* Inverted pendulum

To design a controller to stabilize the inverted pendulum at the up-position.

1. Math plant

The math plant of an Inverted Pendulum will is (ref. 2022\_Week\_6\_EngineeringModel)

And

It is difficult to find a controller using non-linear for the math pant, and there may be no-non-linear controller to stabilize (except Reinforcement Learning), we may use an approximated math plant using linearization.

1. Linearization

* Lyapunov’s linearization theorem

Consider . Suppose is an equilibrium point, i.e., , Define

So that the linearized system

* Theorem:
* If , is asymptotically stable, then the non linear system is locally asymptotically stable, If , not locally asymptotically stable
* If , it may not determine the stability.

Proof: let’s assume , using Taylor’s theory

Suppose, A is stable, then

Select a Lyapunov function

So for

Since

So that

Implies is asymptotically stable at

* Example

Assume the pendulum is initially near at the up-position, i.e.,

Then the linearizes plant, we may define the linearized state as

Then one of the linearized system may be

For simple notation, , the state space model may be

Here are non-zero elements.

1. Design a linear state feedback

Now we may apply a plenty of linear control theories

* 1. Check the stability

Find e-values of A

* 1. Check the controllability
  2. Pole placements

Hopefully it is controllable, hence you may find a feedback gain such that the closed loop linearized system has the poles as you assign

* 1. Linear quadratic controller design in the finite time interval

Let define the cost function in the finite time interval as

1. Choose “S”,”Q”, “R” as

Here the variable , is the maximum allowable magnitude of the control input.

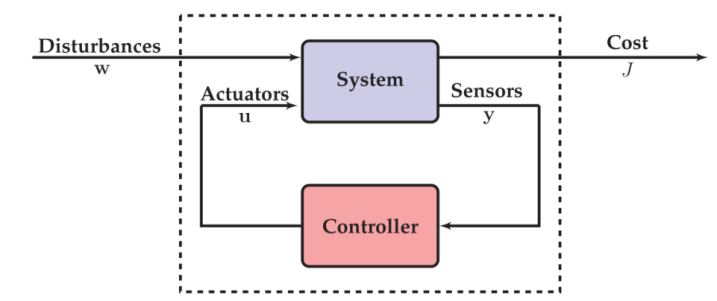
1. Given finite time interval[ 0 T], we may find the time varying linear feedback gain .You may code as the textbook
2. Check the closed loop optimal trajectory when . Select a good
   1. Suboptimal controller design –linear quadratic Regulator

Here in this case, , and

1. Steady state the optimal gain

The algebraic Riccati equation can be used. Or in matlab, lqr commnd may be used

1. Compare the optimal trajectory between finite time and infinite time.
2. Check the closed loop poles.
   1. Simulate the closed loop system as



Here we may simulate

System : the math plant

Actuators : used the designed the optimal controller

And different initial points as

With what initial values, the pendulum may not be stabilized?

* Some facts for Lyapunov function

Then

Where is the solution to the Lyapunov equation

Proof : First the solution of the Lyapunov equation is

Since

Now

1. Dynamic Programming

* Introduction :

Let

And the cost

To find an optimal controller, in the previous chapter, we try to solve the analytic solutions,

The other method is “Dynamic Programming” to solve it. Since we may have powerful computation power, why not utilize it? However, even if the power is bigger and bigger, still it is difficult to solve it. Dynamic Programming is one method to reduce the computational burdens. It was initiated by Bellman, he said,

“An optimal policy has the property that no matter what the previous decision(i.e., controls) has been, the remaining decisions must an optimal policy with regard to the state resulting from those previous decision.”

* limit the number of the potentially optimal control strategies that must be investigated.
* The optimal strategies must be determined by working “backward” from the final stage.

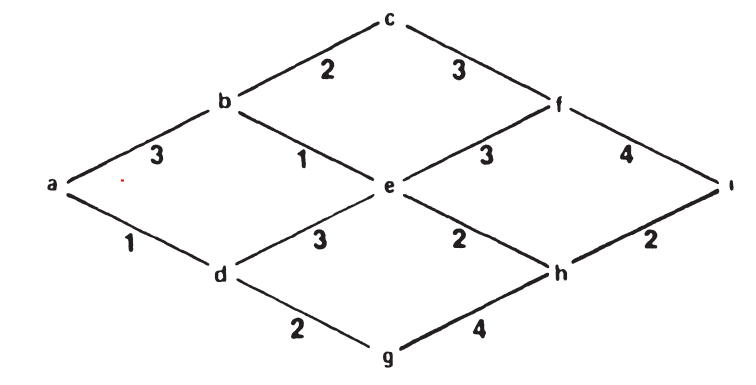
%%% Kim’s comment

Nowadays, machine learning, or AI adapt this basic principle since it reduces to calculation time !!”

%%%

One example is

An airplane from (a) to (i) to minimize the fuel, which way is the optimal?



1. Forward method is

From (a) to (i)

(2) x(4) x(4)x(2) = 64 ways

2) From the Bellmans principle

From (f,h) ; find

From (c,e,g) : find

From (b,d): find

🡪 the calculation number is less than the forward method

OK… now how about continuous system? The number of state at different time is infinite, to discretization is good to the solve this.

**The backward is the principle concept to solve the optimal control problems..(up to now).**

**Anyway I want to skip this chapter..**

1. Optimal Control for polynomial systems --skip
2. Output Feedback and Structured Control

8.1 Linear Quadratic Regulator with Output Feedback

1) Problem

with output feedback

To minimize the cost

1. Cost with the output feedback

The closed loop equation is

Plug (8.1-3) the cost is

Hence the problem is to find “K” to minimize (8.1-8) with the path constraint (8.1-7)

1. Matrix minimization

Suppose , a

Then the cost may be

Hence

For every should be satisfied, i.e., (necessary condition)

With this eq. (8.1-11)

Where

1. Solution

The problem is to find to minimize the cost

subject to

**Hence the Hamiltonian is**

Here is a symmetric matrix Lagrange multiplier

%%% Kim’s comment

Since %%%

The necessary conditions are

%%% Kim’ comment: matrix derivative

* Matrix derivative, :

And some strange notation as (see googling)

%%%

Hence the optimal gain is

where are the solution to (8.1-20,21)

%%% Kim’s comment

* Verify (8.1-23) is equivalent to the state feedback in the steady state(i.e., .

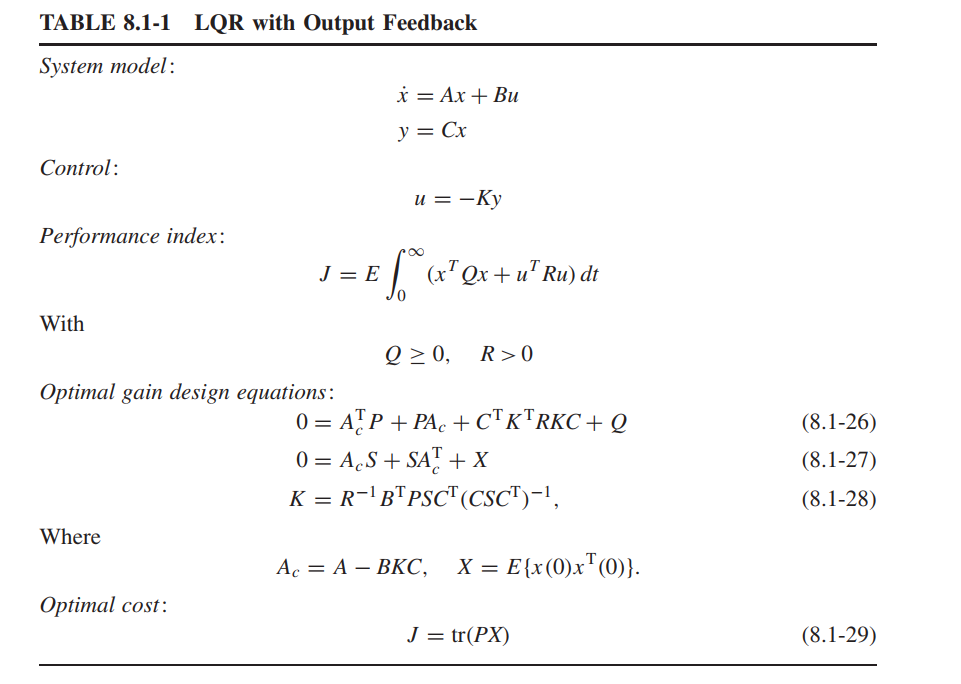
Proof: in case of the state feedback, so that

And is the solution to

which is the equivalent to ARE of the state feedback. What else about (8.1-21)? We will see later.

QED

Table 8.1 (p.303)



:

1. Necessary conditions on the existence of the optimal gain

* such that is stable
* ) is detectable

1. Numerical Solution

From the table 8.1-1. To find , in advance, to find , the solution to the Lyapunov equation, and which is also a lyapunov equation w.r.t the initial points. So, there are three unknown vector / matrix with three equation. But thee are no formal analytic solution. Hence numerical methods should be adopted. See. Table 8,1-2

8.2 ~ 8.6 tracking with the output –skip

The optimal output feedback without observers is a new concept. The solutions are generally related to matrix optimization which are solvable in numerically, up to now. There are several unsolved problems until now. Next chapter, one of the brilliant idea by “Kalman” to design observer. We will study it.

* HW\_Week\_7.1

PID controller, usually output feedback, is popular in industry society. Some papers are concerned about the relation between PID and the state feedback. One paper is “PID Equivalent of Optimal Regulator”, published 1978.

1. Review and summarize the paper.
2. Consider the plant , mass-spring-damper

<https://ctms.engin.umich.edu/CTMS/index.php?example=Introduction&section=ControlPID>

And PID controller is

2.1) find the closed loop poles

2.2)This problem will be considered in the frame of optimal controller

With the cost

Where,

Using the paper results,

a) find the optimal feedback gain

b) find the closed loop poles

c) compare the outputs between 2.1) and c) when the input is step.