- Continuous to Discrete : ref. Discretization: wikipedia

Consider a conituous system

$$\dot{x} = \mathrm{Ax} + \mathrm{Bu} + w\,, \quad w \sim N(0,Q), \text{white noise}$$
$$y = \mathrm{Cx} + \mathrm{Du} + v\,, \quad v \sim N(0,R), \text{white noise}$$

to be discretized i with zero-order hold for the input into

$$x_{k+1} = A_d x_k + B_d u_k + w_k$$
$$y_k = C_d x_k + D_d u_k + v_k$$

with

$$w_k \sim N(0, Q_d)$$
$$v_k \sim N(0, R_d)$$

1. Derivation without noises:

Observe the matrixes C_d, D_d is the same. Continuous system with input

$$\dot{x} = \mathrm{Ax} + \mathrm{Bu}$$

has the solution as

$$x(t) = e^{\mathrm{At}} x(0) + \int_0^t e^{A(t-\tau)} \mathrm{Bu}(\tau) d\tau$$

Then the sampled value of the solution is

$$x_{t+\Delta t} = e^{\mathrm{At}} x_t + \left(\int_t^{t+\Delta t} e^{A\tau} \mathrm{Bu}(\tau) d\tau \right)$$

Since

$$e^{\mathrm{AT}} \equiv \sum_{k=0}^{\infty} \frac{1}{k!} (\mathrm{AT})^k$$

Approximating it upto the first order gives

$$e^{\mathrm{At}} \approx I + \mathrm{At}$$

2. The first order approximation with $T = \Delta t$

$$A_d = e^{A\Delta t} = I + A\Delta t$$

If the input is constant during $t \in [t, t+\Delta t)$, i.e., using a zero-order hold for the input, then $u(t) = u_t$ which is constant,

$$B_d = \left(\int_t^{t+\Delta t} e^{A\tau}\mathrm{Bu}_t d\tau\right) = \left(\int_0^{\Delta t} e^{A\tau}\mathrm{Bd}\tau\right)u_t$$

Hence

$$x_{k+1} = A_d x_k + B_d u_k$$

with (A_d, B_d).

3. Example by matlab

If (A, B) are given as in the following code,

```
clear all
syms dt real
A = [0 1;0 0];
B = [0;1];
Ad= expm(A*dt)
Bd =int(Ad,dt)*B
A =[0 1 0;0 0 1;0 0 0];
B= [0;0;1];
Ad= expm(A*dt)
Bd =int(Ad,dt)*B
```

4. Comments

- Without the input, if a system is

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

Then as small time interval it is approximated to

$$\begin{bmatrix} x(t+\Delta t\ ) - x(t) \\ v(t+\Delta t) - v(t) \end{bmatrix} = \begin{bmatrix} 0 & \Delta t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$$

so that

$$\begin{bmatrix} x(t+\Delta t\ ) \\ v(t+\Delta\ ) \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$$

So that it is simple.However if there is an input (not only a deterministic signal or noise) the input matix is quite different as in the previous example,

$$B_d\ = \begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix}$$

Why this happen? Since at time interval the state is moving due to the input.

- How about in non-linear case? it is difficult and there are several methods. Think about it.

- In matlab,in a linear time invariant system, a simple commnad as "c2d"

4. **Discretization of process noise**

Now consider a corrupted noise as

$$\dot{x} = \mathrm{Ax} + \mathrm{Bw}, w \sim \text{white noise } (0, q_c)$$

The Noise covariance with a input matirx *B* is

$$Q_c = E[(\mathrm{Bw})(\mathrm{Bw})^T] = \mathrm{Bww}^T B,$$

For example if

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

Then $$Q_c = E[(\mathrm{Bw})(\mathrm{Bw})^T] = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{ww}^T [0 \quad 1] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} q_c,$$

Now the R.P. x(t) solution is (in engineering model)

$$x(t) = e^{\mathrm{At}} x(0) + \int_0^t e^{A(t-\tau)} w(\tau) d\tau$$

There are two methods for the discretization of the noise.

1) **Noise as a constant between the sampling time**

As a deterministic zero order hold case, the white noise w(t) is constant

between $t \in [t \;\; t + \Delta t]$, then

$$A_d = e^{A \Delta t},$$

$$B_d = \left(\int_t^{t+\Delta t} e^{A\tau} w_t d\tau \right) = \left(\int_0^{\Delta t} e^{A\tau} \mathrm{B} d\tau \right) w_t$$

which are the same to the deterministic case. And its covariance is

$$Q = B_d w_t w_t^T B_d^T$$

Let us an example

```
clear all
syms dt sigma real
%2nd order
A = [0 1; 0 0];   %system matrix
B =[0;1];
Ad =expm(A*dt);
Bd =int(Ad*B,dt)
```

```
Bd =
```

$$\begin{pmatrix} \frac{dt^2}{2} \\ dt \end{pmatrix}$$

```
Bd*sigma*sigma'*Bd'  % covariance
```

ans =

$$\begin{pmatrix} \frac{dt^4 \sigma^2}{4} & \frac{dt^3 \sigma^2}{2} \\ \frac{dt^3 \sigma^2}{2} & dt^2 \sigma^2 \end{pmatrix}$$

```
% 3rd order
A =[0 1 0; 0 0 1;0 0 0];
B =[0;0;1];
Ad =expm(A*dt)
```

Ad =

$$\begin{pmatrix} 1 & dt & \frac{dt^2}{2} \\ 0 & 1 & dt \\ 0 & 0 & 1 \end{pmatrix}$$

```
Bd =int(Ad,dt)*B
```

Bd =

$$\begin{pmatrix} \frac{dt^3}{6} \\ \frac{dt^2}{2} \\ dt \end{pmatrix}$$

```
Bd*sigma*sigma'*Bd'  % covariance
```

ans =

$$\begin{pmatrix} \frac{\mathrm{dt}^6\,\sigma^2}{36} & \sigma_1 & \sigma_2 \\ \sigma_1 & \frac{\mathrm{dt}^4\,\sigma^2}{4} & \sigma_3 \\ \sigma_2 & \sigma_3 & \mathrm{dt}^2\,\sigma^2 \end{pmatrix}$$

where

$$\sigma_1 = \frac{\mathrm{dt}^5\,\sigma^2}{12}$$

$$\sigma_2 = \frac{\mathrm{dt}^4\,\sigma^2}{6}$$

$$\sigma_3 = \frac{\mathrm{dt}^3\,\sigma^2}{2}$$

2) **During the time interval, noise is still active**

consider $\dot{x} = \mathrm{Ax} + w, w \sim \mathrm{White}(0, Q_c) \;\; - \to x_{k+1} = \mathrm{Fx}_k + \Gamma\, w_k$

The discretization noise covariance Q

$$Q = \int_0^{\Delta t} A_d Q_c A_d^T \,\mathrm{dt}$$

Comaring to the piecewise case, calculate it analytically.

For example assume

$$\begin{bmatrix} \dot{x} \\ \dot{v} \\ \dot{a} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w$$

If the system matrix in discrete form is

$$A_d = \begin{bmatrix} 1 & \Delta t & \frac{\Delta t^2}{2} \\ 0 & 1 & \Delta t \\ 0 & 0 & 1 \end{bmatrix}, Q_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sigma_v^2$$

or the second order

$$A_d = \begin{bmatrix} 1 & \Delta \\ 0 & 1 \end{bmatrix}, Q_c = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \sigma_v^2$$

Then the covariancer Q is

$$Q = \int_0^{\Delta t} A_d Q_c A_d^T \, \mathrm{dt}$$

```
clear all
syms dt real
%2nd order
A = [0 1; 0 0];   %system matrix
B =[0;1];
Qc =B*B';     % assume : sigma = 1
Ad =expm(A*dt)
Q =int(Ad*Qc*Ad',dt)
%3rd order
A =[0 1 0; 0 0 1;0 0 0];
Ad =expm(A*dt)
B =[0;0;1];
Qc = B*B';
Q2 =int(Ad*Qc*Ad', dt)
```

3) Comments

- Here there two covariances in the same SDE. Which one is correct or to apply?

- If the SDE is continuous, let us say, then the covariance is the result of 2) , not 1)

```
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