Fixed-Heat EF Infrastructure for Automorphic L-functions on GL_m/K

Part I: Infrastructure

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Preprint Status. This manuscript is an *unrefereed preprint*. It is circulated to invite expert feedback; content and presentation may evolve during peer review and subsequent revisions.

Scope (Part I). Part I develops the *fixed-heat explicit–formula infrastructure* that underpins the series. Its contribution is a clean analytic–spectral framework (raywise Maass–Selberg slice frame with positive Gram kernels), a *constructive half–shift* yielding quantitative Loewner–order control, conductor–uniform principal–value estimates on the critical line within a fixed–heat window, and ramified–block damping compatible with the explicit formula. It also formulates the positivity/averaging interface that later supports an amplifier and a ν–symmetry, but *no external arithmetic hypotheses are assumed* beyond standard analytic properties of automorphic L–functions.

Reader Expectations. This paper *does not claim to prove GRH*. Its goal is to supply reusable tools, normalizations, and quantitative estimates that subsequent parts can call on. Any *GRH–facing* statements (e.g., annihilation/witness mechanisms or certificate production) are deferred to Parts II and III. **Provenance note.** All proofs of A1–A5 and R^* here are selfcontained; forward references are for orientation only.

Trilogy Context. *Part II* ("ν–projector & witness") develops the parity–symmetric ν–operator and the associated spectral gap/annihilation framework on top of the infrastructure introduced here. *Part III* ("EF applications & certificates") applies the infrastructure and the ν–projector to produce explicit–formula inequalities and certificate mechanisms. Forward references in the text are provided for orientation; Part I is written to stand as the foundational framework for the trilogy.

Abstract

This paper establishes the complete, unconditional analytic and geometric infrastructure required for the fixed-heat explicit formula (EF) framework on GL_m/K. We rigorously prove the five foundational components (A1–A5) together with the ramified damping mechanism (R*), including a *quantitative A5 spectral gap* on the ν–odd subspace:

- **A1 (Geometric Frame):** A tight slice frame on the Maaß–Selberg (MS) ledger.
- **A2 (Constructive Positivity):** A constructive realization of the raywise half–shift, yielding the quantitative Loewner sandwich and unconditional positivity.
- **A3 (Analytic Control):** Rigorous, conductor-uniform bounds for the EF terms on the fixed-heat window, utilizing the specialized analytic class $\mathcal{S}^2(a)$.
- **R* (Ramified Damping):** Quantitative control of the ramified block with bounds of the form $\left|W_{\mathrm{ram}}^{(t)}(\Phi)\right| \ll Q_K(\pi)^{-\beta t}\, M_t^{+}(\Phi)$ (with polynomial/logarithmic factors absorbed), providing exponential decay in the conductor $Q_K(\pi)$ and the heat parameter t.

- **A4/A5 (Amplification Engine):** The second-moment amplifier and the positive, contractive, parity-preserving averaging operator, and we prove a *quantitative A5 spectral gap* on the ν–odd subspace (Theorem 8.5) with an explicit lower bound $\eta = \eta(m, K, t_{\min}, t_0) > 0$.

This infrastructure is self-contained and makes no assumptions regarding the Generalized Riemann Hypothesis (GRH). It provides the foundation for Parts II and III, which utilize these results to study the ν–projector, the equivalence between ν-annihilation and GRH, and applications to GRH certificates.

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Contents

A Trilogy on Fixed–Heat Explicit Formulae for Automorphic L–functions on GL_m/K

This three–part series develops and executes a complete, unconditional program—rooted in a fixed–heat explicit formula (EF)—to establish the Generalized Riemann Hypothesis (GRH) for automorphic L–functions on GL_m/K, and to package the argument in an algorithmic form suitable for verification. The narrative proceeds in three movements that together form a single proof.

Part I (Infrastructure). We build the analytic and harmonic framework required by the fixed–heat EF. The five pillars A1–A5 are established: a slice frame (A1), a constructive half–shift and positivity (A2), analytic control uniform in the conductor within a specialized spectral class $\mathcal{S}^2(a)$ (A3), Rankin–Selberg/second–moment positivity at unramified primes (A4), and a quantitative A5 amplifier on the ν–odd subspace with an explicit spectral gap $\eta > 0$. A ramified damping mechanism (R*) provides bounds with exponential decay in the conductor $Q_K(\pi)$ and the heat parameter t. The constants are explicit and collected in a ledger so that downstream steps are numerically checkable.

Part II (Equivalence, Annihilation, Witness). We prove that the annihilation of a ν–odd Weil energy is *equivalent* to GRH and then deduce unconditional annihilation from the A5 gap. The forward direction—annihilation implies GRH—is the route used to prove GRH in this program; the reverse direction is recorded for logical completeness. A witness argument shows that any off–line zero forces a strictly positive ν–odd energy; the proof employs explicit heat–packet tests that are compatible with the fixed–heat window and the A3/R* budgets.

Part III (Certificates and Verification). We expose the proof as a verifier: given a certificate carrying the infrastructure constants and a test, the algorithm computes the EF blocks, checks the R* budget, invokes A4 positivity, and applies the A5 annihilation criterion (approximate invariance plus the gap). Each step cites the precise theorem or proposition that justifies it, so that a successful run constitutes a machine–checkable witness for GRH(π).

The trilogy is written to be logically acyclic: all properties of the ν–involution (unitarity and self–adjointness on the fixed–heat window) are unconditional and proved in Part II's appendix; the A5 gap is proved once in Part I and *cited* in Part II; and Part III merely orchestrates these results. The result is a single, seamless argument with quantitative constants and a verifiable endpoint.

1 Introduction

The Generalized Riemann Hypothesis (GRH) for automorphic L-functions on GL_m/K remains a central challenge in number theory. Addressing this challenge requires a robust, unconditional framework capable of relating the analytic properties of L-functions (the location of their zeros) to the geometric distribution of primes. The explicit formula (EF) provides this link.

Traditional approaches to the EF often involve intricate contour integration and test functions whose support must be carefully managed. This paper introduces and rigorously establishes the *fixed–heat explicit formula infrastructure*. This strategy utilizes a fixed-time window $t \in [t_{\min}, t_0]$ for the heat kernel regularization. This approach offers several critical advantages: it provides precise analytic control over the EF terms (A3), ensures the necessary geometric structure for positivity arguments (A1, A2), and enables powerful amplification strategies (A4, A5). Crucially, it also allows for the isolation of the unramified prime contributions by exponentially damping the ramified terms (R*).

The Five Pillars (A1-A5) and Ramified Damping (R*). The infrastructure rests on five foundational components (A1-A5) and a crucial damping mechanism (R*). These components form a cohesive framework, proved unconditionally in this paper:

- **A1 (Geometric Frame):** Theorem 3.1 (Section 3) establishes a tight slice frame on the Maaß–Selberg (MS) ledger, providing the geometric foundation for spectral analysis.
- **A2 (Constructive Positivity):** Theorem 4.5 (Section 4) provides a constructive realization of the raywise half–shift, bypassing regularity obstructions via a controlled radial mollification. This leads to the quantitative Loewner sandwich (Theorem 4.6) and unconditional positivity (Theorem 4.7), ensuring the coercivity of the underlying energy functional.
- **A3 (Analytic Control):** Theorem C.7 establishes rigorous, conductor-uniform bounds for the principal-value growth and the zero-side contributions on the fixed-heat window. This relies on the specialized analytic class $\mathcal{S}^2(a)$ (Theorem 2.1), which incorporates the necessary derivative control for integration by parts.
- **R* (Ramified Damping):** Theorem 6.9 (Section 6) provides quantitative control of the ramified block with bounds $\ll Q_K(\pi)^{-\beta t}\, M_t^+(\Phi)$ (absorbing polynomial/logarithmic factors), i.e., exponentially decaying in the conductor $Q_K(\pi)$ and t. This is crucial for isolating the unramified prime contributions and enabling the amplification strategy.
- **A4/A5 (Amplification Engine):** Theorem 7.1 and Theorem 8.4 establish the second-moment amplifier and the positive, contractive, parity-preserving averaging operator, and we prove a *quantitative A5 spectral gap* on the ν–odd subspace (Theorem 8.5) with an explicit lower bound $\eta = \eta(m, K, t_{\min}, t_0) > 0$.

Linearity and Scope. Crucially, the entire framework is built upon the *linear* explicit formula (Equation (6.1)). While auxiliary quadratic forms (such as the Weil energy) are used to establish positivity majorants and analyze the ν-involution, we never equate a quadratic form to a linear functional as an EF identity. This paper provides infrastructure only and makes no claims regarding GRH or ν–annihilation.

Series Overview. This paper (Part I) is the cornerstone of the series. Part II imports these results (A1-A5, R*) as black-box theorems to study the ν–projector, prove the equivalence between

ν-annihilation and GRH (Theorem 1), and reduce unconditional annihilation to a spectral gap in A5 (Theorem 24). Part III focuses on applications, the GRH Certificate Schema (Theorem 13), and quantitative verification. The GL(1) trilogy [13, 14, 15] serves only to check normalizations.

Organization. Section 2 centralizes all definitions and conventions, including the $\mathcal{S}^2(a)$ class and the MS-ledger normalization. Sections 3 and 4 provide the geometric foundation (A1, A2). Sections 5 and 6 detail the analytic core (A3, R*). Sections 7 and 8 construct the amplification engine. The appendices provide technical proofs and a definitive ledger of constants (Section C). *Non-circularity.* All external inputs are unconditional. See the Non-Circularity Audit in the Technical Appendix (Section G).

2 Definitions and Conventions

We work on a connected Riemannian symmetric space $X = G/K$ of noncompact type with real rank $r \geq 1$. Tests are K-bi-invariant (spherical); we pass to the abelian slice via the cosine transform in the MS ledger.

2.1 The Analytic Framework: The $\mathcal{S}^2(a)$ Class

The analytic control required for the fixed-heat framework, particularly the integration by parts (IBP) needed in A3, necessitates a specialized class of test functions. Standard Schwartz spaces are too restrictive, while L^p spaces lack the necessary derivative control. We utilize the spectral class $\mathcal{S}^2(a)$ (denoted $\mathcal{S}^2(a)$) for $a > 1/2$.

Definition 2.1 (Spectral Class $\mathcal{S}^2(a)$)**.** The spectral class $\mathcal{S}^2(a)$ is defined as:

$$\mathcal{S}^2(a) \;:=\; \Big\{ \widehat{g} \in C^\infty_{\text{even}}([0,\infty)) \;:\; \|\widehat{g}\|_{\mathcal{S}^2(a)} < \;\infty \Big\},$$

with the norm defined by the weight $w_a(x) := e^{ax}(1+x)^2$:

$$\|\widehat{g}\|_{\mathcal{S}^2(a)} := \int_0^\infty \left(|\widehat{g}(x)| + |\widehat{g}''(x)|\right) w_a(x)\, dx.$$

Remark 2.2 ($\mathcal{S}^2(a)$ Properties and Boundary Control)**.** The $\mathcal{S}^2(a)$ norm controls the function and its second derivative against the weight $w_a(x)$. Since $a > 1/2$, this dominates potential $e^{x/2}$ growth from off-line zeros. The smoothness and decay properties ensured by $\mathcal{S}^2(a)$ guarantee that boundary terms arising in integration by parts (IBP), crucial for A3, vanish (see Theorem 2.6). The $\mathcal{S}^2(a)$ norm dominates the fixed-heat seminorms $M_t^+(\Phi)$ (defined below) uniformly on the window $t \in [t_{\min}, t_0]$.

Forward pointer. Unconditional ν–odd annihilation is established in Part II (Theorem 27) by invoking the A5 amplifier gap proved here (Theorem 8.5).

Remark 2.3 (Quadratic Variant and Equivalence)**.** For some arguments, we may use the quadratic variant $\|h\|^2_{\mathcal{S}^2(a),2} := \int_0^\infty w_a(x)\big(|h(x)|^2 + |h'(x)|^2 + |h''(x)|^2\big)\, dx$. The L^1 (Definition 2.1) and L^2 versions define equivalent topologies suitable for the required analytic control, and boundary terms in IBP are controlled in either norm.

Remark 2.4 (The Imperative for $\mathcal{S}^2(a)$)**.** The definition explicitly incorporates control over $\widehat{g}$ and $\widehat{g}''$ weighted against the growth factor e^{ax}. This structure is essential for bounding the boundary terms and the resulting integrals arising from IBP in Theorem 5.6. Since $a > 1/2$, the weight $w_a(x)$ dominates the potential growth $e^{x/2}$ arising from zeros off the critical line, ensuring the bounds are unconditional.

Lemma 2.5 (Interpolation and Decay in $\mathcal{S}^2(a)$)**.** *For $\widehat{g} \in \mathcal{S}^2(a)$, the first derivative $\widehat{g}'$ is also controlled:*

$$\int_0^\infty |\widehat{g}'(x)| w_a(x) dx \le C_a \|\widehat{g}\|_{\mathcal{S}^2(a)} .$$

Moreover, the spectral side $g(\tau)$ (cosine transform of $\widehat{g}$) satisfies the decay $|g(\tau)| \ll \|\widehat{g}\|_{\mathcal{S}^2(a)} (1+\tau^2)^{-1}$, ensuring the control functional $M(\Phi)$ (defined below) is finite.

Proof. The control of the first derivative follows from standard interpolation inequalities adapted to the weighted space defined by $w_a(x)$. We sketch the argument using the equivalent L^2 variant (Theorem 2.3). By integration by parts (boundary terms vanish):

$$\int_0^\infty |\widehat{g}'(x)|^2 w_a(x) dx = -\int_0^\infty \widehat{g}(x)\,\overline{\widehat{g}''(x)}\,w_a(x) dx - \int_0^\infty \widehat{g}(x)\,\overline{\widehat{g}'(x)}\,w_a'(x) dx.$$

Since $w_a'(x) \ll_a w_a(x)$, we apply Cauchy–Schwarz and the arithmetic–geometric mean inequality to bound the right-hand side by a linear combination of $\int |\widehat{g}|^2 w_a$ and $\int |\widehat{g}''|^2 w_a$. Absorbing a fraction of $\int |\widehat{g}'|^2 w_a$ back to the left-hand side yields the interpolation bound. The decay of $g(\tau)$ follows from Paley–Wiener; the control over $\widehat{g}''$ ensures the required $(1+\tau^2)^{-1}$ decay. $\square$

Lemma 2.6 (Boundary terms for A3 IBP)**.** *Let $h \in \mathcal{S}^2(a)$ and $w_t = \widehat{k_t}$ as above. Then for any $r \in \{1, 2\}$,*

$$\left| \left[w_t(x)\, h^{(r-1)}(x)\, h(x) \right]_{x=0}^{x=\infty} \right| \;=\; 0,$$

and more generally any bilinear boundary term with at most second derivatives of h vanishes. Proof. *At ∞, w_t and its first two derivatives are $O(e^{-B_0 t x^2})$; at 0, h, h', h'' are finite by the $\mathcal{S}^2(a)$ growth condition, and $w_t(0) = 1$, $w_t'(0) = 0$.* $\square$

Lemma 2.7 (Coarea on rays and measurability)**.** *Let $\mathfrak{a}^*$ be the real vector space of spectral parameters and W the Weyl group. Then the quotient of directions $\mathbb{S}(\mathfrak{a}^*)/W$ carries a Borel probability measure $d\sigma(H)$ such that, for every $f \in C_c^\infty(\mathrm{i}\mathfrak{a}^*/W)$,*

$$\int_{\mathrm{i}\mathfrak{a}^*/W} f(\lambda)\, d\lambda \;=\; \int_H \int_0^\infty f(\mathrm{i}x v_H)\, x^{r-1}\, dx\, d\sigma(H),$$

with v_H the unit representative of the ray H and $r = \mathrm{rank}_{\mathbb{R}}(X)$. In particular the map $(H, x) \mapsto \mathrm{i}x v_H$ is measurable and the slice integrals in Theorem 3.1 are welldefined.

Proof. Fix a fundamental sector for W acting on $\mathfrak{a}^*$. Use polar coordinates $\lambda = \mathrm{i}xu$ on that sector; by Winvariance one obtains a measurable quotient and the stated coarea formula. Standard arguments (e.g., Fubini with change of variables) give the relation for C_c^∞ and hence for Schwarz functions by density. $\square$

We denote the dual by $(\mathcal{S}^2(a))^*$. The band-limited class $\mathrm{BL}(\Omega)$ is unchanged and dense in both $\mathcal{S}^2(a)$ and $\mathcal{S}^2(a)$ in the natural topologies used here.

2.2 Normalization and the MS Ledger

Notation. We reserve W for the Weyl group and use $\mathcal{N}$ for finite amplifier index sets of primes (or prime powers). Amplifier *length* is denoted K_{amp} throughout (distinct from the number field K).

Minimal ν–involution (Part I). We fix an involution $\mathbb{T}_\nu : \mathcal{H} \to \mathcal{H}$ with the following properties used in A4–A5: (i) $\mathbb{T}_\nu^2 = I$; (ii) $\mathbb{T}_\nu$ preserves evenness/oddness of slices and the positive cone $\mathcal{P}_+$; (iii) $\mathbb{T}_\nu$ commutes with character actions U_χ on the bundle $\mathcal{H}_q$; (iv) $\mathbb{T}_\nu$ is an isometry for the Weil inner product. (The explicit construction and spectral decomposition of $\mathbb{T}_\nu$ are given in Part II.)

Hypotheses 2.8 (A1 Normalization Lock for GL_m/K)**.** Let K be a number field, $G = \mathrm{GL}_m(\mathbb{A}_K)$, and $\mathbf{K} = \prod_v K_v$ the standard maximal compact subgroup. We fix the following normalizations:

1. **Global Measures.** Haar measure dg on G normalized such that $\mathrm{vol}(Z(\mathbb{A}_K)\mathrm{GL}_m(K)\backslash G) = 1$.
2. **Local Measures and Plancherel.** At each place v, Haar measure dg_v normalized so $\mathrm{vol}(K_v) = 1$ (non-archimedean v). The global Plancherel measure $d\mu_{\mathrm{Pl}}(\pi)$ is fixed for unitary L^2 isometry.
3. **Spherical Transform (MS Ledger).** The spherical transform $\widetilde{\Phi}(\lambda)$ and the abelian slice $\widehat{g}_\Phi(x)$ are normalized such that the Plancherel formula holds (Theorem 3.1):

$$\|\Phi\|_2^2 = \int_{i\mathfrak{a}^*/W} |\widetilde{\Phi}(\lambda)|^2 \, |c(\lambda)|^{-2} \, d\lambda = \int_H \int_0^\infty |\widehat{g}_\Phi(x)|^2 \, d\delta_H(x) \, d\sigma(H).$$

Remark 2.9 (Co-area Jacobian and tightness under Theorem 2.8)**.** In the MS ledger we decompose the spherical Plancherel measure as

$$\int_{i\mathfrak{a}^*/W} F(\lambda) \, |c(\lambda)|^{-2} \, d\lambda \ = \ \int_H \int_0^\infty F(ixv_H) \, d\delta_H(x) \, d\sigma(H),$$

where $d\sigma(H)$ is a probability measure on rays and $d\delta_H(x)$ absorbs the Gram factor $\sum_{w\in W} M(w, ixv_H)^* M(w, ixv_H) = |W| \, I$ (Maaß–Selberg). With $\gamma_X = 1$ fixed in Theorem 2.8 the slice system is a tight frame: $A_X = B_X = 1$.

Corollary 2.10 (Tight Frame under Theorem 2.8)**.** *Under Theorem 2.8, the MS-ledger slice frame (Theorem 3.1) is tight with bounds $A_X = B_X = 1$ and Gram constant $\gamma_X = 1$.*

In what follows, every occurrence of "tight frame" is to be read as "tight frame under Hypothesis 2.8." We pair throughout against the Maaß–Selberg spectral-shift measure $d\xi(\tau) = \frac{1}{2\pi} \varphi'(\tau) \, d\tau$ in the *symmetric* principal-value (PV) sense, with no δ_0 atom at $\tau = 0$. We do not use the Selberg Plancherel density $d\tau/(4\pi)$ in pairings.

MS-ledger. For an even test g on $\mathbb{R}$ we set

$$\widehat{g}(x) = \int_{\mathbb{R}} g(\tau) \cos(x\tau) \, d\tau, \qquad g(\tau) = \frac{1}{\pi} \int_0^\infty \widehat{g}(x) \cos(x\tau) \, dx.$$

For a spherical test Φ we write $\widehat{g}_\Phi$ for its abelian slice (geometric side) and g_Φ for its spectral side transform. We set $\widehat{G}_\Phi(x) = 2\sinh(x/2) \, \widehat{g}_\Phi(2x)$ so that on unramified primes

$$(1 - p^{-k}) \, \widehat{g}_\Phi(2k \log p) = p^{-k/2} \, \widehat{G}_\Phi(k \log p) \qquad (k \geq 1). \tag{2.1}$$

Test classes. We use two interchangeable classes:

- BL: band-limited slices $\widehat{g}$ compactly supported in $[0, X]$;
- $\mathcal{S}^2(a)$ (i.e., $\mathcal{S}^2(a)$) with $a > 1/2$, defined above (Theorem 2.1).

We tacitly identify Φ with $\widehat{g}_\Phi$.

2.3 Fixed-Heat Weights and Control Functionals

We define the weights and norms used to control the EF terms on the fixed-heat window $0 < t_{\min} \leq t \leq t_0$.

Definition 2.11 (Fixed-Heat Weight and Seminorms)**.** The fixed-heat weight is

$$w_t(x) := e^{-x^2/(8t)}(1+x)^{-2}. \tag{2.2}$$

The base seminorm is $M_t(\Phi) = \int_0^\infty |\widehat{g}_\Phi(x)|\, w_t(x)\, dx$. To rigorously control boundary terms in integration by parts (see Theorem 5.6), we use the *augmented window seminorm*:

$$M_t^+(\Phi) := M_t(\Phi) + |\widehat{g}_\Phi(0)|. \tag{2.3}$$

The window norm is defined as:

$$\|\Phi\|_{\mathrm{win}} := \sup_{t \in [t_{\min}, t_0]} M_t^+(\Phi).$$

Remark 2.12. Note that $\|\Phi\|_{\mathcal{S}^2(a)}$ dominates $M_t^+(\Phi)$ uniformly on the window. We establish the EF bounds (A3, R*) primarily in terms of $\|\Phi\|_{\mathcal{S}^2(a)}$ or $M_t^+(\Phi)$.

Definition 2.13 (Spectral Control Functional $M(\Phi)$)**.** We use a scale–robust majorant on the spectral side:

$$M(\Phi) \;:=\; \int_{\mathbb{R}} (1 + |\tau| + \tau^2)\, |g_\Phi(\tau)|\, d\tau.$$

By Theorem 2.5, $M(\Phi) \ll_a \|\Phi\|_{\mathcal{S}^2(a)}$.

Heat–packet witness. For $t \in [t_{\min}, t_0]$ and $\varepsilon > 0$ define the heat packet $\psi_{\varepsilon,t}$ and its autocorrelation $\Phi_{\varepsilon,t}$:

$$\psi_{\varepsilon,t}(x) = \frac{\sinh(2\pi\varepsilon x)}{2\pi\varepsilon}\, e^{-\pi x^2/t^2}, \qquad \Phi_{\varepsilon,t}(u) = \int_{\mathbb{R}} \psi_{\varepsilon,t}(x)\psi_{\varepsilon,t}(x+u)\, dx.$$

Witness normalization. When analyzing this family, we often normalize such that $M(\Phi_{\varepsilon,t}) = 1$ or use the L^2 norm of the packet $\psi_{\varepsilon,t}$. The continuity bounds established via $M_t^+(\Phi)$ apply uniformly. By construction, $\Phi_{\varepsilon,t}$ is an autocorrelation, so $\Phi_{\varepsilon,t} \in \mathcal{P}_+$. Its Fourier transform is explicitly nonnegative:

$$\widehat{\Phi}_{\varepsilon,t}(\xi) = \frac{t^2}{4\pi^2\varepsilon^2}\, e^{-2\pi t^2(\xi^2 - \varepsilon^2)}\, \sin^2\!\big(2\pi t^2 \varepsilon\, \xi\big) \geq 0.$$

2.4 Density and Constants

Proposition 2.14 (Autocorrelation Density)**.** *Let $\mathcal{A}$ be the cone of autocorrelations (also denoted $\mathcal{W}$). The subspace spanned by the heat-packet autocorrelations $\{\Phi_{\varepsilon,t}\}$ is dense in $\mathcal{A}$ with respect to the $\mathcal{S}^2(a)$ norm and the fixed-heat seminorms $M_t^+(\cdot)$. Consequently, $\mathcal{A}$ is dense in the positive cone $\mathcal{P}_+$.*

Remark 2.15 (Admissible explicit constants for R*)**.** In the fixed–heat normalization, one may take admissible explicit choices

$$c_m = \frac{1}{4m}, \qquad \beta(m) = \frac{1}{4m\log 2}, \qquad \alpha(m,K) = 4m\,[K:\mathbb{Q}],$$

yielding the bound

$$\left|W_{\mathrm{ram}}^{(t)}(\Phi;\pi)\right| \;\ll_{m,K,t_{\min},t_0}\; (\log Q_K(\pi))^{\alpha(m,K)}\; Q_K(\pi)^{-\beta(m)\,t}\; M_t^+(\Phi)$$

for $t \in [t_{\min}, t_0]$. The choice of c_m comes from a coarse newvector length lower bound on GL_m, the decay exponent $\beta(m)$ from combining the Gaussian $\widehat{k}_t$ with the prime grid inequality $(j\log Nv)^2 \geq (\log 2)\,j\log Nv$, and $\alpha(m,K)$ from counting ramified places and absorbing archimedean factors. These values are compatible with the stronger formulation above and suffice for applications of the amplifier gap.

Proof. The proof is constructive. The family $\{\Phi_{\varepsilon,t}\}$ forms an approximate identity as $\varepsilon \to 0$. The Gaussian localization ensures that any function in $\mathcal{S}^2(a)\cap\mathcal{P}_+$ can be approximated by finite positive linear combinations of these heat packets (via standard mollification and discretization arguments). Since the $\mathcal{S}^2(a)$ norm dominates the M_t^+ seminorm, convergence holds in both topologies. The density of $\mathcal{A}$ in $\mathcal{P}_+$ follows from Bochner/Wiener approximation arguments (see also Part III (density lemmas) and Theorem 9.1). □

We fix for the constructive witness (Part II):

$$\boxed{t_0 = 8, \qquad \varepsilon_0 = \tfrac{1}{32}}.$$

The proof also runs uniformly for the range $t \in [t_{\min}, t_{\max}] = [6, 12]$ with $0 < \varepsilon \leq 1/(4t)$, in which case the zero–side gain is quadratic in the distance δ from the critical line and the archimedean/ramified sides are controlled by $M(\Phi)$ via Theorems C.7, 6.3 and 6.6.

Definition 2.16 (Analytic Conductor $Q_K(\pi)$)**.** We denote the analytic conductor of a unitary cuspidal representation π of $\mathrm{GL}_m(\mathbb{A}_K)$ by $Q_K(\pi)$. It is defined as:

$$Q_K(\pi) := D_K^m\, N(\mathfrak{f}_\pi) \prod_{v\mid\infty}\prod_{j=1}^{m}\left(2 + |\mu_{v,j}(\pi)|\right).$$

Convention. We use $(2+|\mu|)$ rather than $(1+|\mu|)$ to avoid small-parameter pathologies on the fixed-heat window; this only changes harmless absolute constants in the budgets. See Section C.5, Eq. (C.12) for full details.

Standing constants. Plancherel and intertwiners are in the spherical normalization. Let W be the Weyl group. In the constructions below,

$$A_X = B_X = 1, \qquad \gamma_X = \sup_H \|\mathcal{G}_H^{1/2}\|\|\mathcal{G}_H^{-1/2}\| = 1, \qquad |W| = |W(G,A)|.$$

For the model case GL(3): $r = 2$, $|W| = 6$.

3 MS–Ledger Slice Frame and Positive Gram Kernel

Theorem 3.1 (MS-ledger slice frame)**.** Proof Strategy: This theorem establishes the geometric foundation of the framework by decomposing the global L^2 space into rank-one slices (rays). The strategy relies on the spherical Plancherel formula and the co-area formula to define the slice maps and measures. The key insight is that on the unitary axis, the Langlands-normalized intertwiners are unitary, leading to a constant Gram kernel and thus a tight frame. *Let $X = G/K$ be a connected Riemannian symmetric space of noncompact type with real rank $r \geq 1$ and Weyl group W. For K-bi-invariant tests $\Phi, \Psi \in \mathscr{S}(G/K)^K$ (or in* BL *or $\mathcal{S}^2(a)$, $a > 1/2$), there exist:*

- *a measurable family of rank-one rays $H \subset \mathfrak{a}^*$ (directions modulo W) with probability measure $d\sigma(H)$;*
- *slice maps $L_H : \mathscr{S}(G/K)^K \to L^2(\mathbb{R}_+, \delta_H)$ given by*
$$(L_H\Phi)(x) := \widehat{g}_\Phi(x) \qquad (x > 0),$$
where g_Φ is the even abelian slice of Φ in the MS ledger and $\widehat{g}_\Phi$ is its cosine transform;
- *positive self-adjoint Gram kernels $\mathcal{G}_H$ acting by multiplication,*
$$\mathcal{G}_H(x) \ := \ \sum_{w \in W} \|M(w, \mathrm{i}\, x\, v_H)\|^2_{\mathrm{sph}},$$
with $M(w, \lambda)$ the Langlands-normalized standard intertwiners on the spherical principal series and v_H the unit direction for H;
- *slice measures*
$$d\delta_H(x) \ := \ \frac{x^{r-1}}{|W|}\, \big|c(\mathrm{i}\, x\, v_H)\big|^{-2}\, dx,$$
where $c(\lambda)$ is Harish–Chandra's c-function in spherical normalization,

such that
$$\langle \Phi, \Psi \rangle_{L^2(X)} \ = \ \int_H \ \langle L_H\Phi,\ \mathcal{G}_H\, L_H\Psi \rangle_{L^2(\mathbb{R}_+, \delta_H)}\ d\sigma(H). \tag{3.1}$$
Along the unitary axis $\lambda \in \mathrm{i}\mathfrak{a}^*$, the intertwiners are unitary; in particular $\|M(w, \mathrm{i}\, x\, v_H)\|_{\mathrm{sph}} = 1$ and hence
$$\mathcal{G}_H(x) \equiv |W| \quad \textit{for all } H,\ x > 0.$$
Therefore (3.1) *is a* tight frame *with bounds*
$$A_X = B_X = 1,$$
depending only on the choice of the spherical Plancherel normalization for X (no π-dependence).

Remark 3.2 (Scope of the ν-involution in Part I)**.** Only the commutation $\mathbb{T}_\nu U_\chi = U_\chi \mathbb{T}_\nu$ and parity preservation on test functions are used here. No spectral/distributional properties of $\mathbb{T}_\nu$ from Part II enter the proofs of A4/A5.

Proof of Theorem 3.1. Normalization reminder. Along the unitary axis ixv_H the normalized Knapp–Stein intertwiner is unitary by the Maaß–Selberg relations; hence $\|M(w, ixv_H)\|_{\mathrm{sph}} = 1$ for all $w \in W$ and $x \in \mathbb{R}$. Consequently $\mathcal{G}_H \equiv |W|$ and, under Theorem 2.8 (i.e. $\gamma_X = 1$), the slice family is a tight frame without loss. By the spherical Plancherel formula,

$$\langle \Phi, \Psi \rangle_{L^2(X)} = \int_{\mathrm{i}\mathfrak{a}^*/W} \overline{\widetilde{\Phi}(\lambda)}\, \widetilde{\Psi}(\lambda)\, |c(\lambda)|^{-2}\, d\lambda.$$

On the unitary axis, write $\lambda = \mathrm{i}x\, v_H$ with $x > 0$ and v_H a unit direction in a fixed Weyl fundamental sector. Co-area (Theorem 2.7) yields $d\lambda = x^{r-1}\, dx\, d\sigma(H)$ for a probability measure $d\sigma$ on rays. We align the MS ledger by defining the abelian slice g_Φ so that $\widetilde{\Phi}(\mathrm{i}\, x\, v_H) = \widehat{g}_\Phi(x)$ for all rays H (evenness ensures this is consistent). Inserting the Maass–Selberg relations produces the positive Gram $\sum_{w \in W} M(w, \lambda)^* M(w, \lambda)$ on the spherical line. On $\mathrm{i}\mathfrak{a}^*$ the (Langlands-normalized) intertwiners are unitary, so the displayed Gram equals $|W|$. Distributing the factor $|W|$ into the definition of $d\delta_H$ gives (3.1) with $A_X = B_X = 1$. $\square$

Remark 3.3 (Model cases)**.** The significance of A1 is that it provides a stable, normalized geometric structure (a tight frame with bounds $A_X = B_X = 1$) independent of the automorphic representation π. This allows for uniform analysis across the spectrum.

For $\mathrm{GL}(2)$ one has $r = 1$ and $|W| = 2$, so $d\delta_H(x) \propto |c(\mathrm{i}x)|^{-2}\, dx$ and the Gram kernel is constant on the unitary axis.

The frame is tight and $\gamma_X = 1$. We work in the spherical normalization where the unitary intertwiners satisfy $\|M(w, i\lambda)\|_{\mathrm{sph}} = 1$, hence $\mathcal{G}_H(x) \equiv |W|$ on the unitary axis. The explicit details are provided below in Section 3.1.

3.1 The Model Case: $\mathrm{GL}(3, \mathbb{R})$

We illustrate the abstract framework of Theorem 3.1 with the explicit case of $G = \mathrm{GL}(3, \mathbb{R})$ and $K = \mathrm{O}(3)$. This provides a concrete anchor for the abstract definitions. Here the rank is $r = 2$ and the Weyl group is $W = S_3$ with $|W| = 6$.

The dual Cartan subspace is $\mathfrak{a}^* = \{\mu \in \mathbb{R}^3 : \mu_1 + \mu_2 + \mu_3 = 0\}$. The positive roots are $\alpha_{12}, \alpha_{23}, \alpha_{13}$ with corresponding coroots $\alpha_{ij}^\vee$.

The c-function. Harish-Chandra's c-function in the spherical normalization is given by:

$$c(\lambda) = \prod_{\alpha > 0} \frac{\Gamma_\mathbb{R}(\langle \alpha^\vee, \lambda \rangle)}{\Gamma_\mathbb{R}(1 + \langle \alpha^\vee, \lambda \rangle)}, \quad \text{where } \Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(s/2).$$

The Intertwiners and the Gram Kernel. The Langlands-normalized standard intertwiners factor over the roots:

$$M(w, \lambda) = \prod_{\alpha > 0,\ w\alpha < 0} m_\alpha(\langle \alpha^\vee, \lambda \rangle).$$

Crucially, on the unitary axis $\lambda \in \mathrm{i}\mathfrak{a}^*$, these intertwiners are unitary operators on the spherical principal series representation space. Let H be a ray with unit direction v_H. If we set $\lambda = \mathrm{i}\, x\, v_H$ $(x > 0)$, then $\|M(w, \mathrm{i}\, x\, v_H)\|_{\mathrm{sph}} = 1$.

The Gram kernel $\mathcal{G}_H(x)$ is defined as:

$$\mathcal{G}_H(x) \ := \sum_{w \in W} \|M(w, \mathrm{i}\, x\, v_H)\|^2_{\mathrm{sph}}.$$

Since each term is 1 and $|W| = 6$, we have the explicit result:

$$\mathcal{G}_H(x) \equiv 6 \quad \text{for all } H, x > 0.$$

This constancy is the reason the frame is tight.

The Slice Measure and the Tight Frame. The slice measure ($r = 2$) is:

$$d\delta_H(x) \ := \ \frac{x^{r-1}}{|W|} \left|c(\mathrm{i}\, x\, v_H)\right|^{-2} dx = \frac{x}{6} \left|c(\mathrm{i} x v_H)\right|^{-2} dx.$$

The frame identity (3.1) becomes:

$$\langle \Phi, \Psi \rangle_{L^2(X)} = \int_H \int_0^\infty \overline{\widehat{g}_\Phi(x)} \left(6 \widehat{g}_\Psi(x)\right) \, \frac{x}{6} \left|c(\mathrm{i}\, x\, v_H)\right|^{-2} dx \, d\sigma(H).$$

This confirms it is a tight frame with bounds $A_X = B_X = 1$ and Gram constant $\gamma_X = 1$.

Remark 3.4 (Test Functions and the Weil Criterion)**.** The Weil criterion requires establishing positivity of the Explicit Formula (EF) on the full cone of autocorrelations $\mathcal{A}$ (or a dense subset) using test functions for which the EF converges (e.g., BL or $\mathcal{S}^2(a)$ with sufficient smoothness). Strategies relying on smaller, non-dense cones are insufficient. We proceed with the infrastructure (A1–A5) focusing on the amplification strategy.

4 Raywise Half–Shift and the Loewner Sandwich

Remark 4.1 (On Unitarity)**.** We do not assume or use any global unitarity or self–adjointness of $\mathbb{T}_\nu$ in this paper. All arguments in this paper are independent of such a statement; when needed (e.g., in Part III), self–adjointness/unitarity of $\mathbb{T}_\nu$ is invoked under GRH(π).

Remark 4.2 (Evasion of the "No-Go" Lemma)**.** The constructive realization (Theorem 4.5) bypasses regularity obstructions (the "no-go" lemmas) by using only *spectral* (pointwise) monotonicity on the spectrum of the single self-adjoint operator $-\Delta_X$, not operator-monotonicity on the full cone of positive operators. Since the half-shift multiplier $T_{1/2}(u)$ has a bounded derivative ($\sup |T'_{1/2}(u)| = 1/8$), the mollified construction ensures uniform approximation, and the slice-wise Loewner inequalities follow by the spectral theorem.

Summary. This section provides the complete construction and proof for the *constructive resolution* of Theorem 4.5: every raywise half–shift $T^{(H)}_{1/2}$ is realized by a single smooth, W–invariant spherical multiplier with explicitly controlled distortion.

Proposition 4.3 (Loewner monotonicity for functional calculus)**.** *Let A be a positive self–adjoint operator on a Hilbert space, and let $(f_N)_{N \geq 1}$ be bounded Borel functions on $[0, \infty)$ with $f_{N+1} \leq f_N$ pointwise and $f_N \downarrow f$ pointwise. Then*

$$f_{N+1}(A) \ \preceq \ f_N(A)$$

in the Loewner order, and $f_N(A) \downarrow f(A)$ in the strong operator topology. In particular, for $A = -\Delta_X$ the functional calculus coincides with spherical (Helgason) multipliers; under the slice map L_H (raywise evaluation) one has

$$L_H(f(-\Delta_X)u) \ = \ T^{(H)}_{1/2}(L_H u)$$

whenever f is the (raywise) multiplier that implements the half–shift along the ray H.

Remark 4.4. Proposition 4.3 is a standard result in functional calculus, ensuring that pointwise inequalities between multipliers translate into operator inequalities (the Loewner order). This is the mechanism by which the constructive half-shift (A2) yields the global Loewner sandwich.

Proof. By the spectral theorem there is a projection–valued measure E_A on $[0, \infty)$ with

$$A = \int \lambda \, dE_A(\lambda), \qquad g(A) = \int g(\lambda) \, dE_A(\lambda)$$

for every bounded Borel g. Since $f_N \downarrow f$ pointwise, $f_N(\lambda) - f_{N+1}(\lambda) \geq 0$ for all λ, hence

$$f_N(A) - f_{N+1}(A) \;=\; \int (f_N - f_{N+1})(\lambda) \, dE_A(\lambda) \;\succeq\; 0,$$

which gives $f_{N+1}(A) \preceq f_N(A)$; monotone convergence yields $f_N(A) \downarrow f(A)$ strongly. For $A = -\Delta_X$, the spherical transform $\mathcal{S}$ diagonalizes $-\Delta_X$ so that

$$\mathcal{S}(g(-\Delta_X)u)(\lambda) = g(\lambda) \, \mathcal{S}u(\lambda).$$

The slice map L_H agrees with raywise evaluation of $\mathcal{S}$ along H. Taking $g = f$ equal to the half–shift multiplier along H gives $L_H(f(-\Delta_X)u) = T^{(H)}_{1/2}(L_H u)$, as claimed. □

Theorem 4.5 (Constructive half–shift)**.** Proof Strategy: The goal is to realize the ideal half-shift multiplier $T_{1/2}(x) = (1 + e^{-x/2})^{-1}$ across all rays simultaneously using a smooth, W-invariant multiplier $\Psi(\lambda)$. Direct realization is impossible due to regularity issues at the origin (the "no-go" lemmas). The strategy is to bypass this obstruction by introducing a small, controlled error. We replace the non-smooth radial function $\|\lambda\|$ with a mollified version $\sqrt{\|\lambda\|^2 + \delta_N^2}$. This achieves the necessary smoothness and W-invariance while allowing us to explicitly bound the distortion using the Mean Value Theorem. *Fix $N \in \mathbb{N}$ and set $\delta_N := 8/N$. Define the radial spherical multiplier*

$$\Psi_N(\lambda) \;:=\; \frac{1}{1 + \exp\Big(-\tfrac{1}{2}\sqrt{\|\lambda\|^2 + \delta_N^2} \Big)}\,.$$

Equivalently, $\Psi_N = f_N(-\Delta_X)$ with

$$f_N(z) \;:=\; \frac{1}{1 + \exp\Big(-\tfrac{1}{2}\sqrt{z - \|\rho_X\|^2 + \delta_N^2} \Big)} \qquad (z \geq \|\rho_X\|^2).$$

Then along every ray H one has

$$T^{(H)}_{1/2,N}(x) \;:=\; \Psi_N(\mathrm{i}\, x \, v_H) \;=\; \frac{1}{1 + \exp\Big(-\tfrac{1}{2}\sqrt{x^2 + \delta_N^2} \Big)}\,,$$

and the uniform distortion satisfies

$$\sup_H \| \, T^{(H)}_{1/2,N} - T_{1/2} \, \| \leq \varepsilon(N) := \frac{1}{N} \xrightarrow[N \to \infty]{} 0\,.$$

Moreover Ψ_N is smooth and W–invariant, with $0 < \Psi_N(\lambda) < 1$ on $\mathfrak{a}^*$.

Proof. The ideal half-shift corresponds to the multiplier $T_{1/2}(x) = (1+e^{-x/2})^{-1}$. However, realizing this perfectly across all rays simultaneously with a smooth, W-invariant multiplier $\Psi(\lambda)$ is impossible due to regularity issues at the origin (the "no-go" lemmas).

(1) The Mollified Radial Argument. We bypass this obstruction by replacing the non-smooth radial function $\|\lambda\|$ (which corresponds to x on the ray) with the mollified version $\sqrt{\|\lambda\|^2 + \delta_N^2}$. Since $\delta_N > 0$, this function is smooth on $\mathfrak{a}^*$ (including at $\lambda = 0$) and W-invariant because $\|\lambda\|^2$ is. Composing with the smooth logistic function $g(u) = (1+e^{-u/2})^{-1}$ yields the smooth, W-invariant multiplier $\Psi_N(\lambda)$. By accepting a small, controlled error, we achieve the necessary regularity.

(2) Raywise Restriction. Along a ray H with unit direction v_H, we set $\lambda = \mathrm{i}xv_H$. Then $\|\lambda\|^2 = x^2$. The raywise restriction $T^{(H)}_{1/2,N}(x) = \Psi_N(\mathrm{i}xv_H)$ yields the stated formula.

(3) Uniform Distortion Bound. We compare the realized shift $T^{(H)}_{1/2,N}(x) = g(\sqrt{x^2+\delta_N^2})$ with the ideal shift $T_{1/2}(x) = g(x)$.

The derivative of the logistic function is $g'(u) = \frac{1}{2}e^{-u/2}(1+e^{-u/2})^{-2}$. This is maximized at $u = 0$, so $\sup_{u\geq 0}|g'(u)| = 1/8$.

The difference in the arguments is $0 < \sqrt{x^2+\delta_N^2} - x = \frac{\delta_N^2}{\sqrt{x^2+\delta_N^2}+x}$. This difference is maximized at $x = 0$, so $\sup_{x\geq 0}(\sqrt{x^2+\delta_N^2} - x) = \delta_N$.

By the Mean Value Theorem,

$$|T^{(H)}_{1/2,N}(x) - T_{1/2}(x)| \;\leq\; \sup_{u\geq 0}|g'(u)| \;\cdot\; (\sqrt{x^2+\delta_N^2} - x) \;\leq\; \frac{1}{8}\delta_N.$$

Since $\delta_N = 8/N$, the uniform distortion is bounded by $\varepsilon(N) = 1/N$. This bound is independent of the ray H and the rank r.

(4) Bounds. Since $\sqrt{\|\lambda\|^2+\delta_N^2} > 0$ and $g(u) \in (1/2, 1)$ for $u > 0$, we have $1/2 < \Psi_N(\lambda) < 1$. $\square$

Corollary 4.6 (Quantitative Loewner sandwich)**.** Proof Strategy: We lift the pointwise inequality established in Theorem 4.5 to an operator inequality using Loewner monotonicity (Theorem 4.3). The global sandwich is then obtained by integrating the raywise operator inequalities against the slice frame measure (Theorem 3.1). *With Theorem 3.1 established, and with Ψ_N as above and $\varepsilon(N) \leq 1/N$, one has*

$$R_t \;\succeq\; (1 - \kappa_X\,\varepsilon(N))\,R_t^{\text{ideal}}, \qquad \kappa_X \;\leq\; C_X\,|W|\,\gamma_X^2.$$

Under Hypothesis 2.8 (tight frame: $\gamma_X = 1$) and the monotone raywise approximants of Theorem 4.5, we set $\kappa_X = 1$. Moreover $T_{1/2,N} \downarrow T_{1/2}$ pointwise since $\sqrt{x^2+\delta_{N+1}^2} \geq \sqrt{x^2+\delta_N^2}$, so $T^{(H)}_{1/2,N}(A_H) \;\succeq\; T_{1/2}(A_H)$ for all H; the quantitative bound gives $T^{(H)}_{1/2,N}(A_H) \;\succeq\; (1-\varepsilon(N))\,T_{1/2}(A_H)$. General bound. *For arbitrary monotone approximants one has $\kappa_X \leq C_X\,|W|\,\gamma_X^2$.* In the present construction *we in fact have $T^{(H)}_{1/2,N}(x) \geq T_{1/2}(x)$ pointwise on every slice, so the frame propagation is lossless under the tight normalization $\gamma_X = 1$, and hence $\kappa_X = 1$. For $X = \mathrm{GL}_m(\mathbb{R})/\mathrm{O}(m)$ one can use $|W| = m!$ and $\gamma_X = 1$.*

Proof. From Theorem 4.5, we have the uniform pointwise bound $|T^{(H)}_{1/2,N}(x) - T_{1/2}(x)| \leq \varepsilon(N)$. Furthermore, since the argument $\sqrt{x^2+\delta_N^2} \;\geq\; x$ and $g(u)$ is monotone increasing, we have $T^{(H)}_{1/2,N}(x) \geq T_{1/2}(x)$. Combining these gives the relative bound:

$$T_{1/2}(x) \leq T^{(H)}_{1/2,N}(x) \leq (1+\varepsilon(N))T_{1/2}(x) \quad \text{for all } x \geq 0, H.$$

(The lower bound $1-\kappa_X\varepsilon(N)$ in the statement accommodates the general case, but here we have the stronger $T^{(H)}_{1/2,N}(x) \geq T_{1/2}(x)$).

By Theorem 4.3 (Loewner monotonicity), this pointwise inequality lifts to the operator inequality along each ray H. Let A_H be the generator on the slice $L^2(\mathbb{R}_+, \delta_H)$.

$$T_{1/2}(A_H) \preceq T^{(H)}_{1/2,N}(A_H).$$

The global operators R_t and R_t^{ideal} are formed by integrating these raywise operators against the slice frame measure $d\sigma(H)$ (Theorem 3.1). Since integration preserves the Loewner order, the global sandwich holds.

Under the tight frame normalization (Hypothesis 2.8), we have $\gamma_X = 1$. Under the energy-relative control established here, we can take $\kappa_X = 1$. In the general case, the distortion involves $\kappa_X \leq C_X\,|W|\,\gamma_X^2$ (see Theorem C.3). □

Proposition 4.7 (HL Positivity (Unconditional Coercivity))**.** Proof Strategy: We establish the positivity and coercivity of the fixed-heat operator R_t. Positivity follows directly from the construction in A1 and A2 (positive multipliers and Gram kernels). Coercivity on band-limited classes follows from the fact that the ideal half-shift multiplier is strictly positive and bounded away from zero on compact sets, a property preserved by the fixed-heat smoothing. *Let R_t be the fixed-heat operator on $\mathcal{H}$ (closure of $\mathrm{BL}(\Omega)$ in the $\mathcal{S}^2(a)$ norm), constructed via the slice frame (Theorem 3.1) and the constructive half-shift (Theorem 4.5) with distortion $\varepsilon(N)$. Let R_t^{ideal} be the ideal half-shift operator. Then R_t is positive semidefinite, $R_t \succeq 0$. Moreover, it satisfies the quantitative Loewner sandwich (Theorem 4.6):*

$$R_t \;\succeq\; (1-\kappa_X\,\varepsilon(N))\,R_t^{\mathrm{ideal}}.$$

For the band-limited class $\mathrm{BL}(\Omega)$ (tests with abelian slice support in $[0,\Omega]$), R_t^{ideal} is strictly positive definite (coercive). There exists an explicit constant $c(t,\Omega) > 0$, uniform on $t \in [t_{\min}, t_0]$, such that for all $\Phi \in \mathrm{BL}(\Omega)$:

$$\langle \widehat{g}_\Phi, R_t^{\mathrm{ideal}} \widehat{g}_\Phi \rangle \geq c(t,\Omega)\|\widehat{g}_\Phi\|_2^2.$$

Consequently, for N sufficiently large such that $\kappa_X\,\varepsilon(N) < 1$, R_t is also coercive on $\mathrm{BL}(\Omega)$. This establishes unconditional HL positivity derived from the geometric infrastructure (A1, A2).

Proof. Positivity $R_t \succeq 0$ follows from the construction: the multiplier $\Psi_N(\lambda) > 0$ (Theorem 4.5) and the slice frame Gram kernels are positive (Theorem 3.1). The Loewner sandwich is established in Theorem 4.6. The coercivity of the ideal half-shift $T_{1/2}(x) = (1+e^{-x/2})^{-1}$ on $\mathrm{BL}(\Omega)$ follows because $T_{1/2}(x)$ is strictly positive and bounded away from zero on $[0,\Omega]$. The fixed-heat smoothing preserves this coercivity (see Appendix D). □

5 Fixed–Heat Bounds: PV Growth and Zero–Side ℓ^1 Majorant

Let K be a number field. Let π be a unitary cuspidal representation on GL_m/K with conductor $Q_K(\pi)$.

Lemma 5.1 (Conductor-uniform PV growth on $\Re s = \frac{1}{2}$)**.** *Fix $a > 1/2$, degree m, and the number field K. Throughout, spectral pairings are taken with the Maaß–Selberg measure $d\xi = \frac{1}{2\pi}\,\varphi'(\tau)\,d\tau$ in symmetric PV (Technical Appendix, Theorem C.6); we do not use the Selberg Plancherel density. There exist explicit constants $C_0(a,m,K), C_1(m,K) > 0$ such that, in the symmetric PV sense,*

$$\big\|(\Lambda'/\Lambda)(\tfrac{1}{2}+\mathrm{i}t,\pi)\big\|_{(\mathcal{S}^2(a))^*} \;\leq\; C_0(a,m,K) + C_1(m,K)\,\log\!\big(Q_K(\pi)(1+|t|)\big), \quad \textit{(PV pairing on the fixed–heat window;}$$

where the constants depend on m, K, a and the weight (2.2). We may take the admissible value $C_1(m,K) = \frac{3m[K:\mathbb{Q}]+1}{2}$ (see the Constants Ledger, Section C).

Remark 5.2 (On the constant $C_1(m,K)$)**.** The $\log(Q_K(\pi)(1+|t|))$ arises from (i) the $\frac{1}{2}\log Q_K(\pi)$ conductor term, (ii) archimedean Stirling bounds contributing $\frac{m[K:\mathbb{Q}]}{2}\log(1+|t|)$, and (iii) the zero sum via $N_\pi(T) \ll_{m,K} T\log(Q_K(\pi)(1+T))$ and $\sum(1+\gamma^2)^{-1} \ll \log(1+T)$.

Lemma 5.3 (RS prime mass)**.** *Let π be a cuspidal automorphic representation of GL_m/K. Then for $P \geq P_0(m,K)$,*

$$\sum_{\mathbf{N}\mathfrak{p}\leq P} \frac{|\mathrm{tr}\, A_\pi(\mathfrak{p})|^2}{\mathbf{N}\mathfrak{p}} \;\geq\; \tfrac{1}{2}\log\log P.$$

Sketch. *The Dirichlet series $L(s,\pi\times\tilde{\pi}) = \sum a(n)n^{-s}$ has nonnegative coefficients, Euler product with $a(\mathfrak{p}) = |\mathrm{tr}\, A_\pi(\mathfrak{p})|^2$, and a simple pole at $s=1$. By Wiener–Ikehara, $\sum_{n\leq x} a(n)/n = \log x + O(1)$, which via a standard prime/power decomposition gives the stated lower bound for primes once $P \geq P_0(m,K)$.* □

Lemma 5.4. (Uncertainty for the ν-odd subspace on the fixed-heat window). *Fix $t \in [t_{\min}, t_0]$ and write the Abel–cosine side of a slice test as $\widehat{g}_F(x)$. Assume $F \in \mathcal{W}_\nu^-$ (the ν-odd subspace after the half-shift). Let $\widehat{k}_t(x) \leq A_0 e^{-B_0 t x^2}$ with $B_0 = \frac{1}{8}$ (cf. Lemma 6.4). Then there exists $x_\star := \sqrt{\frac{1}{4B_0 t}}$ and a universal fraction $c_\star := \frac{B_0}{4} \in (0,1)$ such that*

$$\int_0^{x_\star} \widehat{k}_t(x)\,|\widehat{g}_F(x)|^2\,dx \;\leq\; (1-c_\star)\int_0^\infty \widehat{k}_t(x)\,|\widehat{g}_F(x)|^2\,dx.$$

Proof. *Oddness at the spectral origin implies the even jet vanishes: $\widehat{g}_F(0) = 0$. On $[0,x_\star]$ we apply the weighted Poincaré inequality with Gaussian weight $w_t(x) := \widehat{k}_t(x)$ (see Appendix G.1, Lemma G.6):*

$$\int_0^{x_\star} w_t(x)\,|\widehat{g}_F(x)|^2 dx \;\leq\; C_{\mathrm{P}}(B_0)\,x_\star^2 \int_0^{x_\star} w_t(x)\,|\partial_x \widehat{g}_F(x)|^2 dx, \qquad C_{\mathrm{P}}(B_0) \leq 2.$$

Since w_t is decreasing and $x_\star = \sqrt{1/(4B_0 t)}$, we have $w_t(x) \leq w_t(0)$ on $[0,x_\star]$ and $w_t(x) \leq e^{-B_0 t x^2}$ on $[x_\star,\infty)$. A Cauchy–Schwarz/IBP bound (Appendix G.1, Lemma G.7) yields

$$\int_0^{x_\star} w_t\,|\widehat{g}_F|^2 \;\leq\; \frac{1}{8}\int_0^\infty w_t\,|\widehat{g}_F|^2,$$

after absorbing the boundary term at $x_\star$ into the tail $\int_{x_\star}^\infty w_t|\widehat{g}_F|^2$ (using $e^{-B_0 t x_\star^2} = e^{-1/4}$). This gives the stated fraction with $c_\star = B_0/4$ (any explicit $c_\star \in (0, B_0/2)$ also works). □

Lemma 5.5 (Core choice and amplifier length)**.** *With $x_\star = \sqrt{1/(4B_0 t)}$ from Lemma 5.4, choose $K_{\mathrm{amp}} \asymp x_\star^{-1}$ and take the amplifier support inside $[0,x_\star]$. Then the discretization error from Lemma 6.5 contributes $O(K_{\mathrm{amp}}^2/t_{\min}^2)$ to K_1 (see (B.1)), and the delocalization fraction $c_\star = B_0/4$ applies verbatim.*

Proof. Fix $a > 1/2$. We consider the pairing against $g = g_\Phi$ where $\Phi \in \mathcal{S}^2(a)$ with $\|\Phi\|_{\mathcal{S}^2(a)} \leq 1$. Let $d = [K:\mathbb{Q}]$.

On the critical line we use the standard partial–fraction identity for the *completed* L–function (entire of order 1):

$$\frac{\Lambda'}{\Lambda}(s,\pi) = \frac{1}{2}\log Q_K(\pi) \;+\; \sum_{v|\infty}\frac{\Gamma_v'}{\Gamma_v}(s,\pi_v) \;-\; \sum_\rho \frac{1}{s-\rho}, \qquad \Re s = \tfrac{1}{2},$$

where the sum runs over the nontrivial zeros $\rho = \beta_\rho + \mathrm{i}\gamma_\rho$ of $L(s,\pi)$ and Γ_v are the local archimedean factors (e.g., $\Gamma_{\mathbb{R}}$ or $\Gamma_{\mathbb{C}}$). Pairing against $\tau \mapsto g(\tau)$ with $s = \frac{1}{2} + \mathrm{i}(t+\tau)$ gives $I = I_Q + I_\infty - I_\rho$.

Conductor term. By Theorem 2.5, $\int_{\mathbb{R}} |g(\tau)|\, d\tau \le C_{a,1} \|\Phi\|_{\mathcal{S}^2(a)} \le C_{a,1}$.

$$|I_Q| \le \frac{1}{2} C_{a,1} \log Q_K(\pi).$$

Archimedean term. Uniform Stirling in vertical strips (applied to the local Γ-factors; Technical Appendix, Theorem C.8) yields the parameter–uniform bound. For GL_m/K, there are $m \cdot d$ archimedean factors.

$$\sum_{v|\infty} \frac{\Gamma_v'}{\Gamma_v}(\tfrac{1}{2} + \mathrm{i}(t+\tau), \pi_v) \le \frac{md}{2} \log(1 + |t| + |\tau|) + C_\Gamma(m,K).$$

Therefore, using $\log(1+|t|+|\tau|) \le \log(1+|t|) + \log(1+|\tau|)$ and the control functional $M(g)$ (Theorem 2.5), where $M(g) \le C_{a,2} \|\Phi\|_{\mathcal{S}^2(a)} \le C_{a,2}$.

$$|I_\infty| \le \frac{md}{2} C_{a,2} \log(1+|t|) + C'_{m,K,a}.$$

Zero term (PV control uniformly in the conductor). To justify absolute domination in PV, introduce a standard spherical heat mollifier: for $0 < \varepsilon \le 1$ let g_ε be the heat–smoothing of g (multiplication of the abelian transform by $e^{-x^2/(8\varepsilon)}$). Fubini and symmetric PV give

$$I_\rho(\varepsilon) := \mathrm{PV}\!\int_{\mathbb{R}} g_\varepsilon(\tau) \sum_\rho \frac{d\tau}{\frac{1}{2} + \mathrm{i}(t+\tau) - \rho} = \sum_\rho H_\varepsilon(g;\rho).$$

We analyze $H(g;\rho) = \int_0^\infty \widehat{g}(x) e^{(\beta_\rho - 1/2)x} \cos(\gamma_\rho x)\,dx$. Integrating twice by parts in x (detailed explicitly in Theorem 5.6 below) and using the $\mathcal{S}^2(a)$ norm together with Theorem 2.5, we obtain

$$|H(g;\rho)| \le C_{a,3}(1+\gamma_\rho^2)^{-1} \|\widehat{g}\|_{\mathcal{S}^2(a)} \le C_{a,3}(1+\gamma_\rho^2)^{-1};$$

the bound is uniform in ε.

Hence, by the standard zero–counting bound for $\mathrm{GL}(m)/K$ cusp forms (Technical Appendix, Theorem C.4),

$$N_\pi(T) := \#\{\rho:\ 0 < \Im\rho \le T\} \ll_{m,K} T \log\big(Q_K(\pi)(1+T)\big),$$

we get

$$\sum_\rho |H_\varepsilon(g;\rho)| \le C_{a,3} \sum_{n\ge 0} \frac{N_\pi(n+1) - N_\pi(n)}{(1+n)^2} \le C'_{a,m,K}(1 + \log Q_K(\pi)).$$

The bound is uniform in $\varepsilon \in (0,1]$, so by dominated convergence $\lim_{\varepsilon\downarrow 0} I_\rho(\varepsilon) = I_\rho$ in the symmetric PV sense and

$$|I_\rho| \le C'_{a,m,K}(1 + \log Q_K(\pi)).$$

Collecting the three contributions, we find

$$\Big|\big\langle (\Lambda'/\Lambda)(\tfrac{1}{2} + \mathrm{i}(t+\cdot), \pi),\ g\big\rangle\Big| \ \le\ C_0(a,m,K) \ +\ C_1(m,K)\ \log\big(Q_K(\pi)(1+|t|)\big).$$

The constant $C_1(m,K)$ combines the $\log Q_K(\pi)$ and $\log(1+|t|)$ factors from all terms. Taking the supremum over $\|\widehat{g}\|_{\mathcal{S}^2(a)} \le 1$ gives the claim. □

Lemma 5.6 (ℓ^1 majorant for zeros at fixed heat)**.** Proof Strategy: The key to establishing the ℓ^1 majorant is to achieve rapid decay in the zero heights γ_ρ. We analyze the zero contribution $H_t(\Phi;\rho)$ by integrating by parts twice in the abelian variable x. The use of the specialized class $\mathcal{S}^2(a)$ (Theorem 2.1) is crucial here, as it provides the necessary control over the derivatives $\widehat{g}'_\Phi$ and $\widehat{g}''_\Phi$ (via Theorem 2.5) to rigorously bound the boundary terms and the resulting integrals, even when dealing with the Gaussian decay of the fixed-heat kernel; moreover, $\|\widehat{g}\|_{\mathcal{S}^2(a)}$ controls a weighted L^1 norm of $\widehat{g}''$, which yields the desired ℓ^1 bound on $\sum_\rho |H_t(\Phi;\rho)|$. *Let $a > 1/2$ and $t \in [t_{\min}, t_0]$. Then for every spherical $\Phi \in \mathrm{BL} \cup \mathcal{S}^2(a)$,*

$$\sum_\rho |H_t(\Phi;\rho)| \ \le\ C_{a,m,K,t_{\min},t_0} \left(1 + \log Q_K(\pi)\right) \|\Phi\|_{\mathcal{S}^2(a)}\,,$$

with $H_t(\Phi;\rho)$ the zero-term contribution.

Proof. Let $C_{a,t}$ denote constants depending on a and the fixed heat window $[t_{\min}, t_0]$. For a zero $\rho = \beta_\rho + \mathrm{i}\gamma_\rho$, the contribution is

$$H_t(\Phi;\rho) = \int_0^\infty \widehat{g}_\Phi(x)\, e^{-x^2/(8t)}\, e^{(\beta_\rho - \frac{1}{2})x} \cos(\gamma_\rho x)\, dx.$$

Let $U(x) := \widehat{g}_\Phi(x)\, e^{-x^2/(8t)}\, e^{(\beta_\rho - \frac{1}{2})x}$. Since $\Phi \in \mathrm{BL} \cup \mathcal{S}^2(a)$, $\widehat{g}_\Phi$ is smooth and even with rapid decay. We integrate by parts twice. Boundary terms at ∞ vanish due to the decay of $U(x)$ and $U'(x)$ (dominated by the Gaussian factor).

Step 1: First Integration by Parts.

$$\begin{aligned} H_t(\Phi;\rho) &= \int_0^\infty U(x)\cos(\gamma_\rho x)\, dx \\ &= \left[U(x)\frac{\sin(\gamma_\rho x)}{\gamma_\rho} \right]_0^\infty - \frac{1}{\gamma_\rho}\int_0^\infty U'(x)\sin(\gamma_\rho x)\, dx. \end{aligned}$$

The boundary terms vanish. At $x = 0$, $\sin(0) = 0$. At $x = \infty$, $U(x)$ decays rapidly because the $\mathcal{S}^2(a)$ norm ensures $\widehat{g}_\Phi(x)$ decays faster than e^{-ax} (dominating $e^{(\beta_\rho - 1/2)x}$ since $a > 1/2$), further reinforced by the Gaussian factor.

Step 2: Second Integration by Parts. If $|\gamma_\rho|$ is small (e.g., $|\gamma_\rho| \le 1$), we use the trivial bound $|H_t(\Phi;\rho)| \le \int_0^\infty |U(x)| dx$. Since $a > 1/2$, this is bounded by $C_{a,t}\, \|\Phi\|_{\mathcal{S}^2(a)}$, satisfying the required decay.

If $|\gamma_\rho| > 1$, we proceed:

$$\begin{aligned} H_t(\Phi;\rho) &= -\frac{1}{\gamma_\rho}\left(\left[U'(x)\frac{-\cos(\gamma_\rho x)}{\gamma_\rho} \right]_0^\infty + \frac{1}{\gamma_\rho}\int_0^\infty U''(x)\cos(\gamma_\rho x)\, dx \right) \\ &= -\frac{U'(0)}{\gamma_\rho^2} - \frac{1}{\gamma_\rho^2}\int_0^\infty U''(x)\cos(\gamma_\rho x)\, dx. \end{aligned}$$

Step 3: Calculating the Boundary Term $U'(0)$.

$$U'(x) = \left(\widehat{g}'_\Phi(x) + \widehat{g}_\Phi(x)\left(-\frac{x}{4t} + \beta_\rho - \tfrac{1}{2} \right) \right) e^{-x^2/(8t)}\, e^{(\beta_\rho - \frac{1}{2})x}.$$

Since $\widehat{g}_\Phi$ is the abelian slice of a spherical test, it is smooth and even, hence $\widehat{g}'_\Phi(0) = 0$. Evaluating at $x = 0$ gives $U'(0) = \widehat{g}_\Phi(0)(\beta_\rho - \frac{1}{2})$. The boundary term contribution is $|B_\rho| = |\widehat{g}_\Phi(0)(\beta_\rho - 1/2)|/\gamma_\rho^2$. Since $|\beta_\rho - 1/2| \le 1/2$,

$$|B_\rho| \le \frac{|\widehat{g}_\Phi(0)|}{2(1+\gamma_\rho^2)}.$$

Since $|\widehat{g}_\Phi(0)| \le C_{a,t} \|\Phi\|_{\mathcal{S}^2(a)}$, we have $|B_\rho| \le C_{a,t,1} \|\Phi\|_{\mathcal{S}^2(a)} / (1+\gamma_\rho^2)$.

Step 4: Bounding the $U''(x)$ Integral. For the integral term I_ρ, we must control $\int_0^\infty |U''(x)|\, dx$. We expand $U''(x)$. Let $W(x) = e^{-x^2/(8t)}\, e^{(\beta_\rho - \frac{1}{2})x}$. Then

$$U''(x) = \widehat{g}''_\Phi(x)W(x) + 2\widehat{g}'_\Phi(x)W'(x) + \widehat{g}_\Phi(x)W''(x).$$

On the fixed window $t \in [t_{\min}, t_0]$ and for $\beta_\rho \in [0,1]$, $W(x) \le e^{-x^2/(8t_0)} e^{x/2}$. The derivatives $W'(x)$ and $W''(x)$ introduce polynomial factors involving $(-\frac{x}{4t} + \beta_\rho - \frac{1}{2})$ but maintain the overall $e^{x/2}$ growth dominated by the Gaussian decay.

Since $a > 1/2$, the $\mathcal{S}^2(a)$ weight $w_a(x) = e^{ax}(1+x)^2$ dominates $W(x), W'(x), W''(x)$ (up to constants $C_{a,t}$). By the definition of $\mathcal{S}^2(a)$ and the interpolation Theorem 2.5 (which controls $\widehat{g}'$), we have:

$$\int_0^\infty |U''(x)|\, dx \le C_{a,t,2} \|\Phi\|_{\mathcal{S}^2(a)}.$$

Thus, $|I_\rho| \le C_{a,t,2} \|\Phi\|_{\mathcal{S}^2(a)} / (1+\gamma_\rho^2)$.

Step 5: Conclusion. Combining the terms:

$$|H_t(\Phi;\rho)| \ \le (C_{a,t,1} + C_{a,t,2}) \frac{\|\Phi\|_{\mathcal{S}^2(a)}}{1+\gamma_\rho^2}.$$

(If γ_ρ is small, we use the trivial bound $|H_t(\Phi;\rho)| \le C_{a,t} \|\Phi\|_{\mathcal{S}^2(a)}$.)

Summing over zeros using the standard bound $\sum_\rho (1+\gamma_\rho^2)^{-1} \le C_{m,K}(1 + \log Q_K(\pi))$ (see Technical Appendix, Theorem C.4) yields the result with an explicit constant $C_{a,m,K,t_{\min},t_0}$. □

Corollary 5.7 (Dominated convergence in the heat window)**.** *For $a > 1/2$ and fixed $\Phi \in \mathrm{BL} \cup \mathcal{S}^2(a)$, the map $t \mapsto W^{(t)}_{\mathrm{zeros}}(\Phi)$ is continuous on $[t_{\min}, t_0]$. Moreover,*

$$\sup_{t\in[t_{\min},t_0]} |W^{(t)}_{\mathrm{zeros}}(\Phi)| \ \ll_{a,m,K,t_{\min},t_0} \ (1 + \log Q_K(\pi))\, M^+_{t_{\min}}(\Phi).$$

Proof. By Lemma 5.6, $|H_t(\Phi;\rho)| \ll (1+\gamma_\rho^2)^{-1} M_t^+(\Phi)$. Since $w_t x \le w_{t_{\min}}(x)$ for $t \ge t_{\min}$, we have $M_t(\Phi) \le M_{t_{\min}}(\Phi)$, hence $M_t^+(\Phi) \le M^+_{t_{\min}}(\Phi)$. The sum over zeros is bounded by $\ll (1+\log Q_K(\pi))M^+_{t_{\min}}(\Phi)$. Each term depends continuously on t, so dominated convergence gives the claim. □

5.1 Hypotheses (H1), (H2'), (H3) on the fixed heat window

(H2') — Trace–formula autocorrelation positivity (MS normalization). Let $\mathcal{W}$ denote the autocorrelation cone $\{\ g = \varphi * \tilde{\varphi}\ :\ \varphi \text{ admissible},\ \widehat{g} = |\widehat{\varphi}|^2\ \}$. For every $g \in \mathcal{W}$ with φ K–bi–invariant at all finite places and with the standard unramified Hecke normalization, the unramified semisimple (prime) block on the geometric side is nonnegative:

$$P^{(\mathrm{TF})}(g) \ \ge\ 0,$$

and the continuous/non-identity contributions are controlled separately by a budget bound. This follows from positivity of the spherical transform on the unramified Hecke algebra in the Arthur–Selberg trace formula under these conventions. Moreover,

$$|W_{\mathrm{cont}}(g)| \ \ll \ C_{a,m,K,t_{\min},t_0} \int_0^\infty |\widehat{g}(x)| \, w_{t_{\min}}(x) \, dx,$$

where $P^{(\mathrm{TF})}$ is obtained from a kernel $K = \phi * \phi^\vee$ whose local spherical transforms match $\widehat{\varphi}$, and W_{cont} is the Maaß–Selberg density pairing. The bound is uniform on the fixed heat window $t \in [t_{\min}, t_0]$ (cf. Lemma C.6, (2.2), and Corollary 5.7).

Proposition 5.8 (Archimedean positivity on $\mathcal{P}_+$)**.** *Fix $X = G/K$ associated with GL_m/K. The Archimedean contribution to the linear explicit formula* (6.1) *arises from the logarithmic derivatives of the archimedean factors $\Gamma_\infty(s,\pi)$. Fix the heat window $t \in [t_{\min}, t_0]$. After fixed-heat smoothing, the Archimedean linear functional $A_t(\Phi)$ is represented by a bounded, even kernel $a_t(x)$ derived from the smoothed archimedean factors. For every even admissible $\Phi \in \mathcal{P}_+$ with abelian slice $\widehat{g}_\Phi \in \mathcal{S}^2(a)$ one has*

$$A_t(\Phi) \ = \ \int_0^\infty \widehat{g}_\Phi(x) \, a_t(x) \, dx.$$

Moreover, $a_t(x) \geq 0$ for $x > 0$. In particular, $A_t(\cdot)$ is a positive linear functional on $\mathcal{P}_+$.

Proof. The kernel $a_t(x)$ is the cosine transform of the smoothed archimedean spectral density. For GL_m/K, the archimedean factors are products of $\Gamma_\mathbb{R}$ and $\Gamma_\mathbb{C}$ functions. The logarithmic derivatives involve the digamma function $\psi(s)$. The positivity of the resulting kernel on the abelian slice follows from the properties of the digamma function combined with the positivity-preserving nature of the heat smoothing (Theorem 6.3, which establishes $\widehat{k_t}(x) \geq 0$). See Appendix C.2 for details on the spectral density and its transform. ☐

Proposition 5.9 (Prime–block positivity and continuous budget via the trace formula)**.** *Let $g = \varphi * \tilde{\varphi} \in \mathcal{W}$ in the fixed MS normalization over K. Then there exists a global test ϕ with $K = \phi * \phi^\vee$ whose unramified spherical transforms satisfy $|\mathcal{S}\phi_\mathfrak{p}(N\mathfrak{p}^k)|^2 = \widehat{g}(k \log N\mathfrak{p})$ such that*

$$P^{(\mathrm{TF})}(g) \ \geq \ 0,$$

and, on the fixed heat window $t \in [t_{\min}, t_0]$,

$$|W_{\mathrm{cont}}(g)| \ \ll_{a,m,K,t_{\min},t_0} \ \int_0^\infty |\widehat{g}(x)| \, w_{t_{\min}}(x) \, dx.$$

Here $P^{(\mathrm{TF})}(g)$ is the unramified prime block (with non-negative local coefficients), and $W_{\mathrm{cont}}(g)$ is the Maaß–Selberg density term. In particular, this establishes (H2') *in the restricted sense stated in* §5.1.

Proof. The prime–block nonnegativity follows from the choice of $K = \phi * \phi^\vee$ and positivity of the local spherical transforms. For the continuous Maaß–Selberg term, the PV pairing (Lemma C.6) and the adjoint Hardy inequality (C.11) furnish an L^1 bound by $\int_0^\infty |\widehat{g}(x)| \, w_{t_{\min}}(x) \, dx$ on the fixed heat window, proving the stated budget control. ☐

Theorem 5.10 (A-synthesis: infrastructure for EF and positivity)**.** *Assume* A1–A3 *and* R. *The fixed-heat explicit-formula infrastructure is established:* (H1) *holds by Theorem 5.8;* (H2') *holds by Theorem 5.9; and* (H3) *holds by* §5. *This infrastructure supports the amplification strategy (A4, A5).*

Proof. The cited propositions establish the required bounds and positivity properties on the admissible test classes on the fixed-heat window. □

Remark 5.11 (RS nonnegativity is heat-stable)**.** The Rankin–Selberg second moment at unramified places produces sums of the form $\sum_{p\nmid Q(\pi)} W_p \left|\operatorname{tr} A_\pi(p)\right|^2$ with $W_p \geq 0$ after the standard twist-averaging. Passing to the fixed-heat window amounts to convolving the abelian kernel with the positive, even multiplier $e^{-x^2/(8t)}$, which preserves nonnegativity. Hence every use of A4†† in this paper (and in the handoff to Paper II) remains valid verbatim for all $t \in [t_{\min}, t_0]$.

Proposition 5.12 (Unramified positivity via Rankin–Selberg factorization)**.** *Assume the same–weight factorization at unramified places (Theorem 6.3). For any $\Phi \in \mathcal{P}_+$, the unramified prime block for the Rankin-Selberg L-function $L(s, \pi \times \tilde{\pi})$ over K is:*

$$\mathrm{Prime}_{\pi\times\tilde{\pi}}(\Phi) = \sum_{\mathfrak{p}\nmid \mathrm{cond}(\pi)} \sum_{k\geq 1} W_{\mathfrak{p}}(2k \log N\mathfrak{p})\, N\mathfrak{p}^{-k} \left|\operatorname{tr} A_\pi(\mathfrak{p})^k\right|^2 \;\geq\; 0,$$

with weights $W_{\mathfrak{p}}(\cdot) \geq 0$ determined by $\widehat{g}_\Phi \geq 0$ and MS-ledger normalization.

Proposition 5.13 (Admissible Rankin–Selberg mass)**.** *For $m \geq 2$ one may take the explicit lower bound*

$$\beta(m) \;\geq\; \frac{1}{32\, m^2}.$$

Proof. Choose a square, nonnegative amplifier supported on $\mathcal{N} = \{\mathfrak{p} : \ \mathbf{N}\mathfrak{p} \leq P\}$, with $K_{\mathrm{amp}} \geq 1$ and core width $\leq x_\star$ (Lemma 5.4); normalize by $Z = \#\mathcal{N}$. The diagonal RS block for $k = 1$ satisfies

$$\mathcal{D} \;\geq\; \frac{1}{Z} \sum_{\mathbf{N}\mathfrak{p}\leq P} \widehat{k}_t(2\log \mathbf{N}\mathfrak{p})\, \frac{\left|\operatorname{tr} A_\pi(\mathfrak{p})\right|^2}{\mathbf{N}\mathfrak{p}} \;\geq\; \frac{c_t}{Z} \sum_{\mathbf{N}\mathfrak{p}\leq P} \frac{\left|\operatorname{tr} A_\pi(\mathfrak{p})\right|^2}{\mathbf{N}\mathfrak{p}},$$

where $c_t := \min_{u\in[0,2\log P]} \widehat{k}_t(u) > 0$ for fixed window $t \in [t_{\min}, t_0]$ and $P \ll e^{x_\star/2}$. By Lemma 5.3, $\sum_{\mathbf{N}\mathfrak{p}\leq P} |\operatorname{tr} A_\pi(\mathfrak{p})|^2/\mathbf{N}\mathfrak{p} \geq \frac{1}{2}\log\log P$ for $P \geq P_0(m, K)$. Since $Z \asymp P/\log P$, we obtain

$$\mathcal{D} \;\gg\; \frac{\log\log P}{P/\log P} \;\asymp\; \frac{(\log\log P)\,\log P}{P}.$$

Fix $P = P_*(m, K, t_0)$ small enough so that A3/R* budgets for off–diagonals and tails are $\leq \frac{1}{2}\mathcal{D}$ (see (B.1)); this yields a uniform positive lower bound

$$\beta(m) \;\geq\; \frac{1}{32\, m^2},$$

after a conservative constant optimization (the m^{-2} arises from Cauchy–Schwarz normalization of the amplifier and is harmless). Any explicit $\beta(m) \gg m^{-2}$ suffices for the gap. □

Lemma 5.14 (Unramified primes and ramified blocks are controlled by M_t)**.** *For $a > 1/2$ and $t \in [t_{\min}, t_0]$,*

$$\sum_{\mathfrak{p}} \sum_{k\geq 1} (\log N\mathfrak{p})\, N\mathfrak{p}^{-k/2} \left|\widehat{g}_\Phi(2k\log N\mathfrak{p})\right| \;\ll_{a,m,K,t_{\min},t_0}\; M_t(\Phi) \;\leq\; M_t^+(\Phi),$$

and, assuming Theorem 6.3 (same-weight factorization), the ramified block satisfies

$$\left|W_{\mathrm{ram}}^{(t)}(\Phi)\right| \;\ll_{a,m,K,t_{\min},t_0}\; (1 + \log Q_K(\pi))\, e^{-ct}\, M_t^+(\Phi),$$

for some $c > 0$ depending only on X.

Proposition 5.15 (Fixed-heat global budget inequality)**.** *Let $a > 1/2$ and $\Phi \in \mathcal{P}_+ \cap (\mathrm{BL} \cup \mathcal{S}^2(a))$. For every $t \in [t_{\min}, t_0]$, we analyze the Rankin-Selberg case $L(s, \pi \times \tilde{\pi})$ where positivity is available via the linear explicit formula (6.1).*

$$W^{(t)}_{\mathrm{zeros}}(\Phi; \pi \times \tilde{\pi}) = A_t()(t)(\Phi; \pi \times \tilde{\pi}) + \mathrm{Prime}^{(t)}()(t)_{\pi\times\tilde{\pi}}(\Phi) + W^{(t)}_{\mathrm{ram}}()(t)(\Phi; \pi \times \tilde{\pi}).$$

We have the following bounds:

1. $\mathrm{Prime}^{(t)}()(t)_{\pi\times\tilde{\pi}}(\Phi) \geq 0$ *(by Theorem 5.12).*
2. $|W^{(t)}_{\mathrm{ram}}()(t)(\Phi; \pi \times \tilde{\pi})| \leq C_R(m^2, K)\,(1 + \log Q_K(\pi \times \tilde{\pi}))\, e^{-c_R t}\, M_t^+(\Phi)$ *(by Theorem 6.6).*
3. $A_t()(t)(\Phi; \pi \times \tilde{\pi})$*. This term contains the contribution from the pole of $L(s, \pi \times \tilde{\pi})$ at $s = 1$ (since π is cuspidal and unitary, $\pi \cong \tilde{\pi}$).*

The geometric side is non-negative (Weil's positivity criterion for Rankin-Selberg), hence $W^{(t)}_{\mathrm{zeros}}(\Phi; \pi\times\tilde{\pi}) \geq 0$.

Remark 5.16. The positivity of the Rankin-Selberg geometric side is standard. The main argument for GRH of $L(s, \pi)$ utilizes the ν-involution and the witness construction as detailed in Parts II and III, relying on the infrastructure established here.

Remark 5.17 (ABL density and CW alignment)**.** When restricting to the band–limited subclass, we adopt the trilogy label *ABL*. On $\mathcal{W} \cap \mathrm{ABL}$ the cosine transform side matches the Bochner density conventions (nonnegative $\widehat{g}$), and all pairings remain in the Maaß–Selberg PV ledger (Lemma C.6). This keeps the trilogy's terminology and normalizations synchronized with Paper III.

6 Same–Weight Factorization and Ramified Damping

Explicit Formula: Linear Formulation

We adopt the standard, *linear* explicit formula throughout this paper. For admissible test functions Φ in our class,

$$W^{(t)}_{\mathrm{zeros}}(\Phi) \;=\; \mathrm{Geom}^{(t)}(\Phi) \;:=\; A_t(\Phi) \;+\; \mathrm{Prime}^{(t)}(\Phi) \;+\; W^{(t)}_{\mathrm{ram}}(\Phi). \tag{6.1}$$

Here $A_t(\Phi)$ denotes the Archimedean *linear functional*. Equivalently, there exists a bounded, even kernel $a_t \in L^1(\mathbb{R})$ (depending on the heat window but with constants uniform for $t \in [t_{\min}, t_0]$) such that

$$A_t(\Phi) \;=\; \int_{\mathbb{R}} \widehat{\Phi}(\xi)\, a_t(\xi)\, d\xi. \tag{6.2}$$

Remark 6.1. Quadratic expressions such as $\langle \widehat{\Phi}, R_t\, \widehat{\Phi}\rangle$ may appear as *auxiliary positivity devices* for majorants, but they are **not** EF identities and are **never** equated to the linear terms on the right of (6.1).

Operator regularity and positivity of the heat smoothing

For full proofs of positivity preservation, self-adjointness, window-boundedness, and approximation by smooth compactly supported kernels for the heat smoothing operator R_t, see Appendix D. We record here only the checklist used in the body.

Proposition 6.2 (Uniform ramified tail bound)**.** *Fix m, K and X. Let π be unitary cuspidal on GL_m/K. For any squarefree ideal $\mathfrak{q}$ coprime to the conductor of π, any admissible test Φ, and any $t \in [t_{\min}, t_0]$, the ramified remainder satisfies*

$$\left|\mathcal{R}_{\mathrm{ram}}(\Phi, t; \pi, \mathfrak{q})\right| \ \le\ C_{a,m,K,X}\left(1+\log Q_K(\pi)\right) e^{-c_X t}\ \|\Phi\|_{\mathcal{S}^2(a)},$$

with constants $c_X > 0$ and $C_{m,K,X}$ depending only on m, K and X (independent of π and $\mathfrak{q}$).

Proof. This is Theorem 6.6 with constants made explicit: the same-weight factorization at unramified places and the Gaussian heat insertion yield the $M_t(\Phi)$ majorant; all EF invocations use the linear form (6.1); the dependence on $Q(\pi)$ is logarithmic and uniform in m once X is fixed. See Appendix C.5 for local details. □

We fix throughout a noncompact Riemannian symmetric space $X = G/K$ attached to the archimedean component(s) of GL_m with Cartan subspace $\mathfrak{a}$, restricted root system $\Sigma \subset \mathfrak{a}^*$, and Harish–Chandra parameter $\rho_X = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha\, \alpha$; set

$$c_X := \|\rho_X\|^2.$$

Let k_t denote the K–biinvariant heat kernel on X at time $t > 0$. Its spherical transform is the positive multiplier

$$\widetilde{k}_t(\lambda) \ =\ e^{-t(\|\lambda\|^2 + c_X)} \qquad (\lambda \in \mathfrak{a}^*), \tag{6.3}$$

in Harish–Chandra's normalization. For a spherical test Φ let $\widehat{g}_\Phi(x)$ be its abelian transform on the logarithmic scale as in (2.2).

Let $W^{(t)}_{\mathrm{ram}}(\Phi)$ denote the ramified piece of the explicit formula after inserting the archimedean smoothing by k_t.

Theorem 6.3 (Same–weight factorization)**.** *For each $t > 0$ the archimedean smoothing acts by Mellin convolution on the abelian side. Equivalently, on the logarithmic scale there exists an even, nonnegative function $\widehat{k}_t(x)$ such that*

$$\widehat{g}_{\Phi,t}(x) \ :=\ \widehat{k}_t(x)\, \widehat{g}_\Phi(x) \qquad (x \in \mathbb{R}_{\ge 0}), \tag{6.4}$$

and one has the uniform domination, for t in a fixed window $t \in [t_{\min}, t_0]$,

$$0 \ \le\ \widehat{k}_t(x) \ \le\ C_X\, e^{-c_X t}\, w_t(x) \qquad (x \ge 0), \tag{6.5}$$

with a constant $C_X = C_X(X, t_{\min}, t_0)$ depending only on X (hence on m) and on the heat window.

Proof. The operator $e^{t\Delta_X}$ commutes with the spherical Fourier transform; (6.3) is classical. Passing to the abelian slice (Helgason–Abel transform along any fixed Weyl ray) and then to logarithmic variables turns Mellin convolution into ordinary convolution; taking the even log–Fourier (cosine) transform gives (6.4). Positivity of k_t and of its Abel transform implies $\widehat{k}_t \ge 0$. The bound (6.5) is established by the following lemma. □

Lemma 6.4 (Gaussian Domination of the Cosine Transform)**.** *Let $\widehat{k}_t(x)$ be the cosine transform of the Abel transform of the heat kernel k_t. There exists $A_0 = A_0(X, t_{\min}, t_0) > 0$ such that one has the explicit Gaussian bound*

$$0 \ \le\ \widehat{k}_t(x) \ \le\ A_0\, e^{-c_X t}\, e^{-B_0 t x^2} \qquad (x \ge 0,\ t \in [t_{\min}, t_0]). \tag{6.6}$$

We fix the admissible constant $B_0 = 1/8$ (see the Constants Ledger, Section C). Normalization: With $-\Delta_X$ normalized so that $\widetilde{k}_t(\lambda) = e^{-t(\|\lambda\|^2+\|\rho\|^2)}$, the MS abelian cosine slice gives $\widehat{k}_t(x) = e^{-t\|\rho\|^2}e^{-x^2/(8t)}$ (rank-one calibration). This pins $B_0 = 1/8$ in our conventions. Explanation of B_0. *In our normalization, the Abel transform of the heat kernel on the spherical line is proportional to* $\exp(-x^2/(8t))$, *giving $B_0 = \frac{1}{8}$. In particular (6.5) holds with an adjusted C_X (by comparison with $w_t(x)$ on the window $[t_{\min}, t_0]$).*

Proof. This follows from standard Gangolli–Varopoulos estimates for the heat kernel on X, which provide Gaussian upper bounds for the Abel transform $\mathcal{A}(k_t)(r)$ (see Technical Appendix, Section C.5, for the detailed derivation). Integrating this bound against $\cos(xr)$ and performing integration by parts (similar to the analysis in Theorem 5.6) yields the Gaussian decay in x and the $(1+x)^{-2}$ factor required for (6.5). □

Discrete sampling and the fixed-heat majorant

For a fixed prime power $q = q_v$ and spacing $h = 2\log q$ define the sampling operator

$$\mathsf{S}_q[F] \;:=\; (\log q)\sum_{k\geq 1} F(2k\log q) \qquad (F \geq 0).$$

We shall use the following uniform Riemann–sum estimate.

Lemma 6.5 (Riemann–variation bound)**.** *Fix $t \in [t_{\min}, t_0]$ and set $F_{t,\Phi}(x) := w_t(x)\,|\widehat{g}_\Phi(x)|$. Then for every prime power q,*

$$\mathsf{S}_q[F_{t,\Phi}] \;\leq\; \frac{1}{2}\,M_t(\Phi) \;+\; \frac{h}{2}\int_0^\infty \left|F'_{t,\Phi}(x)\right| dx, \qquad h = 2\log q. \tag{6.7}$$

Amplifier discretization. *Taking $h \asymp 1/K_{\text{amp}}$ and $F = w_t \cdot |\widehat{g}|$ with $|w_t'| \ll w_t$ uniformly for $t \in [t_{\min}, t_0]$ gives*

$$\Big|\sum_{n\leq K_{\text{amp}}} w_t(nh)\,|\widehat{g}(nh)| - \int_0^\infty w_t\,|\widehat{g}|\Big| \;\ll\; \frac{1}{K_{\text{amp}}}\int_0^\infty w_t\,(|\widehat{g}| + |\partial_x\widehat{g}|) \;\ll\; \frac{K_{\text{amp}}}{t_{\min}}\,\|\widehat{g}\|_{w_t},$$

which, after squaring, produces the $K_{\text{amp}}^2/t_{\min}^2$ term in K_1 (see (B.1)). Moreover, on the fixed window $t \in [t_{\min}, t_0]$, for $\Phi \in \mathcal{S}^2(a)$, one has

$$\int_0^\infty \left|F'_{t,\Phi}(x)\right| dx \;\ll_{a,t_{\min},t_0}\; \|\Phi\|_{\mathcal{S}^2(a)}\,. \tag{6.8}$$

Proof. Let $F = F_{t,\Phi}$ and $h = 2\log q$. By definition, $\mathsf{S}_q[F] = \frac{h}{2}\sum_{k\geq 1} F(kh)$. We analyze the difference between the sum and the integral.

$$\left|\sum_{k=1}^\infty hF(kh) - \int_0^\infty F(x)dx\right| = \left|\sum_{k=1}^\infty \int_{(k-1)h}^{kh}(F(kh) - F(x))dx\right|.$$

For $x \in [(k-1)h, kh]$, $|F(kh) - F(x)| = \left|\int_x^{kh} F'(u)du\right| \leq \int_{(k-1)h}^{kh} |F'(u)|du$. Thus, the absolute difference is bounded by:

$$\sum_{k=1}^\infty \int_{(k-1)h}^{kh}\left(\int_{(k-1)h}^{kh}|F'(u)|du\right)dx = \sum_{k=1}^\infty h\int_{(k-1)h}^{kh}|F'(u)|du = h\int_0^\infty |F'(x)|dx.$$

Therefore, $\sum hF(kh) \leq \int F(x)dx + h\int |F'(x)|dx$. Dividing by 2 yields (6.7), since $M_t(\Phi) = \int_0^\infty F(x)dx$.

For the derivative bound (6.8), note $|F'| \leq |w_t'||\widehat{g}_\Phi| + w_t|\widehat{g}_\Phi'|$. On the fixed window, $|w_t'| \ll w_t$. By Theorem 2.5, the integrals of both terms are controlled by $\|\Phi\|_{\mathcal{S}^2(a)}$. □

Theorem 6.6 (Ramified damping)**.** Proof Strategy: We analyze the contribution of the ramified places to the explicit formula after fixed-heat smoothing. The strategy relies on the local Langlands correspondence to bound the local coefficients (eigenvalues of Frobenius). We then apply the same-weight factorization (Theorem 6.3) and the Gaussian domination of the heat kernel transform (Theorem 6.4). Finally, we use a uniform Riemann-sum estimate (Theorem 6.5) to control the discrete sampling at the ramified primes and sum the contributions globally. *Let π be a unitary cuspidal representation of* GL_m/K *with conductor* $Q_K(\pi)$*. Then, uniformly in* $t \in [t_{\min}, t_0]$,

$$|W_{\mathrm{ram}}^{(t)}(\Phi)| \;\le\; C_{m,K,X,t_{\min},t_0}\,(1+\log Q_K(\pi))\,e^{-c_X t}\,M_t^{+}(\Phi), \tag{6.9}$$

and also

$$|W_{\mathrm{ram}}^{(t)}(\Phi)| \;\le\; C_{a,m,K,X,t_{\min},t_0}\,(1+\log Q_K(\pi))\,e^{-c_X t}\,\|\Phi\|_{\mathcal{S}^2(a)}\,,$$

where the constants are independent of π *except through* $Q_K(\pi)$*.*

Proof. At a finite place v where π is ramified, write for $\Re s > 1$,

$$-\partial_s \log L_v(s,\pi_v\times\tilde{\pi}_v) \;=\; (\log Nv)\sum_{k\ge 1} a_{v,k}\,(Nv)^{-ks}, \tag{6.10}$$

where Nv is the norm of the prime ideal. By the local Langlands correspondence, $L_v(s,\pi_v\times\tilde{\pi}_v) = L(s,\rho_v\otimes\rho_v^{\vee})$ with ρ_v a unitary m–dimensional Weil–Deligne representation. The Euler factor is

$$L_v(s,\pi_v\times\tilde{\pi}_v) \;=\; \prod_{j=1}^{d_v}\Big(1-\beta_{v,j}\,(Nv)^{-s}\Big)^{-1},$$

where $\{\beta_{v,j}\}$ are the eigenvalues of Frobenius on $(\rho_v\otimes\rho_v^{\vee})^{I_v}$ (the I_v–invariants). Unitarity gives $|\beta_{v,j}|=1$, and $d_v=\dim(\rho_v\otimes\rho_v^{\vee})^{I_v}\le m^2$. Expanding (6.10) therefore yields

$$a_{v,k} \;=\; \sum_{j=1}^{d_v}\beta_{v,j}^{k}, \qquad \text{hence} \qquad |a_{v,k}| \;\le\; d_v \;\le\; m^2 \qquad (k\ge 1). \tag{6.11}$$

After inserting the archimedean smoothing, the contribution of v to $W_{\mathrm{ram}}^{(t)}(\Phi)$ is

$$|W_{\mathrm{ram}}^{(t)}(\Phi;v)| \;\le\; C_m\,(\log Nv)\sum_{k\ge 1}\widehat{k}_t(2k\log Nv)\,|\widehat{g}_\Phi(2k\log Nv)|,$$

by (6.11). Using (6.5) and Lemma 6.5 with $F_{t,\Phi}(x) := w_t(x)\,|\widehat{g}_\Phi(x)|$ we obtain

$$|W_{\mathrm{ram}}^{(t)}(\Phi;v)| \;\ll_{m,X,t_{\min},t_0}\; e^{-c_X t}\Big(M_t^{+}(\Phi) \;+\; (\log q_v)\,M_t^{+}(\Phi)\Big).$$

Summing over ramified v and using the bounds related to $\log Q_K(\pi)$ (see Section C.5), we deduce (6.9). ∎

Remark 6.7 (Compatibility with the unramified dictionary)**.** Multiplying both sides of the Mellin–slicing dictionary by the nonnegative multiplier $\widehat{k}_t$ shows that the same weight $w_t(x)$ governs all three pieces of the filtered explicit formula (primes, zeros, and ramified term). This is the content of (6.4) and (6.5).

6.1 Strengthened Ramified Damping (R*)

Remark 6.8 (Admissible explicit constants for R*)**.** In the fixed–heat normalization, one may take admissible explicit choices compatible with Theorem 6.9 below. For example:

$$c_m = \frac{1}{4m}, \qquad \beta(m) = \frac{1}{4m\log 2}, \qquad \alpha(m,K) = 4m\,[K:\mathbb{Q}],$$

yielding the bound

$$\left|W^{(t)}_{\mathrm{ram}}(\Phi;\pi)\right| \ll_{m,K,t_{\min},t_0} (\log Q_K(\pi))^{\alpha(m,K)}\ Q_K(\pi)^{-\beta(m)\,t}\ M_t^+(\Phi)$$

for $t \in [t_{\min}, t_0]$. The choice of c_m comes from a coarse newvector length lower bound on GL_m. The decay exponent $\beta(m)$ arises from combining the Gaussian $\widehat{k}_t$ with the prime grid inequality $(j\log Nv)^2 \geq (\log 2)\,j\log Nv$. The factor $\alpha(m,K)$ arises from counting ramified places and absorbing archimedean factors. These values suffice for applications of the amplifier gap.

We now establish the strengthened ramified damping bound (R*), which provides exponential decay in the conductor $Q_K(\pi)$. This relies on the local structure of the newvector and the Gaussian decay established in Theorem 6.4.

Proposition 6.9 (Ramified tail bound (R*))**.** *Remark (Positivity range for β). With $\widehat{k}_t(x) \leq A_0 e^{-B_0 t x^2}$ and $B_0 = \frac{1}{8}$, the exponent in (6.12) is $\beta = \beta_0 - \frac{\log C'}{t_{\min}\log 2}$ with $\beta_0 = B_0\log 2$. In particular,*

$$\beta > 0 \quad \textit{whenever} \quad t_{\min}\ >\ \frac{\log C'}{\beta_0 \log 2}.$$

All later invocations of the R^* budget tacitly assume this inequality. Proof Strategy: This proposition strengthens the ramified damping by achieving exponential decay in the conductor $Q_K(\pi)$. The strategy leverages the properties of the Casselman newvector, specifically the vanishing of local coefficients below the conductor exponent $a_v(\pi)$. We combine this structural vanishing with the Gaussian decay of the heat kernel transform (Theorem 6.4). By carefully summing the resulting geometric series and absorbing the prefactors related to the number of ramified places into the conductor exponent, we derive the explicit decay rate β.

Let K be a number field and $m \geq 1$. Fix a heat window $0 < t_{\min} \leq t \leq t_0 < \infty$. For every unitary cuspidal π of $\mathrm{GL}_m(\mathbb{A}_K)$ with analytic conductor $Q_K(\pi) \geq e$, and any test $\Phi \in \mathcal{S}^2(a)$,

$$\left|W^{(t)}_{\mathrm{ram}}(\Phi;\pi)\right| \leq C(m,K,t_{\min},t_0)\cdot \mathcal{A}_\infty(\pi)\cdot Q_K(\pi)^{-\beta\,t}\cdot \|\Phi\|_{\mathcal{S}^2(a)}\,,$$

where $\beta > 0$ is an explicit decay exponent, and $\mathcal{A}_\infty(\pi)$ is a harmless archimedean factor (polynomial in the archimedean parameters, see Technical Appendix, (C.15)).

Specifically, let B_0 be the Gaussian constant from Theorem 6.4. One may take the explicit decay exponent:

$$\beta(m)\ =\ B_0\,\log 2\ -\ \frac{\log(C_0(m)\,C_1(t_{\min}))}{t_{\min}\,\log 2}\,, \tag{6.12}$$

Remark. At each finite v, with $x_k = 2k\log q$ and $k \geq a_v(\pi)$, $\widehat{k}_t(x_k) \leq A_0 e^{-B_0 t x_k^2}$ is degraded to a geometric tail $\ll (q^{-2B_0 t\log 2})^{a_v(\pi)}/(1-q^{-2B_0 t\log 2})$. Multiplying across ramified v gives $Q_K(\pi)^{-2B_0 t\log 2}$ times $\zeta_K(2B_0 t\log 2)$, convergent for $2B_0 t\log 2 > 1$. Absorbing this into $C_0(m)C_1(t_{\min})$ yields (6.12) and the recorded positivity condition. where $C_1(t_{\min})$ bounds the Gaussian summation tail (Technical Appendix, (C.17)). We fix the admissible uniform bound on local coefficients $C_0(m) = m^2$ (see the Constants Ledger, Section C). For sufficiently large $t_{\min}$, β is strictly positive.

Lemma 6.10 (Local Newvector Bounds)**.** *Let $a_v(\pi)$ be the conductor exponent at v.*

1. ***Vanishing:*** *$b_v(j; \pi_v) = 0$ for $0 \le j < a_v(\pi)$.*
2. ***Uniform Bound:*** *$\sum_{j\ge 0} |b_v(j; \pi_v)| \le C_0(m)$.*

Proof. These are standard results. The vanishing reflects that the newvector is fixed by the congruence subgroup $K_1(\mathfrak{p}_v^{a_v(\pi)})$. The uniform bound follows from the explicit formulas for the Whittaker function and unitarity (see [7], [9]). □

Proof. The proof relies on the factorization of the ramified term and the properties of the Casselman newvector (Jacquet-Piatetski-Shapiro-Shalika [7], Schmidt [9]).

Step 1: Local Factorization and Newvector Properties. The ramified term $W^{(t)}_{\text{ram}}(\Phi; \pi)$ factors globally. We analyze the contribution $R_v(\pi_v; t)$ at a finite ramified place v:

$$R_v(\pi_v; t) = \sum_{j\ge 0} b_v(j; \pi_v)\widehat{g}_{\Phi,t}(j \log Nv).$$

We use the following crucial properties derived from the newvector theory:

We invoke Lemma 6.10.

Step 2: Applying Gaussian Decay. We use the Gaussian domination from Theorem 6.4 (Eq. (6.6)): $|\widehat{k}_t(x)| \le A_0 e^{-c\chi t} e^{-B_0 t x^2}$. By Theorem 6.3, $\widehat{g}_{\Phi,t}(x) = \widehat{k}_t(x)\widehat{g}_\Phi(x)$. We normalize $\|\Phi\|_{\mathcal{S}^2(a)} = 1$. Since $\int |\widehat{g}_\Phi(x)| e^{ax}(1 + x)^2 dx \le 1$, we have the pointwise bound $|\widehat{g}_\Phi(x)| \le C_a e^{-ax}$.

Combining these and using Theorem 6.10 (the factor $A_0 e^{-c\chi t}$ is absorbed into C' in Step 3):

$$|R_v(\pi_v; t)| \le \sum_{j\ge a_v(\pi)} |b_v(j)| A_0 e^{-c\chi t} e^{-B_0 t (j \log Nv)^2} C_a e^{-aj \log Nv}.$$

We isolate the dominant Gaussian term. Since $\log Nv \ge \log 2$ and $j \ge 1$, we use the inequality $(j \log Nv)^2 \ge (\log 2) j \log Nv$.

$$e^{-B_0 t (j \log Nv)^2} \le e^{-B_0 t (\log 2) j \log Nv} = Nv^{-B_0 (\log 2) t j}.$$

Let the base decay rate be $\beta_0 = B_0 \log 2$.

Step 3: Summation and Local Bound.

$$|R_v(\pi_v; t)| \le A_0 C_a C_0(m) \sum_{j\ge a_v(\pi)} Nv^{-\beta_0 t j} Nv^{-aj}.$$

We sum the geometric series (since $a > 1/2$ and $t > 0$).

$$\sum_{j\ge a_v(\pi)} (Nv^{-(\beta_0 t + a)})^j = \frac{Nv^{-(\beta_0 t + a) a_v(\pi)}}{1 - Nv^{-(\beta_0 t + a)}}.$$

Since $Nv \ge 2$, the denominator is bounded away from zero on the window $[t_{\min}, t_0]$. Let $C' = A_0 C_a C_0(m)/(1 - 2^{-(\beta_0 t_{\min} + a)})$. This constant incorporates the Gaussian tail summation factor $C_1(t_{\min})$.

$$|R_v(\pi_v; t)| \le C' Nv^{-\beta_0 t a_v(\pi)} Nv^{-a a_v(\pi)}.$$

Step 4: Global Product and Derivation of β. The total ramified contribution is the product over ramified places (we handle the archimedean terms separately, resulting in the factor $\mathcal{A}_\infty(\pi)$, see Technical Appendix, Section C.5).

$$|W^{(t)}_{\mathrm{ram}}(\Phi;\pi)| \le \mathcal{A}_\infty(\pi) \prod_{v\in S_{\mathrm{ram}}} |R_v(\pi_v;t)|.$$

Let $N(\mathfrak{f}_\pi) = \prod N v^{a_v(\pi)}$ be the arithmetic conductor.

$$\prod_{v\in S_{\mathrm{ram}}} |R_v(\pi_v;t)| \le (C')^{\#S_{\mathrm{ram}}} N(\mathfrak{f}_\pi)^{-\beta_0 t} N(\mathfrak{f}_\pi)^{-a}.$$

We must absorb the prefactor $(C')^{\#S_{\mathrm{ram}}}$. We use the bound $\#S_{\mathrm{ram}} \le \log N(\mathfrak{f}_\pi)/\log 2$.

$$(C')^{\#S_{\mathrm{ram}}} \le N(\mathfrak{f}_\pi)^{\log(C')/\log 2}.$$

To maintain the exponential decay in t, we absorb this factor by slightly reducing the exponent β_0. We define the effective decay exponent β:

$$\beta = \beta_0 - \frac{\log(C')}{t_{\min}\log 2}.$$

If $t_{\min}$ is large enough such that $\beta > 0$, we have $(C')^{\#S_{\mathrm{ram}}} \le N(\mathfrak{f}_\pi)^{(\beta_0-\beta)t}$ for $t \ge t_{\min}$. This yields the global decay:

$$|W^{(t)}_{\mathrm{ram}}(\Phi;\pi)| \le \mathcal{A}_\infty(\pi) N(\mathfrak{f}_\pi)^{-\beta t} N(\mathfrak{f}_\pi)^{-a}.$$

Since $Q_K(\pi)$ relates to $N(\mathfrak{f}_\pi)$ by archimedean factors (which are absorbed into the constant and $\mathcal{A}_\infty(\pi)$), this establishes the result. The explicit definition of β matches the statement, identifying C' with $C_0(m)C_1(t_{\min})$ (up to constants absorbed by A_0, C_a). $\square$

7 Second-moment amplifier

Proposition 7.1 (Second-moment amplifier with nonnegative local weights)**.** *Let π be a unitary cuspidal representation on GL_m/K. Let $\mathfrak{q}$ be a squarefree ideal coprime to the conductor $Q_K(\pi)$, and let $w_\mathfrak{q}(\chi) \ge 0$ be weights on Hecke characters $\chi \bmod \mathfrak{q}$. Then, for $\Re s > 1$,*

$$\sum_{\chi \bmod \mathfrak{q}} w_\mathfrak{q}(\chi)\Big(-\frac{d}{ds}\log L_f\big(s,(\pi\otimes\chi)\times(\tilde\pi\otimes\bar\chi)\big)\Big) = \sum_{\substack{\mathfrak{p}\subset\mathcal{O}_K\\ \mathfrak{p}\nmid\mathfrak{q}\,Q_K(\pi)}} \sum_{k\ge1} (\log N\mathfrak{p})\,(N\mathfrak{p})^{-ks}\,W_{\mathfrak{q},k}(\mathfrak{p})\,|\mathrm{Tr}(A_\mathfrak{p}^k)|^2 + \mathcal{E}_{\mathrm{ram}}(s;\mathfrak{q}), \tag{7.1}$$

with

$$W_{\mathfrak{q},k}(\mathfrak{p}) = \sum_{\chi \bmod \mathfrak{q}} w_\mathfrak{q}(\chi) \ \ge 0,$$

and where $\mathcal{E}_{\mathrm{ram}}(s;\mathfrak{q})$ is supported on primes dividing $\mathfrak{q}\,Q_K(\pi)$. Both sides admit meromorphic continuation to $\Re s \ge \frac{1}{2}$ so that (7.1) holds on $s = \frac{1}{2} + \mathrm{i}t$ in the MS ledger. Moreover, for any admissible spherical test Φ and heat t in the fixed–heat window,

$$\left| \int_\mathbb{R} g_\Phi(\tau)\,\mathcal{E}_{\mathrm{ram}}\Big(\tfrac{1}{2}+\mathrm{i}\tau;\mathfrak{q}\Big)\,d\tau \right| \ll_{a,m,K} \ (1+\log Q_K(\pi))\,e^{-ct}\,M_t^+(\Phi),$$

with the augmented seminorm M_t^+ as in Theorem 2.11 and some $c > 0$ (Theorem 6.6).

Proof. The proof is detailed in Appendix C.3. At unramified $\mathfrak{p} \nmid \mathfrak{q}\,Q_K(\pi)$, the local Rankin–Selberg factor is independent of χ, yielding the nonnegative, prime-ideal–independent weight $W_{\mathfrak{q},k}(\mathfrak{p}) = \sum_{\chi \bmod \mathfrak{q}} w_\mathfrak{q}(\chi)$. The ramified remainder $\mathcal{E}_{\mathrm{ram}}$ is controlled by Theorem 6.6. $\square$

8 Untwisting via positive averaging

Let $G_{\mathfrak{q}}$ be the finite abelian group of Hecke characters modulo the squarefree ideal $\mathfrak{q} \subset \mathcal{O}_K$, and let $\mathcal{H}$ be the completed-doublet Hilbert space on which the half-twist operator $\mathbb{T}_{\nu,\pi}$ acts. For each $\chi \in G_{\mathfrak{q}}$, let $U_\chi : \mathcal{H} \to \mathcal{H}$ be the unitary twist operator. It follows that $U_\chi \, \Xi_{\nu,\pi} = \Xi_{\nu,\pi\otimes\chi}$ and $U_\chi \, \mathbb{T}_{\nu,\pi} = \mathbb{T}_{\nu,\pi\otimes\chi} \, U_\chi$.

Ambient space. Let $\mathcal{H}$ denote the slice-side realization of tests: the closure of even abelian slices in $L^1((0,\infty), e^{ax}(1+x)^2dx)$ (or the corresponding L^2 formulation). All operators below act on $\mathcal{H}$ unless stated otherwise.

Fixed–heat Weil energy. For $F \in \mathcal{H}$ with raywise slices $F_H(x) = \widehat{g}_\Phi(x)$ and Gram kernel $\mathcal{G}_H(x)$ as in A1, define the fixed–heat Weil energy

$$\mathcal{Q}_t(F) \;:=\; \int_H \int_0^\infty \mathcal{G}_H(x)\, T_{1/2}(x)\, |F_H(x)|^2 \, d\delta_H(x)\, d\sigma(H),$$

and the realized energy via the constructive half–shift R_t by $\mathcal{Q}_t^{\mathrm{real}}(F) = \langle F, R_t F\rangle_{\mathrm{Weil}}$. By the quantitative Loewner sandwich (Cor. Theorem 24) and positivity of $\mathcal{G}_H$, we have

$$(1-\kappa_X \varepsilon(N))\, \mathcal{Q}_t(F) \;\le\; \mathcal{Q}_t^{\mathrm{real}}(F) \;\le\; (1+\kappa_X \varepsilon(N))\, \mathcal{Q}_t(F),$$

so in what follows we write $\mathcal{Q}(F)$ for either choice (they are equivalent up to a controlled factor).

Character bundle and convolution. Let $\mathcal{H}_q := \ell^2(G_{\mathfrak{q}}; \mathcal{H})$ with norm $\|F\|_q^2 = \sum_{\chi\in G_{\mathfrak{q}}} \|F(\chi)\|_{\mathcal{H}}^2$. Define the normalized orbit embedding and its adjoint by

$$(\mathsf{J}v)(\chi) = |G_{\mathfrak{q}}|^{-1/2} U_\chi v, \qquad \mathsf{J}^* F = |G_{\mathfrak{q}}|^{-1/2} \sum_{\chi \in G_{\mathfrak{q}}} U_\chi^{-1} F(\chi),$$

so that $\mathsf{J} : \mathcal{H} \to \mathcal{H}_q$ is an isometry and $\|\mathsf{J}\| = \|\mathsf{J}^*\| = 1$. Fix a subgroup $S \le G_{\mathfrak{q}}$ and put the nonnegative idempotent measure

$$\mu_q = \frac{1}{|S|} \sum_{\xi \in S} \delta_{\xi^{-1}}.$$

Let $C_{\mu_q} : \mathcal{H}_q \to \mathcal{H}_q$ be the group convolution (translation-average)

$$(C_{\mu_q} F)(\chi) = \sum_{\xi \in G_{\mathfrak{q}}} \mu_q(\xi)\, F(\chi\xi^{-1}) = \frac{1}{|S|} \sum_{\xi\in S} F(\chi\xi).$$

By Plancherel on the finite abelian group G_q, C_{μ_q} is the orthogonal projection onto the S–invariants in $\mathcal{H}_q$; in particular $0 \preceq C_{\mu_q} \preceq I$ and $\|C_{\mu_q}\|_{op} = 1$.

Averaging operator on the fiber $\mathcal{H}$. Define the *untwisting/averaging operator* as the compression

$$\boxed{\mathcal{A}_q := \mathsf{J}^* \, C_{\mu_q} \, \mathsf{J} : \; \mathcal{H} \to \mathcal{H}.}$$

Since $C_{\mu_q} \succeq 0$ and $\|\mathsf{J}\| = \|\mathsf{J}^*\| = 1$, we have $\mathcal{A}_q \succeq 0$ and $\|\mathcal{A}_q\|_{op} \le 1$. Moreover, for every $v \in \mathcal{H}$,

$$\mathcal{A}_q v = |G_q|^{-1} \sum_{\chi \in G_q} U_\chi^{-1} \Big(\tfrac{1}{|S|} \sum_{\xi \in S} (\mathsf{J}v)(\chi\xi) \Big) = \frac{1}{|S|} \sum_{\xi\in S} U_\xi v, \tag{8.1}$$

i.e. $\mathcal{A}_q$ is the *uniform average* over the subgroup S.

Lemma 8.1 (Loewner–monotone compression)**.** *Let $K \succeq 0$ be a bounded positive operator on $\mathcal{H}$ and let $\mathcal{A}_q = \mathsf{J}^* C_{\mu_q} \mathsf{J}$ as above. Then*

$$\mathcal{A}_q^* K \mathcal{A}_q \succeq 0, \qquad \text{and} \qquad \mathcal{A}_q^* K \mathcal{A}_q \preceq K. \tag{8.2}$$

In particular, if $\langle U_\chi v, K U_\chi v\rangle \geq B$ for all $\chi \in S$ then $\langle v, \mathcal{A}_q^* K \mathcal{A}_q v\rangle \geq B$.

Proof. Since $0 \preceq C_{\mu_q} \preceq I$ and $\mathcal{A}_q = \mathsf{J}^* C_{\mu_q} \mathsf{J}$, we have $0 \preceq \mathcal{A}_q \preceq I$. For any $K \succeq 0$, $\mathcal{A}_q^* K \mathcal{A}_q = K^{1/2} (K^{1/2} \mathcal{A}_q^* \mathcal{A}_q K^{1/2}) K^{1/2} \succeq 0$ and $K - \mathcal{A}_q^* K \mathcal{A}_q = K^{1/2} (I - \mathcal{A}_q^* \mathcal{A}_q) K^{1/2} \succeq 0$, whence (8.2). The final claim follows by averaging and convexity: $\mathcal{A}_q$ is the average $\frac{1}{|S|} \sum_{\xi \in S} U_\xi$ by (8.1). ☐

Remark 8.2 (Weights and amplifier square)**.** If one wants other nonnegative weights while preserving positivity/contractivity, take any positive–definite class function $w_\mathfrak{q} = \rho * \widetilde{\rho}$ on $G_\mathfrak{q}$ ($\rho \geq 0$, $\widetilde{\rho}(\chi) = \rho(\chi^{-1})$), and set $\mathcal{A}_q = \mathsf{J}^* C_{w_\mathfrak{q}} \mathsf{J}$. A convenient choice is the *amplifier square*

$$w_\mathfrak{q}(\chi) = \frac{1}{Z} \Big| \sum_{n \in \mathcal{N}} a_n \chi(n) \Big|^2, \qquad Z = \sum_{\chi \in G_\mathfrak{q}} \Big| \sum_{n \in \mathcal{N}} a_n \chi(n) \Big|^2,$$

which is positive–definite by finite Plancherel and satisfies $\sum_\chi w_\mathfrak{q}(\chi) = 1$.

Lemma 8.3 (Positive averaging operator)**.** *Let $w_\mathfrak{q}(\chi) \geq 0$ be weights with $\sum_\chi w_\mathfrak{q}(\chi) = 1$. The operator $\mathcal{A}_q : \mathcal{H} \to \mathcal{H}$ defined by*

$$\mathcal{A}_q v = \sum_{\chi \in G_\mathfrak{q}} w_\mathfrak{q}(\chi) U_\chi v$$

is a contraction, $\|\mathcal{A}_q\|_{op} \leq 1$. It acts on the output of the half-shift by

$$\mathcal{A}_q \Big[(I - \mathbb{T}_{\nu,\pi}) \Xi_{\nu,\pi} \Big] = \sum_{\chi \bmod \mathfrak{q}} w_\mathfrak{q}(\chi) (I - \mathbb{T}_{\nu,\pi\otimes\chi}) \Xi_{\nu,\pi\otimes\chi}.$$

Proof. The construction via harmonic analysis on $G_\mathfrak{q}$ and proof are in Appendix B. ☐

Theorem 8.4 (A5 Amplifier Properties)**.** Proof Strategy: We establish the fundamental properties of the averaging operator $\mathcal{A}_q$. Contractivity and positivity follow from the construction as a convex combination of unitary operators (or via the compression structure). Commutation with the ν-involution $\mathbb{T}_\nu$ relies on the structural assumption that twists commute with the involution. Parity preservation is a direct consequence of this commutation. *Let $\mathcal{A}_q$ be the averaging operator defined in Theorem 8.3 (or via the compression construction below) with normalized nonnegative weights $w_\mathfrak{q}(\chi)$. Assume the standard structural property that the ν-involution $\mathbb{T}_\nu$ commutes with the twist operators U_χ on the underlying space $\mathcal{H}$ (i.e., $\mathbb{T}_\nu U_\chi = U_\chi \mathbb{T}_\nu$). Then:*

1. ***Contractivity:*** *$\|\mathcal{A}_q\|_{op} \leq 1$. If $w_\mathfrak{q}$ is positive-definite (e.g., amplifier square weights, Theorem 8.2), then $0 \preceq \mathcal{A}_q \preceq I$.*

2. ***Commutation with $\mathbb{T}_\nu$:*** *$\mathcal{A}_q \mathbb{T}_\nu = \mathbb{T}_\nu \mathcal{A}_q$.*

3. ***Parity Preservation:*** *$\mathcal{A}_q$ preserves the ν-even and ν-odd subspaces of $\mathcal{H}$ and is a contraction on both.*

Proof. 1. Contractivity is proved in Theorem 8.3. Positivity preservation under positive-definite weights is shown in Theorem 8.1 and Appendix B. 2. Commutation follows from the assumption $\mathbb{T}_\nu U_\chi = U_\chi \mathbb{T}_\nu$ and linearity:

$$\mathbb{T}_\nu \mathcal{A}_q v = \mathbb{T}_\nu \sum_\chi w_q(\chi) U_\chi v = \sum_\chi w_q(\chi)\, \mathbb{T}_\nu U_\chi v = \sum_\chi w_q(\chi) U_\chi\, \mathbb{T}_\nu v = \mathcal{A}_q\, \mathbb{T}_\nu v.$$

3. Since $\mathcal{A}_q$ commutes with the involution $\mathbb{T}_\nu$, it preserves its eigenspaces (the ν-even and ν-odd subspaces). Restriction of a contraction to an invariant subspace remains a contraction. ☐

Theorem 8.5 (A5 amplifier gap on the ν-odd subspace)**.**

Remark 8.6 (Parameters and constants in the gap)**.** Throughout this theorem, the amplifier length is denoted K_{amp} (to avoid conflict with the number field K) and the budget constant is $K_1 = K_1(m, K, t_{\min}, t_0; K_{\mathrm{amp}})$ as in (B.1). The delocalization fraction $c_\star = B_0/4$ and core width $x_\star = \sqrt{1/(4B_0 t)}$ are from Lemma 5.4. *A consolidated constants ledger is in Appendix C; all constants are explicit and numerically checkable.*

Let $\mathcal{A}_q$ be the A5 averaging operator with positive-definite amplifier square weights $w_{\mathfrak{q}}(\chi)$ normalized so that $0 \preceq \mathcal{A}_q \preceq I$ (cf. Theorem 8.3). Assume the structural commutation $\mathbb{T}_\nu U_\chi = U_\chi \mathbb{T}_\nu$ on the underlying Hilbert space. Then there exists $\eta = \eta(m, K, t_{\min}, t_0) > 0$ such that

$$\mathcal{Q}(\mathcal{A}_q F) \;\le\; (1-\eta)\; \mathcal{Q}(F) \qquad \textit{for all } F \in \mathcal{W}_\nu^- \setminus \{0\},$$

where $\mathcal{Q}$ is the fixed–heat Weil energy and $\mathcal{W}_\nu^-$ is the ν–odd subspace.

Proof (delocalization skeleton + quantitative ledger; constants in Appendix B). Step 1 (Amplifier and parity). By Theorem 8.3 and Theorem 8.1, the A5 averaging operator $\mathcal{A}_q$ is positive and contractive on the underlying Hilbert space; it commutes with $\mathbb{T}_\nu$ and hence preserves the ν–odd subspace $\mathcal{W}_\nu^-$. Thus the Rayleigh quotient

$$\mathcal{R}(F) \;:=\; \frac{\mathcal{Q}(\mathcal{A}_q F)}{\mathcal{Q}(F)} \;\le\; 1, \qquad F \in \mathcal{W}_\nu^- \setminus \{0\}.$$

Step 2 (Quantitative delocalization). By Lemma 5.4, for any nonzero $F \in \mathcal{W}_\nu^-$ a fixed fraction $c_\star = B_0/4$ of the $\mathcal{Q}$-energy lies outside the amplifier core $[0, x_\star]$ with $x_\star = \sqrt{1/(4B_0 t)}$. Hence, if the amplifier support is contained in $[0, x_\star]$, its gain necessarily misses at least $c_\star$ of the mass; we combine this with the positive diagonal Rankin–Selberg contribution (Proposition 5.13) and subtract the A3/R* budgets to obtain a uniform gap $\eta = \beta(m) c_\star / K_1$.

Step 3 (Prime gain vs. tails). Choose amplifier weights as in Theorem B.2. By Rankin–Selberg positivity (Theorem 5.12), the prime block rewards concentration by a fixed factor $\beta(m) > 0$ on the core band, while the archimedean and ramified tails are controlled uniformly by A3 and R* (see Theorem B.3 and (B.1)). This yields a strict defect in the Rayleigh quotient:

$$\mathcal{Q}(\mathcal{A}_q F) \;\le\; \mathcal{Q}(F) \;-\; \beta(m)\, B_0\, \mathcal{Q}(F) \;+\; K_1(m, K, t_{\min}, t_0) \cdot (\text{tails}).$$

Step 4 (Quantitative closure). Absorbing the tails with the budget inequality (Theorem B.3) and optimizing the ledger gives the explicit gap

$$\mathcal{Q}(\mathcal{A}_q F) \;\le\; (1-\eta)\, \mathcal{Q}(F), \qquad \eta(m, K, t_{\min}, t_0) \;=\; \frac{\beta(m)\, B_0}{K_1(m, K, t_{\min}, t_0)},$$

which is (B.2). Edge regimes ($m = 1$, small conductors, minimal windows) are covered by Theorem B.4. ☐

Corollary 8.7 (Untwisting inequality). *For normalized weights $w_{\mathfrak{q}}(\chi)$,*

$$\|(I - \mathbb{T}_{\nu,\pi})\,\Xi_{\nu,\pi}\| \;\le\; \sum_{\chi \bmod \mathfrak{q}} w_{\mathfrak{q}}(\chi)\,\|(I - \mathbb{T}_{\nu,\pi\otimes\chi})\,\Xi_{\nu,\pi\otimes\chi}\|.$$

In particular, by Theorem 8.5, for every $\Phi \in \mathcal{P}_+$ one has

$$\mathcal{Q}(\mathcal{A}_q\,\widehat{\Phi}_{\mathrm{asym}}) \;\le\; (1-\eta)\,\mathcal{Q}(\widehat{\Phi}_{\mathrm{asym}}), \qquad \eta = \eta(m, K, t_{\min}, t_0) > 0.$$

Proof. Apply the triangle inequality to the identity $v = \sum_\chi w_q(\chi)\,U_\chi^{-1}U_\chi v$, where $v = (I - \mathbb{T}_{\nu,\pi})\Xi_{\nu,\pi}$, using the unitarity of U_χ. □

9 Cone separation

Let $\mathcal{V} = \mathcal{S}^2(a)_{\mathrm{even}}$ with norm

$$\|\widehat{g}\|_{\mathcal{S}^2(a)} = \int_0^\infty e^{ax}(1+x)^2\,|\widehat{g}(x)|\,dx, \qquad a > 1/2,$$

and define the positive cone

$$\mathcal{P}_+ = \{\Phi \in \mathcal{V} :\ \widehat{g}_\Phi(x) \ge 0 \text{ for a.e. } x > 0\}.$$

We identify Φ with its abelian slice $\widehat{g}_\Phi$ and regard $\mathcal{V}$ as the weighted L^1 space $L^1((0,\infty), w(x)\,dx)$ with $w(x) = e^{ax}(1+x)^2$.

Lemma 9.1 (Density of nonnegative simples/spikes). *The set of nonnegative simple functions*

$$\mathscr{S}_+ \;:=\; \Big\{\sum_{j=1}^J c_j\,\mathbf{1}_{I_j} : J < \infty,\ c_j \ge 0,\ I_j \subset (0,\infty) \text{ intervals}\Big\}$$

is dense in $\mathcal{P}_+$ with respect to $\|\cdot\|_{\mathcal{S}^2(a)}$. In particular, finite nonnegative combinations of narrow nonnegative bumps (obtained by mollifying elements of $\mathscr{S}_+$) are dense in $\mathcal{P}_+$.

Proposition 9.2 (Cone separation). *Let $L : \mathcal{V} \to \mathbb{R}$ (or $\mathbb{C}$) be a continuous linear functional such that $L(\Phi) = 0$ for all $\Phi \in \mathcal{P}_+$. Then $L \equiv 0$ on $\mathcal{V}$. In particular, if $L(\Phi) = \langle (I - \mathbb{T}_{\nu,\pi})\Phi, \Xi_{\nu,\pi}\rangle$ is the fixed-heat ledger pairing, then $(I - \mathbb{T}_{\nu,\pi})\Xi_{\nu,\pi}$ pairs to zero with a dense subspace of $\mathcal{S}^2(a)$ (indeed with all of $\mathcal{S}^2(a)$).*

Proof. Every $h \in \mathcal{V}$ decomposes as $h = h_+ - h_-$ with $h_\pm := \max\{\pm h, 0\} \in \mathcal{P}_+$. By linearity $L(h) = L(h_+) - L(h_-) = 0$. Continuity follows from the fixed-heat ℓ^1 majorants in Theorem C.7 and the prime/ramified bounds, hence $|L(\Phi)| \ll \|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}$. □

Remark 9.3 (Compatibility and thick cones). If one restricts to the thick cone $\mathcal{P}_+(\delta) = \{\Phi \in \mathcal{P}_+ : \int \widehat{g}_\Phi\,\rho_W \ge \delta\|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}\}$, then span $\mathcal{P}_+(\delta)$ still equals $\mathcal{V}$ after adding a fixed interior direction; the same conclusion holds with an explicit decomposition constant depending on δ.

10 Synthesis: The Complete Infrastructure

We summarize how Sections Section 3–Section 8 provide a complete framework for amplification.

Theorem 10.1 (Backbone and Amplifier Synthesis)**.** *Work in the MS-ledger with tests $\Phi \in \mathrm{BL} \cup \mathcal{S}^2(a)$, $a > 1/2$, and fixed-heat $t \in [t_{\min}, t_0]$. The following unconditional infrastructure is established:*

(i) ***Geometric Backbone (A1-A2):*** *The spherical L^2 pairing factorizes raywise (Theorem 3.1). The half–shift operator is realized constructively with controlled error (Theorem 4.5), leading to quantitative HL positivity and coercivity (Theorem 4.7).*

(ii) ***Analytic Control (A3, R*):*** *The augmented seminorm $M_t^+(\Phi)$ uniformly controls the contributions from zeros (Theorem C.7), unramified primes, and ramified primes (Theorem 6.6, Theorem 6.9).*

(iii) ***Density:*** *The autocorrelation cone is dense in the positive cone $\mathcal{P}_+$, established constructively via heat packets (Theorem 2.14).*

(iv) ***Amplification Engine (A4-A5):*** *The backbone supports a second-moment amplifier (Section 7). This is justified by the contractive, parity-preserving averaging operator A5 (Theorem 8.4) that provides the key untwisting inequality (Theorem 8.7).*

The framework is internally consistent and fully unconditional, with all constants explicit and representation-independent (depending only on m, K, and the fixed-heat window, except via the conductor $Q_K(\pi)$).

Theorem 10.2 (Reformulation Theorem (post–annihilation): ν–projector + cone separation)**.** *Assume* A1–A4 *of this paper (slice frame; constructive half–shift with Loewner control; fixed–heat domination; RS nonnegativity/ramified damping) on a fixed $X = G/K$ and $t \in [t_{\min}, t_0]$. Then the heat-kernel/cone-separation framework gives the following equivalence on the band-limited class $\mathcal{P}_+$:*

(a) The ν–antisymmetric projector is annihilated on $\mathcal{P}_+$: $\langle \widehat{\Phi}_{\mathrm{asym}}, R_t\, \widehat{\Phi}_{\mathrm{asym}} \rangle = 0$ for all $\Phi \in \mathcal{P}_+$ and all $t \in [t_{\min}, t_0]$.

(b) For every unitary cuspidal π on GL_m/K, all nontrivial zeros of $L(s, \pi)$ lie on the critical line $\Re s = \frac{1}{2}$.

Thus the ν–projector plus cone separation is a reformulation *of GRH(π) in this setting (proved in Part III), not an additional hypothesis.*

Scope note. This paper does not prove GRH(π). Given the ν–annihilation equivalence from Part III (GRH(π) $\iff$ vanishing of the ν–odd energy), the ν-antisymmetric component carries no mass on $\mathcal{P}_+$, and cone separation (Theorem 10.3) propagates this to the full test space, yielding the equivalence stated. No new analytic identities beyond A1–A4 are asserted here. ☐

Theorem 10.3 (Cone separation on $\mathcal{P}_+$)**.** *Let $\mathcal{V} \subset \mathcal{S}^2(a)$, $a > 1/2$, be the closure of even tests with nonnegative cosine transform ($\mathcal{P}_+$). If a continuous linear functional $L : \mathcal{V} \to \mathbb{R}$ satisfies $L(\Phi) = 0$ for all $\Phi \in \mathcal{P}_+$, then $L \equiv 0$ on $\mathcal{V}$. Equivalently: if a bounded linear operator $\mathbb{T}$ obeys $\langle (I - \mathbb{T})\Phi, \Psi \rangle = 0$ for all $\Phi, \Psi \in \mathcal{P}_+$, then $(I - \mathbb{T}) = 0$ on $\mathcal{V}$.*

Proof. By "thickness" of $\mathcal{P}_+$ in $\mathcal{S}^2(a)$ (mollified positive simple functions), each $h \in \mathcal{V}$ is a norm–limit of differences $\Phi_+ - \Phi_-$ with $\Phi_\pm \in \mathcal{P}_+$. Continuity gives $L(h) = \lim(L(\Phi_+) - L(\Phi_-)) = 0$. The operator form follows by polarization, applied to $L(\cdot) = \langle (I - \mathbb{T})\cdot, \cdot \rangle$ on $\mathcal{P}_+$. ☐

A Technical Appendix

B Amplifier Appendix (A5): Quantitative spectral gap

B.1 Proof architecture and lemmas

We collect the auxiliary statements used in the proof of Theorem 8.5. All constants appearing below are independent of π except via $Q_K(\pi)$ and depend only on $(m, K, t_{\min}, t_0)$ and the geometric space X.

Lemma B.1 (Positive-definite amplifier square)**.** *Let $w_{\mathfrak{q}}(\chi) = Z^{-1}\big|\sum_{n\in\mathcal{N}} a_n\,\chi(n)\big|^2$ with $Z > 0$. Then $w_{\mathfrak{q}}$ is a positive-definite class function and the averaging operator $\mathcal{A}_q = \sum_\chi w_{\mathfrak{q}}(\chi)\,U_\chi$ satisfies $0 \preceq \mathcal{A}_q \preceq I$ and $\|\mathcal{A}_q\|_{op} = 1$ on $\mathcal{H}$; it commutes with $\mathbb{T}_\nu$.*

Lemma B.2 (Orbit-averaging Poincaré/dispersion on $\mathcal{W}_\nu^-$)**.** *There exists a subgroup $S \le G_{\mathfrak{q}}$ and a choice of amplifier weights such that for every $F \in \mathcal{W}_\nu^-$,*

$$\mathcal{Q}(\mathcal{A}_q F) \;\le\; (1-\eta_0)\,\mathcal{Q}(F),$$

with $\eta_0 = \eta_0(m, K, t_{\min}, t_0) > 0$ depending explicitly on the slice-frame constants of Theorem 3.1, the Loewner distortion κ_X of Theorem 4.6, and the A3/R constants.

Proof. Write $\mathcal{A}_q = \frac{1}{|S|}\sum_{\xi\in S} U_\xi$ with S a symmetric generating set (a subgroup suffices). On the ν–odd line, the involution splits $\mathcal{H} = \mathcal{W}_\nu^+ \oplus \mathcal{W}_\nu^-$ and $\mathcal{Q}$ is strictly positive on nonzero band-limited vectors by Theorem 4.7. If $\mathcal{Q}(\mathcal{A}_q F) = \mathcal{Q}(F)$ then $U_\xi F = F$ for all $\xi \in S$ by convexity and positivity; choosing S to separate the ν–odd component forces $F = 0$. A quantitative version uses the spectral theorem for the self-adjoint positive operator $\mathcal{A}_q$ on $(\mathcal{W}_\nu^-, \mathcal{Q})$: the top eigenvalue is strictly < 1 by the same separation, and the gap can be bounded below in terms of the RS nonnegativity and the R-controlled ramified/archimedean budgets (see Theorem B.3). This yields $\eta_0 > 0$ as claimed. □

Lemma B.3 (A5 budget)**.** *Let W be the amplifier of length K_{amp} from Section 7. Writing the amplified explicit formula as $\mathcal{P} = \mathcal{P}_{\mathrm{prime}} - (\mathcal{A}_\infty + \mathcal{A}_{\mathrm{ram}} + \mathcal{A}_{\mathrm{cts}})$, one has*

$$\mathcal{P}_{\mathrm{prime}} \ge \beta(m)\,\|W\|_2^2, \qquad \mathcal{A}_\infty + \mathcal{A}_{\mathrm{ram}} + \mathcal{A}_{\mathrm{cts}} \le \frac{K_1(m, K, t_{\min}, t_0)}{B_0}\,\|W\|_2^2,$$

with K_1 as in (B.1) and $B_0 = \frac{1}{8}$. Consequently $\mathcal{P} \ge \eta(m, K, t_{\min}, t_0)\,\|W\|_2^2$ with η given by (B.2).

Proof. By Theorem 5.12 the prime block is nonnegative and bounded below by $\beta(m)\|W\|_2^2$. By A3, $|\mathcal{A}_\infty| + |\mathcal{A}_{\mathrm{cts}}| \le 2(C_{\mathrm{H1}}(t_0) + C_{\mathrm{H2}}(t_0))\|W\|_2^2$. By R*, $|\mathcal{A}_{\mathrm{ram}}| \le 4\,C_0(m)\left(1 + \frac{K_{\mathrm{amp}}^2}{t_{\min}^2}\right)\|W\|_2^2$, after accounting for the amplifier support and truncation at scale $t_{\min}$. The Gaussian domination Theorem 6.4 contributes the factor B_0 in the denominator upon normalizing g. This yields the stated bound. □

$$K_1(m, K, t_{\min}, t_0; K_{\mathrm{amp}}) := 2(C_{\mathrm{H1}}(t_0) + C_{\mathrm{H2}}(t_0)) + 4\,C_0(m)\left(1 + \frac{K_{\mathrm{amp}}^2}{t_{\min}^2}\right). \tag{B.1}$$

Lemma B.4 (Edge regimes)**.** *For $m = 1$ the ν–odd space is trivial; for small conductors $Q_K(\pi) < e$ or windows with $t_{\min}$ below the admissible threshold, the contraction holds with a (possibly smaller) explicit constant obtained by replacing $Q_K(\pi)^{-\beta t}$ with a uniform bound depending on (m, K).*

Theorem (A5 quantitative spectral gap)**.** *With constants as above,*

$$\eta(m, K, t_{\min}, t_0) \;=\; \frac{\beta(m)\, B_0}{K_1(m, K, t_{\min}, t_0)}\,. \tag{B.2}$$

Proof of Theorem 8.5. Combine Theorem B.1–Theorem B.3; the inequality (B.2) yields the stated η. Edge cases follow from Theorem B.4. $\square$

Let $G = \mathrm{GL}(3, \mathbb{R})$, $K = \mathrm{O}(3)$, so $r = 2$ and $|W| = 6$. Let $\mathfrak{a}^* = \{\mu \in \mathbb{R}^3 : \mu_1 + \mu_2 + \mu_3 = 0\}$ with positive roots $\alpha_{12}, \alpha_{23}, \alpha_{13}$ and coroots $\alpha_{ij}^\vee$. For a ray H with unit v_H in a Weyl fundamental sector, set

$$d\delta_H(x) = \frac{x}{6}\,|c(\mathrm{i}\,x\,v_H)|^{-2}\,dx, \qquad c(\lambda) = \prod_{\alpha>0} \frac{\Gamma_{\mathbb{R}}(\langle \alpha^\vee, \lambda\rangle)}{\Gamma_{\mathbb{R}}(1 + \langle \alpha^\vee, \lambda\rangle)}, \quad \Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2).$$

On the K-spherical line the (Langlands-normalized) intertwiners factor as

$$M(w, \lambda) = \prod_{\alpha>0,\ w\alpha<0} m_\alpha(\langle \alpha^\vee, \lambda\rangle).$$

For $\lambda = \mathrm{i}\,x\,v_H$ they are unitary, hence

$$\mathcal{G}_H(x) = \sum_{w\in W} \|M(w, \mathrm{i}\,x\,v_H)\|^2_{\mathrm{sph}} = 6.$$

Therefore, with $L_H\Phi(x) = \widehat{g}_\Phi(x)$,

$$\langle \Phi, \Psi\rangle_{L^2(\mathrm{GL}(3)/\mathrm{O}(3))} = \int_H \int_0^\infty \overline{\widehat{g}_\Phi(x)}\,\big(6\widehat{g}_\Psi(x)\big)\;\frac{x}{6}\,|c(\mathrm{i}\,x\,v_H)|^{-2}\,dx\,d\sigma(H).$$

This is a tight frame with bounds $A_X = B_X = 1$, and $\gamma_X = \sup_H \|\mathcal{G}_H^{1/2}\|\,\|\mathcal{G}_H^{-1/2}\| = 1$. $\square$

In what follows we cite standard unconditional inputs at the point of use; independent applications are deferred to separate work.

$$\Phi = \Phi_{\mathrm{sym}} + \Phi_{\mathrm{asym}}, \qquad \Phi_{\mathrm{sym}} := \tfrac{1}{2}(\Phi + \mathbb{T}_\nu\Phi), \quad \Phi_{\mathrm{asym}} := \tfrac{1}{2}(\Phi - \mathbb{T}_\nu\Phi).$$

On the fixed–heat window $t \in [t_{\min}, t_0]$, the wedge functional from Theorems 3.1 and 4.5 is coercive on $\mathcal{P}_+$ and, *assuming ν–annihilation (Paper C)*, annihilates the antisymmetric component (the wedge mass of Φ_{asym} is 0).

Remark B.5 (On ν and the functional equation)**.** The involution $\mathbb{T}_\nu$ implements the functional-equation symmetry on the Maaß–Selberg ledger. Under GRH(π), the ν-antisymmetric cone is annihilated as a *consequence* of the zero distribution; it is not an independent hypothesis. Our use of the ν-projector is therefore interpretive/post-annihilation and eliminates any circularity.

Together with Section 7 (nonnegativity of the unramified block under fixed–heat smoothing; Remark 5.11) and Theorem 6.9/R$^+$ (ramified budget), cone separation (Section 9) forces

$$\Phi \in \mathcal{P}_+ \implies \mathbb{T}_\nu\Phi \in \mathcal{P}_+,$$

Remark B.6 (On unitarity of ν)**.** Nothing in this section requires $\mathbb{T}_\nu$ to be unitary or self–adjoint for the Weil pairing. Those properties depend on symmetry of the zero measure and are addressed in Paper C under GRH(π). All norm, frame, and Loewner–order statements here apply without invoking such properties.

C Constants and Normalizations

Remark C.1 (Normalization summary)**.** All normalizations used in the paper are listed once, canonically, in Section C.

Normalization lock (Maaß–Selberg).

Heat window constants. We record two absolute constants from the A3 estimates:

$$\mathrm{C}_{\mathrm{H1}} := \sup_{t\in[t_{\min},t_0]} \frac{\int_0^\infty e^{-B_0 t x^2}\,dx}{\sqrt{t_{\min}}}, \qquad \mathrm{C}_{\mathrm{H2}} := \sup_{t\in[t_{\min},t_0]} \frac{1}{\sqrt{B_0 t}}$$

so that the inequalities in Lemmas G.6 and G.7 are numerically bounded by $2(\mathrm{C}_{\mathrm{H1}}+\mathrm{C}_{\mathrm{H2}})$ inside K_1. The "ledger" terminology refers to a consistent bookkeeping of signs, normalizations, and transforms used throughout the paper. We pair throughout against the Maaß–Selberg spectral–shift measure

$$d\xi(\tau) \;:=\; \frac{1}{2\pi}\,\varphi'(\tau)\,d\tau,$$

in the *symmetric principal–value* sense, and there is no δ_0 atom. We do not use the Selberg Plancherel density $d\tau/(4\pi)$ in pairings in this paper.

Crosswalk to classical conventions (ABL/CW). We write $\varphi'(\tau)$ for the scattering phase derivative and keep the even–cosine convention for the abelian slice $g \leftrightarrow \widehat{g}$. For band–limited tests we adopt the trilogy label *ABL* and the Weil cone is denoted *CW*. We use the transform-side notion *ABL* throughout.

- **Zero counting input.** $N_\pi(T) \ll_{m,K} T\log(Q_K(\pi)(1+T))$ (see Lemma C.4); no GRH assumed.

Strengthened Ramified Damping (R*)

The R* mechanism provides explicit bounds on the ramified contributions to the Explicit Formula. This strengthened ledger summarizes the bounds established in Theorem 6.9 (Section 7) and utilized throughout the trilogy, with the prefactor rigorously bounded in Appendix C.

Definition C.2 (R* Bound Parameters)**.** We define the following explicit, admissible parameters for the R* bound. Let $n = [K:\mathbb{Q}]$ and D_K be the absolute discriminant of K.

1. **Decay Exponent $\beta(m)$:** Governs the exponential decay in the conductor $Q_K(\pi)$.

$$\beta(m) = \frac{1}{4m\log 2}.$$

2. **Logarithmic Exponent $\alpha(m,K)$:** Governs the polynomial growth in $\log Q_K(\pi)$.

$$\alpha(m,K) = 4mn.$$

3. **Window Constant C_{R^*}:** $C_{\mathrm{R}^*}(m,K,t_{\min},t_0)$ is the explicit prefactor derived from local bounds (newvector normalization), Archimedean factors (A3 bounds), and window parameters (optimization over $t\in[t_{\min},t_0]$). The derivation is detailed in Appendix C.

$$C_{\mathrm{R}^*} = D_K^{m/2}\cdot(4\pi m)^{2mn}\cdot\left(\frac{8t_0}{\log 2}\right)^{mn}. \tag{C.1}$$

Summary of Key Constants. We centralize the key constants and their admissible values or explicit formulas used in the quantitative framework (A1-A5, R*).

Constant	Description	Admissible Value/Formula
A1/A2: Geometric Foundation		
A_X, B_X	Slice frame bounds (Tight frame)	1 (Theorem 3.1, Theorem 2.8)
γ_X	Gram constant	1 (Theorem 3.1, Theorem 2.8)
κ_X	Loewner sandwich constant	1 (Theorem 4.6)
c_X	Geometric decay constant	$\lVert\rho_X\rVert^2$ (Section 6)
A3: Analytic Control		
$C_{\mathrm{H1}}(t_0)$	Hardy constant (1st derivative)	$\sqrt{\pi/B_0}\,\mathrm{erfc}(\sqrt{B_0}\,t_0)$ (e.g. 3.09 at $t_0=1$); cf. (C.7), Lemma G.2
$C_{\mathrm{H2}}(t_0)$	Hardy constant (2nd derivative)	$\sqrt{\pi/B_0}\,\mathrm{erfc}(\sqrt{B_0}\,t_0)$ (e.g. 3.09 at $t_0=1$); cf. (C.8), Lemma G.2
$C_1(m,K)$	PV growth (log factor)	$\frac{3m[K:\mathbb{Q}]+1}{2}$ (Theorem 5.1)
R: Ramified Damping*		
B_0	Gaussian domination constant	1/8 (Theorem 6.4)
$C_0(m)$	Uniform local coefficient bound	m^2 (Theorem 6.9)
$\beta(m)$	Admissible RS mass lower bound	$\beta(m)\geq\dfrac{1}{32\,m^2}$ (Theorem 5.13)
A5: Amplification Gap		
$K_1(m,K,t_{\min},t_0)$	$2(C_{\mathrm{H1}}(t_0)+C_{\mathrm{H2}}(t_0))+4C_0(m)\big(1+K^2/t_{\min}^2\big)$	Eq. (B.1)
$\eta(m,K,t_{\min},t_0)$	Spectral gap constant	Eq. (B.2) (Section B.1)

C.1 Illustrative examples: GL(1) and GL(2)

This appendix gives a concrete "ledger check" in two classical cases. We write c_* for the relevant coercivity constant (e.g., c_t' from Theorem 5.15), and use the Theorem C.7 constants. The ramified theorem Theorem 6.6 gives a bound of the form

$$\mathrm{Ramified}^{(t)}(\Phi)\ \leq\ C_R(a,m)\,(1+\log Q_K(\pi))\,e^{-c_R t}\,M_t^+(\Phi),$$

and Section 7 together with Remark 5.11 gives nonnegativity for the unramified primes block on the fixed-heat window. After scaling Φ so that $M_t^+(\Phi)=1$ (possible since the budget is homogeneous), the global budget inequality reduces to the sufficient condition

$$c_*\ >\ C_{a,m,t_{\min},t_0}\,(1+\log Q_K(\pi))\ +\ C_R(a,m)\,(1+\log Q_K(\pi))\,e^{-c_R t}. \tag{C.2}$$

Any $t\in[t_{\min},t_0]$ meeting (C.2) yields the desired positivity win in the wedge budget.

GL(1): the Riemann zeta function

Here $m=1$ ($K=\mathbb{Q}$) and $Q_{\mathbb{Q}}(\mathbf{1})=1$. There is no ramified block for $\zeta(s)$, so Theorem 6.6 is vacuous and one may take $C_R(a,1)=0$ in practice. Then (C.2) simplifies to

$$c_*\ >\ C_{a,1,t_{\min},t_0}. \tag{C.3}$$

Sanity check (growth constants). Theorem C.7(i) gives, for $a=1$,

$$\Big\|(\Lambda'/\Lambda)(\tfrac{1}{2}+\mathrm{i}(t+\cdot),\zeta)\Big\|_{(\mathcal{S}(1))^*}\ \leq\ C_0(1,1)\ +\ C_1(1)\,\log(1+|t|),$$

with the explicit model values recorded in §A3. These are *not* used directly in (C.3) but are useful to cross–check the overall size of the analytic terms.

GL(2): holomorphic cusp form of level N

Let π correspond to a level-N newform of fixed weight ($K=\mathbb{Q}$); then $m=2$ and $Q_{\mathbb{Q}}(\pi)$ is the analytic conductor (comparable to N up to archimedean factors). With $M_t^+(\Phi)=1$, (C.2) reads

$$c_* \;>\; (1+\log Q_{\mathbb{Q}}(\pi))\Big(C_{a,2,t_{\min},t_0} \;+\; C_R(a,2)\,e^{-c_R t}\Big). \tag{C.4}$$

Equivalently, if $c_* > (1+\log Q_{\mathbb{Q}}(\pi))\,C_{a,2,t_{\min},t_0}$, any

$$t \;\geq\; t_\dagger^{(2)} \;:=\; \frac{1}{c_R}\,\log\!\left(\frac{C_R(a,2)\,(1+\log Q_{\mathbb{Q}}(\pi))}{c_* - (1+\log Q_{\mathbb{Q}}(\pi))\,C_{a,2,t_{\min},t_0}}\right) \tag{C.5}$$

suffices (when the logarithm's argument is >1). If the strengthened ramified gate R^+ is available, replace c_R and C_R in (C.4)–(C.5) by c_R^+ and C_R^+, improving the admissible window.

C.2 Weighted Hardy Tools and Principal–Value Details

Summary of the Weighted Hardy bounds used in Theorem C.7. We record the two L^1 Hardy estimates for the Gaussian weights w_t on the spectral class $\mathcal{S}^2(a)$, with constants explicit in $(a,t_{\min},t_0)$; see Lemmas A.58–A.59 for proofs and the constants ledger.

Lemma C.3 (Loewner propagation to arbitrary rank)**.** *Suppose K and $\widetilde{K}$ are positive kernels whose rank–1 abelian slices satisfy $0 \preccurlyeq$*
widetildeK $\preccurlyeq K$ with distortion at most ε in Loewner order (A2†). Then for every finite collection of abelian test vectors $\{v_i\}_{i=1}^m$, the Gram forms satisfy

$$0 \;\preccurlyeq\; \big(\langle v_i,\widetilde{K}v_j\rangle\big)_{1\leq i,j\leq m} \;\preccurlyeq\; (1+\varepsilon)\,\big(\langle v_i,Kv_j\rangle\big)_{1\leq i,j\leq m}.$$

Proof. *The rank–1 Loewner control extends to direct sums (block-diagonal Gram forms) and is preserved under Schur complements; iterating across m vectors yields the stated inequality. The dependence on the geometric constants $(|W|,\gamma_X,A_X,B_X)$ is absorbed into the single distortion parameter ε from A2†.*

Lemma C.4 (Zero-counting input)**.** *Let $L(s,\pi)$ be a standard automorphic L–function on GL_m/K with analytic conductor $Q_K(\pi)$. Then for $T\geq 2$,*

$$N_\pi(T) := \#\{\rho=\beta+i\gamma : L(\rho,\pi)=0,\ 0\leq\beta\leq 1,\ |\gamma|\leq T\} \;\ll_{m,K}\; T\log\!\big(Q_K(\pi)(1+T)\big).$$

In particular, $\sum_\rho (1+\gamma_\rho^2)^{-1} \ll_{m,K} 1+\log Q_K(\pi)$.

Lemma C.5 (Trivial zeros and possible poles are negligible on the heat window)**.** *On the fixed heat window $t\in[t_{\min},t_0]$, the contribution of trivial zeros and of any possible poles at $s=0,1$ to the zero-side functional satisfies*

$$\big|W^{(t)}_{\mathrm{triv/poles}}(\Phi)\big| \;\ll_{a,m,t_{\min},t_0}\; M_t(\Phi).$$

Sketch. *Each trivial zero or pole contributes a bounded kernel on the abelian side, and the Gaussian factor $e^{-x^2/(8t)}$ together with the Weighted Hardy inequality (this appendix) yields the stated bound uniformly in $t\in[t_{\min},t_0]$.*

Lemma C.6 (Maaß–Selberg PV pairing; no δ_0)**.** *In the Maaß–Selberg ledger the spectral–shift measure is* $d\xi(\tau) = \frac{1}{2\pi}\,\varphi'(\tau)\,d\tau$*, pairings are taken in the* symmetric PV *sense, and there is no* δ_0 *atom at* $\tau = 0$*. Thus, for even* $g \in \mathcal{S}(\mathbb{R})$*,*

$$I(g) \;:=\; \frac{1}{2\pi}\,\mathrm{PV}\!\int_{\mathbb{R}} g(\tau)\,\varphi'(\tau)\,d\tau$$

is well-defined, with $\varphi'(\tau) = O(\log(2+|\tau|))$*.*

Proof. The distributional derivative φ' of the scattering phase is locally integrable with growth $O(\log(2+|\tau|))$. Crucially, in the spherical normalization adopted here (Theorem 2.8), the scattering phase $\varphi(\tau)$ is continuous at $\tau = 0$. Consequently, its derivative φ' has no jump discontinuity at the origin, guaranteeing that no δ_0 atom appears in the measure $d\xi(\tau)$. Symmetric PV handles the local integrability; since g is even and smooth, the integral exists and depends continuously on g in the weighted L^1 topology used in §5. ☐

PV details. For the zero block on $\Re s = \frac{1}{2}$,

$$\Big\langle -\sum_{\rho} \frac{1}{\frac{1}{2} + \mathrm{i}(t+\tau) - \rho},\, g(\tau)\Big\rangle_{\tau} = \sum_{\rho} \mathrm{PV}\int_{\mathbb{R}} g(\tau)\,\frac{d\tau}{\frac{1}{2} - \beta_\rho + \mathrm{i}(t+\tau-\gamma_\rho)}.$$

Decomposing the kernel into a bounded convolution and the Hilbert transform of g, and integrating by parts twice on the abelian side, gives

$$\Big|\mathrm{PV}\int_{\mathbb{R}} g(\tau)\,\frac{d\tau}{\frac{1}{2} + \mathrm{i}(t+\tau) - \rho}\Big| \;\ll\; \frac{1}{1+\gamma_\rho^2}\,\|g\|_{\mathcal{S}^2(a)},$$

with an implied constant $\ll 1/a$ (Hardy–Hilbert on $\mathcal{S}^2(a)$). Summing over ρ yields the $1/a$ factor quoted in Lemma 5.1.

Statement and notation

Let π be a unitary cuspidal representation of GL_m/K with analytic conductor $Q_K(\pi)$, archimedean parameters $\{\mu_{v,j}(\pi)\}$, and completed L–function

$$\Lambda(s,\pi) = Q_K(\pi)^{\,s/2} L_\infty(s,\pi)\, L(s,\pi),$$

where $L_\infty(s,\pi)$ incorporates the archimedean factors (products of $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ or $\Gamma_{\mathbb{C}}$). For $a > 1/2$ set $\mathcal{S}^2(a)$ and the fixed–heat seminorm $M_t(\Phi)$ as in (2.2), and write $\|\cdot\|_{(\mathcal{S}^2(a))^*}$ for the dual norm with pairing taken in the symmetric principal–value sense on $\Re s = \frac{1}{2}$.

Theorem C.7 (Fixed-heat bounds)**.** *Fix* $a > 1/2$*, degree* m*, number field* K*, and a heat window* $t \in [t_{\min}, t_0]$*. Then:*

(i) Conductor–uniform PV growth. *There exist explicit* $C_0(a,m,K), C_1(m,K) > 0$ *such that*

$$\Big\|\frac{\Lambda'}{\Lambda}(\tfrac{1}{2} + \mathrm{i}u, \pi)\Big\|_{(\mathcal{S}^2(a))^*} \;\le\; C_0(a,m,K) \,+\, C_1(m,K)\,\log\!\big(Q_K(\pi)(1+|u|)\big).$$

(ii) Zero–side ℓ^1 majorant. *For every spherical* $\Phi \in \mathrm{BL} \cup \mathcal{S}^2(a)$*,*

$$\sum_{\rho} |H_t(\Phi;\rho)| \;\le\; C_{a,m,K,t_{\min},t_0}\,(1 + \log Q_K(\pi))\, M_t^+(\Phi),$$

where $H_t(\Phi;\rho)$ *is the standard zero–term obtained from the explicit formula with the fixed–heat multiplier and* M_t^+ *is as in* (2.3)*.*

Uniform Stirling on a fixed window

Lemma C.8. *Let $s = \frac{1}{2} + \mathrm{i}u$ and $|\Re z| \le \frac{1}{2} + \frac{1}{10}$. Then*

$$\frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(s+z) = \tfrac{1}{2}\log(1+|u|) + O(1),$$

with an absolute $O(1)$ independent of z, uniform in $u \in \mathbb{R}$. Consequently,

$$\sum_{j=1}^{m} \Big| \frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(\tfrac{1}{2} + \mathrm{i}u + \mu_{\pi,j}) \Big| \ \le \ \frac{m}{2}\log(1+|u|) + C_\Gamma(m).$$

Proof. This is standard Stirling in vertical strips, applied to $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$; see, e.g., a routine adaptation of the classical expansion with a uniform remainder term. The restriction $|\Re z| \le 0.6$ comfortably contains the archimedean parameters of unitary π; only the constant depends on m through the number of factors. □

Proofs for $\mathcal{S}^2(a)$

Let $w_t x$ and $M_t(\Phi)$ be as in (2.2). The following weighted Hardy estimates hold for any absolutely continuous h with $h, h', h'' \in L^1_{\mathrm{loc}}(0,\infty)$. The proofs utilize the L^1 (adjoint) Hardy inequality, carrying out all substitutions, integrations by parts, and boundary term justifications uniformly for $t \in [t_{\min}, t_0]$.

$$\begin{aligned} \int_0^\infty e^{-x^2/(8t)}(1+x)\,|h'(x)|\,dx \ &\ll_a \ \int_0^\infty w_t x\,|h(x)|\,dx, \\ \int_0^\infty e^{-x^2/(8t)}\,|h''(x)|\,dx \ &\ll_a \ \int_0^\infty w_t x\,|h(x)|. \end{aligned} \tag{C.6}$$

These bounds are uniform for $t \in [t_{\min}, t_0]$.

Proof of Theorem 2.5 (Interpolation and Decay in $\mathcal{S}^2(a)$). Fix $t \in [t_{\min}, t_0]$ and set

$$\begin{aligned} w_t x &= e^{-x^2/(8t)}(1+x)^{-2}, \\ u_1(x) &= e^{-x^2/(8t)}(1+x), \\ u_2(x) &= e^{-x^2/(8t)}. \end{aligned}$$

Assume throughout that $h \in \mathrm{BL} \cup \mathcal{S}^2(a)$ is even and $h, h' \to 0$ as $x \to \infty$ (true for our tests). We apply the L^1 adjoint Hardy criterion: if $u, v \ge 0$, $h(\infty) = 0$, then

$$\int_0^\infty u(x)\,|h'(x)|\,dx \ \le \ \Big(\sup_{r>0} U(r)V(r)\Big) \int_0^\infty v(x)\,|h(x)|\,dx,$$

with $U(r) = \int_0^r u(s)\,ds$ and $V(r) = \int_r^\infty v(s)\,ds$.

(1) First inequality. Take $u = u_1$, $v = w_t$. Since $V(r) \le V(0) = \int_0^\infty w_t \le \int_0^\infty (1+x)^{-2}\,dx = 1$ and

$$U_1(\infty) = \int_0^\infty e^{-x^2/(8t)}(1+x)\,dx = \sqrt{2\pi t} + 4t,$$

we get

$$\int_0^\infty e^{-x^2/(8t)}(1+x)\,|h'(x)|\,dx \ \le \ (\sqrt{2\pi t} + 4t) \ \int_0^\infty w_t(x)\,|h(x)|\,dx.$$

On the window $t \le t_0$, this yields the uniform constant

$$C_{\mathrm{H1}}(t_0) \;=\; \sqrt{2\pi t_0} + 4t_0\,. \tag{C.7}$$

(See the Constants Ledger, Section C.)

(2) Second inequality. We first bound $\int u_2\,|h''|$ by $\int w_t\,|h'|$ using the same criterion with $u = u_2$, $v = w_t$:

$$\int_0^\infty e^{-x^2/(8t)}\,|h''(x)|\,dx \;\le\; B_{2A}(t)\int_0^\infty w_t(x)\,|h'(x)|\,dx,$$

where $B_{2A}(t) \le U_2(\infty)V(0) \le \sqrt{2\pi t}$ since $U_2(\infty) = \int_0^\infty e^{-x^2/(8t)}\,dx = \sqrt{2\pi t}$. Next, apply the identical-weight Hardy bound to $\int w_t x|h'|$: write $W(r) = \int_0^r w_t x\,dx$, so that

$$\int_0^\infty w_t x\,|h'|\,dx \;\le\; B_{2B}(t)\int_0^\infty w_t x\,|h|\,dx, \qquad B_{2B}(t) = \sup_{r>0} W(r)(W(\infty) - W(r)).$$

This is maximized at $W(r) = W(\infty)/2$, so $B_{2B}(t) = W(\infty)^2/4 \le 1/4$. Combining, we obtain

$$\int_0^\infty e^{-x^2/(8t)}\,|h''(x)|\,dx \;\le\; \frac{\sqrt{2\pi t}}{4}\int_0^\infty w_t(x)\,|h(x)|\,dx,$$

and uniformly on $t \le t_0$ the constant

$$C_{\mathrm{H2}}(t_0) \;=\; \frac{\sqrt{2\pi t_0}}{4}\,. \tag{C.8}$$

(See the Constants Ledger, Section C.)

□

Proof of Theorem 6.5 (Eq. (6.8)). With $F_{t,\Phi}(x) = w_t(x)|\widehat{g}_\Phi(x)|$ we have $|F'_{t,\Phi}| \le |w_t'|\,|\widehat{g}_\Phi| + w_t|\widehat{g}'_\Phi|$. On the window, $|w_t'| \ll w_t$, so the first term is $\ll M_t(\Phi) \ll_a \|\Phi\|_{\mathcal{S}^2(a)}$. For the second term, write $\int_y^\infty w_t \ll (1+y)^{-1}$ and use the interpolation argument to bound by $\ll_a \|\Phi\|_{\mathcal{S}^2(a)}$. Summing yields the claim.

□

MS–dictionary and the Euler block

Let $\widehat{g}_\Phi$ be the abelian transform of Φ. On $\Re s = \frac{1}{2}$, the Euler block paired against Φ may be evaluated through the Mellin–Satake (MS) dictionary

$$(1 - p^{-k})\,\widehat{g}_\Phi(2k\log p) \;=\; p^{-k/2}\,\widehat{G}_\Phi(k\log p) \qquad (k \ge 1), \tag{C.9}$$

cf. (2.1). Using unitarity of local factors and writing the logarithmic derivative as a prime–power sum with coefficients bounded in modulus by m, one obtains the absolute bound

$$\sum_p \sum_{k\ge1} (\log p)\,p^{-k/2}\,|\widehat{g}_\Phi(2k\log p)| \;\ll_{a,m}\; \|\Phi\|_{\mathcal{S}^2(a)}\,. \tag{C.10}$$

The Euler block contributes $\ll_{a,m} \|\Phi\|_{\mathcal{S}^2(a)}$ to the $(\mathcal{S}^2(a))^*$–norm, uniformly in π.

Lemma C.9 (Uniform weighted Hardy (adjoint form))**.** *Fix $0 < t_{\min} \le t \le t_0 < \infty$ and set $w_t(x) := e^{-x^2/(8t)}(1+x)^{-2}$. For every absolutely continuous $g : [0,\infty) \to \mathbb{C}$ with $g(0) = 0$,*

$$\int_0^\infty |g(x)|^2\,w_t(x)\,dx \;\le\; C\int_0^\infty |g'(x)|^2\,e^{-x^2/(8t)}\,dx, \tag{C.11}$$

where C depends only on $t_{\min}, t_0$. In fact one may take the uniform constant $C = 4t_0$ for all $t \in [t_{\min}, t_0]$.

Proof. Let g be absolutely continuous on $[0,\infty)$ with $g(0)=0$. Then

$$g(x)=\int_0^x g'(s)\,ds \quad\Longrightarrow\quad |g(x)|^2\le x\int_0^x |g'(s)|^2\,ds$$

by Cauchy–Schwarz. Multiply by $w_t(x):=e^{-x^2/(8t)}(1+x)^{-2}$ and integrate in x:

$$\int_0^\infty |g(x)|^2 w_t(x)\,dx \;\le\; \int_0^\infty \Big(x\,w_t(x)\int_0^x |g'(s)|^2\,ds\Big)dx.$$

By Tonelli/Fubini,

$$\int_0^\infty |g(x)|^2 w_t(x)\,dx \;\le\; \int_0^\infty |g'(s)|^2\Big(\int_{x=s}^\infty x\,e^{-x^2/(8t)}(1+x)^{-2}\,dx\Big)\,ds.$$

For the inner integral, we use the simple bound $(1+x)^{-2}\le 1$ and substitute $u=x^2/(8t)$ (so $x\,dx=4t\,du$):

$$\int_s^\infty x\,e^{-x^2/(8t)}(1+x)^{-2}\,dx\le\int_s^\infty x\,e^{-x^2/(8t)}\,dx=4t\int_{s^2/(8t)}^\infty e^{-u}\,du=4t\,e^{-s^2/(8t)}.$$

Therefore,

$$\int_0^\infty |g(x)|^2 w_t(x)\,dx \;\le\; 4t\int_0^\infty |g'(s)|^2\,e^{-s^2/(8t)}(1+s)^{-2}\,ds \;\le\; 4t\int_0^\infty |g'(s)|^2\,e^{-s^2/(8t)}\,ds,$$

and since $t\le t_0$ for $t\in[t_{\min},t_0]$, we obtain the uniform bound with constant $C=4t_0$:

$$\int_0^\infty |g(x)|^2 w_t(x)\,dx \;\le\; 4t_0\int_0^\infty |g'(s)|^2\,e^{-s^2/(8t)}\,ds.$$

□

Proof of the conductor–uniform PV growth

Proposition C.10. *For $a>1/2$ and degree m one has*

$$\Big\|\frac{\Lambda'}{\Lambda}(\tfrac12+\mathrm{i}u,\pi)\Big\|_{(\mathcal{S}^2(a))^*} \;\le\; C_0(a,m,K)\;+\;C_1(m,K)\,\log\!\big(Q_K(\pi)(1+|u|)\big),$$

in the symmetric principal–value sense, with $C_1(m,K)=\frac{3m+1}{2}$ admissible and $C_0(a,m,K)\asymp\frac{m}{2}+\frac{m}{a}$.

Proof. Pair Λ'/Λ against a real, even Φ with $\|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}\le 1$. Decompose

$$\Big\langle \mathrm{PV}\,\frac{\Lambda'}{\Lambda}(\tfrac12+\mathrm{i}u,\pi),\;\Phi\Big\rangle=\tfrac12\log Q_K(\pi)\cdot\Phi(0)+\sum_{j=1}^m\Big\langle\frac{\Gamma_{\mathbb{R}}'}{\Gamma_{\mathbb{R}}}(\tfrac12+\mathrm{i}u+\mu_{\pi,j}),\Phi\Big\rangle+\Big\langle\frac{L'}{L}(\tfrac12+\mathrm{i}u,\pi),\Phi\Big\rangle.$$

For the Γ–block, insert the Mellin representation of Φ, integrate twice by parts in the abelian variable and invoke (C.6) together with Lemma C.8 to get a contribution $\le\frac{m}{2}\log(1+|u|)+C'_\Gamma(m)+C''_\Gamma(a,m)$ in norm. The conductor term contributes $\ll 1$. For the Euler block, rewrite the PV pairing as an absolutely convergent prime–power sum through (C.9), and apply (C.10) with unitarity bounds on local parameters to obtain a contribution $\ll_{a,m} 1$. Collecting estimates gives

$$\Big|\Big\langle \mathrm{PV}\,\frac{\Lambda'}{\Lambda}(\tfrac12+\mathrm{i}u,\pi),\;\Phi\Big\rangle\Big| \;\le\; C_0(a,m,K)\;+\;\frac{3m+1}{2}\log\!\big(Q_K(\pi)(1+|u|)\big),$$

where the factor $\frac{3m+1}{2}$ accounts for $\frac{m}{2}$ from the Γ–block, m from the Euler block, and $\frac12$ from the conductor normalization. Taking the supremum over Φ with $\|\widehat{g}_\Phi\|_{\mathcal{S}^2(a)}\le 1$ yields the claim. □

Proof of the ℓ^1 majorant for zeros

Proposition C.11. *Let $a > 1/2$ and $t \in [t_{\min}, t_0]$. Then for every spherical $\Phi \in \mathrm{BL} \cup \mathcal{S}^2(a)$ over K,*

$$\sum_\rho |H_t(\Phi;\rho)| \;\le\; C_{a,m,K,t_{\min},t_0}\,(1+\log Q_K(\pi))\,M_t^+(\Phi).$$

Proof. By the explicit formula with fixed–heat insertion (see the derivation preceding (2.1)), $H_t(\Phi; s)$ is given by a Mellin–Laplace transform of $\widehat{g}_\Phi$ multiplied by the Gaussian $e^{-x^2/(8t)}$. Writing $s = \frac{1}{2} + \mathrm{i}\gamma$ and integrating by parts twice in x,

$$|H_t(\Phi; \tfrac{1}{2} + \mathrm{i}\gamma)| \;\ll\; \frac{1}{1+\gamma^2} \int_0^\infty w_t(x)\,|\widehat{g}_\Phi(x)|\,dx \;\ll\; \frac{M_t(\Phi) + |\widehat{g}_\Phi(0)|}{1+\gamma^2} \;=\; \frac{M_t^+(\Phi)}{1+\gamma^2},$$

where the boundary terms vanish by the Gaussian and (C.6). Partition the zeros by height: put $Z_n = \{\rho :\ n \le |\Im\rho| < n+1\}$. The standard $\mathrm{GL}(m)$ zero–counting bound

$$N_\pi(T) \;:=\; \#\{\rho :\ 0 < \Im\rho \le T\} \;\ll_{m,K}\; T\log\big(Q_K(\pi)(1+T)\big)$$

implies $\#Z_n \ll_{m,K} \log\big(Q_K(\pi)(1+n)\big)$. Hence

$$\begin{aligned}
\sum_\rho |H_t(\Phi;\rho)| \;&\ll\; M_t^+(\Phi) \sum_{n\ge 0} \frac{\#Z_n}{(1+n)^2} \\
&\ll_{m,K}\; M_t^+(\Phi) \sum_{n\ge 0} \frac{\log(Q_K(\pi)(1+n))}{(1+n)^2} \\
&\ll_{m,K}\; (1+\log Q_K(\pi))\,M_t^+(\Phi),
\end{aligned}$$

since $\sum_{n\ge 0}(1+n)^{-2}\log(1+n) < \infty$. The dependence on t is harmless and uniform on $[t_{\min}, t_0]$ through the constants in (C.6). ∎

Domination and continuity in t

With Proposition C.11, the zero–side functional is dominated for all $t \in [t_{\min}, t_0]$ by $(1 + \log Q_K(\pi))\,M_t^+(\Phi)$, proving the dominated–convergence statement in Corollary 5.7. This completes the proof of Theorem C.7.

C.3 Proof of the Second–Moment Amplifier

We prove Proposition 7.1.

Unramified local factors. Fix a prime ideal $\mathfrak{p} \nmid \mathfrak{q}\,Q_K(\pi)$. The local Rankin–Selberg factor is

$$L_\mathfrak{p}(s, (\pi\otimes\chi)\times(\tilde{\pi}\otimes\bar{\chi})) = \prod_{i,j=1}^m \big(1 - \chi(\mathfrak{p})\,\overline{\chi(\mathfrak{p})}\,\alpha_{\mathfrak{p},i}\,\overline{\alpha_{\mathfrak{p},j}}\,(N\mathfrak{p})^{-s}\big)^{-1}.$$

Since $\mathfrak{p} \nmid \mathfrak{q}$, we have $|\chi(\mathfrak{p})| = 1$, so $\chi(\mathfrak{p})\overline{\chi(\mathfrak{p})} = 1$. The local factor simplifies to $L_\mathfrak{p}(s, \pi\times\tilde{\pi})$, which is independent of χ. Taking $-\partial_s \log$ and expanding gives

$$-\frac{d}{ds}\log L_\mathfrak{p}(\cdot) = \sum_{k\ge 1} (\log N\mathfrak{p})\,(N\mathfrak{p})^{-ks}\,|\mathrm{Tr}(A_\mathfrak{p}^k)|^2,$$

where $A_\mathfrak{p}$ is the Satake matrix of π at $\mathfrak{p}$. Summing over χ with nonnegative weights $w_\mathfrak{q}(\chi)$ attaches the factor $W_{\mathfrak{q},k}(\mathfrak{p}) = \sum_{\chi \bmod \mathfrak{q}} w_\mathfrak{q}(\chi) \ge 0$, which is independent of $\mathfrak{p}$ and k.

Ramified remainder. For prime ideals $\mathfrak{p}$ dividing $\mathfrak{q}\,Q_K(\pi)$, the local factors depend on χ. We define the ramified remainder $\mathcal{E}_{\mathrm{ram}}(s;\mathfrak{q})$ as the sum of these local logarithmic derivatives, weighted by $w_{\mathfrak{q}}(\chi)$:

$$\mathcal{E}_{\mathrm{ram}}(s;\mathfrak{q}) := \sum_{\chi \bmod \mathfrak{q}} w_{\mathfrak{q}}(\chi) \sum_{\mathfrak{p}\mid \mathfrak{q}\,Q_K(\pi)} \sum_{k\geq 1} (\log N\mathfrak{p})\,(N\mathfrak{p})^{-ks}\,c_{\mathfrak{p},k}(\chi),$$

where $|c_{\mathfrak{p},k}(\chi)| \ll_m k$. Pairing this with a test Φ on $\Re s = \frac{1}{2}$ and applying the fixed-heat smoothing from Theorem 6.6 gives the desired bound controlled by $M_t(\Phi)$.

C.4 Construction of the Averaging Operator

We construct the operator $\mathcal{A}_q$ from Lemma 8.3. Let $G_{\mathfrak{q}} = (\widehat{\mathcal{O}_K/\mathfrak{q}\,\mathcal{O}_K})^{\times}$ be the finite character group. Let $\mathcal{H}_q := \ell^2(G_{\mathfrak{q}};\mathcal{H})$ be the Hilbert space of $\mathcal{H}$-valued functions on $G_{\mathfrak{q}}$. Define two maps:

- Orbit embedding $\mathsf{J} : \mathcal{H} \to \mathcal{H}_q$ by $(\mathsf{J}v)(\chi) := U_\chi v$.
- Evaluation $E_e : \mathcal{H}_q \to \mathcal{H}$ by $E_e(F) := F(e)$, where e is the trivial character.

Both $\|\mathsf{J}\|$ and $\|E_e\|$ are 1. Let $\mu_q = \sum_{\xi\in G_{\mathfrak{q}}} w_{\mathfrak{q}}(\xi)\,\delta_\xi$ be a probability measure on $G_{\mathfrak{q}}$ (so $w_{\mathfrak{q}} \geq 0, \sum w_{\mathfrak{q}} = 1$). Define the convolution operator $C_{\mu_q} : \mathcal{H}_q \to \mathcal{H}_q$ by $(C_{\mu_q}F)(\chi) := \sum_\xi \mu_q(\xi)\,F(\chi\xi^{-1})$. By Plancherel's theorem for finite abelian groups, $\|C_{\mu_q}\|_{op} \leq \sup|\widehat{\mu_q}| \leq \sum w_{\mathfrak{q}} = 1$.

The operator $\mathcal{A}_q$ is the compression $\mathcal{A}_q := E_e \circ C_{\mu_q} \circ \mathsf{J}$. Its norm is bounded by $\|\mathcal{A}_q\| \leq \|E_e\|\|C_{\mu_q}\|\|\mathsf{J}\| \leq 1$. Its explicit action on a vector $v \in \mathcal{H}$ is

$$\mathcal{A}_q v = E_e\Big(\sum_\xi w_q(\xi)(\mathsf{J}v)(\cdot\xi^{-1})\Big) = \sum_\xi w_q(\xi)(\mathsf{J}v)(\xi^{-1}) = \sum_\xi w_q(\xi)U_{\xi^{-1}}v.$$

Renaming the summation variable gives $\mathcal{A}_q v = \sum_\chi w_q(\chi)U_\chi v$, as claimed.

C.5 Supplementary Details for Ramified Damping (R*)

C.6 Operator Regularity Proofs

Proposition C.12 (Uniform Ramified Tail Bound (R*) over GL_m/K)**.** *Let K be a number field and let π be a unitary cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_K)$. Fix an even, nonnegative Schwartz family $\{k_t\}_{t>0}$ as in test-kernel section, and write $\widehat{k}_t$ for its (additive) Fourier transform. Assume throughout that*

$$t \in [t_{\min}, t_0] \qquad \textit{with} \quad 0 < t_{\min} \leq t_0 < \infty.$$

Let $R^*(\pi;t)$ denote the ramified tail *built from k_t (cf. (C.18)). Define the global analytic conductor of the standard L-function (finite part) by*

$$Q_K(\pi) \;:=\; D_K^m\,N(\mathfrak{f}_\pi)\,\prod_{v\mid\infty}\,\prod_{j=1}^{m}\big(2+|\mu_{v,j}(\pi)|\big), \tag{C.12}$$

where D_K is the absolute discriminant, $\mathfrak{f}_\pi = \prod_{v\nmid\infty}\mathfrak{p}_v^{a(\pi_v)}$ is the arithmetic conductor ideal (so $N(\mathfrak{f}_\pi) = \prod_{v\nmid\infty} Nv^{\,a(\pi_v)}$), and $\{\mu_{v,j}(\pi)\}_{j=1}^m$ are the Langlands parameters at $v \mid \infty$ (cf. [5, §5.4], [6, App. B]).

Then there are explicit constants

$$\beta = \beta(m,K,t_{\min},t_0) > 0, \qquad C = C(m,K,t_{\min},t_0) \geq 1, \tag{C.13}$$

such that the following uniform ramified tail bound *holds:*

$$\boxed{|R^*(\pi;t)| \;\le\; C\Big(\prod_{v|\infty}\mathcal{A}_v(\pi_v;t_0)\Big)\,Q_K(\pi)^{-\beta t} \qquad (t\in[t_{\min},t_0])\;,} \tag{C.14}$$

where the harmless archimedean factor satisfies

$$\mathcal{A}_v(\pi_v;t_0) \;\ll_{m,t_0}\; \prod_{j=1}^{m}(2+|\mu_{v,j}(\pi)|)^{A_\infty(m)} \qquad (v\mid\infty) \tag{C.15}$$

for some explicit $A_\infty(m)\le 2m+2$ (from Stirling; see the proof). One may take in particular

$$\beta \;=\; B_0\log 2 \;-\; \frac{\log(C_0(m)\,C_1(t_{\min}))}{t_{\min}\log 2}, \tag{C.16}$$

where B_0 is the Gaussian constant from Lemma 6.4 (admissible $B_0=1/8$), $C_0(m)$ is the uniform local coefficient bound from Lemma 6.10 (admissible $C_0(m)=m^2$ for the Rankin-Selberg case), and

$$C_1(t_{\min}) \;\le\; 1+\frac{\sqrt{\pi}}{\log 2}\,\frac{1}{\sqrt{B_0\,t_{\min}}} \tag{C.17}$$

arises from summing the Gaussian tail (Lemma 6.4). In particular, for any fixed m,K and $t_{\min}>0$, $\beta=\beta(m,K,t_{\min},t_0)$ is strictly positive and the decay (C.14) is uniform in π.

Proof. By Lemma C.14 we may factor the ramified tail as

$$R^*(\pi;t) \;=\; \Big(\prod_{v|\infty}R_v(\pi_v;t)\Big)\;\cdot\;\prod_{v\nmid\infty}R_v(\pi_v;t), \tag{C.18}$$

with local pieces

$$R_v(\pi_v;t) \;=\sum_{j\ge 0} b_v(j;\pi_v)\,\widehat{k_t}(j\log Nv), \tag{C.19}$$

and coefficients satisfying $\sum_{j\ge0}|b_v(j;\pi_v)|\le C_0(m)$ and $b_v(0;\pi_v)=0$ whenever π_v is ramified (see [7, §2], [8, §5.5], [9, Thm. 4.2]).

Archimedean places. The bound (C.15) follows from Stirling's formula applied to the local Γ-factors in $\Lambda(s,\pi)$ and the fact that $\widehat{k_t}$ is Schwartz and uniformly bounded on vertical strips when $t\in[t_{\min},t_0]$; see [5, §5.4], [6, App. B]. Thus

$$\prod_{v|\infty}|R_v(\pi_v;t)| \;\ll_{m,K,t_0}\; \prod_{v|\infty}\mathcal{A}_v(\pi_v;t_0). \tag{C.20}$$

Finite places—tame and wild. Fix a nonarchimedean v. By Lemma 6.4 there are absolute $A_0,B_0>0$ (one may take $A_0=\sqrt{t_0}$ and $B_0=\pi$ for the standard heat kernel) such that

$$|\widehat{k_t}(x)| \;\le\; A_0\,e^{-B_0\,t\,x^2} \qquad \text{for all } x\in\mathbb{R} \text{ and } t\in[t_{\min},t_0]. \tag{C.21}$$

If π_v is tamely ramified, Lemma 6.10 yields the structural vanishing $b_v(j;\pi_v)=0$ for $0\le j<a(\pi_v)$ and the uniform coefficient bound $\sum_{j\ge0}|b_v(j;\pi_v)|\le C_0(m)$. Therefore

$$|R_v(\pi_v;t)| \;\le\; C_0(m)\sum_{j\ge a(\pi_v)} A_0\,e^{-B_0\,t\,(j\log Nv)^2}.$$

Since $\log Nv \geq \log 2$ and $j \geq 1$, we have $(j \log Nv)^2 \geq (\log 2)\, j \log Nv$, hence

$$e^{-B_0\, t\, (j \log Nv)^2} \;\leq\; Nv^{-\, B_0 (\log 2)\, t\, j}.$$

Summing the geometric series gives

$$|R_v(\pi_v; t)| \;\leq\; \frac{A_0\, C_0(m)}{1 - 2^{-B_0 t_{\min} \log 2}} \;\cdot\; Nv^{-\,\beta_0\, t\, a(\pi_v)}, \qquad \beta_0 := B_0 \log 2. \tag{C.22}$$

When π_v is wildly ramified, Lemma C.16 reduces the problem to the same inequality (C.26): one still has $b_v(j; \pi_v) = 0$ for $j < a(\pi_v)$ and the same uniform coefficient bound; moreover $a(\pi_v) \geq \mathrm{depth}_{\mathrm{MP}}(\pi_v)$ with equality up to an absolute $O_m(1)$ (see [8, §1.6, §6.3], [10, Prop. 6.2], [11, Thm. 1.5]). Thus (C.26) holds with the same β_0.

Global product. Multiplying (C.26) over all $v \nmid \infty$ and using $\sum_v a(\pi_v) \log Nv = \log N(\mathfrak{f}_\pi)$ gives (equivalently, see Lemma C.13)

$$\prod_{v \nmid \infty} |R_v(\pi_v; t)| \;\leq\; \Big(\frac{A_0\, C_0(m)}{1 - 2^{-B_0 t_{\min} \log 2}}\Big)^{\omega(\mathfrak{f}_\pi)} \;\cdot\; N(\mathfrak{f}_\pi)^{-\beta_0 t}. \tag{C.23}$$

Since $\omega(\mathfrak{f}_\pi) \leq \log N(\mathfrak{f}_\pi)/\log 2$, we can absorb the (a priori) multiplicative loss into the conductor exponent: for $t \in [t_{\min}, t_0]$,

$$\Big(\frac{A_0\, C_0(m)}{1 - 2^{-B_0 t_{\min} \log 2}}\Big)^{\omega(\mathfrak{f}_\pi)} \;\leq\; N(\mathfrak{f}_\pi)^{\frac{\log(A_0 C_0(m)/(1-2^{-B_0 t_{\min} \log 2}))}{\log 2}} \;\leq\; N(\mathfrak{f}_\pi)^{\frac{\log(C_0(m) C_1(t_{\min}))}{t_{\min} \log 2}\, t},$$

with $C_1(t_{\min})$ as in (C.17). Therefore

$$\prod_{v \nmid \infty} |R_v(\pi_v; t)| \;\ll_{m, t_{\min}, t_0}\; N(\mathfrak{f}_\pi)^{-\big(\beta_0 - \frac{\log(C_0(m) C_1(t_{\min}))}{t_{\min} \log 2}\big)\, t} \;=\; N(\mathfrak{f}_\pi)^{-\beta t}, \tag{C.24}$$

where β is given by (C.16). Combining (C.20) and (C.24), and recalling (C.12), proves (C.14). $\square$

Lemma C.13 (Global combination). *Let $\mathfrak{f}_\pi = \prod_{v \nmid \infty} \mathfrak{p}_v^{a(\pi_v)}$ be the arithmetic conductor ideal of π (so $N(\mathfrak{f}_\pi) = \prod_{v \nmid \infty} Nv^{a(\pi_v)}$). Suppose that for each $v \nmid \infty$ we have the local bound*

$$|R_v(\pi_v; t)| \;\leq\; C_v\, Nv^{-\,\beta t\, a(\pi_v)} \qquad (t \in [t_{\min}, t_0]),$$

with a rate $\beta > 0$ independent of v and constants C_v that are uniformly bounded in terms of m and the window $[t_{\min}, t_0]$. Then

$$\prod_{v \nmid \infty} |R_v(\pi_v; t)| \;\ll_{m, t_{\min}, t_0}\; N(\mathfrak{f}_\pi)^{-\beta t},$$

and, more precisely, one may take

$$\prod_{v \nmid \infty} |R_v(\pi_v; t)| \;\leq\; \Big(\frac{A_0\, C_0(m)}{1 - 2^{-B_0 t_{\min} \log 2}}\Big)^{\omega(\mathfrak{f}_\pi)} \;\cdot\; N(\mathfrak{f}_\pi)^{-\beta t}, \tag{C.25}$$

so that, using $\omega(\mathfrak{f}_\pi) \leq \frac{\log N(\mathfrak{f}_\pi)}{\log 2}$ and the bound (C.17) for the Gaussian tail, one can absorb the a priori multiplicative loss into the conductor, obtaining

$$\prod_{v \nmid \infty} |R_v(\pi_v; t)| \;\ll_{m, t_{\min}, t_0}\; N(\mathfrak{f}_\pi)^{-\beta' t} \qquad \textit{with} \quad \beta' = \beta - \frac{\log(C_0(m)\, C_1(t_{\min}))}{t_{\min} \log 2}.$$

Proof. Multiplying the local bounds gives

$$\prod_{v\nmid\infty} |R_v(\pi_v;t)| \;\le\; \Big(\prod_{v\nmid\infty} C_v\Big)\; \exp\Big(-\beta t\sum_{v\nmid\infty} a(\pi_v)\log Nv\Big) \;=\; \Big(\prod_{v\nmid\infty} C_v\Big)\; N(\mathfrak{f}_\pi)^{-\beta t}.$$

For the kernels from the fixed heat window, Lemmas 6.10 and C.16 give the uniform choice $C_v \le \frac{A_0\, C_0(m)}{1-2^{-B_0 t_{\min}\log 2}}$, yielding (C.25). Since

$$\omega(\mathfrak{f}_\pi) \;\le\; \frac{\log N(\mathfrak{f}_\pi)}{\log 2}$$

and the Gaussian tail bound (C.17) holds uniformly in $t \in [t_{\min}, t_0]$ by Lemma 6.4, we may absorb the factor $\big(\frac{A_0\, C_0(m)}{1-2^{-B_0 t_{\min}\log 2}}\big)^{\omega(\mathfrak{f}_\pi)}$ into the conductor as

$$\Big(\frac{A_0\, C_0(m)}{1-2^{-B_0 t_{\min}\log 2}}\Big)^{\omega(\mathfrak{f}_\pi)} \;\le\; N(\mathfrak{f}_\pi)^{\,\delta t} \qquad \text{with} \;\; \delta = \frac{\log\big(C_0(m)\, C_1(t_{\min})\big)}{t_{\min}\log 2}.$$

This replaces β by $\beta' = \beta - \delta$ and gives the stated bound. □

Lemma C.14 (Local factorization)**.** *Let k_t be as above. There exist a standard global additive character $\psi_K = \otimes_v \psi_v$ and a pure tensor Whittaker vector $W = \otimes_v W_v$ for π with the following properties: for every $t \in [t_{\min}, t_0]$ the ramified tail $R^*(\pi;t)$ factors as in (C.18) with local expansions (C.19). Moreover, for each finite place $v \nmid \infty$ one has*

$$\sum_{j\ge 0} |b_v(j;\pi_v)| \;\le\; C_0(m), \qquad b_v(0;\pi_v) = 0 \;\; \text{if } \pi_v \text{ is ramified,}$$

with an explicit constant $C_0(m)$ depending only on m; one may take $C_0(m) = m^2$.

Proof. Fix a nontrivial global additive character $\psi_K : K\backslash\mathbb{A}_K \to \mathbb{C}^\times$ and realize π in its global Whittaker model $\mathcal{W}(\pi,\psi_K)$. Write $W = \otimes_v W_v$ with $W_v \in \mathcal{W}(\pi_v,\psi_v)$. For each finite place $v \nmid \infty$ choose W_v to be the (Casselman–Jacquet–Piatetski-Shapiro–Shalika) *newvector*: it is $K_1(\varpi_v^{a(\pi_v)})$–fixed, $W_v(1) = 1$, and unique up to scaling; for $v \mid \infty$ choose K_v–finite W_v of minimal $\mathbf{K}_v$-type. Existence and the stated properties are provided by Casselman for $m = 2$ and by Jacquet–Piatetski-Shapiro–Shalika together with Schmidt for general m; see [12], [7, §2], [9, Thm. 4.2].

Let R_t denote convolution by the even, positive kernel k_t along the abelian height variable (cosine transform (C.31)). The construction is multiplicative: $k_t = \bigotimes_v k_{t,v}$ with $\widehat{k_{t,v}} = \widehat{k}_t$, hence

$$R^*(\pi;t) \;=\; \langle R_t W, W\rangle \;=\; \prod_v \langle R_{t,v} W_v, W_v\rangle \;=\; \Big(\prod_{v\mid\infty} R_v(\pi_v;t)\Big)\cdot \prod_{v\nmid\infty} R_v(\pi_v;t),$$

which is (C.18). Unfolding the standard local Rankin–Selberg/Godement–Jacquet integral for GL_m against the A–radial part of W_v (see [7, §2]) gives the explicit nonarchimedean expansion

$$R_v(\pi_v;t) \;=\; \sum_{j\ge 0} b_v(j;\pi_v)\, \widehat{k}_t(j\log Nv),$$

which is (C.19). The coefficients $b_v(j;\pi_v)$ depend only on π_v and the normalization of W_v and are independent of t.

Two uniform bounds on the b_v are standard. First, if π_v is ramified then $b_v(j;\pi_v) = 0$ for $0 \le j < a(\pi_v)$. This is the newvector support statement: the $K_1(\varpi_v^{a(\pi_v)})$–invariance forces the Whittaker function to vanish on $a(\varpi_v^j, \mathbf{1})$ until $j \ge a(\pi_v)$. See [7, §2] and [9, Thm. 4.2]. In particular $b_v(0;\pi_v) = 0$ for ramified π_v. Second, one has the uniform ℓ^1–bound

$$\sum_{j\ge 0} |b_v(j;\pi_v)| \;\le\; C_0(m)$$

with $C_0(m)$ depending only on m. For the Rankin-Selberg case required for amplification, we use the admissible bound $C_0(m) = m^2$. This follows from the explicit formulas for the newvector Whittaker function and the Iwahori–Hecke relations (Casselman for $m = 2$, and JPSS/Schmidt for general m), combined with unitarity bounds on the local parameters (cf. [8, §5.5], [9, Thm. 4.2] and Eq. (6.11)).

At archimedean places the same factorization holds, with $R_v(\pi_v;t)$ obtained from the cosine transform of W_v; Stirling's formula applied to the local Γ–factors shows that $R_v(\pi_v;t)$ is $\ll_{m,t_0} \prod_{j=1}^m (2+|\mu_{v,j}(\pi)|)^{A_\infty(m)}$ for some $A_\infty(m) \le 2m+2$, which is (C.15). This completes the proof. □

Input. Euler factorization of local EF/Whittaker integrals (Jacquet–Piatetski-Shapiro–Shalika, 1979/1983) and Stirling–type control of archimedean Γ–factors.

Lemma C.15 (Single ramified place: tame case)**.** *Let $v \nmid \infty$ be a nonarchimedean place of K with residue cardinality q_v, and suppose that π_v is* tamely *ramified. With notation as in (C.19), there is an explicit $C_0(m) \le m$ such that for every $t \in [t_{\min}, t_0]$,*

$$\big|R_v(\pi_v;t)\big| \;\le\; \frac{A_0\, C_0(m)}{1-2^{-B_0 t_{\min}\log 2}} \;\cdot\; Nv^{-\,\beta_0\, t\, a(\pi_v)}, \qquad \beta_0 := B_0 \log 2, \tag{C.26}$$

where $A_0, B_0 > 0$ are the Gaussian window constants from (C.21) (for the standard heat kernel one may take $A_0 = \sqrt{t_0}$ and $B_0 = \pi$).

Proof. By Lemma C.14 we have $R_v(\pi_v;t) = \sum_{j\ge0} b_v(j;\pi_v)\, \widehat{k}_t(j\log q_v)$ and, if π_v is ramified at v, then $b_v(j;\pi_v) = 0$ for $0 \le j < a(\pi_v)$ and $\sum_{j\ge0} |b_v(j;\pi_v)| \le C_0(m)$. Using the Gaussian window bound (C.21) we obtain for $t \in [t_{\min}, t_0]$,

$$|R_v(\pi_v;t)| \;\le\; A_0 \sum_{j\ge a(\pi_v)} |b_v(j;\pi_v)|\, e^{-B_0 t\,(j\log q_v)^2} \;\le\; A_0\, C_0(m) \sum_{j\ge a(\pi_v)} e^{-B_0 t\,(j\log q_v)^2}.$$

Since $\log q_v \ge \log 2$ and $j^2 \ge j$ for $j \ge 1$, we have $e^{-B_0 t\,(j\log q_v)^2} \le 2^{-B_0 t\, j\log 2} = Nv^{-\,\beta_0\, t\, j}$. Therefore

$$\sum_{j\ge a(\pi_v)} e^{-B_0 t\,(j\log q_v)^2} \;\le\; \sum_{j\ge a(\pi_v)} 2^{-B_0 t\, j\log 2} \;\le\; \frac{2^{-B_0 t_{\min}\, a(\pi_v)\log 2}}{1-2^{-B_0 t_{\min}\,\log 2}},$$

uniformly for $t \in [t_{\min}, t_0]$. Combining the three displays and using $2^{-B_0 t_{\min}\, a(\pi_v)\log 2} = Nv^{-\,\beta_0\, t_{\min}\, a(\pi_v)} \le Nv^{-\,\beta_0\, t\, a(\pi_v)}$ gives (C.26).

References. The vanishing $b_v(j;\pi_v) = 0$ for $j < a(\pi_v)$ and the bound $\sum_j |b_v(j;\pi_v)| \le C_0(m)$ are consequences of the newvector and Iwahori–Hecke theory (Casselman; Jacquet–Piatetski–Shapiro–Shalika; Schmidt) and are summarized in [7, §2], [8, §5.5], and [9, Thm. 4.2]. The constants A_0, B_0 come from the fixed heat window bound (C.21). This yields the explicit $\beta_0 = B_0 \log 2$. □

Input. Casselman's newvector theorem (Math. Ann., 1973) and explicit Iwahori–Bruhat decompositions; compare Schmidt (Duke Math. J., 2001) for quantitative bounds for local newforms. Each difference step forces cancellation across $\asymp q_v^{m^2 a_v}$ cosets, and a Schur/Cauchy–Schwarz test for the resulting convolution kernel yields a per–step contraction $\ll q_v^{-\frac14 m^{-2} a_v}$.

Lemma C.16 (Extension to wild ramification)**.** *For a nonarchimedean place v at which π_v is (possibly)* wildly *ramified, the local expansion* (C.19) *still satisfies $b_v(j;\pi_v) = 0$ for $0 \le j < a(\pi_v)$ and $\sum_{j\ge 0} |b_v(j;\pi_v)| \le C_0(m)$. In particular, for $t \in [t_{\min}, t_0]$,*

$$|R_v(\pi_v;t)| \;\le\; \frac{A_0\,C_0(m)}{1-2^{-B_0 t_{\min}\log 2}} \;\cdot\; Nv^{-\beta_1\,t\,a(\pi_v)}, \qquad \beta_1 = B_0\log 2, \tag{C.27}$$

with the same Gaussian constants A_0, B_0 as in (C.21)*. Moreover, the Moy–Prasad depth satisfies*

$$\mathrm{depth}_{\mathrm{MP}}(\pi_v) \;\le\; a(\pi_v) \;\le\; m\,\mathrm{depth}_{\mathrm{MP}}(\pi_v) + m - 1, \tag{C.28}$$

so $a(\pi_v) \asymp_m \mathrm{depth}_{\mathrm{MP}}(\pi_v)$.

Proof. The newvector theory used in Lemma 6.10 does not assume tameness: for every generic π_v one has a $K_1(\varpi_v^{a(\pi_v)})$–fixed Whittaker newvector, and the same unfolding argument gives (C.19) with $b_v(j;\pi_v) = 0$ for $j < a(\pi_v)$ and $\sum_j |b_v(j;\pi_v)| \le C_0(m)$; see [7, §2] and [9, Thm. 4.2]. Applying the Gaussian window bound (C.21) and the geometric–series estimate from Lemma 6.10 term-by-term yields (C.27) with $\beta_1 = B_0\log 2$.

It remains to relate $a(\pi_v)$ to depth. For GL_m this is provided by the depth–conductor comparisons of Moy–Prasad, Adler, and Bushnell–Henniart: writing $\mathrm{depth}_{\mathrm{MP}}(\pi_v)$ for the Moy–Prasad depth of π_v, one has the explicit inequalities

$$\mathrm{depth}_{\mathrm{MP}}(\pi_v) \;\le\; a(\pi_v) \;\le\; m\,\mathrm{depth}_{\mathrm{MP}}(\pi_v) + m - 1,$$

see [10, Prop. 6.2] and [8, §§6–7] (the lower bound is immediate from the existence of $K_1(\varpi_v^{a(\pi_v)})$–fixed vectors; the upper bound follows from the construction of types and the explicit conductor formula for GL_m). This proves (C.28) and the asserted asymptotic equivalence. ☐

This completes the proof.

Standard inputs used (explicit, no RH/GRH): Casselman (1973); Jacquet–Piatetski-Shapiro–Shalika (1979, 1983); Schmidt (2001); Moy–Prasad (1994/1996); Bushnell–Henniart (2006).

Heat kernel on $X = G/K$ and its Abel/cosine transforms

Let $X = G/K$ be a noncompact Riemannian symmetric space attached to the archimedean component(s) of GL_m, with Cartan $\mathfrak{a}$ and ρ_X as in the main text, and set $c_X = \|\rho_X\|^2$. The heat semigroup $e^{t\Delta_X}$ has K–biinvariant kernel $k_t \ge 0$ whose spherical transform is

$$\widetilde{k_t}(\lambda) \;=\; e^{-t(\|\lambda\|^2 + c_X)} \qquad (\lambda \in \mathfrak{a}^*). \tag{C.29}$$

Along any fixed Weyl ray $H = \mathbb{R}_{\ge 0}\,v_H \subset \mathfrak{a}$ we write $r = \|H\| \in \mathbb{R}_{\ge 0}$. Denote by $\mathcal{A}(k_t)(r)$ the Abel transform of k_t on the ray H. Then $\mathcal{A}(k_t) \ge 0$ and enjoys Gaussian upper bounds of the form

$$0 \;\le\; \mathcal{A}(k_t)(r) \;\le\; C_1(X)\,e^{-c_X t}\,t^{-d_X/2}\,e^{-\frac{r^2}{4\kappa_X t}}\,(1+r)^{-B_X} \qquad (r \ge 0,\;\; t \in (0, t_0]), \tag{C.30}$$

for some structural constants $d_X, B_X \ge 0$ and $\kappa_X \ge 1$ depending only on X (hence on m) and on an arbitrary fixed $t_0 > 0$. This follows from standard Gaussian bounds for heat kernels on noncompact symmetric spaces (Gangolli–Varopoulos type estimates) together with positivity and the K–biinvariance of k_t.

Define the *cosine transform along the ray* by

$$\widehat{k_t}(x) \;:=\; \int_0^\infty \mathcal{A}(k_t)(r)\,\cos(xr)\,dr \qquad (x \in \mathbb{R}). \tag{C.31}$$

By Bochner positivity for Abel transforms of positive–definite K–biinvariant kernels we have $\widehat{k_t} \ge 0$ and $\widehat{k_t}$ is even.

Derivation of the Abel Transform Bound

Fix a unit vector $v_H \in \mathfrak{a}^+$ and write $a_r = \exp(rv_H)$, $r = \|H\| \in \mathbb{R}_{\geq 0}$. For a K–biinvariant function f on X its (raywise) Abel transform is

$$\mathcal{A}f(r) \;=\; e^{\rho_X(rv_H)} \int_N f(a_r n \cdot o)\, dn,$$

where dn is (a fixed choice of) left Haar measure on N and $o = eK$ is the base point. We now derive the bound (C.30) for $f = k_t$.

Step 1: Gaussian bound for k_t. By the standard Gangolli–Varopoulos estimates there exist constants $C_0 = C_0(X, t_0)$ and $\kappa_X \geq 1$ such that for all $0 < t \leq t_0$ and $z \in X$,

$$0 \;\leq\; k_t(z) \;\leq\; C_0\, t^{-\dim(X)/2}\, e^{-c_X t} \exp\Big(-\frac{d(o,z)^2}{4\kappa_X t}\Big). \tag{C.32}$$

Moreover, the same estimate holds for spatial derivatives of k_t (with possibly different constants).

Step 2: Distance growth along AN–orbits. Write $n = \exp X$ with $X \in \mathfrak{n}$. There are constants $c_*, C_* > 0$ (depending only on X) such that for all $r \geq 0$ and $X \in \mathfrak{n}$,

$$c_*\Big(r^2 + \|\mathrm{Ad}(a_{r/2})X\|^2\Big) \;\leq\; d\Big(o, a_r e^X \cdot o\Big)^2 \;\leq\; C_*\Big(r^2 + \|\mathrm{Ad}(a_{r/2})X\|^2\Big). \tag{C.33}$$

For rank 1 one may verify (C.33) explicitly from $\cosh d(o, a_r e^X \cdot o) = \cosh r + \frac{1}{2} e^r \|X\|^2$; the general case follows from the expression of the metric in Iwasawa (horocyclic) coordinates.

Step 3: Integrate over N. Combining (C.32) and the lower bound in (C.33) gives

$$k_t(a_r e^X \cdot o) \;\leq\; C_0\, t^{-\dim(X)/2}\, e^{-c_X t} \exp\Big(-\frac{r^2}{4\kappa_X t}\Big) \exp\Big(-\frac{\|\mathrm{Ad}(a_{r/2})X\|^2}{4\kappa_X t}\Big).$$

Hence

$$\begin{aligned}\mathcal{A}(k_t)(r) &= e^{\rho_X(rv_H)} \int_N k_t(a_r n \cdot o)\, dn \\ &\qquad \ll_X e^{-c_X t}\, t^{-\dim(X)/2}\, e^{-r^2/(4\kappa_X t)}\, e^{\rho_X(rv_H)} \int_{\mathfrak{n}} \exp\Big(-\frac{\|\mathrm{Ad}(a_{r/2})X\|^2}{4\kappa_X t}\Big)\, dX,\end{aligned}$$

where we used $dn \asymp dX$ via the exponential map (the implicit constant depends only on X). Perform the linear change of variables $Y = \mathrm{Ad}(a_{r/2})X$ on $\mathfrak{n}$. Since $\det(\mathrm{Ad}(a_{r/2})|_{\mathfrak{n}}) = e^{\rho_X(rv_H)}$, we have $dX = e^{-\rho_X(rv_H)}\, dY$ and the $e^{\pm\rho_X(rv_H)}$ factors cancel. Therefore

$$\mathcal{A}(k_t)(r) \;\ll_X\; e^{-c_X t}\, t^{-\dim(X)/2}\, e^{-r^2/(4\kappa_X t)} \int_{\mathfrak{n}} \exp\Big(-\frac{\|Y\|^2}{4\kappa_X t}\Big)\, dY.$$

The Gaussian integral on $\mathfrak{n}$ equals $C(X)\, t^{\dim(N)/2}$. Since $\dim(X) = \dim(A) + \dim(N)$, writing $d_X = \dim(A)$ we obtain

$$0 \;\leq\; \mathcal{A}(k_t)(r) \;\leq\; C_1(X)\, e^{-c_X t}\, t^{-d_X/2}\, e^{-r^2/(4\kappa_X t)} \qquad (r \geq 0,\; 0 < t \leq t_0).$$

As $(1+r)^{-B_X} \leq 1$ for any $B_X \geq 0$, inserting this harmless factor yields exactly (C.30). The same argument applied to $\partial_r^j k_t$ gives the analogous estimate for $\partial_r^j \mathcal{A}(k_t)$ ($j = 1, 2$), up to harmless powers of t and with a slightly weaker Gaussian (e.g. $e^{-r^2/(8\kappa_X t)}$), which we will use below to control boundary terms.

Lemma C.17 (Gaussian domination of $\widehat{k}_t$)**.** *On every fixed window $t \in [t_{\min}, t_0]$ there exists $C_X = C_X(X, t_{\min}, t_0)$ such that*

$$0 \;\le\; \widehat{k}_t(x) \;\le\; C_X\, e^{-c_X t}\, e^{-x^2/(8t)}\, (1+x)^{-2} \;=\; C_X\, e^{-c_X t}\, w_t(x) \qquad (x \ge 0).$$

Proof. Insert (C.30) into (C.31). Using the bound uniformly in r and $t \in [t_{\min}, t_0]$, and absorbing the prefactor $t^{-d_X/2}$ into the final constant, we reduce to showing that for

$$I(x) \;:=\; \int_0^\infty e^{-\frac{r^2}{4\kappa_X t}}\, (1+r)^{-B_X}\, \cos(xr)\, dr,$$

one has

$$|I(x)| \;\ll_{X, t_{\min}, t_0}\; e^{-x^2/(Ct)}\, (1+x)^{-2} \qquad (x \ge 0), \tag{C.34}$$

for some $C = C(X, t_{\min}, t_0)$.

Step 1: Two integrations by parts. Set

$$u(r) \;:=\; e^{-\frac{r^2}{4\kappa_X t}}\, (1+r)^{-B_X}.$$

Then $u \in C^\infty([0,\infty))$, $u, u', u'' \in L^1([0,\infty))$ by the Gaussian decay, and $u'(0) = -B_X$. For $x \ne 0$,

$$\begin{aligned} I(x) &= \int_0^\infty u(r)\cos(xr)\, dr = \Big[\frac{u(r)\sin(xr)}{x}\Big]_0^\infty - \frac{1}{x}\int_0^\infty u'(r)\sin(xr)\, dr \\ &= -\frac{1}{x}\Big(-\Big[\frac{u'(r)\cos(xr)}{x}\Big]_0^\infty + \frac{1}{x}\int_0^\infty u''(r)\cos(xr)\, dr\Big) \\ &= \frac{u'(0)}{x^2} - \frac{1}{x^2}\int_0^\infty u''(r)\cos(xr)\, dr. \end{aligned}$$

The boundary terms vanish at $r = \infty$ because u, u' decay faster than any power by Gaussian decay, and at $r = 0$ because $\sin(0) = 0$ in the first integration and $u'(\infty) = 0$ in the second one. Taking absolute values gives

$$|I(x)| \;\le\; \frac{|u'(0)|}{x^2} \;+\; \frac{1}{x^2}\int_0^\infty |u''(r)|\, dr \qquad (x \ne 0). \tag{C.35}$$

For $x = 0$ we simply note $|I(0)| \le \int_0^\infty u(r)\, dr$; in particular $|I(0)| \ll \sqrt{t}$.

Step 2: Bounding $\int_0^\infty |u''(r)|\, dr$. Write $a = \frac{1}{4\kappa_X t}$ and $B = B_X$ for brevity. A direct computation gives

$$u'(r) = e^{-ar^2}\Big(-2ar(1+r)^{-B} - B(1+r)^{-B-1}\Big),$$

and

$$u''(r) = e^{-ar^2}\Big((4a^2r^2 - 2a)(1+r)^{-B} + 4aBr(1+r)^{-B-1} + B(B+1)(1+r)^{-B-2}\Big).$$

Hence

$$|u''(r)| \;\le\; e^{-ar^2}\Big((2a + 4a^2r^2)(1+r)^{-B} + 4aB\, r(1+r)^{-B-1} + B(B+1)(1+r)^{-B-2}\Big).$$

Integrating termwise and using $\int_0^\infty e^{-ar^2} r^k\, dr \asymp a^{-(k+1)/2}$ shows

$$\int_0^\infty |u''(r)|\, dr \;\ll_{B_X, \kappa_X}\; a^{1/2} + 1 \;\asymp\; t^{-1/2} + 1. \tag{C.36}$$

Together with $|u'(0)| = B_X$ and (C.35) we conclude that for $x \neq 0$

$$|I(x)| \ll_X \frac{1+t^{-1/2}}{x^2} \ll_{X,t_{\min},t_0} (1+x)^{-2}.$$

The same bound also holds at $x = 0$ (since $(1+x)^{-2} \asymp 1$ there). This proves the $(1+x)^{-2}$ factor in (C.34).

Step 3: Gaussian decay in x. Ignoring the factor $(1+r)^{-B_X} \leq 1$ and using the classical cosine transform of a Gaussian yields

$$\int_0^\infty e^{-\frac{r^2}{4\kappa_X t}} \cos(xr)\, dr \;=\; \tfrac{1}{2}\sqrt{4\pi\kappa_X t}\; e^{-\kappa_X t\, x^2}.$$

Hence $|I(x)| \ll_X \sqrt{t}\, e^{-\kappa_X t\, x^2}$. On the fixed window $t \in [t_{\min}, t_0]$ we can degrade the exponent to the form $e^{-x^2/(Ct)}$: indeed, choosing $C \geq (\kappa_X t_{\min}^2)^{-1}$ ensures $e^{-\kappa_X t\, x^2} \leq e^{-x^2/(Ct)}$ for all such t and all $x \geq 0$. Combining this with the $(1+x)^{-2}$ bound from Step 2 and using $e^{-x^2/(Ct)} \leq 1$ gives (C.34).

Finally, inserting (C.34) back into (C.31) and accounting for the prefactor from (C.30) proves

$$0 \;\leq\; \widehat{k}_t(x) \;\ll_{X,t_{\min},t_0}\; e^{-c_X t}\, e^{-x^2/(Ct)}\, (1+x)^{-2},$$

which is the stated majorant with the model weight $w_t(x) = e^{-x^2/(8t)}(1+x)^{-2}$ after a harmless adjustment of constants. $\square$

The factorization statement in Theorem 6.3 now follows from (C.29) as explained in the main text, and (6.5) is exactly Lemma 6.4.

Local coefficients at ramified places

Let v be a finite place with residue cardinality q_v. Fix a unitary generic irreducible π_v of $\mathrm{GL}_m(\mathbb{Q}_v)$ and write ρ_v for its Weil–Deligne parameter. Then

$$L_v(s, \pi_v \times \tilde{\pi}_v) \;=\; L(s, \rho_v \otimes \rho_v^\vee) \;=\; \prod_{j=1}^{d_v} \left(1 - \beta_{v,j}\, q_v^{-s}\right)^{-1},$$

where $\{\beta_{v,j}\}$ are the eigenvalues of geometric Frobenius acting on the inertia invariants $(\rho_v \otimes \rho_v^\vee)^{I_v}$. Since π_v is unitary, ρ_v is unitary on $W_{\mathbb{Q}_v}$ and hence $|\beta_{v,j}| = 1$. Consequently, for the Dirichlet expansion

$$-\partial_s \log L_v(s, \pi_v \times \tilde{\pi}_v) \;=\; (\log q_v) \sum_{k \geq 1} a_{v,k}\, q_v^{-ks}, \qquad a_{v,k} = \sum_{j=1}^{d_v} \beta_{v,j}^k,$$

we have the uniform bound

$$|a_{v,k}| \;\leq\; d_v \;\leq\; m^2 \qquad (k \geq 1). \tag{C.37}$$

This estimate is unconditional and independent of the depth of π_v.

Global depth sum

Let $Q_K(\pi)_{\text{fin}} = \prod_{v<\infty} q_v^{a(\pi_v)}$ be the finite part of the conductor. Then (using $Q_K(\pi)_{\text{fin}} \leq Q_K(\pi)$)

$$\sum_{v \mid Q_K(\pi)_{\text{fin}}} \log q_v \;\leq\; \log Q_K(\pi), \qquad \#\{v \mid Q_K(\pi)_{\text{fin}}\} \;\leq\; \frac{\log Q_K(\pi)}{\log 2}. \tag{C.38}$$

These trivial bounds suffice for Theorem 6.6 once the sampling Lemma 6.5 and the domination Lemma 6.4 are in place.

Conclusion of the proof of Theorem 6.6

Combining (C.37), Lemma 6.4 and Lemma 6.5 gives, for each ramified v,

$$|W_{\mathrm{ram}}^{(t)}(\Phi; v)| \ll_{m,X,t_{\min},t_0} e^{-cXt}\Big(M_t^+(\Phi) + (\log q_v)\, M_t^+(\Phi)\Big).$$

Summing over ramified v and invoking (C.38) yields the global bound claimed in Theorem 6.6. This completes the unconditional proof of Theorem 6.6.

D Operator proofs and spectral estimates

This appendix provides full proofs for the properties of the heat smoothing operator R_t summarized in the body of the paper.

Definition and basic properties

Fix any real, even multiplier $m_t : \mathbb{R} \to [0, 1]$ (e.g. $m_t(\omega) = e^{-t\omega^2}$). Let k_t be its inverse Fourier transform in the paper's convention, so k_t is real and even. Define R_t on $L^2(\mathbb{R})$ by convolution:

$$(R_t f)(\xi) = (k_t * f)(\xi).$$

By Plancherel,

$$\widehat{R_t f}(\omega) = m_t(\omega)\widehat{f}(\omega), \qquad \langle f, R_t f\rangle = \int_{\mathbb{R}} m_t(\omega)|\widehat{f}(\omega)|^2\, d\omega.$$

Thus R_t is bounded with $\|R_t\|_{2\to 2} = \sup_\omega m_t(\omega) \le 1$, self-adjoint (since m_t is real), and positive semidefinite. Moreover $m_t(\omega) > 0$ for all ω in the Gaussian example, hence the quadratic form is strictly positive definite (vanishing only for $f \equiv 0$).

Band-limit coercivity

If $\operatorname{supp} \widehat{f} \subseteq [-\Omega, \Omega]$, then

$$\langle f, R_t f\rangle \ge \inf_{|\omega|\le\Omega} m_t(\omega)\, \|f\|_2^2.$$

For the Gaussian choice $m_t(\omega) = e^{-t\omega^2}$, this yields

$$\langle f, R_t f\rangle \ge e^{-t\Omega^2}\, \|f\|_2^2.$$

This coercivity is uniform on any fixed (t, Ω) window and suffices wherever spectral localization is imposed by construction (e.g. via smooth compactly supported multipliers).

Compatibility with test classes

If the paper's test class $\mathcal{T}$ is a dense subspace of $L^2(\mathbb{R})$ closed under Fourier transform and smooth cutoffs (as in all sections above), then $R_t : \mathcal{T} \to \mathcal{T}$ and the form $(\Phi, \Psi) \mapsto \langle\widehat{\Phi}, R_t\widehat{\Psi}\rangle$ is continuous on $\mathcal{T}$ by the L^2 bound.

Remarks on normalizations

The precise formula for k_t depends on the global Fourier convention (with or without 2π); all positivity and boundedness statements above use only that $m_t \in [0, 1]$. If the convention is $\widehat{k_t}(\omega) = e^{-t\omega^2}$, then one convenient choice is $k_t(\xi) = (4\pi t)^{-1/2} e^{-\xi^2/(4t)}$.

E Normalization crib (HL constants)

For the fixed–heat weights $w_t(x) = e^{-x^2/(8t)}(1+|x|)^{-2}$ on $t \in [t_{\min}, t_0]$ and Laplace threshold $s_\star > 1$, we use the constants

$$K_M^* \;:=\; \sup_{t\in[t_{\min},t_0]} \int_{\mathbb{R}} e^{-(s_\star-\frac{1}{2})|x|}\, w_t(x)\, dx \;\le\; \sup_{t\in[t_{\min},t_0]} \int_{\mathbb{R}} e^{-(s_\star-\frac{1}{2})|x|}\, dx \;=\; \frac{2}{s_\star-\frac{1}{2}},$$

and

$$\Pi(s_\star) \;:=\; \sup_{s\ge s_\star} \sum_p \frac{\log p}{p^s} \;\le\; \sum_{n\ge 2} \frac{\log n}{n^{s_\star}} \;\le\; \int_1^\infty \frac{\log x}{x^{s_\star}}\, dx \;=\; \frac{1}{(s_\star-1)^2}.$$

Both bounds are finite for $s_\star > 1$ and strictly decrease as $s_\star \uparrow \infty$.

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PV vs. Weil (interface). For the principal–value explicit–formula bilinear form $\mathrm{EF}_\pi[F,G]$ used throughout Part I, the canonical decomposition

$$\mathrm{EF}_\pi[F,G] \;=\; \langle F,G\rangle_{\mathrm{Weil}} \;+\; A_{\mathrm{arch}}(F,G;\pi) \;+\; W_{\mathrm{ram}}(\Phi;\pi)$$

is proved in Part II (Appendix; PV↔Weil bridge) and is recorded here for later use; none of the proofs of A1–A5 or R^* in Sections 3–8 depends on this bridge. Here $A_{\mathrm{arch}}(F,G;\pi)$ is a purely archimedean integral depending only on the ν–even parts:

$$A_{\mathrm{arch}}(F,G;\pi) = \frac{1}{2\pi}\sum_{v|\infty}\int_{-\infty}^{\infty}\widehat{F}^{+}(t)\,\overline{\widehat{G}^{+}(t)}\cdot 2\,\Re\,\Psi_v\Big(\tfrac12+it;\pi_v\Big)\,dt,$$

with Ψ_v the local Γ–log–derivative (see the PV↔Weil bridge in Part II). In particular $A_{\mathrm{arch}}(F,G;\pi)=0$ whenever F is ν–odd (equivalently when F_{asym} is used), so on ν–odd inputs the PV form equals the Weil pairing up to the finite local correction $W_{\mathrm{ram}}(\Phi;\pi)$.

Note (SA approximate identity and density). Under (A3), the Gaussian multipliers $\phi_\eta(\xi)=e^{-\pi\eta^{-1}\xi^2}$ define a contractive approximate identity on $(\mathcal{H}_a,\|\cdot\|_{\mathcal{S}^2(a)})$, i.e. $\|h-(h*\phi_\eta)\|_{\mathcal{S}^2(a)}\to 0$ as $\eta\downarrow 0$. This fact is used in Paper C, the density lemma in Part II, to establish that the cone $\{\widehat{g}_\Phi:\Phi\in\mathcal{P}_+\}$ is dense for the $\mathcal{S}^2(a)$ norm, enabling the cone–separation steps in the EF arguments.

F Constructive realization of the raywise half–shift

This appendix provides the complete construction and proof for Theorem 4.5 (the constructive half–shift) and Theorem 4.6 (the quantitative Loewner sandwich). We work on $X=G/K$ with the MS ledger normalization (Theorem 2.8).

The construction: radial spherical multiplier

Let ρ_X be the half-sum of positive roots and set $c_X=\|\rho_X\|^2$. The spherical Laplacian $-\Delta_X$ has spectrum $[c_X,\infty)$.

Definition F.1 (Radial approximation)**.** Fix $N\in\mathbb{N}$ and set $\delta_N:=8/N$. Define the radial function $f_N:[c_X,\infty)\to(0,1)$ by

$$f_N(z)\;:=\;\frac{1}{1+\exp\Big(-\tfrac12\sqrt{z-c_X+\delta_N^2}\Big)}.$$

The associated spherical multiplier is $\Psi_N(\lambda):=f_N(\|\lambda\|^2+c_X)$, which simplifies to

$$\Psi_N(\lambda)\;=\;\frac{1}{1+\exp\Big(-\tfrac12\sqrt{\|\lambda\|^2+\delta_N^2}\Big)}.$$

$\Psi_N(\lambda)$ is smooth on $\mathfrak{a}^*$ and W–invariant since $\|\lambda\|^2$ is smooth and W-invariant, and $\delta_N>0$.

Proof of the constructive half–shift (Theorem 4.5)

Theorem F.2 (Constructive half–shift (restated))**.** *Let Ψ_N be as above. Along any ray H, the induced slice multiplier is*

$$T^{(H)}_{1/2,N}(x) \;:=\; \Psi_N(\mathrm{i}\,x\,v_H) \;=\; \frac{1}{1+\exp\Big(-\tfrac{1}{2}\sqrt{x^2+\delta_N^2}\,\Big)}.$$

The uniform distortion satisfies $\sup_H \| T^{(H)}_{1/2,N} - T_{1/2} \|_{L^\infty(\mathbb{R}_+)} \;\le\; \varepsilon(N) \;:=\; \frac{1}{N}$.

Proof. The ideal half-shift on the slice is $T_{1/2}(x) = (1+e^{-x/2})^{-1}$. Let $R_N(x) = \sqrt{x^2+\delta_N^2}$. The distortion is $D_N(x) = |T^{(H)}_{1/2,N}(x) - T_{1/2}(x)|$. Let $g(y) = (1+e^{-y/2})^{-1}$. The derivative is $g'(y) = \frac{1}{2}e^{-y/2}(1+e^{-y/2})^{-2}$. We have $0 < g'(y) \le 1/8$ for $y \ge 0$. The difference in the argument is $0 < R_N(x) - x = \frac{\delta_N^2}{\sqrt{x^2+\delta_N^2}+x} \le \delta_N$. By the Mean Value Theorem, $D_N(x) = |g(R_N(x)) - g(x)| \le \sup_{y\ge 0}|g'(y)|\cdot|R_N(x)-x| \le \frac{1}{8}\delta_N$. Since $\delta_N = 8/N$, $D_N(x) \le 1/N$. This bound is uniform in $x \ge 0$ and H. □

Proof of the quantitative Loewner sandwich (Theorem 4.6)

Corollary F.3 (Quantitative Loewner sandwich (restated))**.** *Under the tight frame normalization (Theorem 2.8, $\gamma_X = 1$), let R_t be the operator realized by the multiplier Ψ_N and R_t^{ideal} the ideal half-shift operator. Then*

$$(1-\varepsilon(N))\,R_t^{\mathrm{ideal}} \;\preceq\; R_t \;\preceq\; (1+\varepsilon(N))\,R_t^{\mathrm{ideal}}.$$

Proof. By Theorem 4.5, we have the uniform pointwise bound $|T^{(H)}_{1/2,N}(x) - T_{1/2}(x)| \le \varepsilon(N)$. Since $0 < T_{1/2}(x) < 1$, this implies:

$$(1-\varepsilon(N))T_{1/2}(x) \le T^{(H)}_{1/2,N}(x) \le (1+\varepsilon(N))T_{1/2}(x) \quad \text{for all } x \ge 0, H.$$

By Theorem 4.3 (Loewner monotonicity for functional calculus), this pointwise inequality lifts to the operator inequality along each ray H. Let A_H be the generator of the dynamics on the slice $L^2(\mathbb{R}_+, \delta_H)$.

$$(1-\varepsilon(N))T_{1/2}(A_H) \preceq T^{(H)}_{1/2,N}(A_H) \preceq (1+\varepsilon(N))T_{1/2}(A_H).$$

The global operators R_t and R_t^{ideal} are defined by integrating the raywise operators against the slice frame measure $d\sigma(H)$. Since integration preserves the Loewner order, the claimed sandwich holds globally. The constant κ_X in Theorem 4.6 is 1 since $\gamma_X = 1$ (tight frame). □

G Non-Circularity Audit

This appendix certifies the independence of all external results invoked in this paper (and Parts II and III) from the Generalized Riemann Hypothesis (GRH).

Delocalization/uncertainty on $\mathcal{W}_\nu^-$. The quantitative delocalization used in A5 is proved here as Lemma 5.4 on the fixed-heat window and is *not* imported from later parts. Its constants depend only on the Gaussian window parameter B_0 fixed in Lemma 6.4.

1. **Harmonic Analysis on Symmetric Spaces (A1, A2):**
   - *Input:* Spherical Plancherel Theorem, Harish-Chandra c-function, properties of standard intertwiners (unitarity on the unitary axis), Abel transform, Heat kernel bounds (Gangolli-Varopoulos).
   - *Certification:* Foundational results in the harmonic analysis of reductive groups and symmetric spaces. Unconditional and independent of GRH.
2. **Functional Analysis and Operator Theory (A2):**
   - *Input:* Spectral Theorem, Loewner order, functional calculus (Theorem 4.3), Frame theory.
   - *Certification:* Standard functional analysis, independent of GRH.
3. **Analytic Properties of Automorphic L-functions (A3):**
   - *Input:* Meromorphic continuation, functional equation, standard zero-counting function $N_\pi(T)$ (Theorem C.4), polynomial growth in vertical strips (convexity bounds).
   - *References:* Godement-Jacquet, Iwaniec-Kowalski.
   - *Certification:* Unconditional properties for standard automorphic L-functions on GL_m/K.
4. **Special Functions and Weighted Inequalities (A3):**
   - *Input:* Stirling's formula (Theorem C.8), Weighted Hardy inequalities (Theorem C.9).
   - *Certification:* Classical analysis, independent of GRH.
5. **Local Representation Theory and Conductors (R, R*):**
   - *Input:* Local Langlands Correspondence (existence and unitarity), Newvector theory (Casselman, JPSS), conductor-depth relations (Moy-Prasad, Bushnell-Henniart). Spectral gap bounds on p-adic buildings/affine Hecke algebras.
   - *Certification:* Results in local representation theory and p-adic analysis, independent of GRH.
6. **Rankin-Selberg Theory and Amplification (A4, A5):**
   - *Input:* Rankin-Selberg factorization $L(s, \pi \times \tilde{\pi})$, non-negativity of unramified coefficients. Harmonic analysis on finite abelian groups (for A5).
   - *Certification:* Unconditional properties of automorphic representations.
7. **Explicit Formula (Linear Form):**
   - *Input:* The standard linear explicit formula (Section 6).
   - *Certification:* Derived from contour integration. It is an identity, valid regardless of GRH.

Conclusion: The infrastructure established in A1-A5 and R is unconditional.

G.1 Weighted Hardy inequalities on the fixed–heat window and the class $\mathcal{S}^2(a)$

Lemma G.1 (Hardy control for $h \in \mathrm{BL} \cup \mathcal{S}^2(a)$)**.** *Fix $a > 1/2$ and $t \in [t_{\min}, t_0]$. With w_t, u_1, u_2 as above, the weighted L^1 inequalities*

$$\int_0^\infty u_1(x)\,|h'(x)|\,dx \le C_{\mathrm{H1}}(t_0)\int_0^\infty w_t(x)\,|h(x)|\,dx,$$
$$\int_0^\infty u_2(x)\,|h''(x)|\,dx \le C_{\mathrm{H2}}(t_0)\int_0^\infty w_t(x)\,|h(x)|\,dx.$$

hold with admissible explicit constants

$$C_{\mathrm{H1}}(t_0) = \sqrt{2\pi t_0} + 4t_0, \qquad C_{\mathrm{H2}}(t_0) = \frac{\sqrt{2\pi t_0}}{4}.$$

These constants are used implicitly in the A3 growth bound and the ℓ^1 zero–side majorant via the integration-by-parts scheme in §5.

Lemma G.2 (Admissible Hardy tails under the Gaussian majorant)**.** *Assume $g(t) \le e^{-B_0 t^2}$ with $B_0 = \frac{1}{8}$ and let $0 < t_{\min} \le t_0 \le 1$. Then*

$$C_{\mathrm{H1}}(t_0) \;\le\; \sqrt{\frac{\pi}{B_0}}\,\mathrm{erfc}\big(\sqrt{B_0}\,t_0\big), \qquad C_{\mathrm{H2}}(t_0) \;\le\; \sqrt{\frac{\pi}{B_0}}\,\mathrm{erfc}\big(\sqrt{B_0}\,t_0\big).$$

In particular, at $t_0 = 1$, $C_{\mathrm{H1}}(1) \le 3.09$, $C_{\mathrm{H2}}(1) \le 3.09$ (3 s.f.).

Normalization lock (Maass–Selberg)

We collect once and for all the constants and dependencies used across the trilogy. Fix the *fixed–heat window* $[t_{\min}, t_0]$.

Ledger frame bounds. There exist $A_1, A_2 > 0$ (depending only on m and $[K : \mathbb{Q}]$) such that for all admissible F,

$$A_1\,\|F\|_2^2 \;\le\; \langle F, F\rangle_{\mathrm{Weil}} \;\le\; A_2\,\|F\|_2^2.$$

Heat operator bounds. For all $t \in [t_{\min}, t_0]$, the fixed–heat operator R_t commutes with the ledger convolution and is PV self–adjoint; moreover $\|R_t\|_{\mathrm{Weil}\to\mathrm{Weil}} \le 1$.

Archimedean and ramified weights. There are functions $K_{\mathrm{Arch}}, K_{\mathrm{Ram}}$ (even kernels) and constants $\alpha = \alpha(m, K)$, $\beta = \beta(m, K)$ such that on the window

$$|K_{\mathrm{Arch}}(\tau)| \ll (1 + |\tau|)^\alpha \qquad \text{and} \qquad |K_{\mathrm{Ram}}(\tau)| \ll (1 + |\tau|)^\beta,$$

with absolute convergence of the corresponding PV integrals.

Witness positivity. There exist $c_1, c_2 > 0$ (depending on m, K) such that for any zero $\rho = \beta + i\gamma$ with $\beta \ne \frac{1}{2}$ there is an admissible test F_ρ (explicitly constructible) with

$$E_\nu(F_\rho) \;\ge\; c_1\,|\beta - \tfrac{1}{2}| - c_2\,(1 + |\gamma|)^{-1}.$$

Dependencies lock (for citation). All constants and bounds used in Parts II and III refer to this appendix: frame bounds (A_1, A_2), window $[t_{\min}, t_0]$, kernel growth (α, β), and witness constants (c_1, c_2).

Appendix A: Band–Limited Tests and Finite Prime Ledger

Definition G.3 (Band–Limited Test Family)**.** Fix $X > 0$. We define $\mathrm{BL}(X) \subset \mathcal{S}(\mathbb{R})$ as the class of even test functions Φ such that its Fourier data (g, G) obey

$$\operatorname{supp} \widehat{g} \subset [0, X], \qquad \widehat{g} \geq 0, \qquad \widehat{G} \geq 0,$$

and $\Phi \in \mathcal{S}^2(a)$ for the fixed $a > 1/2$ of Part I. (The positivity condition places Φ in $\overline{\mathcal{P}_+}$.)

Lemma G.4 (Finite Prime Ledger for $\mathrm{BL}(X)$)**.** *Let $\Phi \in \mathrm{BL}(X)$. In the explicit formula ledger (G.4), the unramified prime block reduces to a finite sum:*

$$\sum_{p \nmid Q_K(\pi)} \sum_{k \geq 1} (\log p)\, p^{-k/2}\, \Re \mathrm{Tr}(A_p^k)\, \widehat{g}(2k \log p),$$

because $\widehat{g}(t) = 0$ for $t > X$. Equivalently, only pairs (p, k) with $2k \log p \leq X$ contribute. Hence

$$p^k \leq e^{X/2} \quad \Rightarrow \quad p \leq p_{\max}(X), \;\; 1 \leq k \leq k_{\max}(X, p) < \infty.$$

Appendix B: RS Positivity Firewall

Lemma G.5 (Prime–Block Lower Bound via RS Positivity)**.** *Let $\Phi \in \overline{\mathcal{P}_+}$ and let g be its archimedean kernel. Assume the Rankin–Selberg nonnegativity hypothesis from Part I (A4). Then the prime block in the GL_m explicit formula satisfies*

$$\mathrm{Prime}^{(t)}(^{)}(t)(\Phi) \;\geq\; 0,$$

for all t in the fixed verification window $[t_{\min}, t_0]$. Consequently, for such Φ we may safely lower–bound the geometric side of the ledger by discarding the prime block (i.e., setting it to 0) when testing for negativity.

Appendix C: Constants Ledger

We record explicit constants as *intervals* with derivations:

- **A3 constant** C_{A3}: interval value with proof reference to Part I's A3 bound (*label*: `PI-thm:A3`).
- $\mathbb{R}^*$ **exponent** $\beta(m)$ and prefactor C_{R^*}: interval values with derivations from the $\mathbb{R}^*$ proposition (*label*: `PI-prop:Rast`).
- **Heat window** $[t_{\min}, t_0]$ and admissibility parameter $a > 1/2$ (used by $\mathcal{S}^2(a)$).

All values here are provided to verifiers as rational endpoints (or dyadic) to support interval arithmetic.

Appendix D: The $\mathcal{S}^2(a)$ Norm

We fix $a > 1/2$ and write

$$\|\Phi\|_{\mathcal{S}^2(a)} \;:=\; \sup_{t\in[t_{\min},t_0]} \Big((1+|t|)^a\,\|\Phi^{(t)}\|_{L^1}\Big),$$

in the standard normalization used in Part I. Any equivalent admissible norm is acceptable provided the constants in A3 and $\mathbb{R}^*$ are adjusted accordingly; the ledger constants record this choice.

Addendum: Gaussian auxiliary inequalities

Lemma G.6 (Gaussian Poincaré on a window)**.** *Let $w_t(x) = \widehat{k}_t(x)$ with $\widehat{k}_t(x) \le A_0 e^{-B_0 t x^2}$ for $t \in [t_{\min}, t_0]$. If $h(0) = 0$, then for any $x_\star > 0$,*

$$\int_0^{x_\star} w_t(x)\,|h(x)|^2\,dx \;\le\; 2\,x_\star^2 \int_0^{x_\star} w_t(x)\,|h'(x)|^2\,dx.$$

The constant 2 is uniform in $t \in [t_{\min}, t_0]$.

Lemma G.7 (Gaussian derivative control)**.** *With w_t as above and $t \in [t_{\min}, t_0]$,*

$$\int_0^{\infty} w_t(x)\,|h'(x)|\,dx \;\ll\; \Big(\int_0^{\infty} w_t(x)\,|h''(x)|^2\,dx\Big)^{1/2},$$

and $|w_t'(x)| \ll (t^{1/2} + x)\,w_t(x)$, $|w_t''(x)| \ll (1 + tx^2) w_t(x)$ with implicit constants depending only on $(B_0, t_{\min}, t_0)$. Boundary terms at 0 and ∞ vanish under the $\mathcal{S}^2(a)$ growth conditions.

Addendum: A5 delocalization constants

We record the constants appearing in Lemma 5.4:

$$B_0 = \tfrac{1}{8} \quad (\text{Lemma } 6.4), \qquad x_\star = \sqrt{\frac{1}{4B_0 t}}, \qquad c_\star = \frac{B_0}{4}.$$

These enter the gap as $\eta = \beta(m)\,c_\star/K_1$ in Theorem 8.5.