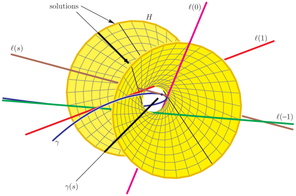
Positivity in real Schubert calculus

Slides available at `snkarp.github.io`



F. Sottile, "Frontiers of reality in Schubert calculus"

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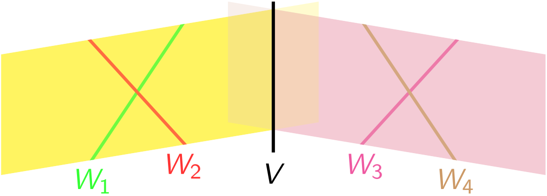
Schubert calculus (1886)

- Divisor Schubert problem: given subspaces $W_1, \ldots, W_{d(m-d)} \subseteq \mathbb{C}^m$ of dimension $m-d$, find all

$$d\text{-subspaces } V \subseteq \mathbb{C}^m \text{ such that } V \cap W_i \neq \{0\} \text{ for all } i.$$

- e.g. $d=2$, $m=4$ (projectivized). Given 4 lines $W_i \subseteq \mathbb{CP}^3$, find all lines $V \subseteq \mathbb{CP}^3$ intersecting all 4. Generically, there are 2 solutions.



$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$

$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$

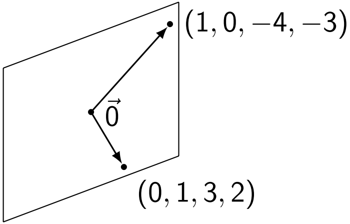
We can see the 2 solutions explicitly when two pairs of the lines intersect.

- If the W_i's are generic, the number of solutions V is $f^{\square}$, the number of *standard Young tableaux* of rectangular shape $d \times (m-d)$.
- Fulton (1984): "The question of how many solutions of real equations can be real is still very much open, particularly for enumerative problems."

The Grassmannian $\mathrm{Gr}_{d,m}(\mathbb{C})$

- The *Grassmannian* $\mathrm{Gr}_{d,m}(\mathbb{C})$ is the set of d-dimensional subspaces of $\mathbb{C}^m$.

$V :=$



$$= \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 2 & 6 & 4 \end{bmatrix} \in \mathrm{Gr}_{2,4}(\mathbb{C})$$

$$\Delta_{1,2} = 1, \quad \Delta_{1,3} = 3, \quad \Delta_{1,4} = 2, \quad \Delta_{2,3} = 4, \quad \Delta_{2,4} = 3, \quad \Delta_{3,4} = 1$$

Plücker relation: $\Delta_{1,3}\Delta_{2,4} = \Delta_{1,2}\Delta_{3,4} + \Delta_{1,4}\Delta_{2,3}$

- Given $V \in \mathrm{Gr}_{d,m}(\mathbb{C})$ as a $d \times m$ matrix, for d-subsets J of $\{1, \ldots, m\}$ let $\Delta_J(V)$ be the $d \times d$ minor of V in columns J. The *Plücker coordinates* $\Delta_J(V)$ are well-defined up to a common scalar.
- $\mathrm{Gr}_{d,m}(\mathbb{C})$ is a projective variety of dimension $d(m - d)$.

Shapiro–Shapiro conjecture

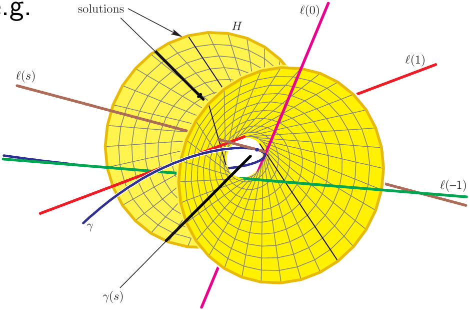
- Do there exist Schubert problems with all real solutions?

Shapiro–Shapiro conjecture (1993)

Let $W_1, \ldots, W_{d(m-d)} \in \mathrm{Gr}_{m-d,m}(\mathbb{R})$ osculate the moment curve $\gamma(t) := \left(\frac{t^{m-1}}{(m-1)!}, \frac{t^{m-2}}{(m-2)!}, \ldots, t, 1\right)$ at real points. Then there exist $f^{\square}$

real $V \in \mathrm{Gr}_{d,m}(\mathbb{R})$ such that $V \cap W_i \neq \{0\}$ for all i.

- e.g.



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- This Schubert problem arises in the study of linear series in algebraic geometry, differential equations, and pole placement problems in control theory.
- Bürgisser, Lerario (2020): a uniformly random Schubert problem over $\mathbb{R}$ has $\approx\sqrt{f^{\square}}$ real solutions.

Shapiro–Shapiro conjecture and secant conjecture

- Sottile (1999) tested the conjecture and proved it asymptotically.
- Eremenko, Gabrielov (2002): cases $d \leq 2$, $m - d \leq 2$.
- Mukhin, Tarasov, Varchenko (2009): full conjecture via the *Bethe ansatz*.
- Levinson, Purbhoo (2021): topological proof of the full conjecture.

Secant conjecture, divisor form (Sottile (2003))

Let $W_1, \ldots, W_{d(m-d)} \in \mathrm{Gr}_{m-d,m}(\mathbb{R})$ be secant to the moment curve $\gamma(t)$ along non-overlapping real intervals. Then there exist $f^{\square}$

$$\textit{real } V \in \mathrm{Gr}_{d,m}(\mathbb{R}) \textit{ such that } V \cap W_i \neq \{0\} \textit{ for all } i.$$

- The Shapiro–Shapiro conjecture is a limiting case of this conjecture.
- Eremenko, Gabrielov, Shapiro, Vainshtein (2006): case $m - d \leq 2$.

Theorem (Karp, Purbhoo (2023))

The divisor form of the secant conjecture is true.

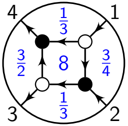
Total positivity

- *Totally positive matrices* (matrices whose minors are all positive) have been studied since the 1930's. Gantmakher and Krein (1937) showed that square totally positive matrices have positive eigenvalues.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \qquad \begin{aligned} \lambda_1 &= 10.6031\cdots \\ \lambda_2 &= 1.2454\cdots \\ \lambda_3 &= 0.1514\cdots \end{aligned}$$

- Lusztig (1994) introduced total positivity for algebraic groups G and flag varieties G/P. An element $V \in \mathrm{Gr}_{d,m}(\mathbb{C})$ is *totally nonnegative* if its Plücker coordinates are all nonnegative.

$$V := \begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 3 & 2 \end{bmatrix} \in \mathrm{Gr}_{2,4}^{\geq 0}$$



$$\Delta_{1,2} = 1,\ \Delta_{1,3} = 3,\ \Delta_{1,4} = 2,\ \Delta_{2,3} = 4,\ \Delta_{2,4} = 3,\ \Delta_{3,4} = 1$$

- Postnikov (2006) parametrized $\mathrm{Gr}_{d,m}^{\geq 0}$ using *plabic graphs*.
- $\mathrm{Gr}_{d,m}^{\geq 0}$ is related to cluster algebras, electrical networks, the KP hierarchy, scattering amplitudes, curve singularities, the Ising model, knot theory, ...

Positive Shapiro–Shapiro conjecture

Positivity conjecture (Mukhin, Tarasov (2017); Karp (2021))

Let $W_1, \ldots, W_{d(m-d)} \in \mathrm{Gr}_{m-d,m}(\mathbb{R})$ osculate the moment curve $\gamma(t)$ at real points $t_1, \ldots, t_{d(m-d)} \geq 0$. Then there exist $f^{\square}$

$$\textit{totally nonnegative } V \in \mathrm{Gr}_{d,m}^{\geq 0} \textit{ such that } V \cap W_i \neq \{0\} \textit{ for all } i.$$

- e.g. $d = 2$, $m = 4$. If $t_3, t_4 \to \infty$, then the 2 solutions $V \in \mathrm{Gr}_{2,4}(\mathbb{C})$ are

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & t_1 t_2 & t_1 + t_2 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{t_1+t_2}{2} & 1 & 0 & 0 \\ -t_1 t_2 & 0 & 2 & 0 \end{bmatrix}.$$

- Karp (2023): the positivity conjecture is equivalent to a conjecture of Eremenko (2015), which implies the divisor form of the secant conjecture.

Theorem (Karp, Purbhoo (2023))

The positivity conjecture is true.

- To prove it, we explicitly solve for the $\Delta_J(V)$'s over $\mathbb{C}[\mathfrak{S}_{d(m-d)}]$.

Universal Plücker coordinates

- Shapiro–Shapiro problem: given $W_1, \ldots, W_{d(m-d)} \in \mathrm{Gr}_{m-d,m}(\mathbb{C})$ which osculate the moment curve $\gamma(t)$ at $t_1, \ldots, t_{d(m-d)} \in \mathbb{C}$, find all

$$V \in \mathrm{Gr}_{d,m}(\mathbb{C}) \text{ such that } V \cap W_i \neq \{0\} \text{ for all } i.$$

Theorem (Karp, Purbhoo (2023))

There exist linear operators $\beta_J = \beta_J(t_1, \ldots, t_{d(m-d)})$ indexed by d-subsets $J \subseteq \{1, \ldots, m\}$ with the following properties.

(i) The β_J's commute and satisfy the Plücker relations.

(ii) There is a bijection between the common eigenspaces of the β_J's and the solutions V above, sending the eigenvalue of β_J to $\Delta_J(V)$.

(iii) If $t_1, \ldots, t_{d(m-d)} \geq 0$, then the β_J's are positive semidefinite.

$$\beta_J := \sum_{\substack{X \subseteq \{1,\ldots,n\}, \\ |X| = |\lambda(J)|}} \Big(\prod_{i \notin X} t_i\Big) \sum_{\pi \in \mathfrak{S}_X} \chi^{\lambda(J)}(\pi)\pi \in \mathbb{C}[\mathfrak{S}_n] \quad (n = d(m-d))$$

Example: $d = 2$, $m = 4$, and $t_3, t_4 \to \infty$

$$\beta_J := \sum_{\substack{X \subseteq \{1,\dots,n\},\\ |X| = |\lambda(J)|}} \Big(\prod_{i \notin X} t_i\Big) \sum_{\pi \in \mathfrak{S}_X} \chi^{\lambda(J)}(\pi)\pi \in \mathbb{C}[\mathfrak{S}_n] \quad (n = 2)$$

- Write $\mathfrak{S}_2 = \{e, \sigma\}$, where e is the identity and $\sigma = (1\ \ 2)$. We have

$$\beta_{1,2} \overset{\varnothing}{=} t_1 t_2 e, \qquad \beta_{1,3} \overset{\square}{=} (t_1 + t_2)e, \qquad \beta_{1,4} \overset{\square\square}{=} e + \sigma, \qquad \beta_{2,3} \overset{\substack{\square\\\square}}{=} e - \sigma,$$

and $\beta_J = 0$ otherwise. The β_J's commute and satisfy the Plücker relation

$$\beta_{1,3}\beta_{2,4} = \beta_{1,2}\beta_{3,4} + \beta_{1,4}\beta_{2,3} \quad \rightsquigarrow \quad 0 = 0 + (e + \sigma)(e - \sigma).$$

- On the eigenspace $\langle e - \sigma \rangle$, the eigenvalues are

$$\beta_{1,2} \rightsquigarrow t_1 t_2, \qquad \beta_{1,3} \rightsquigarrow t_1 + t_2, \qquad \beta_{1,4} \rightsquigarrow 0, \qquad \beta_{2,3} \rightsquigarrow 2,$$

which are the Plücker coordinates of

$$V = \begin{bmatrix} \frac{t_1 + t_2}{2} & 1 & 0 & 0 \\ -t_1 t_2 & 0 & 2 & 0 \end{bmatrix} \in \mathrm{Gr}_{2,4}(\mathbb{C}).$$

Proof 1: KP hierarchy

- The key to the proof is showing that the β_J's satisfy the Plücker relations.
- The *KP equation*

$$\frac{\partial}{\partial x}\left(-4\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}\right)+3\frac{\partial^2 u}{\partial y^2}=0$$

models shallow water waves. It is the first equation in the *KP hierarchy*.

- The solutions to the KP hierarchy are symmetric functions $\tau(\mathbf{x})$ in the variables $\mathbf{x}=(x_1,x_2,\dots)$ satisfying *Hirota's bilinear identity*:

$$[t^{-1}]\exp\Big(\sum_{k\geq 1}\tfrac{t^k}{k}(p_k(\mathbf{x})-p_k(\mathbf{y}))\Big)\exp\Big(\sum_{k\geq 1}-t^{-k}\big(\tfrac{\partial}{\partial p_k(\mathbf{x})}-\tfrac{\partial}{\partial p_k(\mathbf{y})}\big)\Big)\tau(\mathbf{x})\tau(\mathbf{y})=0.$$

- Sato (1981): $\tau(\mathbf{x})$ satisfies Hirota's bilinear identity if and only if its coefficients in the Schur basis $s_\lambda(\mathbf{x})$ satisfy the Plücker relations.
- Karp, Purbhoo (2023): $\sum_J \beta_J s_{\lambda(J)}(\mathbf{x})$ satisfies Hirota's bilinear identity.

Proof 2: higher Gaudin Hamiltonians

- The *higher Gaudin Hamiltonian* associated to the partition λ is

$$T_\lambda = (t_1 + \mathrm{d}_1)\cdots(t_n + \mathrm{d}_n)s_\lambda(h) \in \mathrm{End}((\mathbb{C}^k)^{\otimes n}),$$

where:

  - h is a $k \times k$ matrix;
  - $s_\lambda(h)$ is the Schur polynomial evaluated at the eigenvalues of h; and
  - d_i is the derivative with respect to h^T acting in the ith tensor factor.

Theorem (Alexandrov, Leurent, Tsuboi, Zabrodin (2014))

The T_λ's pairwise commute and satisfy the Plücker relations.

Theorem (Karp, Mukhin, Tarasov (2024))

(i) We have $\beta_J = T_{\lambda(J)}|_{h=0}$.

(ii) If $t_1, \ldots, t_n \geq 0$ and h is positive semidefinite, then so is T_λ.

- Part (ii) gives a positivity theorem for spaces of *quasi-exponentials*.

Future directions

- Further explore the connection to the KP hierarchy.
- What happens to the higher Gaudin Hamiltonian T_λ if s_λ is replaced by a different symmetric function?
- Address generalizations and variations of the Shapiro–Shapiro conjecture: the discriminant conjecture, the general form of the secant conjecture, the monotone conjecture, the total reality conjecture for convex curves, ...

Thank you!