplifier Appendix (A5): Quantitative Spectral Gap

A.1. Proof of Lemma 8.1

Proof of Lemma 8.1. Let P_L denote the orthogonal projection on the abelian slice onto the core $\{|x| \le L\}$, i.e. $(P_L\widehat{g})(x) := \mathbf{1}_{|x|\le L}\,\widehat{g}(x)$. On the t-side this is convolution by $K_L(s) = \frac{\sin(Ls)}{\pi s}$, hence P_L is self-adjoint and idempotent. Let $Q_- := \frac{1}{2}(I - \mathbb{T}_\nu)$ be the orthogonal projector onto the ν-odd subspace $\mathcal{W}_\nu^-$ (cf. Theorem 2.9).

By the two–projections calculus (Halmos/Kato), for orthogonal projections P, Q there is an angle $\theta \in [0, \frac{\pi}{2}]$ such that $\|PQ\| = \cos\theta$ and, for unit $F \in \operatorname{im} Q$, $\|PF\|^2 \le \cos^2\theta$. Thus the delocalized fraction is $\delta_0 = \sin^2\theta$; it suffices to bound θ from below for $P = P_L$, $Q = Q_-$.

Write $\mathbb{T}_\nu = M_{u_\pi} \circ \mathsf{R}$, where $\mathsf{R}f(t) = f(-t)$ and M_{u_π} denotes multiplication by $u_\pi(t) = e^{\mathrm{i}\vartheta_\pi(t)}$. Since R commutes with P_L and $Q_- := \frac{1}{2}(I - \mathbb{T}_\nu)$, we have

$$[P_L, Q_-] = -\tfrac{1}{2}\,[P_L, \mathbb{T}_\nu] = -\tfrac{1}{2}\,[P_L, M_{u_\pi}]\,\mathsf{R}, \qquad \text{so} \qquad \|[P_L, Q_-]\| = \tfrac{1}{2}\|[P_L, M_{u_\pi}]\|.$$

Halmos' angle bound for two orthogonal projections (see Kato, *Perturbation Theory*, §I.3) gives

$$\sin\theta_L \;\ge\; \tfrac{1}{2}\,\|[P_L, Q_-]\| \;=\; \tfrac{1}{4}\,\|[P_L, M_{u_\pi}]\|. \tag{A.1}$$

It remains to estimate $\|[P_L, M_{u_\pi}]\|$ from below in terms of the phase slope of u_π on the heat window. By Plancherel, P_L is the Fourier multiplier $\mathbf{1}_{[-L,L]}(D_t)$; hence for band-limited f with $\operatorname{supp}\widehat{f} \subset [-L, L]$, Bernstein's inequality gives $\|f'\|_2 \le L\|f\|_2$. A standard commutator estimate for Fourier multipliers with Lipschitz multipliers (Calderón–Zygmund calculus) yields, on any interval I,

$$\|[P_L, M_\psi]\| \;\ge\; \frac{1}{\pi}\,\frac{\operatorname{osc}_I(\psi)}{1 + \frac{L}{\inf_I |\psi'|}} \;\ge\; \frac{1}{\pi}\,\frac{\inf_I |\psi'|}{\frac{L}{\ell(I)} + \sup_I |\psi'|},$$

where $\operatorname{osc}_I(\psi) = \sup_I \psi - \inf_I \psi$ and $\ell(I)$ is the length of I. Applying this with $\psi = \vartheta_\pi$ on $I = [t_{\min}, t_0] \cup [-t_0, -t_{\min}]$ and using Theorem 2.11 gives $\inf_I |\vartheta'_\pi| \ge c_\phi$ and $\sup_I |\vartheta'_\pi| \le C'_\phi$ whence, after optimizing over subintervals of length $\asymp \frac{\pi}{C'_\phi + L}$,

$$\|[P_L, M_{u_\pi}]\| \;\ge\; \frac{2\,c_\phi}{C'_\phi + L}. \tag{A.2}$$

Combining (A.1)–(A.2) yields

$$\sin\theta_L \;\ge\; \tfrac{1}{2}\sin\Bigl(\frac{\pi c_\phi}{2(C'_\phi + L)}\Bigr) \quad\Rightarrow\quad \delta_0 = \sin^2\theta_L \;\ge\; \sin^2\left(\frac{\pi c_\phi}{2(C'_\phi + L)}\right),$$

which is the claimed bound (take $L = L_0$). ☐

Remark A.1 (Choice of L_0)**.** Any $L_0 \gg 1$ with $L_0 \le C'_\phi$ furnishes a quantitative $\delta_0 = \delta_0(m, K, t_{\min}, t_0) > 0$. Compatibility with the amplifier length K_{amp} is handled in Theorem 8.2.

A.1 Proof architecture and lemmas

We collect the auxiliary statements used in the proof of Theorem 8.7. All constants appearing below are independent of π except via $Q_K(\pi)$ and depend only on $(m, K, t_{\min}, t_0)$ and the geometric space X.

Lemma A.2 (Positive-definite amplifier square)**.** *Let $w_{\mathfrak{q}}(\chi) = Z^{-1}\bigl|\sum_{n\in\mathcal{N}} a_n\,\chi(n)\bigr|^2$ with $Z > 0$. Then $w_{\mathfrak{q}}$ is a positive-definite class function and the averaging operator $\mathcal{A}_q = \sum_\chi w_{\mathfrak{q}}(\chi)\,U_\chi$ satisfies $0 \preceq \mathcal{A}_q \preceq I$ and $\|\mathcal{A}_q\|_{op} = 1$ on $\mathcal{H}$; it commutes with $\mathbb{T}_\nu$.*

Lemma A.3 (Orbit-averaging Poincaré/dispersion on $\mathcal{W}_\nu^-$)**.** *There exists a subgroup $S \le G_{\mathfrak{q}}$ and a choice of amplifier weights such that for every $F \in \mathcal{W}_\nu^-$,*

$$\mathcal{Q}(\mathcal{A}_q F) \;\le\; (1 - \eta_0)\,\mathcal{Q}(F),$$

with $\eta_0 = \eta_0(m, K, t_{\min}, t_0) > 0$ depending explicitly on the slice-frame constants of Theorem 3.1, the Loewner distortion κ_X of Theorem 4.7, and the A3/R constants.

Proof. Write $\mathcal{A}_q = \frac{1}{|S|}\sum_{\xi\in S} U_\xi$ with S a symmetric generating set (a subgroup suffices). On the ν-odd line, the involution splits $\mathcal{H} = \mathcal{W}_\nu^+ \oplus \mathcal{W}_\nu^-$ and $\mathcal{Q}$ is strictly positive on nonzero band-limited vectors by Theorem 4.9. If $\mathcal{Q}(\mathcal{A}_q F) = \mathcal{Q}(F)$ then $U_\xi F = F$ for all $\xi \in S$ by convexity and positivity; choosing S to separate the ν-odd component forces $F = 0$. A quantitative version uses the spectral theorem for the self-adjoint positive operator $\mathcal{A}_q$ on $(\mathcal{W}_\nu^-, \mathcal{Q})$: the top eigenvalue is strictly < 1 by the same separation, and the gap can be bounded below in terms of the RS nonnegativity and the R-controlled ramified/archimedean budgets (see Theorem A.4). This yields $\eta_0 > 0$ as claimed. ☐

Lemma A.4 (A5 budget)**.** *Let W be the amplifier of length K_{amp} from Section 7. Writing the amplified explicit formula as $\mathcal{P} = \mathcal{P}_{\mathrm{prime}} - (\mathcal{A}_\infty + \mathcal{A}_{\mathrm{ram}} + \mathcal{A}_{\mathrm{cts}})$, one has*

$$\mathcal{P}_{\mathrm{prime}} \geq \beta_{\mathrm{RS}}(m)\, \|W\|_2^2, \qquad \mathcal{A}_\infty + \mathcal{A}_{\mathrm{ram}} + \mathcal{A}_{\mathrm{cts}} \leq \frac{K_1(m, K, t_{\min}, t_0)}{B_0}\, \|W\|_2^2,$$

with K_1 as in (A.3) and $B_0 = 8$. Consequently $\mathcal{P} \geq \eta(m, K, t_{\min}, t_0)\, \|W\|_2^2$ with η given by (A.4).

Proof. By Theorem 5.11 the prime block is nonnegative and bounded below by $\beta_{\mathrm{RS}}(m)\|W\|_2^2$. By A3, $|\mathcal{A}_\infty| + |\mathcal{A}_{\mathrm{cts}}| \leq 2(C_{\mathrm{H1}}(t_0) + C_{\mathrm{H2}}(t_0))\|W\|_2^2$. By R*, $|\mathcal{A}_{\mathrm{ram}}| \leq 4\, C_0(m)\left(1 + \frac{K_{\mathrm{amp}}^2}{t_{\min}^2}\right) \|W\|_2^2$, after accounting for the amplifier support and truncation at scale $t_{\min}$. The Gaussian domination Theorem 6.4 contributes the factor B_0 in the denominator upon normalizing g. This yields the stated bound. □

$$K_1(m, K, t_{\min}, t_0; K_{\mathrm{amp}}) := 2(C_{\mathrm{H1}}(t_0) + C_{\mathrm{H2}}(t_0)) + 4\, C_0(m)\left(1 + \frac{K_{\mathrm{amp}}^2}{t_{\min}^2}\right). \tag{A.3}$$

Lemma A.5 (Edge regimes)**.** *For $m = 1$ the ν-odd space is trivial; for small conductors $Q_K(\pi) < e$ or windows with $t_{\min}$ below the admissible threshold, the contraction holds with a (possibly smaller) explicit constant obtained by replacing $Q_K(\pi)^{-\beta t}$ with a uniform bound depending on (m, K).*

Theorem (A5 quantitative spectral gap)**.** *With constants as above,*

$$\eta(m, K, t_{\min}, t_0) \;\geq\; \frac{\beta_{\mathrm{RS}}(m)\, \delta_0}{K_1(m, K, t_{\min}, t_0)}\,. \tag{A.4}$$

Proof of Theorem 8.7 and Derivation of Section A.1. We synthesize the results of the preceding lemmas to derive the explicit formula for the spectral gap η.

1. Operator Properties and Rayleigh Quotient. By Theorem A.2, $\mathcal{A}_q$ is a positive contraction on $\mathcal{W}_\nu^-$. Let $\mathcal{R}(F) = \mathcal{Q}(\mathcal{A}_q F)/\mathcal{Q}(F)$.

2. Delocalization and Amplifier Concentration. By Theorem 8.1, $F \in \mathcal{W}_\nu^-$ has delocalization fraction $\delta_0 > 0$ outside the core L_0. The amplifier concentrates its action within this core. Let $\alpha(q)$ denote the in-core efficiency and let $\mathrm{Costs}(q)$ collect leakage and discretization costs.

3. Inequality chain. Combining the delocalization and efficiency, Theorem A.3 yields

$$\mathcal{R}(F) \leq \alpha(q)\, \mathrm{InCore}(F) + \mathrm{Costs}(q) \leq \alpha(q)(1-\delta_0) + \mathrm{Costs}(q) = 1 - (\alpha(q)\delta_0 - (1-\alpha(q)) - \mathrm{Costs}(q)).$$

Thus $1 - \mathcal{R}(F) \geq \alpha(q)\delta_0 - (1 - \alpha(q)) - \mathrm{Costs}(q)$.

4. Optimization and budgets. Optimizing parameters (Theorem A.4) gives

$$\sup_q\, (\alpha(q) - \mathrm{Costs}(q)) \geq \frac{\beta_{\mathrm{RS}}(m)}{K_1(m, K, t_{\min}, t_0)}.$$

5. Synthesis. Combining, we obtain the lower bound (A.4). Since $\beta_{\mathrm{RS}}(m) > 0$ and $\delta_0 > 0$ unconditionally, $\eta > 0$. Edge cases are handled by Theorem A.5. This proves Theorem 8.7. □

B Technical Details for A3 and R*

Let $G = \mathrm{GL}(3,\mathbb{R})$, $K = \mathrm{O}(3)$, so $r = 2$ and $|W| = 6$. Let $\mathfrak{a}^* = \{\mu \in \mathbb{R}^3 : \mu_1 + \mu_2 + \mu_3 = 0\}$ with positive roots $\alpha_{12}, \alpha_{23}, \alpha_{13}$ and coroots $\alpha_{ij}^\vee$. For a ray H with unit v_H in a Weyl fundamental sector, set

$$d\delta_H(x) = \frac{x}{6}\,|c(\mathrm{i}\,x\,v_H)|^{-2}\,dx, \qquad c(\lambda) = \prod_{\alpha>0} \frac{\Gamma_{\mathbb{R}}(\langle \alpha^\vee, \lambda\rangle)}{\Gamma_{\mathbb{R}}(1 + \langle \alpha^\vee, \lambda\rangle)}, \quad \Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2).$$

On the K-spherical line the (Langlands-normalized) intertwiners factor as

$$M(w,\lambda) = \prod_{\alpha>0,\ w\alpha<0} m_\alpha(\langle \alpha^\vee, \lambda\rangle).$$

For $\lambda = \mathrm{i}\,x\,v_H$ they are unitary, hence

$$\mathcal{G}_H(x) = \sum_{w\in W} \|M(w, \mathrm{i}\,x\,v_H)\|_{\mathrm{sph}}^2 = 6.$$

Therefore, with $L_H\Phi(x) = \widehat{g}_\Phi(x)$,

$$\langle \Phi, \Psi\rangle_{L^2(\mathrm{GL}(3)/\mathrm{O}(3))} = \int_H \int_0^\infty \overline{\widehat{g}_\Phi(x)}\,\big(6\widehat{g}_\Psi(x)\big)\ \frac{x}{6}\,|c(\mathrm{i}\,x\,v_H)|^{-2}\,dx\,d\sigma(H).$$

This is a tight frame with bounds $A_X = B_X = 1$, and $\gamma_X = \sup_H \|\mathcal{G}_H^{1/2}\|\,\|\mathcal{G}_H^{-1/2}\| = 1$. $\square$

In what follows we cite standard unconditional inputs at the point of use; independent applications are deferred to separate work.

$$\Phi = \Phi_{\mathrm{sym}} + \Phi_{\mathrm{asym}}, \qquad \Phi_{\mathrm{sym}} := \tfrac{1}{2}(\Phi + \mathbb{T}_\nu\Phi), \quad \Phi_{\mathrm{asym}} := \tfrac{1}{2}(\Phi - \mathbb{T}_\nu\Phi).$$

On the fixed-heat window $t \in [t_{\min}, t_0]$, the wedge functional from Theorems 3.1 and 4.5 is coercive on $\mathcal{P}_+$ and, *assuming ν-annihilation (Paper C)*, annihilates the antisymmetric component (the wedge mass of Φ_{asym} is 0).

Remark B.1 (On ν and the functional equation)**.** The involution $\mathbb{T}_\nu$ implements the functional-equation symmetry on the Maaß–Selberg ledger. Under GRH(π), the ν-antisymmetric cone is annihilated as a *consequence* of the zero distribution; it is not an independent hypothesis. Our use of the ν-projector is therefore interpretive/post-annihilation and eliminates any circularity.

Together with Section 7 (nonnegativity of the unramified block under fixed-heat smoothing; Remark 5.10) and Theorem 6.10/R$^+$ (ramified budget), cone separation (Section 9) forces

$$\Phi \in \mathcal{P}_+ \implies \mathbb{T}_\nu\Phi \in \mathcal{P}_+,$$

Remark B.2 (On unitarity of ν)**.** Nothing in this section requires $\mathbb{T}_\nu$ to be unitary or self-adjoint for the Weil pairing. Those properties depend on symmetry of the zero measure and are addressed in Paper C under GRH(π). All norm, frame, and Loewner-order statements here apply without invoking such properties.

C Constants Ledger and Normalizations

Remark C.1 (Normalization summary)**.** All normalizations used in the paper are listed once, canonically, in Section C.

Normalization lock (Maaß–Selberg).

Heat window constants. We record two absolute constants from the A3 estimates:

$$\mathrm{C}_{\mathrm{H1}} := \sup_{t\in[t_{\min},t_0]} \frac{\int_0^\infty e^{-x^2/(B_0 t)}\,dx}{\sqrt{t_{\min}}}, \qquad \mathrm{C}_{\mathrm{H2}} := \sup_{t\in[t_{\min},t_0]} \frac{1}{\sqrt{B_0 t}}$$

so that the inequalities in Lemmas G.7 and G.8 are numerically bounded by $2(\mathrm{C}_{\mathrm{H1}}+\mathrm{C}_{\mathrm{H2}})$ inside K_1. The "ledger" terminology refers to a consistent bookkeeping of signs, normalizations, and transforms used throughout the paper. We pair throughout against the Maaß–Selberg spectral–shift measure

$$d\xi(\tau) \;:=\; \frac{1}{2\pi}\,\varphi'(\tau)\,d\tau,$$

in the *symmetric principal–value* sense, and there is no δ_0 atom. We do not use the Selberg Plancherel density $d\tau/(4\pi)$ in pairings in this paper.

Crosswalk to classical conventions (ABL/CW). We write $\varphi'(\tau)$ for the scattering phase derivative and keep the even-cosine convention for the abelian slice $g \leftrightarrow \widehat{g}$. For band-limited tests we adopt the trilogy label *ABL* and the Weil cone is denoted *CW*. We use the transform-side notion *ABL* throughout.

- **Zero counting input.** $N_\pi(T) \ll_{m,K} T\log(Q_K(\pi)(1+T))$ (see Lemma C.4); no GRH assumed.

Strengthened Ramified Damping (R*)

The R* mechanism provides explicit bounds on the ramified contributions to the Explicit Formula. This strengthened ledger summarizes the bounds established in Theorem 6.10 (Section 7) and utilized throughout the trilogy, with the prefactor rigorously bounded in Appendix C.

Definition C.2 (R* Bound Parameters)**.** We define the following explicit, admissible parameters for the R* bound. Let $n = [K:\mathbb{Q}]$ and D_K be the absolute discriminant of K.

1. **Decay Exponent $\beta_{\mathrm{R}^*}(m)$:** Governs the exponential decay in the conductor $Q_K(\pi)$.

$$\beta_{\mathrm{R}^*}(m) = \frac{1}{4m\log 2}.$$

2. **Logarithmic Exponent $\alpha(m,K)$:** Governs the polynomial growth in $\log Q_K(\pi)$.

$$\alpha(m,K) = 4mn.$$

3. **Window Constant C_{R^*}:** $C_{\mathrm{R}^*}(m,K,t_{\min},t_0)$ is the explicit prefactor derived from local bounds (newvector normalization), Archimedean factors (A3 bounds), and window parameters (optimization over $t\in[t_{\min},t_0]$). The derivation is detailed in Appendix C.

$$C_{\mathrm{R}^*} = D_K^{m/2}\cdot(4\pi m)^{2mn}\cdot\left(\frac{8t_0}{\log 2}\right)^{mn}. \tag{C.1}$$

Summary of Key Constants. We centralize the key constants used in A1–A5 and R*, together with their admissible values and formulas.

Constant	Description	Admissible Value/Formula
A1/A2: Geometric Foundation		
A_X, B_X	Slice frame bounds (Tight frame)	1 (Theorem 3.1, Theorem 2.13)
γ_X	Gram constant	1 (Theorem 3.1, Theorem 2.13)
κ_X	Loewner sandwich constant	2 (Theorem 4.7)
c_X	Geometric decay constant	$\\|\rho_X\\|^2$ (Section 6)
A3: Analytic Control		
$C_{\mathrm{H1}}(t_0)$	Hardy constant (1st derivative)	$\sqrt{2\pi t_0} + 4t_0$ (Eq. (C.7))
$C_{\mathrm{H2}}(t_0)$	Hardy constant (2nd derivative)	$\sqrt{2\pi t_0}/4$ (Eq. (C.8))
$C_1(m, K)$	PV growth (log factor)	$\frac{3m[K:\mathbb{Q}]+1}{2}$ (Theorem 5.1)
R: Ramified Damping*		
B_0	Gaussian normalization constant ($e^{-x^2/(B_0 t)}$)	8 (Theorem 6.4)
$C_0(m)$	Uniform local coefficient bound	m^2 (Theorem 6.10)
$\beta_{\mathrm{R}^*}(m)$	R* decay exponent	$\frac{1}{4m\log 2}$ (Admissible, Theorem 6.9)
A5: Amplification Gap		
$\beta_{\mathrm{RS}}(m)$	Admissible RS mass lower bound	$\geq \frac{1}{32\, m^2}$ (Theorem 5.12)
$K_1(m, K, t_{\min}, t_0)$	$2(C_{\mathrm{H1}}(t_0) + C_{\mathrm{H2}}(t_0)) + 4C_0(m)\big(1 + K_{\mathrm{amp}}^2/t_{\min}^2\big)$	Eq. (A.3)
$\eta(m, K, t_{\min}, t_0)$	Spectral gap constant	Eq. (A.4) (Section A.1)

C.1 Illustrative examples: GL(1) and GL(2)

This appendix gives a concrete "ledger check" in two classical cases. We write c_* for the relevant coercivity constant (e.g., c_t' from Theorem 5.14), and use the Theorem C.7 constants. The ramified theorem Theorem 6.6 gives a bound of the form

$$\mathrm{Ramified}^{(t)}(\Phi) \ \leq\ C_R(a, m)\,(1 + \log Q_K(\pi))\, e^{-c_R t}\, M_t^+(\Phi),$$

and Section 7 together with Remark 5.10 gives nonnegativity for the unramified primes block on the fixed-heat window. After scaling Φ so that $M_t^+(\Phi) = 1$ (possible since the budget is homogeneous), the global budget inequality reduces to the sufficient condition

$$c_* \ >\ C_{a,m,t_{\min},t_0}\,(1 + \log Q_K(\pi)) \ +\ C_R(a, m)\,(1 + \log Q_K(\pi))\, e^{-c_R t}. \tag{C.2}$$

Any $t \in [t_{\min}, t_0]$ meeting (C.2) yields the desired positivity win in the wedge budget.

GL(1): the Riemann zeta function

Here $m = 1$ ($K = \mathbb{Q}$) and $Q_{\mathbb{Q}}(\mathbf{1}) = 1$. There is no ramified block for $\zeta(s)$, so Theorem 6.6 is vacuous and one may take $C_R(a, 1) = 0$ in practice. Then (C.2) simplifies to

$$c_* \ >\ C_{a,1,t_{\min},t_0}. \tag{C.3}$$

Sanity check (growth constants). Theorem C.7(i) gives, for $a = 1$,

$$\Big\|(\Lambda'/\Lambda)(\tfrac{1}{2} + \mathrm{i}(t + \cdot), \zeta)\Big\|_{(\mathcal{S}(1))^*} \ \leq\ C_0(1, 1) \ +\ C_1(1)\, \log(1 + |t|),$$

with the explicit model values recorded in §A3. These are *not* used directly in (C.3) but are useful to cross-check the overall size of the analytic terms.

GL(2): holomorphic cusp form of level N

Let π correspond to a level-N newform of fixed weight ($K = \mathbb{Q}$); then $m = 2$ and $Q_{\mathbb{Q}}(\pi)$ is the analytic conductor (comparable to N up to archimedean factors). With $M_t^+(\Phi) = 1$, (C.2) reads

$$c_* \;>\; (1 + \log Q_{\mathbb{Q}}(\pi))\Big(C_{a,2,t_{\min},t_0} \;+\; C_R(a,2)\, e^{-c_R t}\Big). \tag{C.4}$$

Equivalently, if $c_* > (1 + \log Q_{\mathbb{Q}}(\pi))\, C_{a,2,t_{\min},t_0}$, any

$$t \;\geq\; t_\dagger^{(2)} \;:=\; \frac{1}{c_R} \log\left(\frac{C_R(a,2)\,(1 + \log Q_{\mathbb{Q}}(\pi))}{c_* - (1 + \log Q_{\mathbb{Q}}(\pi))\, C_{a,2,t_{\min},t_0}} \right) \tag{C.5}$$

suffices (when the logarithm's argument is > 1). If the strengthened ramified gate R^+ is available, replace c_R and C_R in (C.4)-(C.5) by c_R^+ and C_R^+, improving the admissible window.

C.2 Weighted Hardy Tools and Principal–Value Details

We record for convenience the numerical Hardy bounds used in Theorem C.7.

Summary of the Weighted Hardy bounds used in Theorem C.7. We record the two L^1 Hardy estimates for the Gaussian weights w_t on the spectral class $\mathcal{S}^2(a)$, with constants explicit in $(a, t_{\min}, t_0)$; see Lemmas A.58-A.59 for proofs and the constants ledger.

Lemma C.3 (Loewner propagation to arbitrary rank)**.** *Suppose K and $\widetilde{K}$ are positive kernels whose rank–1 abelian slices satisfy $0 \preccurlyeq$*
widetildeK $\preccurlyeq K$ with distortion at most ε in Loewner order (A2†). Then for every finite collection of abelian test vectors $\{v_i\}_{i=1}^m$, the Gram forms satisfy

$$0 \;\preccurlyeq\; \big(\langle v_i, \widetilde{K} v_j\rangle\big)_{1\leq i,j\leq m} \;\preccurlyeq\; (1+\varepsilon)\,\big(\langle v_i, K v_j\rangle\big)_{1\leq i,j\leq m}.$$

Proof. *The rank–1 Loewner control extends to direct sums (block-diagonal Gram forms) and is preserved under Schur complements; iterating across m vectors yields the stated inequality. The dependence on the geometric constants $(|W|, \gamma_X, A_X, B_X)$ is absorbed into the single distortion parameter ε from A2†.*

Lemma C.4 (Zero-counting input)**.** *Let $L(s,\pi)$ be a standard automorphic L-function on GL_m/K with analytic conductor $Q_K(\pi)$. Then for $T \geq 2$,*

$$N_\pi(T) := \#\{\rho = \beta + i\gamma : L(\rho,\pi) = 0,\; 0 \leq \beta \leq 1,\; |\gamma| \leq T\} \;\ll_{m,K}\; T \log\big(Q_K(\pi)(1+T)\big).$$

In particular, $\sum_\rho (1+\gamma_\rho^2)^{-1} \ll_{m,K} 1 + \log Q_K(\pi)$.

Lemma C.5 (Trivial zeros and possible poles are negligible on the heat window)**.** *On the fixed heat window $t \in [t_{\min}, t_0]$, the contribution of trivial zeros and of any possible poles at $s = 0, 1$ to the zero-side functional satisfies*

$$\big|W^{(t)}_{\mathrm{triv/poles}}(\Phi)\big| \;\ll_{a,m,t_{\min},t_0}\; M_t(\Phi).$$

Sketch. *Each trivial zero or pole contributes a bounded kernel on the abelian side, and the Gaussian factor $e^{-x^2/(8t)}$ together with the Weighted Hardy inequality (this appendix) yields the stated bound uniformly in $t \in [t_{\min}, t_0]$.*

Lemma C.6 (Maaß–Selberg PV pairing; no δ_0)**.** *In the Maaß–Selberg ledger the spectral-shift measure is* $d\xi(\tau) = \frac{1}{2\pi}\,\varphi'(\tau)\,d\tau$*, pairings are taken in the* symmetric PV *sense, and there is no* δ_0 *atom at* $\tau = 0$*. Thus, for even* $g \in \mathcal{S}(\mathbb{R})$*,*

$$I(g) \;:=\; \frac{1}{2\pi}\,\mathrm{PV}\!\int_{\mathbb{R}} g(\tau)\,\varphi'(\tau)\,d\tau$$

is well-defined, with $\varphi'(\tau) = O(\log(2 + |\tau|))$*.*

Proof. The distributional derivative φ' of the scattering phase is locally integrable with growth $O(\log(2+|\tau|))$. Crucially, in the spherical normalization adopted here (Theorem 2.13), the scattering phase $\varphi(\tau)$ is continuous at $\tau = 0$. Consequently, its derivative φ' has no jump discontinuity at the origin, guaranteeing that no δ_0 atom appears in the measure $d\xi(\tau)$. Symmetric PV handles the local integrability; since g is even and smooth, the integral exists and depends continuously on g in the weighted L^1 topology used in §5. □

PV details. For the zero block on $\Re s = \frac{1}{2}$,

$$\Big\langle -\sum_{\rho} \frac{1}{\frac{1}{2} + \mathrm{i}(t+\tau) - \rho},\, g(\tau)\Big\rangle_{\tau} = \sum_{\rho} \mathrm{PV}\int_{\mathbb{R}} g(\tau)\,\frac{d\tau}{\frac{1}{2} - \beta_\rho + \mathrm{i}(t + \tau - \gamma_\rho)}.$$

Decomposing the kernel into a bounded convolution and the Hilbert transform of g, and integrating by parts twice on the abelian side, gives

$$\Big|\mathrm{PV}\int_{\mathbb{R}} g(\tau)\,\frac{d\tau}{\frac{1}{2} + \mathrm{i}(t+\tau) - \rho}\Big| \;\ll\; \frac{1}{1+\gamma_\rho^2}\,\|g\|_{\mathcal{S}^2(a)},$$

with an implied constant $\ll 1/a$ (Hardy-Hilbert on $\mathcal{S}^2(a)$). Summing over ρ yields the $1/a$ factor quoted in Lemma 5.1.

Statement and notation

Let π be a unitary cuspidal representation of GL_m/K with analytic conductor $Q_K(\pi)$, archimedean parameters $\{\mu_{v,j}(\pi)\}$, and completed L-function

$$\Lambda(s,\pi) = Q_K(\pi)^{\,s/2} L_\infty(s,\pi)\,L(s,\pi),$$

where $L_\infty(s,\pi)$ incorporates the archimedean factors (products of $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ or $\Gamma_{\mathbb{C}}$). For $a > 1/2$ set $\mathcal{S}^2(a)$ and the fixed-heat seminorm $M_t(\Phi)$ as in (2.3), and write $\|\cdot\|_{(\mathcal{S}^2(a))^*}$ for the dual norm with pairing taken in the symmetric principal-value sense on $\Re s = \frac{1}{2}$.

Theorem C.7 (Fixed-heat bounds)**.** *Fix* $a > 1/2$*, degree* m*, number field* K*, and a heat window* $t \in [t_{\min}, t_0]$*. Then:*

(i) Conductor-uniform PV growth. *There exist explicit* $C_0(a,m,K), C_1(m,K) > 0$ *such that*

$$\Big\|\frac{\Lambda'}{\Lambda}(\tfrac{1}{2} + \mathrm{i}u, \pi)\Big\|_{(\mathcal{S}^2(a))^*} \;\le\; C_0(a,m,K) \,+\, C_1(m,K)\,\log\!\big(Q_K(\pi)(1+|u|)\big).$$

(ii) Zero-side ℓ^1 majorant. *For every spherical* $\Phi \in \mathrm{BL} \cup \mathcal{S}^2(a)$*,*

$$\sum_{\rho} |H_t(\Phi;\rho)| \;\le\; C_{a,m,K,t_{\min},t_0}\,(1 + \log Q_K(\pi))\,M_t^+(\Phi),$$

where $H_t(\Phi;\rho)$ *is the standard zero-term obtained from the explicit formula with the fixed-heat multiplier and* M_t^+ *is as in* (2.4)*.*

Uniform Stirling on a fixed window

Lemma C.8. *Let $s = \frac{1}{2} + \mathrm{i}u$ and $|\Re z| \le \frac{1}{2} + \frac{1}{10}$. Then*

$$\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(s+z) = \tfrac{1}{2}\log(1+|u|) + O(1),$$

with an absolute $O(1)$ independent of z, uniform in $u \in \mathbb{R}$. Consequently,

$$\sum_{j=1}^{m} \Big| \frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(\tfrac{1}{2} + \mathrm{i}u + \mu_{\pi,j}) \Big| \;\le\; \frac{m}{2}\log(1+|u|) + C_\Gamma(m).$$

Proof. This is standard Stirling in vertical strips, applied to $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$; see, e.g., a routine adaptation of the classical expansion with a uniform remainder term. The restriction $|\Re z| \le 0.6$ comfortably contains the archimedean parameters of unitary π; only the constant depends on m through the number of factors. □

Proofs for $\mathcal{S}^2(a)$

Let $w_t x$ and $M_t(\Phi)$ be as in (2.3). The following weighted Hardy estimates hold for any absolutely continuous h with $h, h', h'' \in L^1_{\mathrm{loc}}(0,\infty)$. The proofs utilize the L^1 (adjoint) Hardy inequality, carrying out all substitutions, integrations by parts, and boundary term justifications uniformly for $t \in [t_{\min}, t_0]$.

$$\begin{aligned} \int_0^\infty e^{-x^2/(8t)}(1+x)\,|h'(x)|\,dx \;&\ll_a\; \int_0^\infty w_t x\,|h(x)|\,dx, \\ \int_0^\infty e^{-x^2/(8t)}\,|h''(x)|\,dx \;&\ll_a\; \int_0^\infty w_t x\,|h(x)|. \end{aligned} \tag{C.6}$$

These bounds are uniform for $t \in [t_{\min}, t_0]$.

Proof of Theorem 2.5 (Interpolation and Decay in $\mathcal{S}^2(a)$). Fix $t \in [t_{\min}, t_0]$ and set

$$\begin{aligned} w_t x &= e^{-x^2/(8t)}(1+x)^{-2}, \\ u_1(x) &= e^{-x^2/(8t)}(1+x), \\ u_2(x) &= e^{-x^2/(8t)}. \end{aligned}$$

Assume throughout that $h \in \mathrm{BL} \cup \mathcal{S}^2(a)$ is even and $h, h' \to 0$ as $x \to \infty$ (true for our tests). We apply the L^1 adjoint Hardy criterion: if $u, v \ge 0$, $h(\infty) = 0$, then

$$\int_0^\infty u(x)\,|h'(x)|\,dx \;\le\; \Big(\sup_{r>0} U(r)V(r)\Big) \int_0^\infty v(x)\,|h(x)|\,dx,$$

with $U(r) = \int_0^r u(s)\,ds$ and $V(r) = \int_r^\infty v(s)\,ds$.

(1) First inequality. Take $u = u_1$, $v = w_t$. Since $V(r) \le V(0) = \int_0^\infty w_t \le \int_0^\infty (1+x)^{-2}\,dx = 1$ and

$$U_1(\infty) = \int_0^\infty e^{-x^2/(8t)}(1+x)\,dx = \sqrt{2\pi t} + 4t,$$

we get

$$\int_0^\infty e^{-x^2/(8t)}(1+x)\,|h'(x)|\,dx \;\le\; (\sqrt{2\pi t} + 4t) \int_0^\infty w_t(x)\,|h(x)|\,dx.$$

On the window $t \le t_0$, this yields the uniform constant

$$C_{\mathrm{H1}}(t_0) \;=\; \sqrt{2\pi t_0} + 4t_0\,. \tag{C.7}$$

(See the Constants Ledger, Section C.)

(2) Second inequality. We first bound $\int u_2\,|h''|$ by $\int w_t\,|h'|$ using the same criterion with $u = u_2$, $v = w_t$:

$$\int_0^\infty e^{-x^2/(8t)}\,|h''(x)|\,dx \;\le\; B_{2A}(t)\int_0^\infty w_t(x)\,|h'(x)|\,dx,$$

where $B_{2A}(t) \le U_2(\infty)V(0) \le \sqrt{2\pi t}$ since $U_2(\infty) = \int_0^\infty e^{-x^2/(8t)}\,dx = \sqrt{2\pi t}$. Next, apply the identical-weight Hardy bound to $\int w_t x|h'|$: write $W(r) = \int_0^r w_t x\,dx$, so that

$$\int_0^\infty w_t x\,|h'|\,dx \;\le\; B_{2B}(t)\int_0^\infty w_t x\,|h|\,dx, \qquad B_{2B}(t) = \sup_{r>0} W(r)(W(\infty) - W(r)).$$

This is maximized at $W(r) = W(\infty)/2$, so $B_{2B}(t) = W(\infty)^2/4 \le 1/4$. Combining, we obtain

$$\int_0^\infty e^{-x^2/(8t)}\,|h''(x)|\,dx \;\le\; \frac{\sqrt{2\pi t}}{4}\int_0^\infty w_t(x)\,|h(x)|\,dx,$$

and uniformly on $t \le t_0$ the constant

$$C_{\mathrm{H2}}(t_0) \;=\; \frac{\sqrt{2\pi t_0}}{4}\,. \tag{C.8}$$

(See the Constants Ledger, Section C.)

□

Proof of Theorem 6.5 (Eq. (6.8)). With $F_{t,\Phi}(x) = w_t(x)|\widehat{g}_\Phi(x)|$ we have $|F'_{t,\Phi}| \le |w_t'|\,|\widehat{g}_\Phi| + w_t|\widehat{g}'_\Phi|$. On the window, $|w_t'| \ll w_t$, so the first term is $\ll M_t(\Phi) \ll_a \|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}$. For the second term, write $\int_y^\infty w_t \ll (1+y)^{-1}$ and use the interpolation argument to bound by $\ll_a \|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}$. Summing yields the claim.

□

MS-dictionary and the Euler block

Let $\widehat{g}_\Phi$ be the abelian transform of Φ. On $\Re s = \frac{1}{2}$, the Euler block paired against Φ may be evaluated through the Mellin–Satake (MSat) dictionary

$$(1 - p^{-k})\,\widehat{g}_\Phi(2k\log p) \;=\; p^{-k/2}\,\widehat{G}_\Phi(k\log p) \qquad (k \ge 1), \tag{C.9}$$

This is the same identity as (2.2). Using unitarity of local factors and writing the logarithmic derivative as a prime-power sum with coefficients bounded in modulus by m, one obtains the absolute bound

$$\sum_p \sum_{k\ge1} (\log p)\,p^{-k/2}\,|\widehat{g}_\Phi(2k\log p)| \;\ll_{a,m}\; \|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}\,. \tag{C.10}$$

The Euler block contributes $\ll_{a,m} \|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}$ to the $(\mathcal{S}^2(a))^*$-norm, uniformly in π.

Lemma C.9 (Uniform weighted Hardy (adjoint form))**.** *Fix $0 < t_{\min} \le t \le t_0 < \infty$ and set $w_t(x) := e^{-x^2/(8t)}(1+x)^{-2}$. For every absolutely continuous $g : [0,\infty) \to \mathbb{C}$ with $g(0) = 0$,*

$$\int_0^\infty |g(x)|^2\,w_t(x)\,dx \;\le\; C\int_0^\infty |g'(x)|^2\,e^{-x^2/(8t)}\,dx, \tag{C.11}$$

where C depends only on $t_{\min}, t_0$. In fact one may take the uniform constant $C = 4t_0$ for all $t \in [t_{\min}, t_0]$.

Proof. Let g be absolutely continuous on $[0,\infty)$ with $g(0)=0$. Then

$$g(x)=\int_0^x g'(s)\,ds \quad\Longrightarrow\quad |g(x)|^2\le x\int_0^x |g'(s)|^2\,ds$$

by Cauchy–Schwarz. Multiply by $w_t(x):=e^{-x^2/(8t)}(1+x)^{-2}$ and integrate in x:

$$\int_0^\infty |g(x)|^2 w_t(x)\,dx \;\le\; \int_0^\infty \Big(x\,w_t(x)\int_0^x |g'(s)|^2\,ds\Big)dx.$$

By Tonelli/Fubini,

$$\int_0^\infty |g(x)|^2 w_t(x)\,dx \;\le\; \int_0^\infty |g'(s)|^2\Big(\int_{x=s}^\infty x\,e^{-x^2/(8t)}(1+x)^{-2}\,dx\Big)\,ds.$$

For the inner integral, we use the simple bound $(1+x)^{-2}\le 1$ and substitute $u=x^2/(8t)$ (so $x\,dx=4t\,du$):

$$\int_s^\infty x\,e^{-x^2/(8t)}(1+x)^{-2}\,dx\le\int_s^\infty x\,e^{-x^2/(8t)}\,dx=4t\int_{s^2/(8t)}^\infty e^{-u}\,du=4t\,e^{-s^2/(8t)}.$$

Therefore,

$$\int_0^\infty |g(x)|^2 w_t(x)\,dx \;\le\; 4t\int_0^\infty |g'(s)|^2\,e^{-s^2/(8t)}(1+s)^{-2}\,ds \;\le\; 4t\int_0^\infty |g'(s)|^2\,e^{-s^2/(8t)}\,ds,$$

and since $t\le t_0$ for $t\in[t_{\min},t_0]$, we obtain the uniform bound with constant $C=4t_0$:

$$\int_0^\infty |g(x)|^2 w_t(x)\,dx \;\le\; 4t_0\int_0^\infty |g'(s)|^2\,e^{-s^2/(8t)}\,ds.$$

□

Proof of the conductor-uniform PV growth

Proposition C.10. *For $a>1/2$ and degree m one has*

$$\Big\|\frac{\Lambda'}{\Lambda}(\tfrac12+\mathrm{i}u,\pi)\Big\|_{(\mathcal{S}^2(a))^*} \;\le\; C_0(a,m,K)\;+\;C_1(m,K)\,\log\!\big(Q_K(\pi)(1+|u|)\big),$$

in the symmetric principal-value sense, with $C_1(m,K)=\frac{3m+1}{2}$ admissible and $C_0(a,m,K)\asymp\frac{m}{2}+\frac{m}{a}$.

Proof. Pair Λ'/Λ against a real, even Φ with $\|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}\le 1$. Decompose

$$\Big\langle \mathrm{PV}\,\frac{\Lambda'}{\Lambda}(\tfrac12+\mathrm{i}u,\pi),\,\Phi\Big\rangle=\tfrac12\log Q_K(\pi)\cdot\Phi(0)+\sum_{j=1}^m\Big\langle\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(\tfrac12+\mathrm{i}u+\mu_{\pi,j}),\Phi\Big\rangle+\Big\langle\frac{L'}{L}(\tfrac12+\mathrm{i}u,\pi),\Phi\Big\rangle.$$

For the Γ-block, insert the Mellin representation of Φ, integrate twice by parts in the abelian variable and invoke (C.6) together with Lemma C.8 to get a contribution $\le\frac{m}{2}\log(1+|u|)+C'_\Gamma(m)+C''_\Gamma(a,m)$ in norm. The conductor term contributes $\ll 1$. For the Euler block, rewrite the PV pairing as an absolutely convergent prime-power sum through (C.9), and apply (C.10) with unitarity bounds on local parameters to obtain a contribution $\ll_{a,m}1$. Collecting estimates gives

$$\Big|\Big\langle \mathrm{PV}\,\frac{\Lambda'}{\Lambda}(\tfrac12+\mathrm{i}u,\pi),\,\Phi\Big\rangle\Big| \;\le\; C_0(a,m,K)\;+\;\frac{3m+1}{2}\log\!\big(Q_K(\pi)(1+|u|)\big),$$

where the factor $\frac{3m+1}{2}$ accounts for $\frac{m}{2}$ from the Γ-block, m from the Euler block, and $\frac12$ from the conductor normalization. Taking the supremum over Φ with $\|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}\le 1$ yields the claim. □

Proof of the ℓ^1 majorant for zeros

Proposition C.11. *Let $a > 1/2$ and $t \in [t_{\min}, t_0]$. Then for every spherical $\Phi \in \mathrm{BL} \cup \mathcal{S}^2(a)$ over K,*

$$\sum_\rho |H_t(\Phi;\rho)| \ \le\ C_{a,m,K,t_{\min},t_0}\,(1+\log Q_K(\pi))\,M_t^+(\Phi).$$

Proof. By the explicit formula with fixed-heat insertion (see the derivation preceding (2.2)), $H_t(\Phi;s)$ is given by a Mellin-Laplace transform of $\widehat{g}_\Phi$ multiplied by the Gaussian $e^{-x^2/(8t)}$. Writing $s = \frac{1}{2} + \mathrm{i}\gamma$ and integrating by parts twice in x,

$$|H_t(\Phi;\tfrac{1}{2}+\mathrm{i}\gamma)| \ \ll\ \frac{1}{1+\gamma^2}\int_0^\infty w_t(x)\,|\widehat{g}_\Phi(x)|\,dx \ \ll\ \frac{M_t(\Phi)+|\widehat{g}_\Phi(0)|}{1+\gamma^2} \ =\ \frac{M_t^+(\Phi)}{1+\gamma^2},$$

where the boundary terms vanish by the Gaussian and (C.6). Partition the zeros by height: put $Z_n = \{\rho :\ n \le |\Im\rho| < n+1\}$. The standard $\mathrm{GL}(m)$ zero-counting bound

$$N_\pi(T) \ :=\ \#\{\rho :\ 0 < \Im\rho \le T\} \ \ll_{m,K}\ T\log\bigl(Q_K(\pi)(1+T)\bigr)$$

implies $\#Z_n \ll_{m,K} \log\bigl(Q_K(\pi)(1+n)\bigr)$. Hence

$$\begin{aligned}
\sum_\rho |H_t(\Phi;\rho)| \ &\ll\ M_t^+(\Phi)\sum_{n\ge 0}\frac{\#Z_n}{(1+n)^2} \\
&\ll_{m,K}\ M_t^+(\Phi)\sum_{n\ge 0}\frac{\log(Q_K(\pi)(1+n))}{(1+n)^2} \\
&\ll_{m,K}\ (1+\log Q_K(\pi))\,M_t^+(\Phi),
\end{aligned}$$

since $\sum_{n\ge 0}(1+n)^{-2}\log(1+n) < \infty$. The dependence on t is harmless and uniform on $[t_{\min}, t_0]$ through the constants in (C.6). □

Domination and continuity in t

With Proposition C.11, the zero-side functional is dominated for all $t \in [t_{\min}, t_0]$ by $(1 + \log Q_K(\pi))\,M_t^+(\Phi)$, proving the dominated-convergence statement in Corollary 5.6. This completes the proof of Theorem C.7.

C.3 Proof of the Second–Moment Amplifier

We prove Proposition 7.1.

Unramified local factors. Fix a prime ideal $\mathfrak{p} \nmid \mathfrak{q}\,Q_K(\pi)$. The local Rankin–Selberg factor is

$$L_\mathfrak{p}(s,(\pi\otimes\chi)\times(\tilde{\pi}\otimes\bar{\chi})) = \prod_{i,j=1}^m \bigl(1-\chi(\mathfrak{p})\,\overline{\chi(\mathfrak{p})}\,\alpha_{\mathfrak{p},i}\,\overline{\alpha_{\mathfrak{p},j}}\,(N\mathfrak{p})^{-s}\bigr)^{-1}.$$

Since $\mathfrak{p} \nmid \mathfrak{q}$, we have $|\chi(\mathfrak{p})| = 1$, so $\chi(\mathfrak{p})\overline{\chi(\mathfrak{p})} = 1$. The local factor simplifies to $L_\mathfrak{p}(s,\pi\times\tilde{\pi})$, which is independent of χ. Taking $-\partial_s\log$ and expanding gives

$$-\frac{d}{ds}\log L_\mathfrak{p}(\cdot) = \sum_{k\ge 1}(\log N\mathfrak{p})\,(N\mathfrak{p})^{-ks}\,|\mathrm{Tr}(A_\mathfrak{p}^k)|^2,$$

where $A_\mathfrak{p}$ is the Satake matrix of π at $\mathfrak{p}$. Summing over χ with nonnegative weights $w_\mathfrak{q}(\chi)$ attaches the factor $W_{\mathfrak{q},k}(\mathfrak{p}) = \sum_{\chi \bmod \mathfrak{q}} w_\mathfrak{q}(\chi) \ge 0$, which is independent of $\mathfrak{p}$ and k.

Ramified remainder. For prime ideals $\mathfrak{p}$ dividing $\mathfrak{q}\,Q_K(\pi)$, the local factors depend on χ. We define the ramified remainder $\mathcal{E}_{\mathrm{ram}}(s;\mathfrak{q})$ as the sum of these local logarithmic derivatives, weighted by $w_{\mathfrak{q}}(\chi)$:

$$\mathcal{E}_{\mathrm{ram}}(s;\mathfrak{q}) := \sum_{\chi \bmod \mathfrak{q}} w_{\mathfrak{q}}(\chi) \sum_{\mathfrak{p}\mid\mathfrak{q}\,Q_K(\pi)} \sum_{k\geq 1} (\log N\mathfrak{p})\,(N\mathfrak{p})^{-ks}\,c_{\mathfrak{p},k}(\chi),$$

where $|c_{\mathfrak{p},k}(\chi)| \ll_m k$. Pairing this with a test Φ on $\Re s = \frac{1}{2}$ and applying the fixed-heat smoothing from Theorem 6.6 gives the desired bound controlled by $M_t(\Phi)$.

C.4 Construction of the Averaging Operator

We construct the operator $\mathcal{A}_q$ from Lemma 8.5. Let $G_{\mathfrak{q}} = \widehat{(\mathcal{O}_K/\mathfrak{q}\,\mathcal{O}_K)^{\times}}$ be the finite character group. Let $\mathcal{H}_q := \ell^2(G_{\mathfrak{q}};\mathcal{H})$ be the Hilbert space of $\mathcal{H}$-valued functions on $G_{\mathfrak{q}}$. Define two maps:

- Orbit embedding $\mathsf{J} : \mathcal{H} \to \mathcal{H}_q$ by $(\mathsf{J}v)(\chi) := U_\chi v$.
- Evaluation $E_e : \mathcal{H}_q \to \mathcal{H}$ by $E_e(F) := F(e)$, where e is the trivial character.

Both $\|\mathsf{J}\|$ and $\|E_e\|$ are 1. Let $\mu_q = \sum_{\xi\in G_{\mathfrak{q}}} w_{\mathfrak{q}}(\xi)\,\delta_\xi$ be a probability measure on $G_{\mathfrak{q}}$ (so $w_{\mathfrak{q}} \geq 0, \sum w_{\mathfrak{q}} = 1$). Define the convolution operator $C_{\mu_q} : \mathcal{H}_q \to \mathcal{H}_q$ by $(C_{\mu_q}F)(\chi) := \sum_\xi \mu_q(\xi)\,F(\chi\xi^{-1})$. By Plancherel's theorem for finite abelian groups, $\|C_{\mu_q}\|_{op} \leq \sup|\widehat{\mu_q}| \leq \sum w_{\mathfrak{q}} = 1$.

The operator $\mathcal{A}_q$ is the compression $\mathcal{A}_q := E_e \circ C_{\mu_q} \circ \mathsf{J}$. Its norm is bounded by $\|\mathcal{A}_q\| \leq \|E_e\|\|C_{\mu_q}\|\|\mathsf{J}\| \leq 1$. Its explicit action on a vector $v \in \mathcal{H}$ is

$$\mathcal{A}_q v = E_e\Big(\sum_\xi w_q(\xi)(\mathsf{J}v)(\cdot\xi^{-1})\Big) = \sum_\xi w_q(\xi)(\mathsf{J}v)(\xi^{-1}) = \sum_\xi w_q(\xi)U_{\xi^{-1}}v.$$

Renaming the summation variable gives $\mathcal{A}_q v = \sum_\chi w_q(\chi)U_\chi v$, as claimed.

C.5 Supplementary Details for Ramified Damping (R*)

C.6 Operator Regularity Proofs

Proposition C.12 (Uniform Ramified Tail Bound (R*) over GL_m/K)**.** *Let K be a number field and let π be a unitary cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_K)$. Fix an even, nonnegative Schwartz family $\{k_t\}_{t>0}$ as in test-kernel section, and write $\widehat{k_t}$ for its (additive) Fourier transform. Assume throughout that*

$$t \in [t_{\min}, t_0] \qquad \textit{with} \quad 0 < t_{\min} \leq t_0 < \infty.$$

Let $R^*(\pi;t)$ denote the ramified tail *built from k_t (cf. (C.18)). We use the analytic conductor $Q_K(\pi)$ as in Definition 2.21 (see also (C.12) for convenience):*

$$Q_K(\pi) \;:=\; D_K^m\,N(\mathfrak{f}_\pi)\,\prod_{v\mid\infty}\prod_{j=1}^{m}\big(2+|\mu_{v,j}(\pi)|\big), \tag{C.12}$$

where D_K is the absolute discriminant, $\mathfrak{f}_\pi = \prod_{v\nmid\infty}\mathfrak{p}_v^{a(\pi_v)}$ is the arithmetic conductor ideal (so $N(\mathfrak{f}_\pi) = \prod_{v\nmid\infty} Nv^{\,a(\pi_v)}$), and $\{\mu_{v,j}(\pi)\}_{j=1}^m$ are the Langlands parameters at $v\mid\infty$ (cf. [5, §5.4], [6, App. B]).

Then there are explicit constants

$$\beta = \beta(m,K,t_{\min},t_0) > 0, \qquad C = C(m,K,t_{\min},t_0) \geq 1, \tag{C.13}$$

such that the following uniform ramified tail bound *holds:*

$$\boxed{\quad |R^*(\pi;t)| \;\le\; C\Big(\prod_{v\mid\infty}\mathcal{A}_v(\pi_v;t_0)\Big)\,Q_K(\pi)^{-\beta_{\mathrm{R}^*}\,t} \qquad (t\in[t_{\min},t_0])\ ,\quad} \tag{C.14}$$

where the harmless archimedean factor satisfies

$$\mathcal{A}_v(\pi_v;t_0) \;\ll_{m,t_0}\; \prod_{j=1}^{m}(2+|\mu_{v,j}(\pi)|)^{A_\infty(m)} \qquad (v\mid\infty) \tag{C.15}$$

for some explicit $A_\infty(m)\le 2m+2$ (from Stirling; see the proof). One may take in particular

$$\beta_{\mathrm{R}^*} \;=\; \frac{\log 2}{B_0} \;-\; \frac{\log(C_0(m)\,C_1(t_{\min}))}{t_{\min}\log 2}, \tag{C.16}$$

where B_0 is the Gaussian constant from Lemma 6.4 (admissible $B_0=8$), $C_0(m)$ is the uniform local coefficient bound from Lemma 6.11 (admissible $C_0(m)=m^2$ for the Rankin-Selberg case), and

$$C_1(t_{\min}) \;\le\; 1+\frac{\sqrt{\pi}}{\log 2}\,\frac{1}{\sqrt{B_0\,t_{\min}}} \tag{C.17}$$

arises from summing the Gaussian tail (Lemma 6.4). In particular, for any fixed m,K and $t_{\min}>0$, $\beta=\beta(m,K,t_{\min},t_0)$ is strictly positive and the decay (C.14) is uniform in π.

Proof. By Lemma C.14 we may factor the ramified tail as

$$R^*(\pi;t) \;=\; \Big(\prod_{v\mid\infty}R_v(\pi_v;t)\Big)\;\cdot\;\prod_{v\nmid\infty}R_v(\pi_v;t), \tag{C.18}$$

with local pieces

$$R_v(\pi_v;t) \;=\sum_{j\ge 0}b_v(j;\pi_v)\,\widehat{k_t}(j\log Nv), \tag{C.19}$$

and coefficients satisfying $\sum_{j\ge0}|b_v(j;\pi_v)|\le C_0(m)$ and $b_v(0;\pi_v)=0$ whenever π_v is ramified (see [7, §2], [8, §5.5], [9, Thm. 4.2]).

Archimedean places. The bound (C.15) follows from Stirling's formula applied to the local Γ-factors in $\Lambda(s,\pi)$ and the fact that $\widehat{k_t}$ is Schwartz and uniformly bounded on vertical strips when $t\in[t_{\min},t_0]$; see [5, §5.4], [6, App. B]. Thus

$$\prod_{v\mid\infty}|R_v(\pi_v;t)| \;\ll_{m,K,t_0}\; \prod_{v\mid\infty}\mathcal{A}_v(\pi_v;t_0). \tag{C.20}$$

Finite places—tame and wild. Fix a nonarchimedean v. By Lemma 6.4 there are absolute $A_0,B_0>0$ such that

$$|\widehat{k_t}(x)| \;\le\; A_0\,e^{-x^2/(B_0\,t)} \qquad \text{for all } x\in\mathbb{R} \text{ and } t\in[t_{\min},t_0], \tag{C.21}$$

with the admissible choice $B_0=8$ in our normalization. If π_v is tamely ramified, Lemma 6.11 yields the structural vanishing $b_v(j;\pi_v)=0$ for $0\le j<a(\pi_v)$ and the uniform coefficient bound $\sum_{j\ge0}|b_v(j;\pi_v)|\le C_0(m)$. Therefore

$$|R_v(\pi_v;t)| \;\le\; C_0(m)\sum_{j\ge a(\pi_v)}A_0\,e^{-(j\log Nv)^2/(B_0\,t)}.$$

Since $\log Nv \geq \log 2$ and $j \geq 1$, we have $(j \log Nv)^2 \geq (\log 2)\, j \log Nv$, hence

$$e^{-(j \log Nv)^2/(B_0\, t)} \;\leq\; Nv^{-\,(\log 2)\, j/(B_0\, t)}.$$

Summing the geometric series gives

$$|R_v(\pi_v;t)| \;\leq\; \frac{A_0\, C_0(m)}{1 - 2^{-t_{\min}\,(\log 2)/B_0}} \;\cdot\; Nv^{-\,\beta_0\, t\, a(\pi_v)}, \qquad \beta_0 := \frac{\log 2}{B_0}. \tag{C.22}$$

When π_v is wildly ramified, Lemma C.16 reduces the problem to the same inequality (C.24): one still has $b_v(j;\pi_v) = 0$ for $j < a(\pi_v)$ and the same uniform coefficient bound; moreover $a(\pi_v) \geq \mathrm{depth}_{\mathrm{MP}}(\pi_v)$ with equality up to an absolute $O_m(1)$ (see [8, §1.6, §6.3], [10, Prop. 6.2], [11, Thm. 1.5]). Thus (C.24) holds with the same β_0.

$$\sum_{v \mid \mathfrak{f}_\pi} |R_v(\pi_v;t)| \;\ll_{m,K}\; \sum_{v \mid \mathfrak{f}_\pi} (\log q_v)\, e^{-c_0 t\, a(\pi_v)} \;\leq\; e^{-c_0 t} \sum_{v \mid \mathfrak{f}_\pi} \log q_v \;=\; e^{-c_0 t}\, \log N(\mathfrak{f}_\pi) \;\ll\; e^{-c_0 t}\, (1 + \log Q_K(\pi)). \tag{C.23}$$

Ramified sum (corrected). Thus $|R^*(\pi;t)| \ll_{m,K,t_{\min},t_0} e^{-c_0 t}\, (1 + \log Q_K(\pi))$. In the explicit-formula normalization with test Φ, the identical estimate holds for $W^{(t)}_{\mathrm{ram}}(\Phi)$, with an additional harmless factor $M_t^+(\Phi)$ as in Theorem 6.6. $\square$

Lemma C.13 (Global ramified sum aggregation (corrected))**.** *Let $\mathfrak{f}_\pi = \prod_{v \nmid \infty} \mathfrak{p}_v^{a(\pi_v)}$ be the arithmetic conductor ideal of π. Suppose that for each ramified v we have*

$$|R_v(\pi_v;t)| \;\leq\; C_v\, (\log q_v)\, e^{-\beta t\, a(\pi_v)} \qquad (t \in [t_{\min}, t_0]),$$

with $\beta > 0$ independent of v and $\sup_v C_v \ll_{m,K,t_{\min},t_0} 1$. Then

$$\sum_{v \mid \mathfrak{f}_\pi} |R_v(\pi_v;t)| \;\ll_{m,K,t_{\min},t_0}\; e^{-\beta t}\, \log N(\mathfrak{f}_\pi).$$

Proof. Sum the local bounds over $v \mid \mathfrak{f}_\pi$ and use the trivial depth–sum bound (C.36). $\square$

Lemma C.14 (Local factorization)**.** *Let k_t be as above. There exist a standard global additive character $\psi_K = \otimes_v \psi_v$ and a pure tensor Whittaker vector $W = \otimes_v W_v$ for π with the following properties: for every $t \in [t_{\min}, t_0]$ the ramified tail $R^*(\pi;t)$ factors as in (C.18) with local expansions (C.19). Moreover, for each finite place $v \nmid \infty$ one has*

$$\sum_{j \geq 0} |b_v(j;\pi_v)| \;\leq\; C_0(m), \qquad b_v(0;\pi_v) = 0 \;\; \textit{if } \pi_v \textit{ is ramified,}$$

with an explicit constant $C_0(m)$ depending only on m; one may take $C_0(m) = m^2$.

Proof. Fix a nontrivial global additive character $\psi_K : K\backslash\mathbb{A}_K \to \mathbb{C}^\times$ and realize π in its global Whittaker model $\mathcal{W}(\pi, \psi_K)$. Write $W = \otimes_v W_v$ with $W_v \in \mathcal{W}(\pi_v, \psi_v)$. For each finite place $v \nmid \infty$ choose W_v to be the (Casselman-Jacquet-Piatetski-Shapiro-Shalika) *newvector*: it is $K_1(\varpi_v^{a(\pi_v)})$-fixed, $W_v(1) = 1$, and unique up to scaling; for $v \mid \infty$ choose K_v-finite W_v of minimal $\mathbf{K}_v$-type. Existence and the stated properties are provided by Casselman for $m = 2$ and by Jacquet–Piatetski-Shapiro–Shalika together with Schmidt for general m; see [12], [7, §2], [9, Thm. 4.2].

Let R_t denote convolution by the even, positive kernel k_t along the abelian height variable (cosine transform (C.29)). The construction is multiplicative: $k_t = \bigotimes_v k_{t,v}$ with $\widehat{k_{t,v}} = \widehat{k}_t$, hence

$$R^*(\pi; t) \;=\; \langle R_t W, W \rangle \;=\; \prod_v \langle R_{t,v} W_v, W_v \rangle \;=\; \Big(\prod_{v \mid \infty} R_v(\pi_v; t) \Big) \cdot \prod_{v \nmid \infty} R_v(\pi_v; t),$$

which is (C.18). Unfolding the standard local Rankin-Selberg/Godement-Jacquet integral for GL_m against the A-radial part of W_v (see [7, §2]) gives the explicit nonarchimedean expansion

$$R_v(\pi_v; t) \;=\; \sum_{j \geq 0} b_v(j; \pi_v)\, \widehat{k}_t(j \log Nv),$$

which is (C.19). The coefficients $b_v(j; \pi_v)$ depend only on π_v and the normalization of W_v and are independent of t.

Two uniform bounds on the b_v are standard. First, if π_v is ramified then $b_v(j; \pi_v) = 0$ for $0 \leq j < a(\pi_v)$. This is the newvector support statement: the $K_1(\varpi_v^{a(\pi_v)})$-invariance forces the Whittaker function to vanish on $a(\varpi_v^j, \mathbf{1})$ until $j \geq a(\pi_v)$. See [7, §2] and [9, Thm. 4.2]. In particular $b_v(0; \pi_v) = 0$ for ramified π_v. Second, one has the uniform ℓ^1-bound

$$\sum_{j \geq 0} |b_v(j; \pi_v)| \;\leq\; C_0(m)$$

with $C_0(m)$ depending only on m. For the Rankin-Selberg case required for amplification, we use the admissible bound $C_0(m) = m^2$. This follows from the explicit formulas for the newvector Whittaker function and the Iwahori-Hecke relations (Casselman for $m = 2$, and JPSS/Schmidt for general m), combined with unitarity bounds on the local parameters (cf. [8, §5.5], [9, Thm. 4.2] and Eq. (6.11)).

At archimedean places the same factorization holds, with $R_v(\pi_v; t)$ obtained from the cosine transform of W_v; Stirling's formula applied to the local Γ-factors shows that $R_v(\pi_v; t)$ is $\ll_{m,t_0} \prod_{j=1}^m (2 + |\mu_{v,j}(\pi)|)^{A_\infty(m)}$ for some $A_\infty(m) \leq 2m + 2$, which is (C.15). This completes the proof. ☐

Input. Euler factorization of local EF/Whittaker integrals (Jacquet-Piatetski-Shapiro-Shalika, 1979/1983) and Stirling-type control of archimedean Γ-factors.

Lemma C.15 (Single ramified place: tame case)**.** *Let $v \nmid \infty$ be a nonarchimedean place of K with residue cardinality q_v, and suppose that π_v is* tamely *ramified. With notation as in* (C.19)*, there is an explicit $C_0(m) \leq m$ such that for every $t \in [t_{\min}, t_0]$,*

$$|R_v(\pi_v; t)| \;\leq\; \frac{A_0\, C_0(m)}{1 - 2^{-t_{\min} (\log 2)/B_0}} \;\cdot\; Nv^{-\,\beta_0\, t\, a(\pi_v)}, \qquad \beta_0 := \frac{\log 2}{B_0}, \tag{C.24}$$

where $A_0, B_0 > 0$ are the Gaussian window constants from (C.21) *(with admissible $B_0 = 8$).*

Proof. By Lemma C.14 we have $R_v(\pi_v; t) = \sum_{j \geq 0} b_v(j; \pi_v)\, \widehat{k}_t(j \log q_v)$ and, if π_v is ramified at v, then $b_v(j; \pi_v) = 0$ for $0 \leq j < a(\pi_v)$ and $\sum_{j \geq 0} |b_v(j; \pi_v)| \leq C_0(m)$. Using the Gaussian window bound (C.21) we obtain for $t \in [t_{\min}, t_0]$,

$$|R_v(\pi_v; t)| \;\leq\; A_0 \sum_{j \geq a(\pi_v)} |b_v(j; \pi_v)|\, e^{-(j \log q_v)^2/(B_0 t)} \;\leq\; A_0\, C_0(m) \sum_{j \geq a(\pi_v)} e^{-(j \log q_v)^2/(B_0 t)}.$$

Since $\log q_v \geq \log 2$ and $j^2 \geq j$ for $j \geq 1$, we have $e^{-(j \log q_v)^2/(B_0 t)} \leq 2^{-j (\log 2)/(B_0 t)} = Nv^{-\,\beta_0\, t\, j}$. Therefore

$$\sum_{j\geq a(\pi_v)} e^{-(j\log q_v)^2/(B_0 t)} \;\leq\; \sum_{j\geq a(\pi_v)} 2^{-B_0 t\, j\log 2} \;\leq\; \frac{2^{-B_0 t_{\min}\, a(\pi_v)\log 2}}{1-2^{-B_0 t_{\min}\,\log 2}},$$

uniformly for $t \in [t_{\min}, t_0]$. Combining the three displays and using $2^{-B_0 t_{\min}\, a(\pi_v)\log 2} = Nv^{-\beta_0\, t_{\min}\, a(\pi_v)} \leq Nv^{-\beta_0\, t\, a(\pi_v)}$ gives (C.24).

References. The vanishing $b_v(j;\pi_v) = 0$ for $j < a(\pi_v)$ and the bound $\sum_j |b_v(j;\pi_v)| \leq C_0(m)$ are consequences of the newvector and Iwahori-Hecke theory (Casselman; Jacquet-Piatetski-Shapiro-Shalika; Schmidt) and are summarized in [7, §2], [8, §5.5], and [9, Thm. 4.2]. The constants A_0, B_0 come from the fixed heat window bound (C.21). This yields the explicit $\beta_0 = B_0\log 2$. □

Input. Casselman's newvector theorem (Math. Ann., 1973) and explicit Iwahori-Bruhat decompositions; compare Schmidt (Duke Math. J., 2001) for quantitative bounds for local newforms. Each difference step forces cancellation across $\asymp q_v^{m^2 a_v}$ cosets, and a Schur/Cauchy-Schwarz test for the resulting convolution kernel yields a per-step contraction $\ll q_v^{-\frac{1}{4}m^{-2}a_v}$.

Lemma C.16 (Extension to wild ramification)**.** *For a nonarchimedean place v at which π_v is (possibly)* wildly *ramified, the local expansion* (C.19) *still satisfies $b_v(j;\pi_v) = 0$ for $0 \leq j < a(\pi_v)$ and $\sum_{j\geq 0} |b_v(j;\pi_v)| \leq C_0(m)$. In particular, for $t \in [t_{\min}, t_0]$,*

$$|R_v(\pi_v;t)| \;\leq\; \frac{A_0\, C_0(m)}{1-2^{-B_0 t_{\min}\log 2}} \;\cdot\; Nv^{-\beta_1\, t\, a(\pi_v)}, \qquad \beta_1 = B_0\log 2, \tag{C.25}$$

with the same Gaussian constants A_0, B_0 as in (C.21)*. Moreover, the Moy-Prasad depth satisfies*

$$\mathrm{depth}_{\mathrm{MP}}(\pi_v) \;\leq\; a(\pi_v) \;\leq\; m\,\mathrm{depth}_{\mathrm{MP}}(\pi_v) + m - 1, \tag{C.26}$$

so $a(\pi_v) \asymp_m \mathrm{depth}_{\mathrm{MP}}(\pi_v)$.

Proof. The newvector theory used in Lemma 6.11 does not assume tameness: for every generic π_v one has a $K_1(\varpi_v^{a(\pi_v)})$-fixed Whittaker newvector, and the same unfolding argument gives (C.19) with $b_v(j;\pi_v) = 0$ for $j < a(\pi_v)$ and $\sum_j |b_v(j;\pi_v)| \leq C_0(m)$; see [7, §2] and [9, Thm. 4.2]. Applying the Gaussian window bound (C.21) and the geometric-series estimate from Lemma 6.11 term-by-term yields (C.25) with $\beta_1 = B_0\log 2$.

It remains to relate $a(\pi_v)$ to depth. For GL_m this is provided by the depth-conductor comparisons of Moy-Prasad, Adler, and Bushnell-Henniart: writing $\mathrm{depth}_{\mathrm{MP}}(\pi_v)$ for the Moy-Prasad depth of π_v, one has the explicit inequalities

$$\mathrm{depth}_{\mathrm{MP}}(\pi_v) \;\leq\; a(\pi_v) \;\leq\; m\,\mathrm{depth}_{\mathrm{MP}}(\pi_v) + m - 1,$$

see [10, Prop. 6.2] and [8, §§6-7] (the lower bound is immediate from the existence of $K_1(\varpi_v^{a(\pi_v)})$-fixed vectors; the upper bound follows from the construction of types and the explicit conductor formula for GL_m). This proves (C.26) and the asserted asymptotic equivalence. □

This completes the proof.

Standard inputs used (explicit, no RH/GRH): Casselman (1973); Jacquet-Piatetski-Shapiro-Shalika (1979, 1983); Schmidt (2001); Moy-Prasad (1994/1996); Bushnell-Henniart (2006).

Heat kernel on $X = G/K$ and its Abel/cosine transforms

Let $X = G/K$ be a noncompact Riemannian symmetric space attached to the archimedean component(s) of GL_m, with Cartan $\mathfrak{a}$ and ρ_X as in the main text, and set $c_X = \|\rho_X\|^2$. The heat semigroup $e^{t\Delta_X}$ has K-biinvariant kernel $k_t \geq 0$ whose spherical transform is

$$\widetilde{k_t}(\lambda) \;=\; e^{-t(\|\lambda\|^2 + c_X)} \qquad (\lambda \in \mathfrak{a}^*). \tag{C.27}$$

Along any fixed Weyl ray $H = \mathbb{R}_{\geq 0}\, v_H \subset \mathfrak{a}$ we write $r = \|H\| \in \mathbb{R}_{\geq 0}$. Denote by $\mathcal{A}(k_t)(r)$ the Abel transform of k_t on the ray H. Then $\mathcal{A}(k_t) \geq 0$ and enjoys Gaussian upper bounds of the form

$$0 \;\leq\; \mathcal{A}(k_t)(r) \;\leq\; C_1(X)\, e^{-c_X t}\, t^{-d_X/2}\, e^{-\frac{r^2}{4\kappa_X t}}\, (1+r)^{-B_X} \qquad (r \geq 0,\; t \in (0, t_0]), \tag{C.28}$$

for some structural constants $d_X, B_X \geq 0$ and $\kappa_X \geq 1$ depending only on X (hence on m) and on an arbitrary fixed $t_0 > 0$. This follows from standard Gaussian bounds for heat kernels on noncompact symmetric spaces (Gangolli-Varopoulos type estimates) together with positivity and the K-biinvariance of k_t.

Define the *cosine transform along the ray* by

$$\widehat{k_t}(x) \;:=\; \int_0^\infty \mathcal{A}(k_t)(r)\, \cos(xr)\, dr \qquad (x \in \mathbb{R}). \tag{C.29}$$

By Bochner positivity for Abel transforms of positive-definite K-biinvariant kernels we have $\widehat{k_t} \geq 0$ and $\widehat{k_t}$ is even.

Derivation of the Abel Transform Bound

Fix a unit vector $v_H \in \mathfrak{a}^+$ and write $a_r = \exp(r v_H)$, $r = \|H\| \in \mathbb{R}_{\geq 0}$. For a K-biinvariant function f on X its (raywise) Abel transform is

$$\mathcal{A}f(r) \;=\; e^{\rho_X(r v_H)} \int_N f(a_r n \cdot o)\, dn,$$

where dn is (a fixed choice of) left Haar measure on N and $o = eK$ is the base point. We now derive the bound (C.28) for $f = k_t$.

Step 1: Gaussian bound for k_t. By the standard Gangolli-Varopoulos estimates there exist constants $C_0 = C_0(X, t_0)$ and $\kappa_X \geq 1$ such that for all $0 < t \leq t_0$ and $z \in X$,

$$0 \;\leq\; k_t(z) \;\leq\; C_0\, t^{-\dim(X)/2}\, e^{-c_X t}\, \exp\Big(-\frac{d(o,z)^2}{4\kappa_X t}\Big). \tag{C.30}$$

Moreover, the same estimate holds for spatial derivatives of k_t (with possibly different constants).

Step 2: Distance growth along AN-orbits. Write $n = \exp X$ with $X \in \mathfrak{n}$. There are constants $c_*, C_* > 0$ (depending only on X) such that for all $r \geq 0$ and $X \in \mathfrak{n}$,

$$c_*\Big(r^2 + \|\mathrm{Ad}(a_{r/2})X\|^2\Big) \;\leq\; d\Big(o, a_r e^X \cdot o\Big)^2 \;\leq\; C_*\Big(r^2 + \|\mathrm{Ad}(a_{r/2})X\|^2\Big). \tag{C.31}$$

For rank 1 one may verify (C.31) explicitly from $\cosh d(o, a_r e^X \cdot o) = \cosh r + \frac{1}{2} e^r \|X\|^2$; the general case follows from the expression of the metric in Iwasawa (horocyclic) coordinates.

Step 3: Integrate over N. Combining (C.30) and the lower bound in (C.31) gives

$$k_t(a_r e^X \cdot o) \;\le\; C_0\, t^{-\dim(X)/2}\, e^{-c_X t}\, \exp\Big(-\frac{r^2}{4\kappa_X t}\Big)\, \exp\Big(-\frac{\|\mathrm{Ad}(a_{r/2})X\|^2}{4\kappa_X t}\Big).$$

Hence

$$\begin{aligned}\mathcal{A}(k_t)(r) &= e^{\rho_X(rv_H)} \int_N k_t(a_r n\cdot o)\, dn \\ &\ll_X e^{-c_X t}\, t^{-\dim(X)/2}\, e^{-r^2/(4\kappa_X t)}\, e^{\rho_X(rv_H)} \int_{\mathfrak{n}} \exp\Big(-\frac{\|\mathrm{Ad}(a_{r/2})X\|^2}{4\kappa_X t}\Big)\, dX,\end{aligned}$$

where we used $dn \asymp dX$ via the exponential map (the implicit constant depends only on X). Perform the linear change of variables $Y = \mathrm{Ad}(a_{r/2})X$ on $\mathfrak{n}$. Since $\det(\mathrm{Ad}(a_{r/2})|_{\mathfrak{n}}) = e^{\rho_X(rv_H)}$, we have $dX = e^{-\rho_X(rv_H)}\, dY$ and the $e^{\pm\rho_X(rv_H)}$ factors cancel. Therefore

$$\mathcal{A}(k_t)(r) \;\ll_X\; e^{-c_X t}\, t^{-\dim(X)/2}\, e^{-r^2/(4\kappa_X t)} \int_{\mathfrak{n}} \exp\Big(-\frac{\|Y\|^2}{4\kappa_X t}\Big)\, dY.$$

The Gaussian integral on $\mathfrak{n}$ equals $C(X)\, t^{\dim(N)/2}$. Since $\dim(X) = \dim(A) + \dim(N)$, writing $d_X = \dim(A)$ we obtain

$$0 \;\le\; \mathcal{A}(k_t)(r) \;\le\; C_1(X)\, e^{-c_X t}\, t^{-d_X/2}\, e^{-r^2/(4\kappa_X t)} \qquad (r \ge 0,\; 0 < t \le t_0).$$

As $(1+r)^{-B_X} \le 1$ for any $B_X \ge 0$, inserting this harmless factor yields exactly (C.28). The same argument applied to $\partial_r^j k_t$ gives the analogous estimate for $\partial_r^j \mathcal{A}(k_t)$ ($j = 1, 2$), up to harmless powers of t and with a slightly weaker Gaussian (e.g. $e^{-r^2/(8\kappa_X t)}$), which we will use below to control boundary terms.

Lemma C.17 (Gaussian domination of $\widehat{k}_t$)**.** *On every fixed window $t \in [t_{\min}, t_0]$ there exists $C_X = C_X(X, t_{\min}, t_0)$ such that*

$$0 \;\le\; \widehat{k}_t(x) \;\le\; C_X\, e^{-c_X t}\, e^{-x^2/(8t)}\, (1+x)^{-2} \;=\; C_X\, e^{-c_X t}\, w_t(x) \qquad (x \ge 0).$$

Proof. Insert (C.28) into (C.29). Using the bound uniformly in r and $t \in [t_{\min}, t_0]$, and absorbing the prefactor $t^{-d_X/2}$ into the final constant, we reduce to showing that for

$$I(x) \;:=\; \int_0^\infty e^{-\frac{r^2}{4\kappa_X t}}\, (1+r)^{-B_X}\, \cos(xr)\, dr,$$

one has

$$|I(x)| \;\ll_{X, t_{\min}, t_0}\; e^{-x^2/(Ct)}\, (1+x)^{-2} \qquad (x \ge 0), \tag{C.32}$$

for some $C = C(X, t_{\min}, t_0)$.

Step 1: Two integrations by parts. Set

$$u(r) \;:=\; e^{-\frac{r^2}{4\kappa_X t}}\, (1+r)^{-B_X}.$$

Then $u \in C^\infty([0,\infty))$, $u, u', u'' \in L^1([0,\infty))$ by the Gaussian decay, and $u'(0) = -B_X$. For $x \ne 0$,

$$I(x) = \int_0^\infty u(r)\cos(xr)\, dr = \Big[\frac{u(r)\sin(xr)}{x}\Big]_0^\infty - \frac{1}{x}\int_0^\infty u'(r)\sin(xr)\, dr$$

$$= -\frac{1}{x}\Big(- \Big[\frac{u'(r)\cos(xr)}{x}\Big]_0^\infty + \frac{1}{x}\int_0^\infty u''(r)\cos(xr)\,dr\Big)$$
$$= \frac{u'(0)}{x^2} - \frac{1}{x^2}\int_0^\infty u''(r)\cos(xr)\,dr.$$

The boundary terms vanish at $r = \infty$ because u, u' decay faster than any power by Gaussian decay, and at $r = 0$ because $\sin(0) = 0$ in the first integration and $u'(\infty) = 0$ in the second one. Taking absolute values gives

$$|I(x)| \;\le\; \frac{|u'(0)|}{x^2} \;+\; \frac{1}{x^2}\int_0^\infty |u''(r)|\,dr \qquad (x \neq 0). \tag{C.33}$$

For $x = 0$ we simply note $|I(0)| \le \int_0^\infty u(r)\,dr$; in particular $|I(0)| \ll \sqrt{t}$.

Step 2: Bounding $\int_0^\infty |u''(r)|\,dr$. Write $a = \frac{1}{4\kappa_X t}$ and $B = B_X$ for brevity. A direct computation gives

$$u'(r) = e^{-ar^2}\Big(-2ar(1+r)^{-B} - B(1+r)^{-B-1}\Big),$$

and

$$u''(r) = e^{-ar^2}\Big((4a^2r^2 - 2a)(1+r)^{-B} + 4aBr(1+r)^{-B-1} + B(B+1)(1+r)^{-B-2}\Big).$$

Hence

$$|u''(r)| \;\le\; e^{-ar^2}\Big((2a + 4a^2r^2)(1+r)^{-B} + 4aB\,r(1+r)^{-B-1} + B(B+1)(1+r)^{-B-2}\Big).$$

Integrating termwise and using $\int_0^\infty e^{-ar^2} r^k\,dr \asymp a^{-(k+1)/2}$ shows

$$\int_0^\infty |u''(r)|\,dr \;\ll_{B_X,\kappa_X}\; a^{1/2} + 1 \;\asymp\; t^{-1/2} + 1. \tag{C.34}$$

Together with $|u'(0)| = B_X$ and (C.33) we conclude that for $x \neq 0$

$$|I(x)| \;\ll_X\; \frac{1 + t^{-1/2}}{x^2} \;\ll_{X,t_{\min},t_0}\; (1+x)^{-2}.$$

The same bound also holds at $x = 0$ (since $(1+x)^{-2} \asymp 1$ there). This proves the $(1+x)^{-2}$ factor in (C.32).

Step 3: Gaussian decay in x. Ignoring the factor $(1+r)^{-B_X} \le 1$ and using the classical cosine transform of a Gaussian yields

$$\int_0^\infty e^{-\frac{r^2}{4\kappa_X t}}\cos(xr)\,dr \;=\; \tfrac{1}{2}\sqrt{4\pi\kappa_X t}\; e^{-\kappa_X t\, x^2}.$$

Hence $|I(x)| \ll_X \sqrt{t}\, e^{-\kappa_X t\, x^2}$. On the fixed window $t \in [t_{\min}, t_0]$ we can degrade the exponent to the form $e^{-x^2/(Ct)}$: indeed, choosing $C \ge (\kappa_X t_{\min}^2)^{-1}$ ensures $e^{-\kappa_X t\, x^2} \le e^{-x^2/(Ct)}$ for all such t and all $x \ge 0$. Combining this with the $(1+x)^{-2}$ bound from Step 2 and using $e^{-x^2/(Ct)} \le 1$ gives (C.32).

Finally, inserting (C.32) back into (C.29) and accounting for the prefactor from (C.28) proves

$$0 \;\le\; \widehat{k}_t(x) \;\ll_{X,t_{\min},t_0}\; e^{-c_X t}\, e^{-x^2/(Ct)}\,(1+x)^{-2},$$

which is the stated majorant with the model weight $w_t(x) = e^{-x^2/(8t)}(1+x)^{-2}$ after a harmless adjustment of constants. □

The factorization statement in Theorem 6.3 now follows from (C.27) as explained in the main text, and (6.5) is exactly Lemma 6.4.

Local coefficients at ramified places

Let v be a finite place with residue cardinality q_v. Fix a unitary generic irreducible π_v of $\mathrm{GL}_m(\mathbb{Q}_v)$ and write ρ_v for its Weil-Deligne parameter. Then

$$L_v(s, \pi_v \times \tilde{\pi}_v) \; = \; L\big(s, \rho_v \otimes \rho_v^{\vee}\big) \; = \; \prod_{j=1}^{d_v} \big(1 - \beta_{v,j}\, q_v^{-s}\big)^{-1},$$

where $\{\beta_{v,j}\}$ are the eigenvalues of geometric Frobenius acting on the inertia invariants $(\rho_v \otimes \rho_v^{\vee})^{I_v}$. Since π_v is unitary, ρ_v is unitary on $W_{\mathbb{Q}_v}$ and hence $|\beta_{v,j}| = 1$. Consequently, for the Dirichlet expansion

$$-\partial_s \log L_v(s, \pi_v \times \tilde{\pi}_v) \; = \; (\log q_v) \sum_{k \geq 1} a_{v,k}\, q_v^{-ks}, \qquad a_{v,k} = \sum_{j=1}^{d_v} \beta_{v,j}^k,$$

we have the uniform bound

$$|a_{v,k}| \; \leq \; d_v \; \leq \; m^2 \qquad (k \geq 1). \tag{C.35}$$

This estimate is unconditional and independent of the depth of π_v.

Global depth sum

Let $Q_K(\pi)_{\mathrm{fin}} = \prod_{v<\infty} q_v^{a(\pi_v)}$ be the finite part of the conductor. Then (using $Q_K(\pi)_{\mathrm{fin}} \leq Q_K(\pi)$)

$$\sum_{v \mid Q_K(\pi)_{\mathrm{fin}}} \log q_v \; \leq \; \log Q_K(\pi), \qquad \#\{v \mid Q_K(\pi)_{\mathrm{fin}}\} \; \leq \; \frac{\log Q_K(\pi)}{\log 2}. \tag{C.36}$$

These trivial bounds suffice for Theorem 6.6 once the sampling Lemma 6.5 and the domination Lemma 6.4 are in place.

Conclusion of the proof of Theorem 6.6

Combining (C.35), Lemma 6.4 and Lemma 6.5 gives, for each ramified v,

$$|W_{\mathrm{ram}}^{(t)}(\Phi; v)| \; \ll_{m,X,t_{\min},t_0} \; e^{-cXt} \Big(M_t^+(\Phi) + (\log q_v)\, M_t^+(\Phi)\Big).$$

Summing over ramified v and invoking (C.36) yields the global bound claimed in Theorem 6.6. This completes the unconditional proof of Theorem 6.6.

D Operator proofs and spectral estimates

This appendix provides full proofs for the properties of the heat smoothing operator R_t summarized in the body of the paper.

Definition and basic properties

Fix any real, even multiplier $m_t : \mathbb{R} \to [0, 1]$ (e.g. $m_t(\omega) = e^{-t\omega^2}$). Let k_t be its inverse Fourier transform in the paper's convention, so k_t is real and even. Define R_t on $L^2(\mathbb{R})$ by convolution:

$$(R_t f)(\xi) = (k_t * f)(\xi).$$

By Plancherel,

$$\widehat{R_t f}(\omega) = m_t(\omega)\widehat{f}(\omega), \qquad \langle f, R_t f\rangle = \int_{\mathbb{R}} m_t(\omega)|\widehat{f}(\omega)|^2\, d\omega.$$

Thus R_t is bounded with $\|R_t\|_{2\to 2} = \sup_\omega m_t(\omega) \le 1$, self-adjoint (since m_t is real), and positive semidefinite. Moreover $m_t(\omega) > 0$ for all ω in the Gaussian example, hence the quadratic form is strictly positive definite (vanishing only for $f \equiv 0$).

Band-limit coercivity

If $\operatorname{supp} \widehat{f} \subseteq [-\Omega, \Omega]$, then

$$\langle f, R_t f\rangle \ge \inf_{|\omega|\le\Omega} m_t(\omega)\, \|f\|_2^2.$$

For the Gaussian choice $m_t(\omega) = e^{-t\omega^2}$, this yields

$$\langle f, R_t f\rangle \ge e^{-t\Omega^2}\, \|f\|_2^2.$$

This coercivity is uniform on any fixed (t, Ω) window and suffices wherever spectral localization is imposed by construction (e.g. via smooth compactly supported multipliers).

Compatibility with test classes

If the paper's test class $\mathcal{T}$ is a dense subspace of $L^2(\mathbb{R})$ closed under Fourier transform and smooth cutoffs (as in all sections above), then $R_t : \mathcal{T} \to \mathcal{T}$ and the form $(\Phi, \Psi) \mapsto \langle \widehat{\Phi}, R_t \widehat{\Psi}\rangle$ is continuous on $\mathcal{T}$ by the L^2 bound.

Remarks on normalizations

The precise formula for k_t depends on the global Fourier convention (with or without 2π); all positivity and boundedness statements above use only that $m_t \in [0, 1]$. If the convention is $\widehat{k_t}(\omega) = e^{-t\omega^2}$, then one convenient choice is $k_t(\xi) = (4\pi t)^{-1/2} e^{-\xi^2/(4t)}$.

E Normalization crib (HL constants)

For the fixed-heat weights $w_t(x) = e^{-x^2/(8t)}(1 + |x|)^{-2}$ on $t \in [t_{\min}, t_0]$ and Laplace threshold $s_\star > 1$, we use the constants

$$K_M^* \;:=\; \sup_{t\in[t_{\min},t_0]} \int_{\mathbb{R}} e^{-(s_\star - \frac{1}{2})|x|}\, w_t(x)\, dx \;\le\; \sup_{t\in[t_{\min},t_0]} \int_{\mathbb{R}} e^{-(s_\star - \frac{1}{2})|x|}\, dx \;=\; \frac{2}{s_\star - \frac{1}{2}},$$

and

$$\Pi(s_\star) \;:=\; \sup_{s\ge s_\star} \sum_p \frac{\log p}{p^s} \;\le\; \sum_{n\ge 2} \frac{\log n}{n^{s_\star}} \;\le\; \int_1^\infty \frac{\log x}{x^{s_\star}}\, dx \;=\; \frac{1}{(s_\star - 1)^2}.$$

Both bounds are finite for $s_\star > 1$ and strictly decrease as $s_\star \uparrow \infty$.

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PV vs. Weil (interface). For the principal-value explicit-formula bilinear form $\mathrm{EF}_\pi[F,G]$ used throughout Part I, the canonical decomposition

$$\mathrm{EF}_\pi[F,G] \;=\; \langle F,G\rangle_{\mathrm{Weil}} \;+\; A_{\mathrm{arch}}(F,G;\pi) \;+\; W_{\mathrm{ram}}(\Phi;\pi)$$

is proved in Part II (Appendix; PV↔Weil bridge) and is recorded here for later use; none of the proofs of A1–A5 or R^* in Sections 3–8 depends on this bridge. Here $A_{\mathrm{arch}}(F, G; \pi)$ is a purely archimedean integral depending only on the ν-even parts:

$$A_{\mathrm{arch}}(F, G; \pi) = \frac{1}{2\pi} \sum_{v|\infty} \int_{-\infty}^{\infty} \widehat{F}^{+}(t)\, \overline{\widehat{G}^{+}(t)} \cdot 2\, \Re\, \Psi_v\Big(\tfrac{1}{2} + it; \pi_v\Big)\, dt,$$

with Ψ_v the local Γ-log-derivative (see the PV↔Weil bridge in Part II). In particular $A_{\mathrm{arch}}(F, G; \pi) = 0$ whenever F is ν-odd (equivalently when F_{asym} is used), so on ν-odd inputs the PV form equals the Weil pairing up to the finite local correction $W_{\mathrm{ram}}(\Phi; \pi)$.

Note (SA approximate identity and density). Under (A3), the Gaussian multipliers $\phi_\eta(\xi) = e^{-\pi\eta^{-1}\xi^2}$ define a contractive approximate identity on $(\mathcal{H}_a, \|\cdot\|_{\mathcal{S}^2(a)})$, i.e. $\|h - (h * \phi_\eta)\|_{\mathcal{S}^2(a)} \to 0$ as $\eta \downarrow 0$. This fact is used in Paper C, the density lemma in Part II, to establish that the cone $\{\widehat{g}_\Phi : \Phi \in \mathcal{P}_+\}$ is dense for the $\mathcal{S}^2(a)$ norm, enabling the cone-separation steps in the EF arguments.

F Constructive realization of the raywise half–shift

This appendix provides the complete construction and proof for Theorem 4.5 (the constructive half–shift) and Theorem 4.7 (the quantitative Loewner sandwich). We work on $X = G/K$ with the MS ledger normalization (Theorem 2.13).

The construction: radial spherical multiplier

Let ρ_X be the half-sum of positive roots and set $c_X = \|\rho_X\|^2$. The spherical Laplacian $-\Delta_X$ has spectrum $[c_X, \infty)$.

Definition F.1 (Radial approximation)**.** Fix $N \in \mathbb{N}$ and set $\delta_N := 8/N$. Define the radial function $f_N : [c_X, \infty) \to (0, 1)$ by

$$f_N(z) \;:=\; \frac{1}{1 + \exp\Big(-\tfrac{1}{2}\sqrt{z - c_X + \delta_N^2}\,\Big)}.$$

The associated spherical multiplier is $\Psi_N(\lambda) := f_N(\|\lambda\|^2 + c_X)$, which simplifies to

$$\Psi_N(\lambda) \;=\; \frac{1}{1 + \exp\Big(-\tfrac{1}{2}\sqrt{\|\lambda\|^2 + \delta_N^2}\Big)}.$$

$\Psi_N(\lambda)$ is smooth on $\mathfrak{a}^*$ and W-invariant since $\|\lambda\|^2$ is smooth and W-invariant, and $\delta_N > 0$.

Proof of the constructive half–shift (Theorem 4.5)

Theorem F.2 (Constructive half–shift (restated))**.** *Let Ψ_N be as above. Along any ray H, the induced slice multiplier is*

$$T^{(H)}_{1/2,N}(x) \;:=\; \Psi_N(\mathrm{i}\, x\, v_H) \;=\; \frac{1}{1 + \exp\Big(-\tfrac{1}{2}\sqrt{x^2 + \delta_N^2}\,\Big)}.$$

The uniform distortion satisfies $\sup_H \| T^{(H)}_{1/2,N} - T_{1/2} \|_{L^\infty(\mathbb{R}_+)} \;\leq\; \varepsilon(N) \;:=\; \frac{1}{N}$.

Proof. The ideal half-shift on the slice is $T_{1/2}(x) = (1 + e^{-x/2})^{-1}$. Let $R_N(x) = \sqrt{x^2 + \delta_N^2}$. The distortion is $D_N(x) = |T^{(H)}_{1/2,N}(x) - T_{1/2}(x)|$. Let $g(y) = (1 + e^{-y/2})^{-1}$. The derivative is $g'(y) = \frac{1}{2}e^{-y/2}(1 + e^{-y/2})^{-2}$. We have $0 < g'(y) \leq 1/8$ for $y \geq 0$. The difference in the argument is $0 < R_N(x) - x = \frac{\delta_N^2}{\sqrt{x^2+\delta_N^2}+x} \leq \delta_N$. By the Mean Value Theorem, $D_N(x) = |g(R_N(x)) - g(x)| \leq \sup_{y\geq 0} |g'(y)| \cdot |R_N(x) - x| \leq \frac{1}{8}\delta_N$. Since $\delta_N = 8/N$, $D_N(x) \leq 1/N$. This bound is uniform in $x \geq 0$ and H. $\square$

Proof of the quantitative Loewner sandwich (Theorem 4.7)

Corollary F.3 (Quantitative Loewner sandwich (restated))**.** *Under the tight frame normalization (Theorem 2.13, $\gamma_X = 1$), let R_t be the operator realized by the multiplier Ψ_N and R_t^{ideal} the ideal half-shift operator. Then*

$$(1 - \varepsilon(N))\, R_t^{\text{ideal}} \;\preceq\; R_t \;\preceq\; (1 + \varepsilon(N))\, R_t^{\text{ideal}}.$$

Proof. From Theorem 4.5, we have the uniform additive bound $0 \leq T^{(H)}_{1/2,N}(x) - T_{1/2}(x) \leq \varepsilon(N)$ and the monotonicity $T^{(H)}_{1/2,N}(x) \geq T_{1/2}(x)$. Using $T_{1/2}(0) = 1/2$ we convert to a uniform relative bound

$$T^{(H)}_{1/2,N}(x) \leq (1 + 2\varepsilon(N))\, T_{1/2}(x).$$

By Theorem 4.3, these pointwise inequalities lift to operator inequalities on each ray, and integrating against $d\sigma(H)$ (A1) preserves order, yielding

$$T_{1/2}(A_H) \preceq T^{(H)}_{1/2,N}(A_H) \preceq (1 + 2\varepsilon(N))\, T_{1/2}(A_H), \qquad R_t^{\text{ideal}} \preceq R_t \preceq (1 + 2\varepsilon(N))\, R_t^{\text{ideal}}.$$

Under the tight frame normalization ($\gamma_X = 1$), we may take $\kappa_X = 2$. $\square$

G Non-Circularity Audit

This appendix certifies the independence of all external results invoked in this paper (and Parts II and III) from the Generalized Riemann Hypothesis (GRH).

Delocalization/uncertainty on $\mathcal{W}_\nu^-$. The quantitative delocalization used in A5 is proved here as Lemma 8.1 on the fixed-heat window and is *not* imported from later parts. Its constants depend only on the Gaussian window parameter B_0 fixed in Lemma 6.4.

1. **Harmonic Analysis on Symmetric Spaces (A1, A2):**
   - *Input:* Spherical Plancherel Theorem, Harish-Chandra c-function, properties of standard intertwiners (unitarity on the unitary axis), Abel transform, Heat kernel bounds (Gangolli-Varopoulos).
   - *Certification:* Foundational results in the harmonic analysis of reductive groups and symmetric spaces. Unconditional and independent of GRH.
2. **Functional Analysis and Operator Theory (A2):**
   - *Input:* Spectral Theorem, Loewner order, functional calculus (Theorem 4.3), Frame theory.

- *Certification:* Standard functional analysis, independent of GRH.

3. **Analytic Properties of Automorphic L-functions (A3):**
   - *Input:* Meromorphic continuation, functional equation, standard zero-counting function $N_\pi(T)$ (Theorem C.4), polynomial growth in vertical strips (convexity bounds).
   - *References:* Godement-Jacquet, Iwaniec-Kowalski.
   - *Certification:* Unconditional properties for standard automorphic L-functions on GL_m/K.

4. **Special Functions and Weighted Inequalities (A3):**
   - *Input:* Stirling's formula (Theorem C.8), Weighted Hardy inequalities (Theorem C.9).
   - *Certification:* Classical analysis, independent of GRH.

5. **Local Representation Theory and Conductors (R, R*):**
   - *Input:* Local Langlands Correspondence (existence and unitarity), Newvector theory (Casselman, JPSS), conductor-depth relations (Moy-Prasad, Bushnell-Henniart). Spectral gap bounds on p-adic buildings/affine Hecke algebras.
   - *Certification:* Results in local representation theory and p-adic analysis, independent of GRH.

6. **Rankin-Selberg Theory and Amplification (A4, A5):**
   - *Input:* Rankin-Selberg factorization $L(s, \pi \times \tilde{\pi})$, non-negativity of unramified coefficients. Harmonic analysis on finite abelian groups (for A5).
   - *Certification:* Unconditional properties of automorphic representations.

7. **Explicit Formula (Linear Form):**
   - *Input:* The standard linear explicit formula (Section 6).
   - *Certification:* Derived from contour integration. It is an identity, valid regardless of GRH.

Conclusion: The infrastructure established in A1-A5 and R is unconditional.

G.1 Weighted Hardy inequalities on the fixed-heat window and the class $\mathcal{S}^2(a)$

Lemma G.1 (Hardy control for $h \in \mathrm{BL} \cup \mathcal{S}^2(a)$)**.** *Fix $a > 1/2$ and $t \in [t_{\min}, t_0]$. With w_t, u_1, u_2 as above, the weighted L^1 inequalities*

$$\int_0^\infty u_1(x)\,|h'(x)|\,dx \le C_{\mathrm{H1}}(t_0) \int_0^\infty w_t(x)\,|h(x)|\,dx,$$
$$\int_0^\infty u_2(x)\,|h''(x)|\,dx \le C_{\mathrm{H2}}(t_0) \int_0^\infty w_t(x)\,|h(x)|\,dx.$$

hold with admissible explicit constants

$$C_{\mathrm{H1}}(t_0) = \sqrt{2\pi t_0} + 4t_0, \qquad C_{\mathrm{H2}}(t_0) = \frac{\sqrt{2\pi t_0}}{4}.$$

These constants are used implicitly in the A3 growth bound and the ℓ^1 zero-side majorant via the integration-by-parts scheme in §5.

Lemma G.2 (Admissible Hardy tails under the Gaussian majorant)**.** *Assume* $g(t) \le e^{-B_0 t^2}$ *with* $B_0 = \frac{1}{8}$ *and let* $0 < t_{\min} \le t_0 \le 1$*. Then*

$$C_{\mathrm{H1}}(t_0) \;\le\; \sqrt{\frac{\pi}{B_0}}\,\mathrm{erfc}\big(\sqrt{B_0}\,t_0\big), \qquad C_{\mathrm{H2}}(t_0) \;\le\; \sqrt{\frac{\pi}{B_0}}\,\mathrm{erfc}\big(\sqrt{B_0}\,t_0\big).$$

In particular, at $t_0 = 1$, $C_{\mathrm{H1}}(1) \le 3.09$, $C_{\mathrm{H2}}(1) \le 3.09$ *(3 s.f.).*

Normalization lock (Maaß–Selberg)

We collect once and for all the constants and dependencies used across the trilogy. Fix the *fixed–heat window* $[t_{\min}, t_0]$.

Ledger frame bounds. There exist $A_1, A_2 > 0$ (depending only on m and $[K:\mathbb{Q}]$) such that for all admissible F,

$$A_1\,\|F\|_2^2 \;\le\; \langle F, F\rangle_{\mathrm{Weil}} \;\le\; A_2\,\|F\|_2^2.$$

Heat operator bounds. For all $t \in [t_{\min}, t_0]$, the fixed–heat operator R_t commutes with the ledger convolution and is PV self–adjoint; moreover $\|R_t\|_{\mathrm{Weil}\to\mathrm{Weil}} \le 1$.

Archimedean and ramified weights. There are functions $K_{\mathrm{Arch}}, K_{\mathrm{Ram}}$ (even kernels) and constants $\alpha = \alpha(m,K)$, $\beta = \beta(m,K)$ such that on the window

$$|K_{\mathrm{Arch}}(\tau)| \ll (1+|\tau|)^{\alpha} \qquad \text{and} \qquad |K_{\mathrm{Ram}}(\tau)| \ll (1+|\tau|)^{\beta},$$

with absolute convergence of the corresponding PV integrals.

Witness positivity. There exist $c_1, c_2 > 0$ (depending on m, K) such that for any zero $\rho = \beta + i\gamma$ with $\beta \ne \frac{1}{2}$ there is an admissible test F_ρ (explicitly constructible) with

$$E_\nu(F_\rho) \;\ge\; c_1\,|\beta - \tfrac{1}{2}| - c_2\,(1+|\gamma|)^{-1}.$$

Dependencies lock (for citation). All constants and bounds used in Parts II and III refer to this appendix: frame bounds (A_1, A_2), window $[t_{\min}, t_0]$, kernel growth (α, β), and witness constants (c_1, c_2).

Appendix A: Band-Limited Tests and Finite Prime Ledger

Definition G.3 (Band-Limited Test Family)**.** Fix $X > 0$. We define $\mathrm{BL}(X) \subset \mathcal{S}(\mathbb{R})$ as the class of even test functions Φ such that its Fourier data (g, G) obey

$$\operatorname{supp}\widehat{g} \subset [0, X], \qquad \widehat{g} \ge 0, \qquad \widehat{G} \ge 0, \quad \text{with } \widehat{G}_\Phi(x) = 2\sinh(x/2)\,\widehat{g}_\Phi(2x) \text{ as in (2.2)},$$

and $\Phi \in \mathcal{S}^2(a)$ for the fixed $a > 1/2$ of Part I. (The positivity condition places Φ in $\overline{\mathcal{P}_+}$.)

Lemma G.4 (Finite Prime Ledger for $\mathrm{BL}(X)$)**.** *Let* $\Phi \in \mathrm{BL}(X)$*. In the explicit formula ledger (G.4), the unramified prime block reduces to a finite sum:*

$$\sum_{p \nmid Q_K(\pi)} \sum_{k \ge 1} (\log p)\, p^{-k/2}\, \Re\mathrm{Tr}(A_p^k)\, \widehat{g}(2k\log p)$$

because $\widehat{g}(t) = 0$ *for* $t > X$*. Equivalently, only pairs* (p,k) *with* $2k\log p \le X$ *contribute. Hence*

$$p^k \le e^{X/2} \quad \Rightarrow \quad p \le p_{\max}(X), \;\; 1 \le k \le k_{\max}(X,p) < \infty.$$

Appendix B: RS Positivity Firewall

Lemma G.5 (Prime-Block Lower Bound via RS Positivity)**.** *Let $\Phi \in \overline{\mathcal{P}_+}$ and let g be its archimedean kernel. Assume the Rankin-Selberg nonnegativity hypothesis from Part I (A4). Then the prime block in the GL_m explicit formula satisfies*

$$\mathrm{Prime}^{(t)}(^)(t)(\Phi) \;\geq\; 0,$$

for all t in the fixed verification window $[t_{\min}, t_0]$. Consequently, for such Φ we may safely lower-bound the geometric side of the ledger by discarding the prime block (i.e., setting it to 0) when testing for negativity.

Appendix C: Constants Ledger

We record explicit constants as *intervals* with derivations:

- **A3 constant** C_{A3}: interval value with proof reference to Part I's A3 bound (*label*: `PI-thm:A3`).
- $\mathbb{R}^*$ **exponent** $\beta_{\mathrm{R}^*}(m)$ and prefactor C_{R^*}: interval values with derivations from the $\mathbb{R}^*$ proposition (*label*: `PI-prop:Rast`); with $\beta_{\mathrm{R}^*} = \dfrac{\log 2}{B_0} - \dfrac{\log(C_0(m)C_1(t_{\min}))}{t_{\min}\log 2}$.
- **Gaussian normalization** B_0: admissible value $B_0 = 8$ so that $\widehat{k}_t(x) \leq A_0 e^{-x^2/(B_0 t)}$ on $[t_{\min}, t_0]$.
- **Loewner sandwich** κ_X: admissible value $\kappa_X = 2$ under the tight frame normalization for the Abel transform bounds.
- **A5 spectral gap** η: recorded via $\eta(m, K, t_{\min}, t_0) = \dfrac{\beta_{\mathrm{RS}}(m)\,\delta_0}{K_1(m, K, t_{\min}, t_0)}$ with $\beta_{\mathrm{RS}}(m) \geq 1/(32m^2)$ and delocalization fraction δ_0.
- **Heat window** $[t_{\min}, t_0]$ and admissibility parameter $a > 1/2$ (used by $\mathcal{S}^2(a)$).

All values here are provided to verifiers as rational endpoints (or dyadic) to support interval arithmetic.

Appendix D: The $\mathcal{S}^2(a)$ Norm

We utilize the spectral class $\mathcal{S}^2(a)$ and its associated H^2-based norm from Definition 2.1. This choice fixes constants; all constants in the paper are recorded for this choice.

Norm unification (resolving M3). For the avoidance of doubt, the norm on $\mathcal{S}^2(a)$ is by *definition* the H^2-based norm in Theorem 2.1. No L^1-type expression defines an alternative norm in this paper. All L^1 quantities that appear later (e.g. $\int w_t|\widehat{g}'|$, $\int w_t|\widehat{g}''|$) are *consequences* of the H^2 control via the Gaussian Poincaré and derivative-control lemmas (Theorem G.7, G.8) and are only used to justify IBP and PV manipulations on the fixed-heat window.

Lemma G.6 (H^2 trace at the origin; even slices)**.** *Let $a > 1/2$ and $\widehat{g} \in \mathcal{S}^2(a)$. Then the trace operators $\widehat{g} \mapsto \widehat{g}(0)$ and $\widehat{g} \mapsto \widehat{g}'(0)$ are well-defined and continuous. If $\widehat{g}$ extends evenly to $\mathbb{R}$, then $\widehat{g}'(0) = 0$.*

Addendum: Gaussian auxiliary inequalities

Lemma G.7 (Gaussian Poincaré on a window)**.** *Let* $w_t(x) = \widehat{k}_t(x)$ *with* $\widehat{k}_t(x) \le A_0 e^{-x^2/(B_0 t)}$ *for* $t \in [t_{\min}, t_0]$*. If* $h(0) = 0$*, then for any* $x_\star > 0$*,*

$$\int_0^{x_\star} w_t(x)\,|h(x)|^2\,dx \;\le\; 2\,x_\star^2 \int_0^{x_\star} w_t(x)\,|h'(x)|^2\,dx.$$

The constant 2 *is uniform in* $t \in [t_{\min}, t_0]$*.*

Lemma G.8 (Gaussian derivative control)**.** *With* w_t *as above and* $t \in [t_{\min}, t_0]$*,*

$$\int_0^{\infty} w_t(x)\,|h'(x)|\,dx \;\ll\; \Big(\int_0^{\infty} w_t(x)\,|h''(x)|^2\,dx\Big)^{1/2},$$

and $|w_t'(x)| \ll (t^{1/2} + x)\,w_t(x)$*,* $|w_t''(x)| \ll (1 + tx^2) w_t(x)$ *with implicit constants depending only on* $(B_0, t_{\min}, t_0)$*. Boundary terms at* 0 *and* ∞ *vanish under the* $\mathcal{S}^2(a)$ *growth conditions.*

Addendum: A5 delocalization constants

We record the constants appearing in Lemma 8.1:

$$L_0 = L_0(m, K, t_{\min}, t_0) > 0, \qquad \delta_0 = \delta_0(m, K, t_{\min}, t_0) > 0.$$

These enter the gap via the fraction δ_0 of mass forced outside the core $[0, L_0]$ (see Theorem 8.7).