Ch.3

1. Stability and Equilibrium

* Equilibrium points:

Consider an autonomous system

The equilibrium points are .

%% Kim’s comment

1. If
2. In Non-linear, the stability is analyzed in , i.e., ,is converged to ?
3. If is continuous and differentiable, then

Let us define

Let us define a small deviation(or perturbation) of x from such that , then

Define the deviation as . **Since**  is small enough

Here due to **is small, the higher order term converge to zero faster than the first order.** Hence, near

which is linear DE. Be sure that this is a LDE only in

We may write as in the previous lecture.

3.1 Basic Concepts

* Definition 3.1

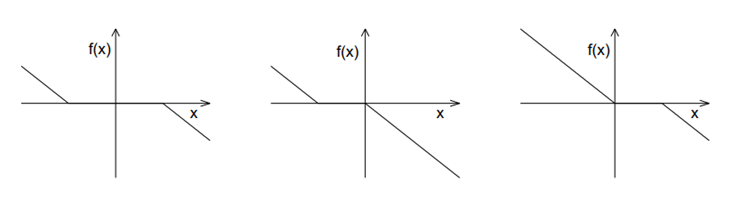
The is

1. stable if for each there is s.t.
2. unstable if it is not stable
3. **asymptotically stable if it is stable** and can be chosen s.t.

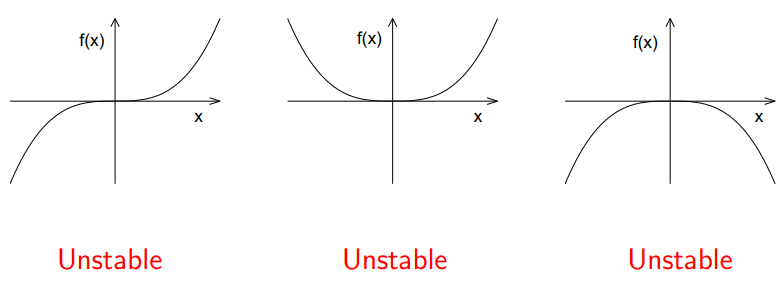
%% Kim’s comment

1. is not mattered as linearized cases.
2. So if there are many , the stability should be analyzed at every

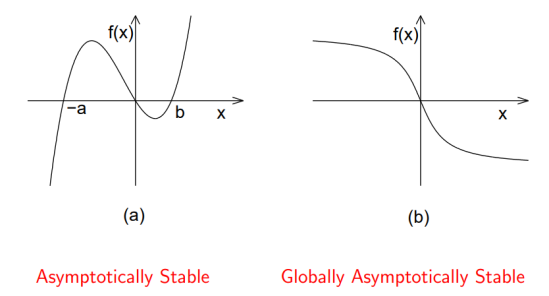
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* Example

1. Stable
2. Unstable



1. Asymptotically stable



* Def.3.2 (The region of attraction)

Let the origin be an asymptotically stable equilibrium point of the system , where f is a locally Lipschitz defined over a domain .

Then The region of attraction (also, the region of asymptotic stability, or domain of attraction, or basin) is the set of all points ,

-END -

%% Kim’s comment:

* One of the BIG problem is how to find the basin? If you linearize a NODE, and at is asymptotic stable. Then how to guarantee with a perturbed initial condition is asymptotic stable? One of the solutions is Lyapunov method which is the nest section.
* **One of the mistakes almost all master students thesis, they linearized a nonlinear system to design a “Linear Control “ for Linearized System not the original non linear system.**

**%%%**

* Def. 3.3

is exponentially stable if

It is globally exponentially stable if the inequality is satisfied for any

* Ex. 3.2

The origin is asymptotic stable since

but is not exponentially stable.

* 1. Linearization

Then the linearized system at is

where

* Theorem 3.2

1. The origin is exponentially stable iff for all e-values.
2. The origin is unstable if for one or more of the e-values

-The End –

* Ex. 3.3 :

1. Find =?
2. Linearizing as

🡪 , we may not say about stability at

* Ex: 3.3 –Kim:

1. Find :
2. Linearizing at each

Since

2.1) 🡪 exponentially stable at

2.2) 🡪 unstable at

2.3) 🡪 unstable at

* Ex.3.4 :Check the stability: the pendulum equation

Sol:

1. Find ?
2. For

Since

The e-values are . Hence for all , 🡪exponentially stable

1. For

The e-values are . Here one e-value has 🡪unstable stable

* 1. Lyapunov Method
* Terminology

Def: Positive definite function

is a positive definite function if

1. Continuously differentiable over

%% Kim’s Comment

* Careful think over the condition 2). It is positive for all

-. : a PDF

-. a PDF in

-. is not a positive definite function since

* A common PDF is quadratic form as

If and , then is a positive definite matrix.

Or equivalently

If and all the eigenvalues are positive, then is a positive definite matrix.

%%% Kim’s comment : PDF

1. it may be transformed into a quadratic forum as

Then check the e-values of , which are 0, 1 not to be a PD.

%%%

* In Linear system, such that the Lyapunov equation has the solution

Then all e-values of have negative (if it is complex, then real(e-values)).

%%%

* Theorem 3.3

Let is a locally Lipschitz function over , Let be a positive definite

1. , 🡪 at is asymptotically stable
2. , 🡪 at is stable

Proof : I will skip but using the following formula

And -QED-

* Terminology : is called a Lyapunov function.

%% Kim’s comment : How to find a positive definite function?

1. The stability is said, upto now, It is analyzed at the equilibrium point, . In linearizing method, we do not know the “domain of attraction” as , but what is ?
2. If you are lucky to find a Lyapunov function, ,and it is stable or asymptotic stable then , which is one of the domain of attraction.
3. The theorem is important and useful in the stability check since we may not solve the solution but to find a function ( Lyapunov function if it is stable) indirectly.
4. The theorem is a **sufficient condition**, hence a candidate positive function may be negative, you may not conclude it is not stable.
5. Using the theorem, you may try and error to find a good candidate Lyapunov function.

If you find it, you are lucky. In general

1. If is a positive matrix, then a candidate a Lyapunov function in Linear system.
2. Here , so for asymptotic stability one of the sufficient conditions

If

Careful thought needed. means at evenif **, the asymptotic stability is not guaranteed**.

For example assume . Then if , which is not negative. Hence it does not satisfy the condition If .

* Ex.3.6 : The pendulum equation

1. Case 1:

A Lyapunov candidate may be

Check is a positive definite at

* ) = 0
* Positive slope along

Hence , it is stable, however we may not conclude it is asymptotical stable.

1. Case 2: the friction is positive,

If we select a Lyapunov candidate as the same,

Here which is not negative, using this Lyapunov function , we may conclude it is stable at but we may not conclude it is asymptotic stable.

Since in phase plane, at this point it is asymptotic stable(really?), we may conclude in this case **the Lyapunov candidate is not appropriate**.

1. Find an appropriate Lyapunov function guarantee the asymptotic stable for Here we may select another Lyapunov candidate. Choose a which is unknown.
2. Check the positive definite

Since need 🡪

1. The derivative along should be negative for asymptotic stability’
2. For sign definite, are sign indefinite, their coefficient should be zero as

Then

For , if we select , then

1. In conclusion, Let a PDF as

so that

So that is asymptotic stable point.

%% Kim’s comment

1. If are positive definite, then is also a positive definite.
2. In linear algebra or optimization

- matrix is a positive definite matrix if and only if all the eigenvalues are positive.

- matrix is a positive definite matrix if and only if the all the principal minors are positive. Ex.

Principle minor :

1. The first minor : determinant of
2. The second minor : determinant of

You may check positive for matrix using determinant

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