Identity	**Commutative**	**Domination**	**Associative**	**Idempotence**	**Distributive**	**Double Negation**	**Negation**	**Absorption**	**Implication Equivalence**
$p \land T \equiv p$	$p \lor q \equiv q \lor p$	$p \lor T \equiv T$	$(p \lor q) \lor r \equiv p \lor (q \lor r)$	$p \lor p \equiv p$	$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$	$\neg(\neg p) \equiv p$	$p \lor \neg p \equiv T$	$p \lor (p \land q) \equiv p$	$p \to q \equiv \neg p \lor q$
$p \lor F \equiv p$	$p \land q \equiv q \land p$	$p \land F \equiv F$	$(p \land q) \land r \equiv p \land (q \land r)$	$p \land p \equiv p$	$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$		$p \land \neg p \equiv F$	$p \land (p \lor q) \equiv p$	
Modus Ponens	**Modus Tollens**	**Disjunctive Syllogism**	**Hypothetical Syllogism**	**Addition**	**Biconditional Equivalence**	**Conjunction**	**Resolution**	**Simplification**	
p $\underline{p \to q}$ $\therefore q$	$\neg q$ $\underline{p \to q}$ $\therefore \neg p$	$p \lor q$ $\underline{\neg p}$ $\therefore q$	$p \to q$ $\underline{q \to r}$ $\therefore p \to r$	$\underline{p}$ $\therefore p \lor q$	$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$	p $\underline{q}$ $\therefore p \land q$	$p \lor q$ $\underline{\neg p \lor r}$ $q \lor r$	$\underline{p \land q}$ $\therefore p$	

Operator Precedence
$\neg$
$\land$
$\lor$
$\to$
$\leftrightarrow$

Implication Translations	p is hypothesis	q is conclusion
if p, then q	p implies q	if p, q
p is sufficient for q	a sufficient condition for q is p	q if p
q when p	q is necessary for p	a necessary condition for p is q
q unless ¬p	p provided that p	p only if q
q whenever p	q follows from p	to p, it is necessary for q
it is necessary to q to p	q provided that p	

Biconditional Translations
p is necessary and sufficient for q
if p then q, and conversely
p iff q
p exactly when q
p if and only if q

Proposition	**Definitions:** A declarative sentence that is either true or false, but not both **Propositions:** - Toronto is the capital of Canada; 2 + 2 = 3; Bob is smart. **Not Propositions:** - What time is it?; Read this carefully.; x + 1 = 2.; She is smart.
Predicate	A property or characteristic that can be true or false for different values P(x)
Valid Argument	An argument where the conclusion can never be false when the premises are true
Fallacies	**Affirming the Conclusion:** For $p \to q$: If q is true, then p must be true – **invalid argument** **Denying the Hypothesis:** For $p \to q$: If p is false, then q must be false – **invalid argument**

Universal Quantifier (∀)	$\forall x\, P(x)$ – "For all x, P(x) is true."
	- **Anding:**
	  - All P(x) is Q(x) $\equiv \forall x\, P(x) \to Q(x)$
	  - No P(x) is Q(x) $\equiv \forall x\, P(x) \to \neg Q(x)$

Existential Quantifier (∃)	$\exists x\, P(x)$ – "There exists an x such that P(x) is true."
	$\exists ! x\, P(x)$ – "There exists one and only one x such that P(x) is true."
	- **Anding:**
	  - Some P(x) are Q(x) $\equiv \exists x\, P(x) \land Q(x)$
	  - Some P(x) are not Q(x) $\equiv \exists x\, P(x) \land \neg Q(x)$

Negating Quantifiers	- $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$ - $\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$ - $\neg(\forall x\, \exists y\, P(x, y)) \equiv \exists x\, \forall y\, \neg P(x, y)$ - $\neg(\exists x\, \forall y\, P(x, y)) \equiv \forall x\, \exists y\, \neg P(x, y)$
Nested Quantifiers	$\forall x\, \exists y\, P(x, y)$: "For every x, there exists a y such that P(x, y) is true." $\exists x\, \forall y\, P(x, y)$: "There exists an x such that for all y, P(x, y) is true." - $\boldsymbol{\forall x\, \forall y\, L(x, y)}$: "Everybody loves everybody." - $\boldsymbol{\exists x\, \exists y\, L(x, y)}$: "Somebody loves somebody." - $\boldsymbol{\forall x\, \exists y\, L(x, y)}$: "Everybody loves somebody." - $\boldsymbol{\exists x\, \forall y\, L(x, y)}$: "Somebody loves everybody." - $\boldsymbol{\neg \exists x\, \forall y\, L(x, y)}$: "Nobody loves everybody."
Vacuous Proof	Show that p is always false
Trivial Proof	Show that q is always true regardless of p

Direct Proof

- **Definition:** Assume P, and show Q
- **Question:** If n is an even number, then n^2 is also even.
- **Proof**:

1.	n is even	Assumption
2.	n = 2k, where k is an integer	Definition of even (1)
3.	$n^2 = (2k)^2$ $= 4k^2$ $= 2(2k^2)$	Substitution (2k) (2) & Arithmetic
4.	2(W), W is an int since ints are closed under +, *	Substitute ($2k^2$) (3)
5.	n^2 is even	Definition of even based on (6)
6.	The original statement is valid because we assumed the hypothesis and proved the conclusion.	

Proof by Contrapositive

- **Definition:** Prove ¬Q → ¬P instead of P → Q
- **Question:** Show that if n is an integer and $n^3 + 5$ is odd, then n is even.
- **Proof**:

1.	Prove that if n is odd, then $n^3 + 5$ is even	Contrapositive
2.	n is odd	Assumption
3.	n = 2k + 1, where k is an integer	Definition of odd (2)
4.	$n^3 + 5 = (2k + 1)^3 + 5$ $= 8k^3 + 12k^2 + 6k + 6$ $= 2(4k^3 + 6k^2 + 3k + 3)$	Substitution (2k + 1) (3) & Arithmetic
5.	2(W), W is an int since ints are closed under +, *	Substitute ($4k^3 + 6k^2 + 3k + 3$) (4)
6.	$n^3 + 5$ is even	Definition of even based on (6)
7.	The original statement is valid because we found the contrapositive and proved it in a direct proof where we assumed the hypothesis and proved the conclusion.	

Proof by Contradiction

- **Definition:** Assume P ∧ ¬Q (negation of original statement/implication), and derive a contradiction
  - Contradiction does not have to be with original statement, can be with anything
- **Question:** Show that if n is an integer and $n^3 + 5$ is odd, then n is even.
- **Proof**:

1.	$n^3 + 5$ is odd and n is odd	Negation of original statement
2.	$n^3 + 5$ is odd	Simplification (1)
3.	n is odd	Simplification (2)
4.	n = 2k + 1, where k is an integer	Definition of odd (3)
5.	$n^3 + 5 = (2k + 1)^3 + 5$ $= 8k^3 + 12k^2 + 6k + 6$ $= 2(4k^3 + 6k^2 + 3k + 3)$	Substitution (2k + 1) (4) & Arithmetic
6.	2(W), W is an int since ins are closed under +, *	Substitute ($4k^3 + 6k^2 + 3k + 3$) (5)
7.	$n^3 + 5$ is even	Definition of even based on (6)
8.	Contradiction	(3) & (7)
9.	The original statement is valid because we assumed the negation and proved a contradiction.	

Proof by Cases

- **Definition:** A proof broken into distinct cases, where these cases cover all prospects

Quantified Statements Inference

Universal Instantiation (UI): From ∀x P(x), you can conclude P(a), where a is any specific element in the domain
Universal Generalization (UG): If P(a) is true for an arbitrary a, conclude ∀x P(x)
Existential Instantiation (EI): From ∃x P(x), conclude P(c), where c is a newly introduced constant that represents some specific element
Existential Generalization (EG): From P(a), conclude ∃x P(x)
Example: Show that ∀x (P(x) ∧ Q(x)) ≡ ∀x P(x) ∧ ∀x Q(x)
Proof:

1.	∀x (P(x) ∧ Q(x))	Assumption
2.	P(a) ∧ Q(a)	UI (Universal Instantiation) (1)
3.	P(a)	Simplification (2)
4.	Q(a)	Simplification (2)
5.	∀x P(x)	UG (Universal Generalization) (3)
6.	∀x Q(x)	UG (Universal Generalization) (4)
7.	∀x P(x) ∧ ∀x Q(x)	Conjunction (3) & (4)

Example: Show that ∃x (P(x) ∨ Q(x)) ≡ ∃x P(x) ∨ ∃x Q(x)
Proof:

1.	∃x (P(x) ∨ Q(x))	Assumption
2.	P(a) ∨ Q(a)	EI (Existential Instantiation) (1)
3.	∃x P(x) ∨ ∃x Q(x)	EG (Existential Generalization) (2)

Theorems

- **Definition:** A statement that can be shown to be true using:
  - Definitions
  - Other theorems
  - Rules of inference
- Less important theorems are sometimes called propositions
- **Axioms:** Statements which are given as true
- **Lemme:** a 'helping theorem' or a result which is needed to prove a theorem
- **Corollary:** A result which directly follows from a theorem
- **Conjecture:** An unproven claim
  - Becomes a theorem once found

Math & Numbers

Domains**:** n,k,a,b,p,q ∈ Z, q ≠ 0

- Integers are closed on +, -, *
- Rational numbers are closed on +, -, *, /
- Irrational numbers are not closed
- **Even:** n = 2k
- **Odd:** n = 2k + 1
- **Perfect Square:** $a = b^2$
- **Rational Number:** $r = \frac{p}{q}$