**Chap. 4 Time-varying and Perturbed Systems**

* Introduction

Consider of the following system is asymp.stable (say the nominal plant)

Now the real plant is perturbed by some uncertainty

where the perturbation is unknown but may be . What conditions on the perturbation do guarantee the asymp.stable?

%% Kim’s comment

The perturbation is here to be deterministic even if governing equation is not unknown. Remember the last semester in “Stochastic”

There, the real plant is perturbed by a white noise, which is a random process. In this case even if is bounded,

i.e., the magnitude is not bounded in probability. Given the real system, which perturbed model is preferable?

1) In deterministic space: In general many problems are solved. But how large of the magnitude of the perturbation? Moreover, in the big data, you may not know the model. In this case a little bit hard problem

2) In stochastic space: Nowadays, we may have more computing power than ever, hence we may simulate as many times as you want. Look at the Mote Carlo Method. More over Given “BIG” data of the system output, which is not known the dynamics, it is a good method to analyze it. However, analytically it is still difficult to get the probabilistic performances.

3) Here in this course, we will deal the problem in the deterministic space only.

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* 1. Time –varying systems

Consider

This is not autonomous system, so that it is difficult to analyze the stability.

%% Kim’s comment(Frozen coefficients do not determine stability)

<https://www.chebfun.org/examples/ode-linear/FrozenCoeffs.html>

Consider a time varying linear system,

We may conclude the stability if for each fixed time if eigenvalue of has negative real values, it is stable**. But this is not true**. See the example. %%

* Definition of **stability**(2023\_Week\_3, 3.1)

For an autonomous system, , the is stable if

if for each there is s.t.

Now, in a time varying system it should be

So, different initial time, it may be stable or not, which is complicate. To analyze it in a systematic way, in general we may introduce the class or the class

* **Terminology class-K / class-KL function** (Def. 4.1 : pp.88)

1. A belongs to class if

and

and belongs to class if

1. A two variables functional belongs to class if

for fixed s, for fixed , is decreasing w.r.t s and

%% Kim’s comment

These functional are generalized Lyapunov functions. They do not need continuous differentiable neither definite. In addition, is considered to “time”. %%%

* Ex. 4.1

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* Lemma 4.1 (pp.89)

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%% Kim’s comment

-Inverse function: i.e., given ,

Example :

:

-Composite function: (x)

* **Lemma 2**. Let be a PDF. Then such that
* **Def. 4.2 (Uniform Stability)** is

1. uniform stable if and , independent of
2. uniformly asymp. Stable if
3. exponentially stable if

%% Kim’ comment: uniform stable means stable but not vice verva %%

* Theorem .4.1 , Assume (in locally or globally) and

1. is uniform stable (locally or globally) if
2. is uniform asymptotic stable (locally or globally) if
3. is exponentially stable (in locally or globally) if

where

%% Kim’s comment

1. For the exponential stable, the parameter “a” in Positive and in Negative are **equivalent.**
2. Remember “globally” means for all

As %%%

* Ex. 4.2 (scalar)

Selecting ,

By theorem 4.1 since , it is globally uniform asymptotic stable.

%% Kim’s comment

However, we may not conclude it is **globally** exponential stable since

1) for positive but

2)

3) Here is an example that it is asymptotic stable but **not exponentially**.

The solution is with

which is convergent to 0, i.e., , but not exponentially. i.e., there is no such that

-End –

* 1. **Perturbed Systems**

Consider

here, . And it may be considered as

* will be called a nominal system
* : the perturbation: modeling errors, aging, uncertainty, disturbance, …
* Case\_1 Nominal plant is exponentially stable

The problem is at , if the nominal system is **exponentially stable**, what is the condition for **exponential stable** of (4.7)?

1. Simple analysis

Consider (4.7) whose nominal system is **exponentially stable**, i.e.,

such that

which are the conditions for the exponential stability, in addition, for all

for . And for **the perturbation bound**

Then

which is guaranteed the perturbation system **exponential** stability.

Hence if , then implies the perturbed system (4.7) is an **exponentially stable**.

1. Ex. 4.5 : (the Region of attraction). Consider
2. for the nominal plant

Let the Lyapunov equation as

Then

In (a), Then

%% Kim’s comment: For exponential stability, remember our goal is

%%

Since , the line of the RHS above quality,

The perturbation is , which is not global Lipschitz. However we may select a compact domain set as , it is locally Lipschitz. Now in to find the maximum value of is, i.e.,

using change coordinate as , or

%% Kim’s comment: For exponential stability, remember our goal is

%%

Then

Combining this upper bound,

For the negative,

In summary, in the domain is exponentially stable if the magnitude of perturbation is smaller as.

%%% Kim’s comment

Here if the size of is large, i.e., the perturbation magnitude is small. %%

* Case\_2: Nominal plant is asymptotic stable.

1. Simple analysis

If the nominal plant is asymptotic stable, then such that

Then the perturbed system to be asymp.stable,

which is negative definite. In general it is complicate and not satisfied

1. Ex.4.6: Consider

* Nominal plant is a

1. Solution: Assume the initial point the solution is

so that

1. Select a is asymptotic stable.
2. However, , is unstable at the origin

4.3 Boundedness and Ultimate Boundedness

Consider a perturbed scalar system

which is no equilibrium point. The solution is

Then

which is bounded uniformly (independent of )

🡪 skip the others.

4.4 **Input-to-State stability (ISS - stability)**

* Concept: With bounded input state bounded

1. Consider with a bounded continuous input

When the unforced system is globally asymp.stable, the forced system

is stable, i.e., is bounded?

1. Linear system

Then the solution

is bounded

Hence the zero-state response is bounded for every bounded input.

1. Non-linear system

Then is asympt. stable. However with

which is unbounded.

%%% kim’s comment: Why unboundedness in engineering?

One of reasons to study the unboundedness even if bounded domain and bounded time is. In the numerical simulation, 99% you may specify a finite domain and a finite time. During numerical simulation, sometimes the value is excessively too large to be dealt in PC. You may have experiences in matlab as

For the unforced system, the unique solution to an ODE is function of initial. What is the state bounded?

1. , which is bounded
2. , which is bounded even if it is unstable.
3. is bounded . At , it is unbounded
4. is unbounded %%%

%% Kim’s comment: for the forced system, the unique solution to an ODE is function of initial points and input i.e.,

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* Def. 4.4 (input-to-state stable) pp.107

Is input-to-state stable if such that

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Then Input-to-state stable implies

1. , state is bounded
2. **is ultimately bounded by**
3. if converges to zero as so converges to zero
4. The origin of the unforced system is **globally asymp.stable.**

%% Kim’s comment :

1. A little different definition as <https://en.wikipedia.org/wiki/Input-to-state_stability>

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1. For linear system

If is asymptotic stable( so it is exponential stable), then **BIBO** stable (Bounded input to Bounded output). %%%

* Theorem 4.6 Suppose

1. is locally Lipschitz in .
2. is continuously differentiable and

And is continuous differentiable PDF,

Then the system is ISS-stable with

%% kim’s comment: Here with is asymptotic stable.

* Ex. 4.12 : consider

Select , then

Hence the system is ISS-stable - The End –

* Lemma 4.5 : Suppose

1. continuously differentiable , and globally Lipschitz in ,
2. is **globally** exponentially stable,

Then is ISS - The end-