1. Robustness and Multivariable Frequency- Domain Techniques

9.1 ~ 9.3 Introduction / Multivariable Frequency- domain analysis

robustness output feedback design

1) Stability :

BIBO : for any initial points, bounded input results in bounded output.

Lyapunov Stability: Without input.

2) Robustness

The plant will be stable even if the perturbation of the plant exists.

Stability margin: The system is analyzed on the Frequency domain , not time domain

Nyquist Plot

Singular value decomposition :

* 1. Observer design

%%% Kim’s comment : “Kalman “ invented the concept “Observer / estimator” in time domain analysis(state space model). What is the “Observer”? In the LQR , the full state feedback is needed. Consider

Here, the output is measured the state only, however, , hence, if the state is known, the state can be calculated, i.e., it is observable using

If in the frequency domain ( It is linear), the transfer function is a proper rational function w.r.t “s”, then the plant is always controllable and observable. However, a linearized system for the real plant, may be observable or unobservable. **In the pendulum, the output is only one state “position” , Kwarknaak… linearized system is unobservable, however, the class model is observable!!** This is important fact how to linearize a physical system.

And the other things, the optimal state feedback is very nice not only giving minimizing cost but also robustness!! ( you may know it…I think so). The problem is with the insufficient measurements, how to observer / estimate the full state is important.

* Observer design

1. Problem : observe / estimate the state with insufficient information

The plant:

1. **Design an observer**:

%%% kim’s comment

In the stochastic, the estimator ( the conditional expectation ) is the form as

which is the same structure !! %%%

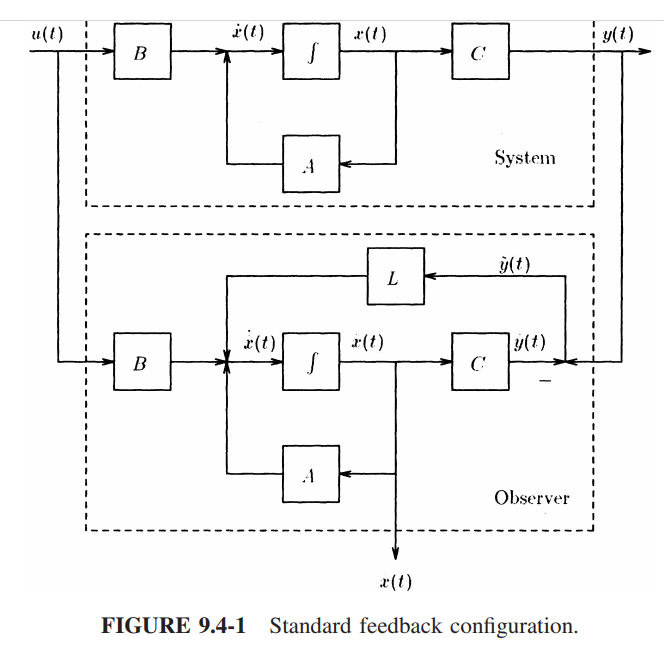
1. Analysis

Let us define the error

Then

Hence if

Then i.e.,



Hence for any stabilizing “observer gain” , the observer error is convergent to zero.

* “Optimal Observer “ - deterministic case
* Linear Quadratic Regulator

1. State feedback
2. Output feedback: design observer for the state feedback

**Observer gain : select such that is asymptotically stable.**

**In other words, the cost is**

**Then**

**There is no optimal choice of**

* “Optimal Observer/estimator” – stochastic case

Consider

here, are noise. In this case, the states are random processes so that there are many estimators as “average”, “moving average”, ‘filtering” and so on. However, We Do know it in the last semester “The best estimator”, or “The minimum min square error estimator” which is related to “Optimal”. **In this stochastic process, we may study optimality.** Here we may design “optimal gain ” depending on the noise characteristics.

So, the problem is to find the best estimator in the stochastic system not only to minimize the cost but also to estimate the perturbed states.

**Probability : Stochastic process**

Introduction:

* Integral:

Any smooth trajectory, whose the first derivative exists, can be integrated. However some noise motion is continuous but not smooth. That is the big problem in the end of 20c. The noise, we may called , Brownian motion. Let us see a deterministic input of a linear differential equation

Then

which is the famous convolution integral. Let us consider the input is noise, i.e., random process.

Here, we do know, as heuristically, is a random process with a pdf. Then how to characterize

in case of a continuous but not differentiable noise…Last semester, we may consider “The best estimator “ , the conditional expectation in discrete system. How about in continuous system? In the optimal controller design of a linear system, which is continuous, we Do know, Lyapunov, or Riccati equation regarding to optimal controller design problem. Let us tackle “ a difficult “ stochastic integral “ **so called Ito integral. The following will be the last semester subject which are skipped.. So the reference is the last semester textbook, “Speyor’s Stochastic,…”**

1. Stochastic Processes and Stochastic Calculus

* Reference “ Optimal control, Lewis Chap.9.4)

%%% Kim’s comments

In the microscopic motion of molecular its position is measureable so that

Now the velocity of the molecular is

* If the noise is a Brownian motion, is not defined at any time.

In fact if you measure something using a continuous sensor, the noise is look like a Brownian motion. Hence the derivative of the output of the sensor is not defined.

Engineers are **approximated this** as a white noise even if the derivative is not exist.

So that

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* 1. Random Walk and Brownian Motion
* Def. 5.1 A scalar Brownian motion process is defined as a process such that

1. is a Gaussian random variable
2. has independent increments, i.e., the increment

is independent

%%% kim’s comment : modelling for a Brownian motion

1. Consider

The Brownian motion is not independent since if it is independent

However the increment is independent by definition, i.e. if the intervals are not overlapped,

The variance of it is

1. Continuity: A R.V is continuous in the mean square sense if its correlation is continuous. Now a Brownian motion’s correlation is

This correction is continuous since

1. White noise : define as the derivative of a Brownian motion

In principle, Brownian motion is not differentiable at any time, in any sense. Despite this, it is simply assert that

Here is denoted as **White Noise**. As you see the white noise correlation is not continuous.

* 1. Mean Square Calculus

Consider the following

Then the solution is

If w is the white noise, it is not integral in common sense.

* 1. Wiener Integrals

Consider

Then

* Are the same to the ordinary integral
  1. integral

where

is defined in terms of its integral representation as

%%% Kim’s comment

Here instead of the derivative of the Brownian motion, the increment of the Brownian is considered. %%%

* stochastic integral: independent
* Remark

%%% Here

%%%

%%% Comment: remember

%%%

* Example 5.20

Let Then

%%% Kim’s Comment

Let us define , 🡪

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* Kim’s comment on (5.40)

1. Eq.(5.40) is not a normal integral There is an additional term.
2. The left and right side of (5.4) are random variables. How to prove the random variables are equal i.e.,

* If 🡪 two random variables are equal in the sense of mean square sense.

1. In the text book, it is proved in the mean square sense.
   1. Second order Ito integral 🡪 skip
   2. Stochastic Differential Equations and Exponentials 🡪 skip
   3. The Ito Stochastic Differential

* Theorem 5.23

Let be the unique solution to the vector Ito stochastic differential equation,

and .

Let be a scalar-valued real function of that is continuously differentiable in and that has continuous second derivatives with respect to . The stochastic differential of is then

%%% Comment:

The difference between the Wiener is

**%%%**

* Example. 5.24 : Consider

Calculate

Sol: Let so that

Using (5.49)

Substitute (5.51) into (5.52)

Take the Expectation on both sides

Hence

%%% Kim’s comment

Consider Eq.5.52, in an ordinary differential equation,

However, the RHS of Eq.(5.52) is different!!

%%%%

* 1. Continuous – Time Gauss-Markov Processes

Consider continuous time Gauss-Markov process(linear system)

1. **The mean of ,**
2. The correlation of

Proof: define

Then and , so that

So that

Let

Similarly

Hence

1. **The covariance of**

Proof: Let . Then

which is same as

Therefore

And its covariance is

%%% Kim’s comment

1. Given a deterministic dynamic system

And the cost

here

which is a Lyapunov equation(remember to solve a) we need boundary conditions)

1. Consider a stochastic differential equation

The covariance of in the steady state is

Compare a) and b), they are similar to each other but the order is different.

And the solutions are different even if they are positive definite.

However consider a stochastic differential equation

The covariance of , in the steady state is

%%%%%%%%

1. Continuous time Gauss-Markov Systems: Continuous Time Kalman Filter, Stationarity, Power Spectral Density and the Wiener Filter
   1. The continuous Time Kalman Filter

* Model

Where

* The innovation process

where the estimator, the estimator error

* The conditional estimator

where is the solution to the **Riccati** equation as

* In continuous time filter, there is no separation as Prediction and estimation.

%%% Kim’s comment

1. The problem is to find the **optimal estimator(optimal filter , optimal observer)** such that

The optimal solution is , which is given in (6.10)

1. **The model may be written in engineering textbook,** as

where as the white noise

Strictly speaking, the derivative of Brownian motion is not defined,(it we measure the position with a Brownian noise, the speed of the position, which is derivative of it, is not defined), hence this model is not correct. However, it is culture to write as it is.

1. In discrete system, it may allowed since the time interval between the sampling is finite,

the difference of the Brownian motion is defined.

1. The Riccati equation is a non-linear differential equation, and in control engineering, it is important for another optimal problem.
2. What is another implication of ? This is the minimum mean variance estimator(or observer) , as