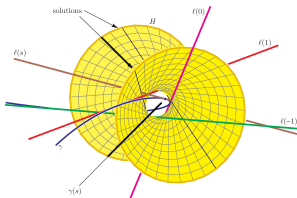


Positivity in real Schubert calculus

Slides available at snkarp.github.io



F. Sottile, “Frontiers of reality in Schubert calculus”

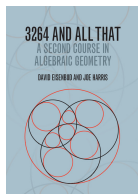


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joint work with Kevin Purbhoo
[arXiv:2309.04645](https://arxiv.org/abs/2309.04645)

April 4, 2024
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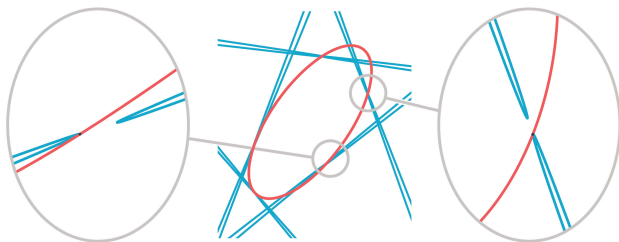
Steiner's conic problem (1848)



- How many conics are tangent to 5 given conics? ~~7776~~.
- de Jonquières (1859): 3264.
- Fulton (1984): “The question of how many solutions of real equations can be real is still very much open, particularly for enumerative problems.”

- Fulton (1986); Ronga, Tognoli, Vust (1997): All 3264 conics can be real.

3264 Conics in a Second



- Breiding, Sturmfels, and Timme (2020) found 5 explicit such conics.

The Grassmannian $\text{Gr}_{d,m}(\mathbb{C})$

- The *Grassmannian* $\text{Gr}_{d,m}(\mathbb{C})$ is the set of d -dimensional subspaces of \mathbb{C}^m .

$$\begin{aligned} V := & \begin{matrix} (1, 0, -4, -3) \\ \nearrow \\ 0 \\ \searrow \\ (0, 1, 3, 2) \end{matrix} = \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \text{Gr}_{2,4}(\mathbb{C}) \\ & = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 2 & 6 & 4 \end{bmatrix} \end{aligned}$$

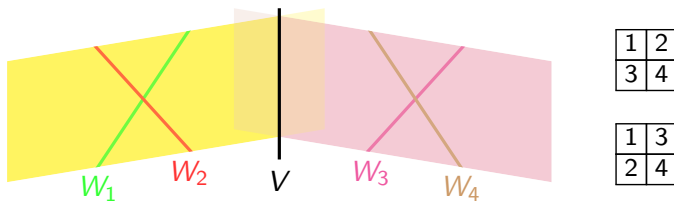
$$\Delta_{1,2} = 1, \quad \Delta_{1,3} = 3, \quad \Delta_{1,4} = 2, \quad \Delta_{2,3} = 4, \quad \Delta_{2,4} = 3, \quad \Delta_{3,4} = 1$$

$$\text{Plücker relation: } \Delta_{1,3}\Delta_{2,4} = \Delta_{1,2}\Delta_{3,4} + \Delta_{1,4}\Delta_{2,3}$$

- $\text{Gr}_{d,m}(\mathbb{C}) \cong \text{GL}_d(\mathbb{C}) \setminus \{d \times m \text{ matrices of rank } d\}$.
- Given $V \in \text{Gr}_{d,m}(\mathbb{C})$ as a $d \times m$ matrix, for d -subsets I of $\{1, \dots, m\}$ let $\Delta_I(V)$ be the $d \times d$ minor of V in columns I . The *Plücker coordinates* $\Delta_I(V)$ are well-defined up to a common scalar.
- $\text{Gr}_{d,m}(\mathbb{C})$ is a projective variety of dimension $d(m-d)$.

Schubert calculus (1886)

- Divisor Schubert problem: given $W_1, \dots, W_{d(m-d)} \in \text{Gr}_{m-d,m}(\mathbb{C})$, find all $V \in \text{Gr}_{d,m}(\mathbb{C})$ such that $V \cap W_i \neq \{0\}$ for all i .
- e.g. $d = 2$, $m = 4$ (projectivized). Given 4 lines $W_i \subseteq \mathbb{P}^3$, find all lines $V \subseteq \mathbb{P}^3$ intersecting all 4. Generically, there are 2 solutions.



We can see the 2 solutions explicitly when two pairs of the lines intersect.

- If the W_i 's are generic, the number of solutions V is

$$\#_{d,m} := \frac{1!2! \cdots (d-1)!}{(m-d)!(m-d+1)! \cdots (m-1)!} (d(m-d))!,$$

the number of *standard Young tableaux* of rectangular shape $d \times (m-d)$.

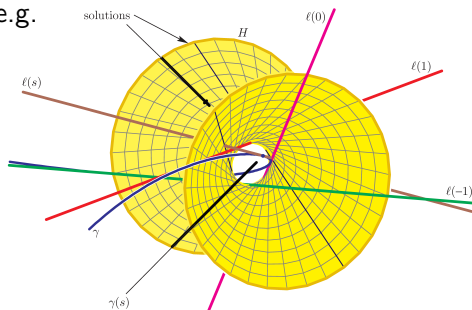
Shapiro–Shapiro conjecture (1993)

- Are there Schubert problems with all solutions real?

Shapiro–Shapiro conjecture (1993)

Let $W_1, \dots, W_{d(m-d)} \in \text{Gr}_{m-d,m}(\mathbb{R})$ osculate the moment curve $\gamma(t) := (1, t, \dots, t^{m-1})$ at real points. Then there are $\#_{d,m}$ real solutions $V \in \text{Gr}_{d,m}(\mathbb{R})$ such that $V \cap W_i \neq \{0\}$ for all i .

- e.g.



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- The Schubert problem above arises in the study of linear series in algebraic geometry, differential equations, and pole placement problems in control theory.
- Bürgisser, Lerario (2020): a 'random' Schubert problem has $\approx \sqrt{\#_{d,m}}$ real solutions.

Shapiro–Shapiro conjecture (1993)

- Sottile (1999) tested the conjecture and proved it asymptotically.
- Eremenko and Gabrielov (2002) proved the conjecture for $d = 2, m - 2$.
- Mukhin, Tarasov, and Varchenko (2009) proved the full conjecture via the *Bethe ansatz*. The proof was restructured in $\mathbb{C}[\mathfrak{S}_n]$ by Purbhoo (2022).
- Levinson and Purbhoo (2021) gave a topological proof of the conjecture.

Secant conjecture, divisor form (Sottile (2003))

Let $W_1, \dots, W_{d(m-d)} \in \text{Gr}_{m-d,m}(\mathbb{R})$ be secant to $\gamma(t) := (1, t, \dots, t^{m-1})$ along non-overlapping intervals. Then there are $\#_{d,m}$ real solutions

$V \in \text{Gr}_{d,m}(\mathbb{R})$ such that $V \cap W_i \neq \{0\}$ for all i .

- The Shapiro–Shapiro conjecture is a limiting case of this conjecture.

Theorem (Karp, Purbhoo (2023))

The divisor form of the secant conjecture is true.

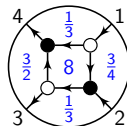
Total positivity

- Total positivity originated in the 1910's from orthogonal polynomials. Gantmakher and Krein (1937) showed that *totally positive matrices* (whose minors are all positive) have positive eigenvalues.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \quad \begin{aligned} \lambda_1 &= 10.6031 \dots \\ \lambda_2 &= 1.2454 \dots \\ \lambda_3 &= 0.1514 \dots \end{aligned}$$

- Lusztig (1994) introduced total positivity for algebraic groups and flag varieties, motivated by quantum groups. An element $V \in \mathrm{Gr}_{d,m}(\mathbb{C})$ is *totally nonnegative* if its Plücker coordinates are all nonnegative.

$$V := \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \mathrm{Gr}_{2,4}^{\geq 0}$$



$$\Delta_{1,2} = 1, \quad \Delta_{1,3} = 3, \quad \Delta_{1,4} = 2, \quad \Delta_{2,3} = 4, \quad \Delta_{2,4} = 3, \quad \Delta_{3,4} = 1$$

- Postnikov (2006) parametrized $\mathrm{Gr}_{d,m}^{\geq 0}$ using *plabic graphs*.
- $\mathrm{Gr}_{d,m}^{\geq 0}$ appears in the study of cluster algebras, electrical networks, the KP hierarchy, scattering amplitudes, curve singularities, the Ising model, ...

Wronski map

- The *Wronskian* of d functions $f_1, \dots, f_d : \mathbb{C} \rightarrow \mathbb{C}$ is

$$\mathrm{Wr}(f_1, \dots, f_d) := \det \begin{bmatrix} f_1 & \cdots & f_d \\ f_1' & \cdots & f_d' \\ \vdots & \ddots & \vdots \\ f_1^{(d-1)} & \cdots & f_d^{(d-1)} \end{bmatrix}.$$

- e.g. $\mathrm{Wr}(f, g) = \det \begin{bmatrix} f & g \\ f' & g' \end{bmatrix} = fg' - f'g = f^2(\frac{g}{f})'$.
- $\mathrm{Wr}(f_1, \dots, f_d) \neq 0$ if and only if f_1, \dots, f_d are linearly independent. Then $\mathrm{Wr}(V)$ is well-defined up to a scalar, where $V := \langle f_1, \dots, f_d \rangle$.
- The monic linear differential operator \mathcal{L} of order d with kernel V is

$$\mathcal{L}(g) = \frac{\mathrm{Wr}(f_1, \dots, f_d, g)}{\mathrm{Wr}(f_1, \dots, f_d)} = g^{(d)} + \dots.$$

- We identify \mathbb{C}^m with the space of polynomials of degree at most $m-1$:

$$\mathbb{C}^m \leftrightarrow \mathbb{C}_{m-1}[u], \quad (a_1, \dots, a_m) \leftrightarrow a_1 + a_2 u + a_3 \frac{u^2}{2} + \cdots + a_m \frac{u^{m-1}}{(m-1)!}.$$

We obtain the *Wronski map* $\mathrm{Wr} : \mathrm{Gr}_{d,m}(\mathbb{C}) \rightarrow \mathbb{P}(\mathbb{C}_{d(m-d)}[u])$.

Shapiro–Shapiro conjecture and positivity conjecture

Shapiro–Shapiro conjecture (Mukhin, Tarasov, and Varchenko (2009))

Let $V \in \text{Gr}_{d,m}(\mathbb{C})$. If all complex zeros of $\text{Wr}(V)$ are real, then V is real.

- e.g. If $\text{Wr}(V) := (u + z_1)^2(u + z_2)^2$, the two solutions $V \in \text{Gr}_{2,4}(\mathbb{C})$ are

$$\begin{aligned} &\langle (u + z_1)(u + z_2), u(u + z_1)(u + z_2) \rangle \quad \text{and} \quad \langle (u + z_1)^3, (u + z_2)^3 \rangle. \\ &= \begin{bmatrix} z_1 z_2 & z_1 + z_2 & 2 & 0 \\ 0 & z_1 z_2 & 2(z_1 + z_2) & 6 \end{bmatrix} \quad = \quad \begin{bmatrix} z_1^3 & 3z_1^2 & 6z_1 & 6 \\ z_2^3 & 3z_2^2 & 6z_2 & 6 \end{bmatrix} \end{aligned}$$

Positivity conjecture (Mukhin, Tarasov (2017); Karp (2021))

Let $V \in \text{Gr}_{d,m}(\mathbb{C})$. If all zeros of $\text{Wr}(V)$ are nonpositive, then $V \in \text{Gr}_{d,m}^{\geq 0}$.

- Karp (2023): The positivity conjecture is equivalent a conjecture of Eremenko (2015), which implies the divisor form of the secant conjecture.
- Karp, Purbhoo (2023): The positivity conjecture is true.

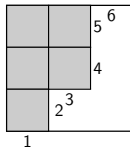
Universal Plücker coordinates

- We want to find all V with $\text{Wr}(V) = (u + z_1) \cdots (u + z_n)$. It suffices to work in $\text{Gr}_{n,2n}(\mathbb{C})$. We construct *universal* Plücker coordinates $\beta^\lambda \in \mathbb{C}[\mathfrak{S}_n]$ indexed by *partitions* λ .
- Partitions inside the $n \times n$ square index n -element subsets of $\{1, \dots, 2n\}$.

$$\lambda = (2, 2, 1)$$

$$|\lambda| = 5$$

$$n = 3$$



$$\leftrightarrow I_\lambda = \{2, 4, 5\}$$

Theorem (Karp, Purbhoo (2023))

- (i) The β^λ 's pairwise commute and satisfy the Plücker relations.
- (ii) There is a bijection between the eigenspaces of the β^λ 's acting on $\mathbb{C}[\mathfrak{S}_n]$ and the elements $V \in \text{Gr}_{n,2n}(\mathbb{C})$ with $\text{Wr}(V) = (u + z_1) \cdots (u + z_n)$, sending the eigenvalue of β^λ to the Plücker coordinate $\Delta_{I_\lambda}(V)$.
- (iii) If $z_1, \dots, z_n \geq 0$, then the β^λ 's are positive semidefinite.

Definition of the universal Plücker coordinates

$$\beta^\lambda := \sum_{\substack{X \subseteq \{1, \dots, n\}, \\ |X| = |\lambda|}} \left(\prod_{i \notin X} z_i \right) \sum_{\pi \in \mathfrak{S}_X} \chi^\lambda(\pi) \pi \in \mathbb{C}[\mathfrak{S}_n]$$

• e.g. $n = 2$. Write $\mathfrak{S}_2 = \{e, \sigma\}$, where e is the identity. We have

$$\beta^\emptyset = z_1 z_2 e, \quad \beta^\square = (z_1 + z_2)e, \quad \beta^{\square\square} = e + \sigma, \quad \beta^{\square\square} = e - \sigma,$$

and $\beta^\lambda = 0$ if $|\lambda| > 2$. On the eigenspace $\langle e - \sigma \rangle$, the eigenvalues are

$$\beta^\emptyset \rightsquigarrow z_1 z_2, \quad \beta^\square \rightsquigarrow z_1 + z_2, \quad \beta^{\square\square} \rightsquigarrow 0, \quad \beta^{\square\square} \rightsquigarrow 2,$$

which are the Plücker coordinates of

$$V = \begin{bmatrix} \frac{z_1+z_2}{2} & 1 & 0 & 0 \\ -z_1 z_2 & 0 & 2 & 0 \end{bmatrix} = \left\langle \frac{z_1+z_2}{2} + u, -z_1 z_2 + u^2 \right\rangle \in \text{Gr}_{2,4}(\mathbb{C}).$$

We can check that

$$\text{Wr}(V) = \det \begin{bmatrix} \frac{z_1+z_2}{2} + u & -z_1 z_2 + u^2 \\ 1 & 2u \end{bmatrix} = (u + z_1)(u + z_2).$$

KP hierarchy

- The *KP equation*

$$\frac{\partial}{\partial x} \left(-4 \frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \right) + 3 \frac{\partial^2 u}{\partial y^2} = 0$$

models shallow water waves. It is the first equation in the *KP hierarchy*.



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- The solutions to the KP hierarchy are symmetric functions $\tau(\mathbf{x})$ in the variables $\mathbf{x} = (x_1, x_2, \dots)$ satisfying the *Hirota bilinear identity*:

$$[t^{-1}] \exp \left(\sum_{k \geq 1} \frac{t^k}{k} (p_k(\mathbf{x}) - p_k(\mathbf{y})) \right) \exp \left(\sum_{k \geq 1} -t^{-k} \left(\frac{\partial}{\partial p_k(\mathbf{x})} - \frac{\partial}{\partial p_k(\mathbf{y})} \right) \right) \tau(\mathbf{x}) \tau(\mathbf{y}) = 0.$$

- Sato (1981): $\tau(\mathbf{x})$ satisfies the bilinear identity if and only if its coefficients in the Schur basis $s_\lambda(\mathbf{x})$ satisfy the Plücker relations.
- A key to our proof is showing $\sum_\lambda \beta^\lambda s_\lambda(\mathbf{x})$ satisfies the bilinear identity.

Future directions

- Further explore the connection to the KP hierarchy.
- Find necessary and sufficient inequalities on the Plücker coordinates of V for all complex zeros of $\text{Wr}(V)$ to be nonpositive, generalizing the Aissen–Schoenberg–Whitney theorem in the case $\dim(V) = 1$. (The positivity conjecture implies that the inequalities $\Delta_I(V) \geq 0$ are necessary.)
- Address generalizations and variations of the Shapiro–Shapiro conjecture: the discriminant conjecture, the general form of the secant conjecture, the monotone conjecture, the total reality conjecture for convex curves, ...

Thank you!