1. Stochastic Processes and Stochastic Calculus

%%% Kim’s comments

In the microscopic motion of molecular its position is measureable so that

Now the velocity of the molecular is

* If the noise is a Brownian motion, is not defined at any time.

In fact if you measure something using a continuous sensor, the noise is look like a Brownian motion. Hence the derivative of the output of the sensor is not defined.

Engineers are approximated this as a white noise even if the derivative is not exist.

So that

%%%%

* 1. Random Walk and Brownian Motion
* Def. 5.1 A scalar Brownian motion process is defined as a process such that

1. is a Gaussian random variable
2. has independent increments
   1. Mean Square Calculus

Consider the following

Then the solution is

If w is the white noise, it is not integral in common sense.

* 1. Wiener Integrals

Consider

Then

* Are the same to the ordinary integral
  1. integral

where

is defined in terms of its integral representation as

* stochastic integral:
* Remark 5.19🡪 see this? Can you prove it?
* Example 5.20

Let Then

%%% Kim’s Comment

Let us define , 🡪

%%%%

* Kim’s comment on (5.40)

1. Eq.(5.40) is not a normal integral There is an additional term.
2. The left and right side of (5.4) are random variables. How to prove the random variables are equal i.e.,

* If 🡪 two random variables are equal in the sense of mean square sense.

1. In the text book, it is proved in the mean square sense.
   1. Second order Ito integral 🡪 skip
   2. Stochastic Differential Equations and Exponentials 🡪 skip
   3. The Ito Stochastic Differential

* Theorem 5.23

Let be the unique solution to the vector Ito stochastic differential equation,

and .

Let be a scalar-valued real function of that is continuously differentiable in and that has continuous second derivatives with respect to . The stochastic differential of is then

* Example. 5.24 : Consider

Calculate

Sol: Let so that

Using (5.49)

Substitute (5.51) into (5.52)

Take the Expectation on both sides

Hence

%%% Kim’s comment

Consider Eq.5.52, in an ordinary differential equation,

However, the RHS of Eq.(5.52) is different!!

%%%%

* 1. Continuous – Time Gauss-Markov Processes

Consider continuous time Gauss-Markov process(linear system)

1. **The mean of ,**
2. The correlation of
3. **The covariance of**

%%% Kim’s comment

From (5.56)

Since

and

Hence

Now however covariance?

1. In linear system, in order to check the stability, one method is Lyapunov.

Given a dynamic system

If there is a , the solution of the following Lyapnov equation

is a positive definite, , then the system is asymptotically stable.

Now the steady state of eq.(5.59) is

Compare a) and b), they are similar to each other but the order is different.

And the solutions are different even if they are positive definite.

%%%%%%%%

1. Continuous time Gauss-Markov Systems: Continuous Time Kalman Filter, Stationarity, Power Spectral Density and the Wiener Filter
   1. The continuous Time Kalman Filter

* Model

Where

* The innovation process

where the estimator, the estimator error

* The conditional estimator

where is the solution to the **Riccati** equation as

* In continuous time filter, there is no separation as Prediction and estimation.

%%% Kim’s comment

1. The problem is to find the **optimal estimator(optimal filter , optimal observer)** such that

The optimal solution is , which is given in (6.10)

1. **The model may be written in engineering textbook,** as

where as the white noise

Strictly speaking, the derivative of Brownian motion is not defined,(it we measure the position with a Brownian noise, the speed of the position, which is derivative of it, is not defined), hence this model is not correct. However, it is culture to write as it is.

1. In discrete system, it may allowed since the time interval between the sampling is finite,

the difference of the Brownian motion is defined.

1. The Riccati equation is a non-linear differential equation, and in control engineering, it is important for another optimal problem.
2. What is another implication of ? This is the minimum mean variance estimator(or observer) , as

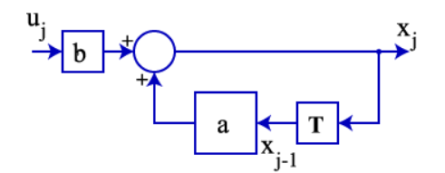
%%% Kim’s comment: Kalman filter – Understanding , Introduction

<http://www.swarthmore.edu/NatSci/echeeve1/Ref/Kalman/ScalarKalman.html>

* Problem – scalar case

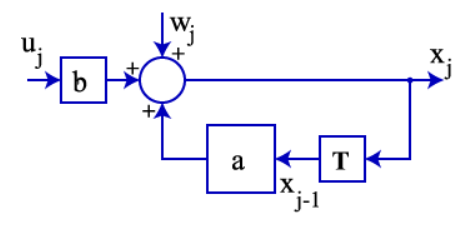
1. Model

* Noise –free dynamic

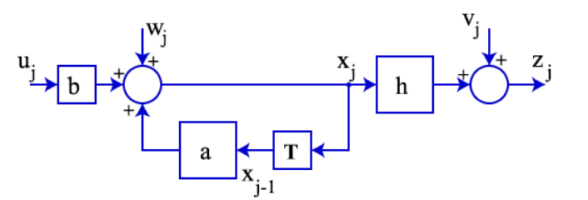


one step delay

* With state noise

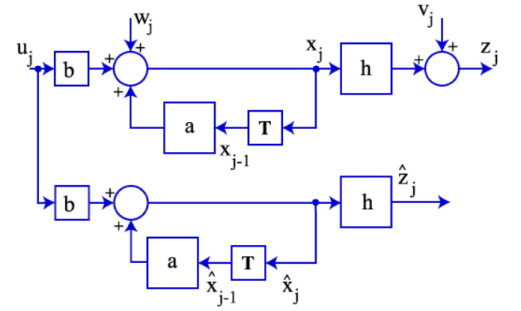


* Measurement noise



What is the best observer / estimator for given ? Remember the noises are random.

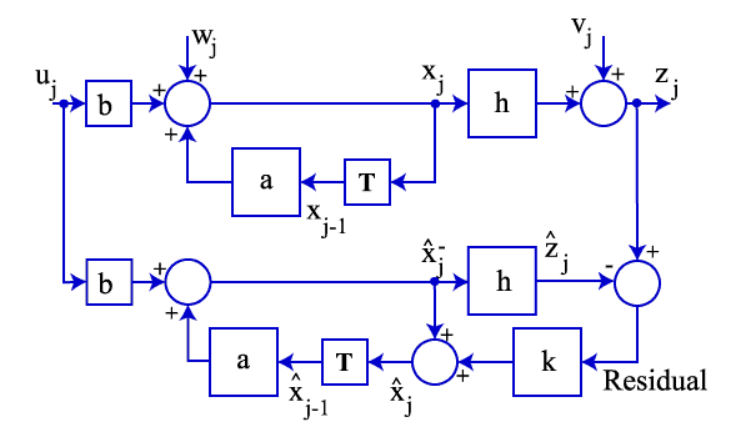
1. The Solution
   1. One solution
   2. Another solution – construct a computer model



1. If or the initial point of is not same to real values
2. If the noises are not compensated.

There are no correction.

* 1. Optimal solution



* Corrections are implemented.
* Residual: Innovation: error or in our textbook as
* The estimator

where kalman gain

1. The equivalent solution but different equations to Kalman Gain

~~~ Check it ~~~

1. The advanced problems
   1. If the system parameters are not equivalent
   2. If the noises’ characteristics are not measured properly
   3. In case of steady state, the transfer function
   4. If the system is time varying, what is the performance?
   5. If the system is non-linear,
2. How about Kalman controller?