Springer fibers and Richardson varieties

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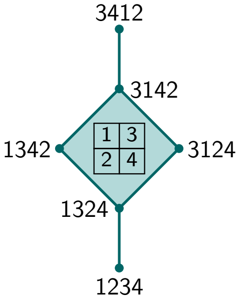

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Flag variety $\mathrm{Fl}_n(\mathbb{C})$

- The *(complete) flag variety* $\mathrm{Fl}_n(\mathbb{C})$ is the set of all flags $F_\bullet$ in $\mathbb{C}^n$:

$$F_\bullet = (0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = \mathbb{C}^n), \quad \dim(F_j) = j \text{ for all } j.$$

Equivalently, $\mathrm{Fl}_n(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C})/B_+$.

- e.g. $n = 3$

$$F_\bullet = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix} \leftrightsquigarrow F_0 = 0,\ F_1 = \left\langle \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \right\rangle,\ F_2 = \left\langle \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle,\ F_3 = \mathbb{C}^3$$

- An $n \times n$ nilpotent matrix M a flag $F_\bullet$ are *compatible* if

$$M(F_j) \subseteq F_{j-1} \quad \text{for all } j \geq 1.$$

- e.g. If $M := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ then M and $F_\bullet$ are compatible.

Springer fiber $\mathcal{B}_\lambda$ (1969)

- Let λ be a *partition* of n (i.e. a weakly decreasing sequence of positive integers summing to n). Let M_λ be the $n \times n$ nilpotent matrix of type λ in Jordan form. Every nilpotent matrix is conjugate to a unique M_λ.

- e.g. $M_{(3)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_{(1,1,1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M_{(2,1)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- The *Springer fiber* $\mathcal{B}_\lambda \subseteq \mathrm{Fl}_n(\mathbb{C})$ is the set of flags compatible with M_λ.

- e.g. $\mathcal{B}_{(3)} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \quad \mathcal{B}_{(1,1,1)} = \mathrm{Fl}_3(\mathbb{C}),$

$$\mathcal{B}_{(2,1)} = \left\{ \begin{bmatrix} a & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right\} \sqcup \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & b & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\} \sqcup \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

- The dimension of $\mathcal{B}_\lambda$ is $\sum_{i \geq 1} (i-1)\lambda_i$.

Why study Springer fibers?

- $\mathcal{B}_\lambda$ is a fiber in the *Springer resolution* of the nilpotent cone of $\mathfrak{sl}_n(\mathbb{C})$.
- Springer (1976): The cohomology $H^*(\mathcal{B}_\lambda)$ is a representation of the symmetric group S_n of all permutations of $\{1, \ldots, n\}$. Moreover, $H^{\text{top}}(\mathcal{B}_\lambda)$ is the irreducible Specht module indexed by λ.
- Springer fibers and generalizations (such as Hessenberg varieties) are related to chromatic quasisymmetric functions and Macdonald polynomials.
- The geometry of irreducible components is related to the combinatorics of Catalan numbers, webs, and standard tableaux.
- A *standard (Young) tableau* σ of shape λ is a filling of the diagram of λ with $1, \ldots, n$ (each used once) which is increasing along rows and columns.
- e.g. $\lambda = (4,2,2) = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & \\ \cline{1-2} & \\ \cline{1-2}\end{array}$ $\quad \sigma = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 \\ \cline{1-2} 7 & 8 \\ \cline{1-2}\end{array}$ $\quad \sigma[3] = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$
- For $1 \leq j \leq n$, let $\sigma[j]$ denote the tableau formed by entries $1, \ldots, j$ of σ.

Irreducible components $\mathcal{B}_\sigma$ of Springer fibers

- Given $F_\bullet \in \mathcal{B}_\lambda$, let $\mathrm{JF}(F_\bullet)$ denote the standard tableau σ such that for all j, the shape of $\sigma[j]$ is the Jordan type of M_λ acting on $\mathbb{C}^n/F_{n-j}$.

- e.g. $\lambda = (2,1)$, $M_\lambda = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $F_\bullet = \begin{bmatrix} 1 & 0 & 0 \\ 0 & b & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$M_\lambda = 0 \text{ on } \mathbb{C}^3/F_1 \;\Rightarrow\; \mathrm{shape}(\sigma[2]) = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \;\Rightarrow\; \mathrm{JF}(F_\bullet) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}$$

- Spaltenstein (1976): The irreducible components of $\mathcal{B}_\lambda$ are precisely

$$\mathcal{B}_\sigma := \overline{\mathcal{B}_\sigma^\circ}, \qquad \text{where } \mathcal{B}_\sigma^\circ := \{F_\bullet \in \mathcal{B}_\lambda : \mathrm{JF}(F_\bullet) = \sigma\}.$$

- e.g. $\mathcal{B}^\circ_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}} = \left\{ \begin{bmatrix} a & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right\}$, $\mathcal{B}^\circ_{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & b & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\} \sqcup \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$

Evacuation on standard tableaux

- Let $\sigma \mapsto \sigma^\vee$ denote *evacuation* (or the *Schützenberger involution*) on standard tableaux. We obtain $\sigma^\vee$ from σ by repeating these steps:
  - delete the box in the top-left corner;
  - slide the empty box to the southeast boundary (via *jeu-de-taquin*);
  - fill the empty box with the largest unused number (starting with n) and freeze it.

- e.g. $\sigma =$

1	3	5	6
2	4		
7	8		

$\rightsquigarrow$ $\sigma^\vee =$

1	2	5	7
3	4		
6	8		

- van Leeuwen (2000): If $F_\bullet \in \mathcal{B}_\sigma^\circ$, then the Jordan type of M_λ acting on F_j is the shape of $\sigma^\vee[j]$, for all j.
- We can similarly recover the *Robinson–Schensted correspondence*.

Total positivity

- *Totally positive matrices* (matrices whose minors are all positive) have been studied since the 1930's. Gantmakher–Krein (1937) showed that square totally positive matrices have positive eigenvalues.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \qquad \begin{aligned} \lambda_1 &= 10.6031\cdots \\ \lambda_2 &= 1.2454\cdots \\ \lambda_3 &= 0.1514\cdots \end{aligned}$$

- Lusztig (1994) introduced total positivity for algebraic groups G and flag varieties G/P. It is connected to representation theory, cluster algebras, electrical networks, combinatorics, topology, Teichmüller theory, tropical and real algebraic geometry, scattering amplitudes, knot theory, ...
- The *totally nonnegative flag variety* $\mathrm{Fl}_n^{\geq 0}$ is the set of flags which have a matrix representative whose left-justified minors are all nonnegative.
- e.g. $F_\bullet = \begin{bmatrix} \boxed{2 \quad -3} & 1 \\ 4 \quad -1 & 0 \\ \boxed{1 \quad\quad 0} & 0 \end{bmatrix} \in \mathrm{Fl}_3^{\geq 0}$ minor $= 3$

Cell decomposition of $\mathrm{Fl}_n^{\geq 0}$

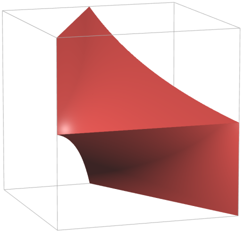
- Lusztig (1994), Rietsch (1999): $\mathrm{Fl}_n^{\geq 0}$ has a cell decomposition

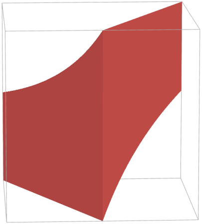
$$\mathrm{Fl}_n^{\geq 0} = \bigsqcup_{v \leq w \text{ in } S_n} R_{v,w}^{>0},$$

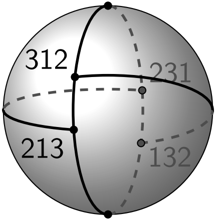
where $R_{v,w}^{>0}$ is the totally positive part of the *Richardson variety*

$$R_{v,w} := \overline{B_- v B_+ / B_+} \cap \overline{B_+ w B_+ / B_+}.$$

- Galashin–Karp–Lam (2022): $\mathrm{Fl}_n^{\geq 0}$ is a regular CW complex.
- e.g. $n = 3$







Totally nonnegative Springer fiber $\mathcal{B}_\lambda^{\geq 0}$

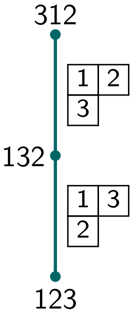
- Lusztig (2020): The *totally nonnegative Springer fiber* is a subcomplex of $\mathrm{Fl}_n^{\geq 0}$:

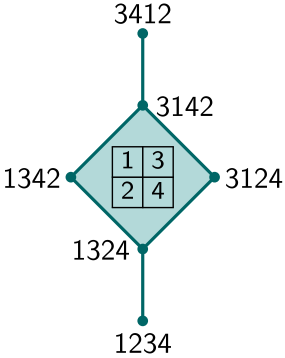
$$\mathcal{B}_\lambda^{\geq 0} := \mathcal{B}_\lambda \cap \mathrm{Fl}_n^{\geq 0} = \bigsqcup_{\substack{v \leq w \text{ in } S_n, \\ R_{v,w} \subseteq \mathcal{B}_\lambda}} R_{v,w}^{>0}.$$

Its top-dimensional cells are indexed by $R_{v,w}$'s which equal some $\mathcal{B}_\sigma$.

- e.g.

$\mathcal{B}_{(2,1)}^{\geq 0}$





$\mathcal{B}_{(2,2)}^{\geq 0}$

Richardson envelope of $\mathcal{B}_\sigma$

Problem

When is the irreducible component $\mathcal{B}_\sigma$ equal to a Richardson variety $R_{v,w}$? That is, describe the top-dimensional cells of $\mathcal{B}_\lambda^{\geq 0}$ in terms of tableaux.

- Additional motivation: much more is known about $R_{v,w}$ than $\mathcal{B}_\sigma$.

Theorem (Karp, Precup (2025+))

There exists a unique minimal Richardson variety R_{v_σ,w_σ} containing $\mathcal{B}_\sigma$.

- e.g.

$$\mathcal{B}_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}} = \overline{\left\{\begin{bmatrix} a & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ \boxed{1} & 0 & 0\end{bmatrix}\right\}} = \overline{\left\{\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} \\ \frac{1}{a} & \boxed{1} & 0\end{bmatrix}\right\}} = R_{132,312}$$

$$\mathcal{B}_{\begin{smallmatrix}1&2\\3&4\end{smallmatrix}} = \overline{\left\{\begin{bmatrix} a & b & \boxed{1} & 0 \\ 0 & a & 0 & \boxed{1} \\ \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0\end{bmatrix}\right\}} = \overline{\left\{\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ \frac{1}{a} & -\frac{b}{a^2} & \boxed{1} & 0 \\ 0 & \frac{1}{a} & 0 & \boxed{1}\end{bmatrix}\right\}} = R_{1234,3412}$$

Reading words of σ

- We can find the Richardson envelope R_{v_σ, w_σ} of $\mathcal{B}_\sigma$ as follows:
  - v_σ^{-1} is the *top-down reading word* of $\sigma^\vee$; and
  - $w_0 w_\sigma^{-1} w_0$ is the *reading word* of σ, where $w_0 := (j \mapsto n+1-j) \in S_n$.
- e.g.

$$\sigma = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & \multicolumn{2}{c}{} \\ \cline{1-2} 7 & 8 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \quad \leftrightsquigarrow \quad \sigma^\vee = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 7 \\ \hline 3 & 4 & \multicolumn{2}{c}{} \\ \cline{1-2} 6 & 8 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$$

$$v_\sigma^{-1} = 12573468 \quad \Rightarrow \quad v_\sigma = 12563748$$

$$w_0 w_\sigma^{-1} w_0 = 78241356$$
$$w_0 = 87654321 \quad \Rightarrow \quad w_\sigma = 78125364$$

- Pagnon (2003), Pagnon–Ressayre (2006): X_{w_σ} is the minimal Schubert variety containing $\mathcal{B}_\sigma$.

Richardson tableaux

- A *Richardson tableau* is a standard tableau σ such that for all $1 \leq j \leq n$, if j appears in row r of σ, then either $r = 1$ or the largest entry of $\sigma[j-1]$ in row $r-1$ is greater than every entry in rows $\geq r$.
- e.g.

$$\sigma = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 6 \\ \hline 2 & 7 \\ \cline{1-2} 5 & 8 \\ \cline{1-2} \end{array} \qquad \tau = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 \\ \cline{1-2} 7 & 8 \\ \cline{1-2} \end{array}$$

Richardson — not Richardson

- All hook-shaped tableaux are Richardson. A two-rowed tableau is Richardson if and only if its second row has no two consecutive entries.

Theorem (Karp, Precup (2025+))

Let σ be a standard tableau of shape λ. The following are equivalent:

(i) $\mathcal{B}_\sigma$ *is equal to some Richardson variety* $R_{v,w}$;

(ii) $\mathcal{B}_\sigma = R_{v_\sigma, w_\sigma}$; *and*

(iii) σ *is a Richardson tableau.*

Slide characterization of Richardson tableaux

Theorem (Karp, Precup (2025+))

A standard tableau σ is Richardson if and only if every evacuation slide of σ (used to calculate $\sigma^\vee$) is L-shaped.

- e.g.

$$\sigma = \begin{array}{|c|c|c|c|c|}\hline 1&4&8&9&11\\\hline 2&5&10\\\cline{1-3} 3&6&12\\\cline{1-3} 7\\\cline{1-1}\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|c|}\hline 2&4&8&9&11\\\hline 3&5&10\\\cline{1-3} 6&12&12\\\cline{1-3} 7\\\cline{1-1}\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|c|}\hline 3&4&8&9&11\\\hline 5&10&11\\\cline{1-3} 6&12&12\\\cline{1-3} 7\\\cline{1-1}\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|c|}\hline 4&8&9&11&10\\\hline 5&10&11\\\cline{1-3} 6&12&12\\\cline{1-3} 7\\\cline{1-1}\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|c|}\hline 5&8&9&11&10\\\hline 6&10&11\\\cline{1-3} 7&12&12\\\cline{1-3} 9\\\cline{1-1}\end{array}$$

$$\rightsquigarrow \begin{array}{|c|c|c|c|c|}\hline 6&8&9&11&10\\\hline 7&10&11\\\cline{1-3} 12&8&12\\\cline{1-3} 9\\\cline{1-1}\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|c|}\hline 7&8&9&11&10\\\hline 10&7&11\\\cline{1-3} 12&8&12\\\cline{1-3} 9\\\cline{1-1}\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|c|}\hline 8&9&11&6&10\\\hline 10&7&11\\\cline{1-3} 12&8&12\\\cline{1-3} 9\\\cline{1-1}\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|c|}\hline 9&11&5&6&10\\\hline 10&7&11\\\cline{1-3} 12&8&12\\\cline{1-3} 9\\\cline{1-1}\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|c|}\hline 10&11&5&6&10\\\hline 12&7&11\\\cline{1-3} 4&8&12\\\cline{1-3} 9\\\cline{1-1}\end{array}$$

$$\rightsquigarrow \begin{array}{|c|c|c|c|c|}\hline 11&3&5&6&10\\\hline 12&7&11\\\cline{1-3} 4&8&12\\\cline{1-3} 9\\\cline{1-1}\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|c|}\hline 12&3&5&6&10\\\hline 2&7&11\\\cline{1-3} 4&8&12\\\cline{1-3} 9\\\cline{1-1}\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|c|}\hline 1&3&5&6&10\\\hline 2&7&11\\\cline{1-3} 4&8&12\\\cline{1-3} 9\\\cline{1-1}\end{array} = \sigma^\vee$$

- Important (non-obvious) fact: if σ is Richardson, then so is $\sigma^\vee$.

Enumeration of Richardson tableaux of fixed shape

Theorem (Karp, Precup (2025+))

The number of Richardson tableaux of shape $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ *is*

$$\binom{\lambda_{\ell-1}}{\lambda_\ell}\binom{\lambda_{\ell-2}+\lambda_\ell}{\lambda_{\ell-1}+\lambda_\ell}\binom{\lambda_{\ell-3}+\lambda_{\ell-1}+\lambda_\ell}{\lambda_{\ell-2}+\lambda_{\ell-1}+\lambda_\ell}\cdots\binom{\lambda_1+\lambda_3+\lambda_4+\cdots+\lambda_\ell}{\lambda_2+\lambda_3+\lambda_4+\cdots+\lambda_\ell}.$$

- e.g. $\lambda = (3,2,1) \rightsquigarrow \binom{2}{1}\binom{3+1}{2+1} = 2 \cdot 4 = 8$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 6 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

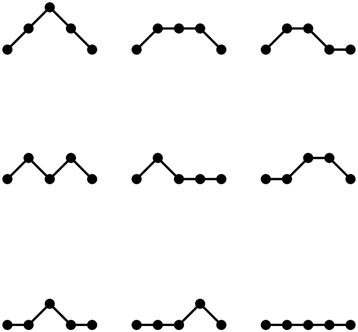
- A q-analogue holds using the major index on standard tableaux.

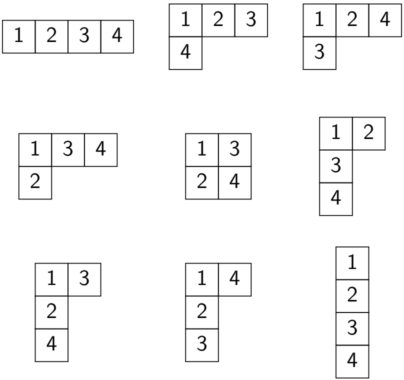
Enumeration of Richardson tableaux of fixed size

Theorem (Karp, Precup (2025+))

The number of Richardson tableaux of size n is the Motzkin number M_n.

- e.g. $n = 4$, $M_4 = 9$





Singular locus and cohomology class of $\mathcal{B}_\sigma$

- There is no known general algorithm to find the singular locus of $\mathcal{B}_\sigma$.
- Billey–Coskun (2012) explicitly describe the singular locus of $R_{v,w}$. Thus we can calculate the singular locus of $\mathcal{B}_\sigma$ is σ is a Richardson tableau.

Problem

For which Richardson tableaux σ is $\mathcal{B}_\sigma$ smooth?

- In all examples of Richardson tableaux σ that we checked (including all such σ with at most two columns or at most three rows), $\mathcal{B}_\sigma$ is smooth.

Problem

Expand the cohomology class $[\mathcal{B}_\sigma]$ in the Schubert basis of $H^*(\mathrm{Fl}_n(\mathbb{C}))$.

- If σ is Richardson, this is equivalent to expanding $\mathfrak{S}_{v_\sigma}\mathfrak{S}_{w_0 w_\sigma}$ in the basis of Schubert polynomials, which was solved by Spink–Tewari (2025+).

Thank you!