1. Optimal Control of Continuous time system
   1. The calculus of variations

* Function variation: Consider a . Then

is called a variation of

A functional is a function of function. Its range is scalar

* Functional :
* Functional Variation :

The problem is to find to minimize

So if minimize the , then it is necessary .

The problem is here , is a function of

%%% Kim’s comment : Leibniz formula –google

If are independent variable, then

* Example

If is dependent of t, , then (it is not formal)

%%%

* Functional variation

A functional is a function of function, its range is scalar

Given a functional

Find the variation of if ,

* 1. **Solution of the general continuous-time optimization problem**
* Plant :
* Performance Index :
* **Final State constraint:**
* Optimal Controller:
* Hamiltonian
* State Equation
* Costate equation
* Stationary condition
* Boundary condition

is given

%%% Kim’s comment: derivation of the solution

* First define the augmented the index

Define the Hamiltonian as

Then

The increment of , is

**To eliminate in integrand, integrate by parts,**

Substituting it into (3.2-7) gives

%%%

* Ex.3.2-1 : Hamiltonian’s Principle in Classical Dynamics

1. Lagrange - Euler equation
2. Problem
3. Necessary condition

%%% Kim’s comment

Why the integrand has ? If are independent, then Euler equation is

In our case looks like..

But are not independent. Hence the term is appeared. %%%

1. Hamiltonian

Let us define . Then the index is

Then the Hamiltonian is

The necessary conditions are

Hence

which is equivalent to Langrange –Euler equation

1. Example : Find the curve that gives the shortest distance between 2 points in a plane and

* Problem
* Using Euler’s condition,

Since is independent variable, substitute , then

1. The first term
2. The second term

* if and 🡪
  1. **Continuous Time Linear Quadratic Regulator**
* Plant :
* Performance Index :
* Continuous Hamiltonian:

%%% Kim’s comment

In the textbook to get the optimal controller, 3- cases are considered

First :

1. zero-input Cost :

No control input case, in this case we do not need to find optimal controller !! However, it is interesting so that the cost is related to “Lyapunov equation”

1. Fixed final state

In this case the optimal controller is “Open loop controller”

1. Free-Final State and closed loop control

In here, the controller is “closed loop controller” and it is solved using “Riccati equation”

%%%

1. **Zero input and Lyapunov equation**

Since there is no control, i.e., the optimal control, however, we may see the performance index.

A.1) Calculate Performance Index :

Assume the index as

Since

The final term is

Then index is

If satisfy the equation

which is called the Lyapunov equation with given

Then the index is

A.2) Continuous Lyapunov equation

The solution to the equation (3.3-9) is

* **HW\_Week\_3.1**

Prove (3.3-14) %%%

* The steady state solution : algebraic Lyapunov equation

If is stable, and is observable, then

%%% Kim’s comment: Observability

The following statements are equivalent

* is observable
* has rank n
* has full column rank
* If ,

1. **Fixed final sate and Open loop control**

B.1) Performance index

Now the final state is fixed,

and let us assume the index as

B.2) Necessary conditions

The costate is

The state equation is

B.3) Boundary condition :to find

where the weighted continuous **reachability gramian** is

Hence with

So that the optimal control is

If , then is the solution to

%%% Kim’s comment : controllability

The following statement’s are equivalent

* is controllable
* has rank
* has full row rank at every eigenvalue
* %%%

1. **Free-Final –state and closed loop control (Riccati equation)**

C.1) Performance Index

C.2) Necessary Conditions

C.3) Boundary conditions

In case of free final state,

C.4) Optimal controllers – Riccati equation

Assume

Then differentiating at each sides,

Rearranging it

which is called as Riccati equation

The optimal controller is

Kalman gain is

where we have

%%% Kim’s comment

1. Zero-input Cost : observability

No control, however the optimal cost is

is the solution to

1. Fixed final state : controllability

With the index

The optimal controller is

And

which is the weighted continuous reachability gramian.

If , then is the solution to

1. Free-Final –state and closed loop control (Riccati equation)

%%%

3.4 Steady-State Closed-Loop Control and Suboptimal Feedback

So far if the final state is free, the optimal controller is given by the solution of the Riccati equation.

* Algebraic Riccati equation

Consider LT system

With constant feedback gain controller

Then its closed loop plant

Consider the algebraic Riccati equation (ARE)

* Theorem 3.4-1

is stabilizable, then ARE has a positive semi-definite solution to ARE

* Theorem 3.4-2

Let . Suppose is observable. Then is stabilizable iff

ARE has the unique solution

* An algebraic solution to the ARE

Consider Hamiltonian of a continuous LTI system as

* FACT : if is an eigenvalue of , then is also eigenvalue of

Proof: Let us define

Then

If is an eigenvalue of with eigenvector ,

So that

Since eigenvalues of are the eigenvalues of , is the eigenvalue

of QED

%%% Kim’s comment : compare the discrete Hamiltonian %%

Now to diagonalize the diagonal matrix and diagonalization matrix are

Where is containing the right half plan ( unstable ! , hence is stable) and is the eigenvector of the stable eigenvalues.

Then

* Design of steady state Regulators

Let be the stable eigenvalues as

Then the steady state optimal feedback gain

So that

Is the optimal trajectory

* **HW\_Week\_3.2 : Prove (3.4-24)**

**3.5 (skip) Frequency domain Results**