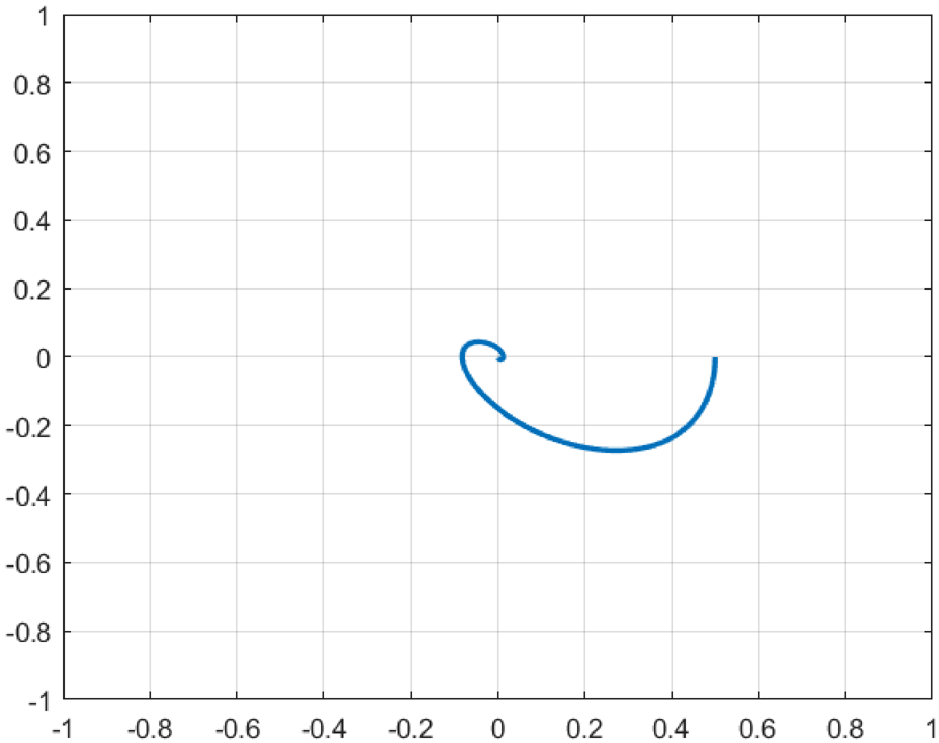
- Lyapunov Function

Lyapunov is an indirect method for the stability status. Consider

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - y \end{aligned} \qquad (x(0), y(0)) = (0.5, 0)$$

Then it is linear, we may get the solution directly

```
clear all;
LW='linewidth';
L = chebop(0, 10);  % time from 0 to 10
L.op =@(t,x,y)[diff(x)-y; diff(y)+x+y];
L.lbc =@(x,y) [x-0.5; y];
rhs =[0;0];
U=L\ rhs;
% plot(U)
plot(U(1), U(2),LW,2);grid on
axis([-1 1 -1 1]);
```



Hence the solution is convergent to zero. OK. without getting the solution, with any initial condition , a famous Lyapunov equation as

```
A =[0 1; -1 -1];
Q =eye(2);
```

```
P = lyap(A',Q)       %A'P + PA = -Q
```

```
P = 2×2
    1.5000    0.5000
    0.5000    1.0000
```

```
eig(P)
```

```
ans = 2×1
    0.6910
    1.8090
```

In matlab, the commanf 'lyap' is

$$\mathrm{AP} + \mathrm{PA}^T = -Q$$

which is the steady state covariance of the state with white noise intensity *Q*, however for the stability

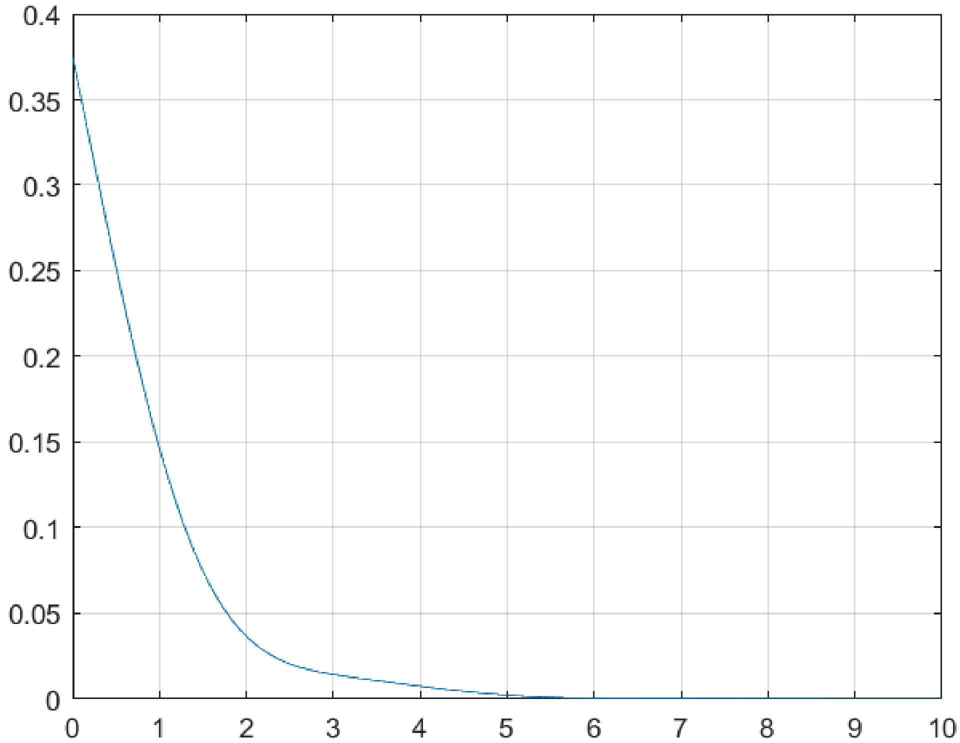
$$A^T P + \mathrm{PA} = -Q$$

Hence to get the solution of the stability, you should write as

P = lyap(A',Q)

Since e-values of *P* are positive the candidate lyapunov $V(x_1, x_2) = x^T \mathrm{Px}$ is truly a lyapunov function. Let us see the time evolution of *V*,since $\dot{V} < 0 \, (x \neq 0)$

```
V = P(1,1)*U(1).^2 + 2*P(1,2)*U(1).*U(2) + P(2,2)*U(2).^2;
plot(V); grid on
```



Hence the $V(x(t), y(t))$ is decreaing as time increasing.

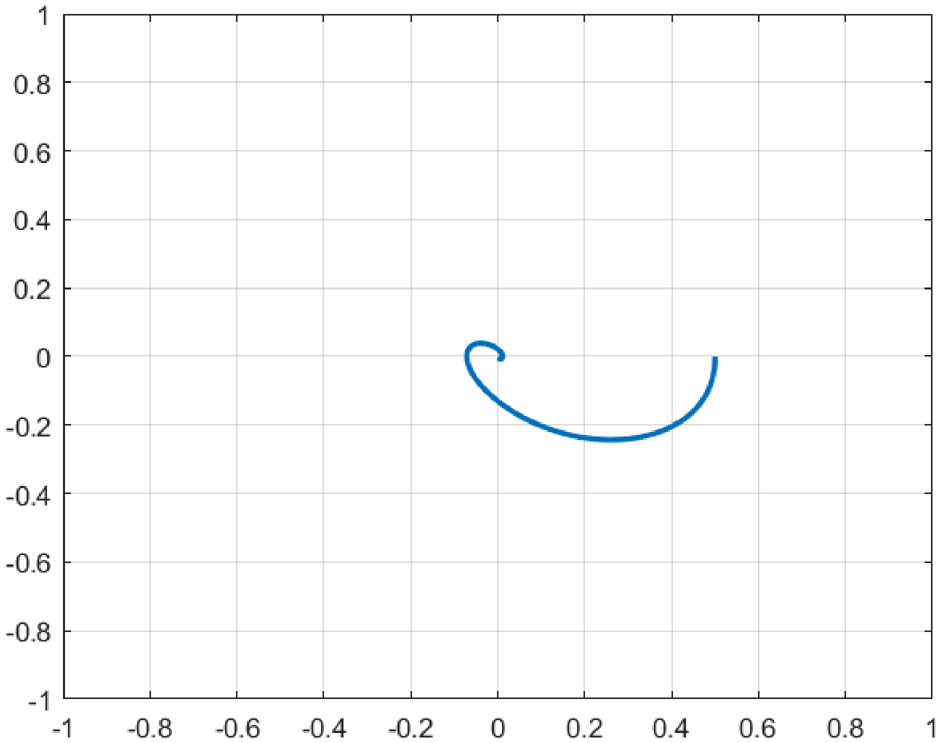
- Non-linear

Now consider the additional non linearity in the above example

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + x^3 - y \end{aligned} \qquad (x(0), y(0)) = (0.5, 0)$$

Given the initial condition, we may get the solution as

```
clear all;
LW='linewidth';
L = chebop(0, 10);
L.op =@(t,x,y)[diff(x)-y; diff(y)+x-x^3+y];
L.lbc =@(x,y) [x-0.5; y];
rhs =[0;0];
U=L\ rhs;
% plot(U)
plot(U(1), U(2),LW,2);grid on
axis([-1 1 -1 1])
```



Comparing the linear system trajectory it is a little bit different. Now without the initila conditions, we may check the stability at the equilibrium points.

1. Linearization at the equilibrium points.

First the jacobian as

$$J = \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1+3x^2 & -1 \end{bmatrix}$$

1) Find equiribrium points: 3points

$(x_e, y_e) = (0,0)\,, (-1,0), (1,0)$

2) Linearization at $(0,0)$ for the system matrix

$$A = J = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

--> $\dot{x_L} = \mathrm{A}x_L$ --> this is the same the previous linear system

3) Linearization at $(-1,0)$ for the system matrix

$$A = J_{(-1,0)} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$$

```
A1 = [0 1;2 -1];
eig(A1)
```

```
ans = 2×1
     1
    -2
```

the e-values of A are positive and negative, so that at this point is a saddle point

4) Linearization at $(1, 0)$ for the system matrix

$$A = J_{(-1,0)} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$$

which is the same to 3) case.

2. Domain of attraction

At each equilibrium points, the only (0,0) is asymptotic stable,hence we may interested in the domian of attraction. The other two points are saddle points, there are no domain of attraction. In linearization, the approximation $x_L \sim N(x_e, r)$ is just near to the x_e , the radius *r* is not defined. Now we may use lyapunov method(This is just a concept). Let us define

$V(x, y) = \frac{1}{2} x^T \mathrm{Px} = \frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, so that P should be positive definite.

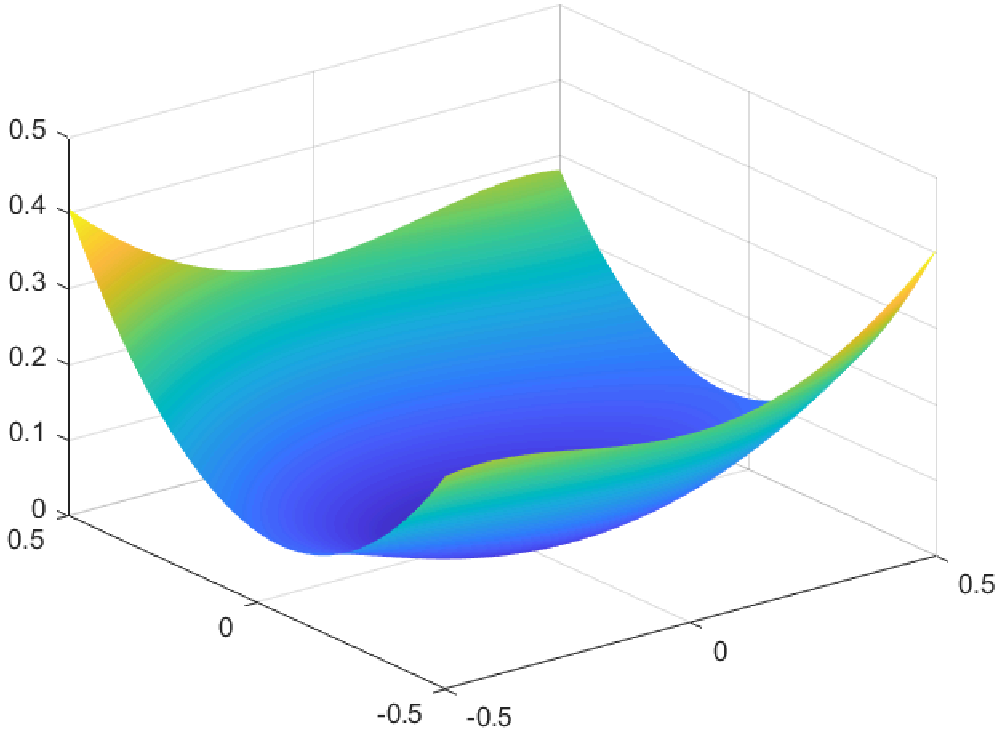
Now

$$\frac{\mathrm{dV}}{\mathrm{dt}} = \frac{\partial y}{\partial x} \frac{\mathrm{dx}}{\mathrm{dt}} = \begin{bmatrix} \frac{3}{2} x + \frac{1}{2} y & \frac{1}{2} x + y \end{bmatrix} \begin{bmatrix} y \\ -x + x^3 - y \end{bmatrix} = \frac{3}{2} \mathrm{xy} + \frac{1}{2} y^2 - \frac{1}{2} x^2 + \frac{1}{2} x^4 - \frac{1}{2} \mathrm{xy} - \mathrm{xy} + x^3 y - y^2$$

$$= -\frac{1}{2} x^2 + \frac{1}{2} x^4 - y^2 + x^3 y$$

For negative definiteness, the difficulity is $x^3 y$. Now is it negative definite in a bounded domain D ? Analytically it is difficult to find D . I will try to find it numerically. Pick up a domain $D = \{(x, y) : \ -0.5 \le x \le 0.5 , \ -0.5 \le y \le 0.5\}$, and find the min of $-V$

```
clear all; clc
D = [-0.5 0.5 -0.5 0.5];
f = chebfun2(@(x,y) 1/2*x^2 - 1/2*x^4 +y^2-x^3*y,D,'vectorize','on');
plot(f)
```




```
min2(f)
```

```
ans = 1.3878e-17
```

Hence, min of $-V(x, y) = 1.310^{-17} \sim 0$ at least in *D*, implies $\dot{V}(x, y) \leq 0$ in *D*

in conclusion with $V(x, y)$ at $x_e = (0, 0)$ the system is asymptotic stable.

We may verify numerically the negativeness of a candidate of Lyapunov function at least a closed bound domain. Next tutorial, tomorrow, we may develope a systematic way to construct a Lyapunov function. Why is it important to find a Lyapunov function? To find a domain of attraction. i.e.,

$\forall\, x(0) \in D - \rightarrow \forall\, x(t) \in D$

Remember given

$\dot{x} = f(x)$

To check the stability

1) Find x_e

2) Linearize the system to check stability at x_e

3) If the system is asymptotic stable at x_e, find donain of attraction as $V(x) \leq c$, $V(x)$ is a Lyapunov function

Test ~~

Draw $f = V(x, y)$ where (x, y) is the solution of

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + x^3 - y \end{aligned} \qquad (x(0), y(0)) = (0.5, 0)$$