Ch.2 Optimal Control of Discrete-Time Systems

2.1 Solution of the general discrete-time optimization problem

* Problem Formulation

Let the plant be described by

And associated scalar performance index,

* Problem solution

1. Define augmented index
2. Define the Hamiltonian

%% Kim’s comment

In the static optimization, the Hamiltonian is

But in the dynamic case, a little bit different should be remembered %%%

1. Rearrange the augmented index with the Hamiltonian
2. The necessary conditions using the gradient w.r.t each variables

where

And so on. Hence

similarly

And at the boundary

1. In summary in Table 2.1-1(p.24)

System model:

Performance Index

Hamiltonian:

Optimal Controller

State equation:

Costate equation:

Stationarity condition:

Boundary conditions:

If final state free

%% Kim’s comment:

* The Lagrange multiplier is not a static but a dynamic system. we may call it as “adjoint system”. In linear algebra, what is the adjoint matrix of A?
* The evolution in time are different from two systems each other. The system is evolved in the forward direction but the adjoint system backward.
* To solve the two dynamic systems, we need two boundary conditions. %%%
* Boundary conditions

1. The initial boundary

If the initial point is given, i.e., fixed as in our case, then

So that there is no constraints on

1. **The final states**

2.1) The final points are fixed

In this case , , which is the same case in the initial state.

2.2) The final points are free

Then from

the necessary condition is

%%% Kim’s comment

At the final points there are two ways of the constraints

1. The final point is fixed : In this case the optimal controller is to minimize the performance index

and also due to fixed it minimize to index

1. The final point are free : Here, in general, the final state term in index may be

such that the optimal should be small enough to have the minimum , i.e., it should be convergent to zero. If the final state needs to be convergent to a reference point then the index may be

Hence the final state is closed to the fixed , but not necessarily same to

1. Some very important control concept. Even if regardless of the constraint of the final state to be unfixed, the final state of the unfixed converges to the fixed state( not exactly same !) But, in general the controller may be open loop for the fixed but the closed loop for the unfixed case. As you are control engineer, closed loop (feedback controller) has many advantages over the open loop controller.
2. If the time horizon is infinite, even if the final state is free, the final state is convergent to a fixed number, such as zero or the reference point, we may define the index as

looks like the fixed final point.

%%%

* Example 2.1-2

Consider the plant is on

Minimize the performance index on ,

1. **Fixed Final State**

Suppose the final state is fixed to i.e.,

i.e., the controller should enforce the state from the initial to the final . There are many controllers , hence we need the optimal controller

* Performance index

Find the optimal minimizing the index as

* Hamiltonian
* The necessary conditions are

From (7)

Together with and the conditions we may get

The costate from the conditions is

From (9) and (11)

The solution to this in terms of is

* **HW\_1\_1**

Prove %%%

* Boundary condition of fixed final state

From

where

Since ,

* Optimal controller

From ,

Here the optimal controller is

where C,D are not dependent on the intermediate state values , so that the optimal controller is **open-loop control**.

* Optimal state

Plugging (20) in the optimal controller into the state equation gives

so that at k

Here the optimal state is independent of the weighting of the performance index and the input scaler b.

* **HW\_1\_2**

Derive (23) from (21) %%

* Optimal(minimal) performance index
* **HW\_1\_3**

Derive (26) %%

1. **Free Final State**

* Performance Index

Since the final state is free, however, we may choose the final reference so that the optimal controller force the final state to be close enough to .

* Hamiltonian

Hamiltonian is the same as (4)

* Necessary conditions

The necessary conditions are same as (5)-(7).

* Boundary conditions

From (2.1-8a), since

From (15),

Solving for

Together (29) and (31),

From the necessary condition, the optimal costate is

From the necessary condition, the optimal controller

Here the optimal controller is dependent of the weighting factor on the energy index.

* Optimal state

Solving this in terms of gives

So that the final optimal state

* Optimal(minimal) performance index
* **HW\_1\_4**

Derive (38) %%

2.2 Discrete time Linear Quadratic Regulator

* The State and Costate Equation
* plant
* Quadratic Index

%% Kim’s comment – for simplicity

Here is time varying, however below I will denote as . %%%

* Hamiltonian
* Necessary conditions

Using (2.2-7)

* Discrete Hamiltonian matrix

**The Hamiltonian equation (2.2-3) with the boundary conditions** can be written as a matrix form as

Here matrix is said to be a **discrete Hamiltonian** type if it satisfies

If define as

Then

%%% Kim’s comment

1. In the continuous case , later in Ch.3 Hamiltonian will be

Do not confuse this with the discrete case

1. Here %%%

Since , and from the necessary condition

**the Hamiltonian matrix can be written in backward**

with is known , but is not known.

**%% Misprint page.34**

**%%**

2.1) In case of Zero-input (no control input) and Lyapunov Equation

In this case, weighting matrixes in the index, First notice the performance index at the final stage

Now to the backward, let

We may define

so that

As proceed to backward, it is clear to define new as

This (2.2-21) is called a **observability Lyapunov equation** in discrete system.

If is found using (2.2-21), then the optimal index is

* Boundary conditions

Here, we assume the steady state case, i.e., , (2.2-21) is

**If the system is**

**+ asymptotically stable**

**+observable**

**Then**

%%% Kim’s comment

Here without controller, we do not need the optimal controller to minimize the performance index value rather than just find the index value. %%%

* 1. **With control and open-loop control (fixed final state)**

The final state

* Performance index

Since the final state is fixed, in the index the final – final state weighting is redundant,

Assume so that the index is ( In fact, we may not assume this, however for the simplicity to get the optimal solution)

Since , from the necessary conditions

The solution in terms of

Then the optimal state equation is

As before this is a difference equation with the input,

To find with , we find

where

Form (2.2-32)

so that the optimal control law is

* **HW\_1\_4 (2.2-40)**

Prove that the optimal law (2.2-38) will bring to

%%% comment on Reachability – Controllability (compare the observability (2.2-21)

The weighting matrix is called the reachability Grammian. Let us define

. Then

If , then if exists, for any initial state and for any final state , there is a controller(in here optimal ) exists.

Once again we may define as

Then with , so to get , solve (2.2-43)

We call (2.2-43) is “reachability Lyapunov equation compared to “observability Lyapunov equation” (2.2-21) -The End --- %%%

* **Example 2.2-1 Open loop Control of a scalar system**

Consider a scalar system

with the index

and the fixed final state , find the optimal controller

1. For reachability check

Since , it is reachable(controllable)

1. From the final condition(boundary) ,

The optimal controller to drive any given to a desired , from (2.2-38)

**-END-**

* 1. **Free-final-state and closed loop control**

**(In this case the performance index )**

* Boundary conditions

Since the final state is free from (2.1-8)

With the given initial state and final costate value this is a two point boundary problem,

**We may assume costate is a linear function of the state**, i.e.,

Then from , and

Solving for

with the costate equation and linear relation between the costate and state (2.2-50)

so that

which should be satisfied for all states,

Using the matrix inversion lemma

or

This (2.2-54) is called **Riccati equation in discrete time**. Remember this is a backward equation with the final value as

%%% Kim’s comment

%%%

* Optimal controller

Solving for

Now define the Kalman gain

Which is linear feedback with gain

For sufficient conditions

* In conclusion

The feedback state equation

with

Recursively with given . And the optimal index

* **Ex. 2.2-2 : Optimal Feedback Control of a scalar system**

The plant is

with the performance index

1. Necessary condition – Riccati equation

Since the plant is linear and the index is quadratic, the necessary condition is to solve Riccati equation. Since from (2.2-54)

so that in this case Riccati equation is

The Kalman gain is

The optimal control is

The optimal value of the index is

The optimal closed loop is

%% Kim’s comment: Now we should find . %%%

1. Case a: No control Weighting

From (4) , Riccati equation,

From (5) and (6)

The performance index is

The closed loop system, from (8)

1. Case b: Very large Control Weighting

Let . From (4) , Riccati equation,

The solution to this (in fact this is a discrete Lyapunov equation)

And the Kalman gain

which implies no control.

1. Case c: No intermediate –state weighting

Let .

From (4)

Solving for

The others may follow… It is complicate !!

**-END-**

2.3 Digital Control of Continuous-Time Systems

* Design of Digital Controls

Let us consider a continuous system

To get an equivalent a discrete version of this system, in the time interval

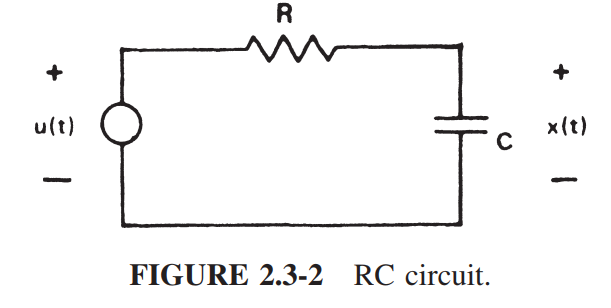
,

Assume , at

%% Kim’ comment: see the last semester 2021\_Week\_11\_Discretization %%%

* **Ex. 2.3-1 Digital control of ab RC Circuit**

Consider the following electrical circuit



The objective is to design the optimal controller such that the capacitance voltage is to a fixed or free in a finite time with the some index.

The governing equation is :

If ,

And the input voltage is discretized with the sampling time

1. Discretization

From (2.3-5)’

1. The performance index

Suppose for any initial voltage , over a 5-sec interval are small. Then and the performance index is selected

1. Necessary conditions - Riccati equation

Since the process is linear and the index is quadratic, the optimal controller is determined by Riccati equation (2.2-54)

(2.2-54)

so that in this case Riccati equation is

The Kalman gain is

The optimal control is

%% Kim’s comment: Now we should find . Also fin the text book is changed to (9) %%%

1. Case a: Free Final State

In this case, from (8), the final boundary condition is given (6). If is large, then the final state converges to zero

1. Case b: Fixed Final State

Here we may want In this case the index will be

Using Ex. 2.2-1, the optimal controller is

where given

%%% Kim’s comment : page 58, (11) is incorrect to change the above (11) %%%

2.4 Steady-State Closed-Loop Control and Suboptimal Feedback

The optimal controller is in general time – varying even if the linear system is time invariant, and the weighting matrixes in the index are constant, we may prefer a constant Kalman gain even if it is not optimal,i.e., it is a suboptimal.

Consider Riccati equation

If , and is convergent on , which is finite, then the time varying Riccati equation is

due to . This is called **the Algebraic Riccati equation.**

Then the Kalman gain is

%% Kim’s comment

Since in the finite horizon, the controller is time varying, it is better to be time-invariant as the solution to (2.4-12). One method to get the time-invariant solution to (2.4-12), is numerically to solve (2.4-11) until is a constant. Or you may solve (2.4-12) directly which is difficult. We will discuss it below .

* Theorem 2.4-1: Let be stablizable. Then the solution to (2.4-2) is bounded.
* Theorem 2.4-2 : Let be a square root of the intermediate-state weighting matrix, so that and suppose . Suppose that Then is stabilizable if anf only if

1. There is a unique solution to Riccati equation.
2. The closed loop plant

is asymptotically stable, where is given (2.4-13). – End-

* Some comment
* If the time interval is large enough, , then the minimum index (cost) is
* Infinite – horizon optimal control problem

Since the optimal value of (2.4-18) is nor dependent of the final state value, The infinite horizon problem is defined as

Is minimized by if be stabilizable and ) is observable.

* Ex.2.4-2. Steady State Control of a scalar System

1. Problem

* The plant :

And

1. ARE equation

ARE is from (2.4-12)

there are two solutions to this equation as

1. Optimal controller

* If the plant is stable, i.e., , the unique non-negative solution is

and the optimal feedback gain is given by (2.4-13)

* If the plant is unstable, i.e., , the unique non-negative solution is
* Analytic solution to the Riccati equation

From (2.2-11)

Define the backward as

with the final costate and the initial state

Then we may get from (2.2-11) as

And

The inverse of the Forward Hamiltonian is

%%%%

* Fact : If is the eigenvalue of , then is also eigenvalue of

Proof : Since is an eigenvalue of , i.e.,

Where is an eigenvector. This can be rearranged to

**The left hand coefficient matrix is from (2.4-26),** which implies is also an eigenvalue of . %%%

%%% Kim’s comment

1. The eigenvalue of the transpose matrix , , is same to that of

🡪 🡪 is an eigenvalue of

Since , is also an eigenvalue of ( but the eigenvector is different)

1. Define a matrix A’ to be changed the row vector of A, Then eigenvalue of the row changed matrix is the same to that of the matrix .

Then ,

%%%

From the fact , has the eigenvalues , which implies can be rearrange in a diagonal matrix such as

where is a diagonal matrix containing **eigenvalues outside the unit circle(i.e., unstable**). In order to diagonalize , there exists a non-singular matrix such that

here  **is a state space transform matrix whose columns are the eigenvectors of .** Define the transformed new states as

Now the transformed new states satisfy the difference equation,

Since is an diagonal matrix, the new states satisfy

Or for stability,

The necessary condition due to **the boundary** at the final time and (2.4-31)

From this solving for

From (2.4-34)

Where

From (2.4-31)

From (2.4-38)

Since this must hold for all ,

%%% Kim’s comment

1. Here is solved not recursive Riccati equation as (2.4-2)
2. is solved in terms of eigenvectors of and some scalar related the final boundary condition, i.e.,

%%%

* The steady state solution to ARE

From (2.4-38), as , then since is stable. Then from (2.4-41),

Here

is unstable eigenvectors of

* Ex. 2.4-4 Analytic Solution to a Scalar (discrete) Riccati Equation
* Problem
* Solution :

1. define the backward Hamiltonian (2.4-21)
2. find eigenvalues of and eigenvectors
3. choose the unstable e-values and the transform matrix

Using the boundary conditions, calculate

1. Find solution to Riccati equation
2. Find the optimal feedback controller

In the steady state , ,

The End of chapter 2.