- Problem:

A random variable Z is a sum of two R.V's (X, Y), we may calculate the conditional expectaion of X, $E[x|z]$

1. First : The pdf of Z

The pdf of Z can be expressed as a convolution integral as

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx$$

Now in chebfun, there is a command to calculate a convolution integral as "con"

for example we may consider the sum of two random variables X, Y as

$$Z = X + Y$$

and corresponding pdf of X, Y which are independent,

$$f_X(x) = \begin{cases} 1/2 & \text{if } 0 < \mathrm{x} < 2 \ . \\ 0, & \text{otherwise.} \end{cases}$$

and

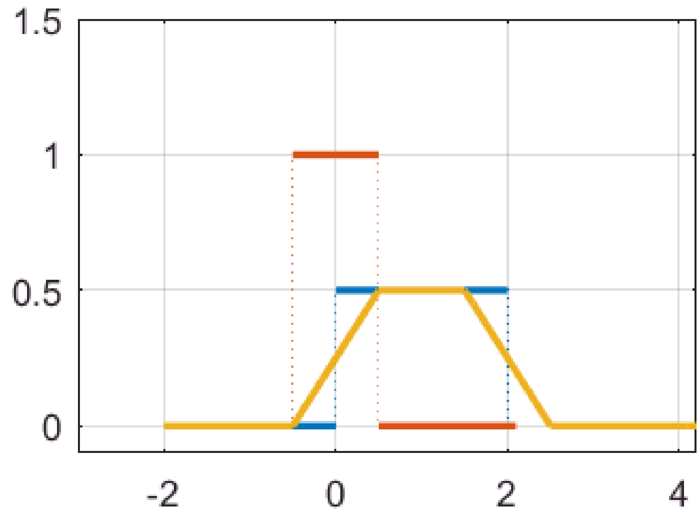
$$f_Y(y) = \begin{cases} 1 & \text{if } -1/2 < x < 1/2. \\ 0, & \text{otherwise.} \end{cases}$$

Then the pdf of Z is

$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx$,

To draw the plot of pdf of Z is

```
LW = 'linewidth';
fx = chebfun({@(x) 0 1/2 0},[-1 0 2 2.1]);
fy = chebfun({@(x) 0 1 0},[ -1 -1/2 1/2 2.1]);
figure('Position', [10 10 300 200])
plot(fx,LW,2); grid on; hold on
axis([-3 3 -0.1 1.5])
plot(fy,LW,2);
fz = conv(fx,fy);
plot(fz, LW,2)
```




```
% hold off
```

Here $f_Z(z) \neq 0$ is in the range of $-0.5 \leq z \leq 2.5$ as expected in the range (x, y).

2. Second we may see the conditional pdf $f(x|z)$.

2.1 In the material in Tutorial,

1) $-1/2 < z < 1/2$

$$f_{X|Z}(x|z) = \begin{cases} \frac{1}{z+1/2} & x \in [0, .z + 1/2] \\ 0, & \text{otherwise.} \end{cases}$$

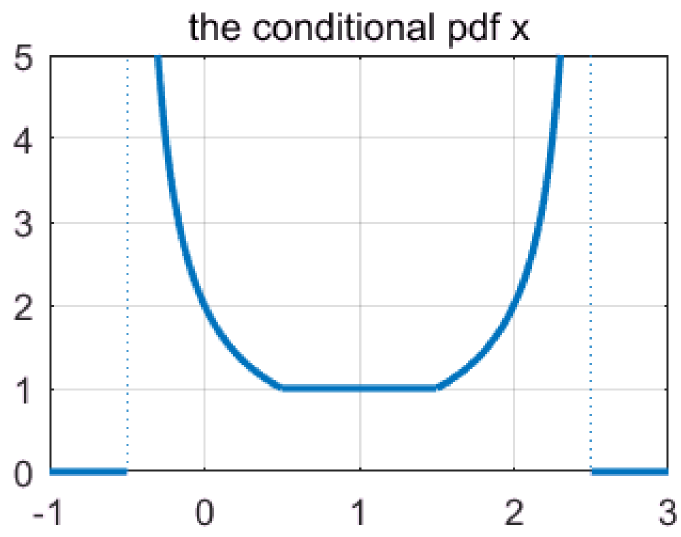
2) $1/2 < z < 3/2$

$$f_{X|Z}(x|z) = \begin{cases} 1 & x \in [z - 1/2, .z + 1/2] \\ 0, & \text{otherwise.} \end{cases}$$

3) $3/2 < z < 5/2$

$$f_{X|Z}(x|z) = \begin{cases} \frac{1}{5/2-z} & x \in [z - 1/2, .2] \\ 0, & \text{otherwise.} \end{cases}$$

```
clear all; clf
LW ='linewidth';
fx_z = chebfun({0,@(z)1/(z+1/2), 1, @(z) 1/(5/2 -z),0 },[-1 -1/2 1/2 3/2 5/2 3], 'splitting',
figure('Position', [10 10 300 200])
plot(fx_z,LW,2); grid on
axis([-1 3 0 5])
title('the conditional pdf x')
```



2.2 Check the legitimacy

So far we get the conditional pdf in the different cases. However, is it a legal pdf? We know for any pdf$g(x)$

should satisfy

$$\int_{-\infty}^{\infty} g(x)dx = 1$$

So let us check this condition. For $z = 0$, then

$$f(x|z = 0) = \frac{1}{1/2+z}\Big|_{z=0} = 2 \;\; x \in [0 \;\; 1/2]$$

which yield to

$$\int_{0}^{1/2} 2dx = 1$$

What is the case of z =1 and 2?

2.3 Some remarks

Here some strange things are happened. at the boundary i.e., $z = -0.5 \;\; and \;\; 2.5$, the $f(x|z) - - > \infty$. Hence if You measure $z = -0.5$, then the only case is

$$x = 0 \;\; and \;\; y = -0.5$$

and vice versa when , **which implies $x = 0$ in pobability 1! . What is the value of the conditional pdf at these points? Hence in order to be "1" at these point, the value of pdf should be infinite!**

3. Finally the conditional expectation;

Now we are ready to calculate the conditional expectation. First we may calculate the unconditional expectaion and it's variance.

3.1 The unconditional expectation and its variance

By the definition, the unconditional expectation is

$$E[x] = \int_{-\infty}^{\infty} xf(x)dx$$

$$= \int_0^2 x(1/2)dx = 1/2\int_0^2 xdx = 1$$

and its variance is

$$E[x - E[x]]^2 = \int_0^2 (x-1)^2(1/2)dx = 1/3$$

3.2 The conditional Expectastion and the variance : the minimum variance estimator

In my material,

1) For $z \in [-1/2 \;\; 1/2]$

$$E[x|z] = \int_0^{z+1/2} xf(x|z)dx = \frac{1}{2}\left(z+\frac{1}{2}\right)$$

$$E\left[(x - E[x|z])^2\right]|z] = \frac{1}{12}\left(z+\frac{1}{2}\right)^2$$

2) For $z \in [1/2, \; 3/2]$

$$E[x|z] = z$$

$$E\left[(x - E[x|z])^2\right]|z] = \frac{1}{12}$$

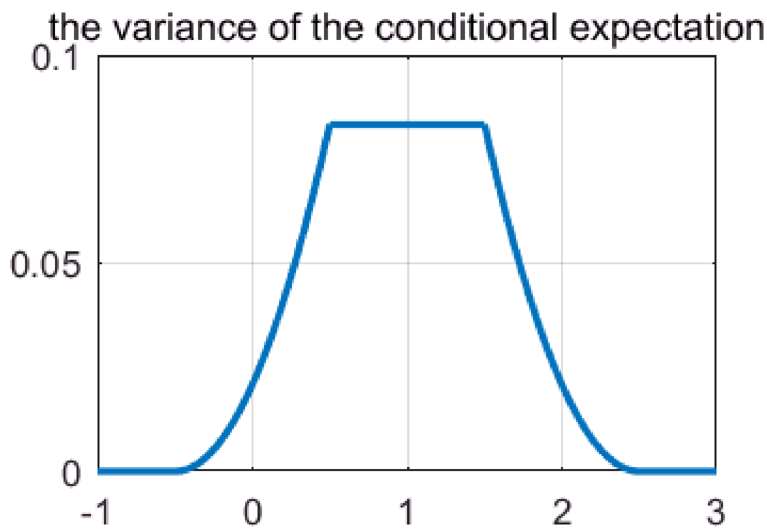
3)For $z \in \left[\frac{3}{2}, \;\; \frac{5}{2}\right]$

$$E[x|z] = \frac{1}{2}\left(z+\frac{3}{2}\right)$$

$$E\left[(x - E[x|z])^2\right]|z] = \frac{1}{12}\left(\frac{5}{2}-z\right)^2$$

Since in my material, there is a graph for $E[x|z]$, i will draw the variance

```
clear all; clf
LW ='linewidth';
var_xz = chebfun({0,@(z) (z+1/2)^2/12 , 1/12, @(z) (5/2-z)^2/12,0},[-1 -1/2 1/2 3/2 5/2 3],...
    'splitting', 'on');
figure('Position', [10 10 300 200])
plot(var_xz, LW,2); grid on
title('the variance of the conditional expectation')
```



3.3 Some remarks

What have we done unitl now?

-. Find the uncondtional expectation $E[x]$, and its variance = 1/3

-. Find the conditional expectation $E[x|z]$ and its variance in the previous figure,

Which one is better estimator? Look at the variance, for all z

$$\sigma^2(x|z) < \sigma^2(x)$$