Unit 2(a) Lecture Notes for MAT224

January 24-26, 2023

§1 1.6 Bases and Dimension Continued...

Write two definitions for a set S to be a basis for a vector space V.

Hint: Definitions are true statements of the form "A if and only if B". For example, "A shape is a triangle if and only if it is closed figure in the plane with three straight sides and three vertices."

Any theorem of the form "A is true if and only if B is true" can also be used as a definition.

Let V be a vector space, and let S be a nonempty subset of V. Then S is a basis of V if and only if every vector $x \in V$may be written uniquely as a linear combination of the vectors in S.

A subset S of a vector space V is called a basis of V if $V = \text{Span}(S)$ and S is linearly independent.

Consider the statements of theorems: 1.6.3, 1.6.6, and 1.6.10, 1.6.14, 1.6.18, Lemma 1.6.8, and Corollary 1.6.11.

How much can they tell you about the answers to the following questions? Let $V = span\{s_1, s_2, \ldots, s_n\}$ for some $n \in \mathbb{N}$

(a) Does V have at least one basis?
Since every subspace has a basis, V have at least one basis.

(b) Do all bases of a given vector space have the same number of elements?
All the bases of a given vector space have the same number of elements.

(c) If a subspace W of vector space V has a basis, can that basis be extended (have vector(s) added to it) to a basis for all of V?
If $x \notin W$, then $W \cup \{x\}$ is a basis of V. The same can be said for all basis of V.

(d) Does it make more sense to talk about **a** dimension of vector space V or **the** dimension of vector space V? In other words, is there more than one candidate for the value of $dim(V)$?
It makes more sense to talk about **the** dimension of vector space V, since the dimension of all bases of V are the same.

§2 2.1 Linear Transformations

Give two definitions for "linear transformation".

Transformation $T : V \to W$ is a linear transformation if and only if for all a and $b \in \mathbb{R}$ and all $\mathbf{u}$ and $\mathbf{v} \in V$

$$T(au + bv) = aT(\mathbf{u}) + bT(\mathbf{v})$$

Transformation $T : V \to W$ is a linear transformation if and only if for all a and $b \in \mathbb{R}$ and all $a_1, \cdots, a_k \in \mathbb{R}$ and for all $v_1, \cdots, v_k \in \mathbb{R}$.

$$T(\sum_{i=1}^{k} a_i v_i) = \sum_{i=1}^{k} a_i T(v_i)$$

Is it possible to determine the pre-image of a vector $\vec{w} \in W$, if you know its image? In other words, let $T : V \to W$ be a linear transformation and let $\vec{w} \in T(W)$. So $T(v) = w$ for some $v \in V$. Can we tell which $v \in V$ is sent to W by V.

It is possible to determine the pre-image of a vector $\vec{w} \in W$ if the image is known. This operation can be done using a invertible linear transformation which undoes the transformation. Consequently, the pre-image can be found.

Consider Proposition 2.1.14. It suggests that, if we know the value of $T(b_i)$ for every $b_1, b_2, \ldots, b_n$ in a basis B of V and linear transformation $T : V \to W$, then we can find the value of $T(v)$ for any $v \in V$. Explain how we would do this.

For example, if $T(1) = 5$, $T(x) = 1 + 2x$, and $T(x^2) = 1 - 2x + 3x^2$, what is the value of $T(2x + 4)$?

$$\begin{aligned} T(2x + 4) &= T(4 + 2x) \\ &= 4T(1) + 2T(x) \\ &= 4(5) + 2(1 + 2x) \\ &= 4x + 22 \end{aligned}$$

§3 2.2 Linear Transformations Between Finite Dimensional Vector Spaces

Give three examples of linear transformations $T : V_1 \to V_2$.
Let $V = W = \mathbb{R}^2$. Choose the standard basis $e_1 = (1,0)$ and $e_2 = (0,1)$ for both V and W. Define T by $Te_1 = e_1 + e_2$ and $Te_2 = 2e_1 - 2e_2$.

Let $V = W = P_2(\mathbb{R})$. Choose the basis $p_0(x) = 1, p_1(x) = x$, and $p_2(x) = x^2$ for both V and W. Define T by $T(p(x)) = x\frac{d}{dx}p(x)$.

Let V = Span$(\{v,, v_2\})$ C R where $v_1 = (1,1,0)$ and $v_2 = (0,1,1)$, and let $W = \mathbb{R}^3$ with the standard basis. Define T by $T(v_1) = e_1 + e_2$ and $T(v_2) = e_2$.

Give an example of a vector space V, two bases $\alpha = \{a_1, a_2, \dots, a_n\}$ and $\beta = \{b_1, b_2, \dots, b_n\}$ of V, a transformation $T : V \to V$, and its matrix forms $[T]_\alpha^\beta$ and $[T]_\beta^\alpha$.

Let $V = W = \mathbb{R}^2$ and we take two bases α and β to be the standard basis: $e_1 = (1,0)$ and $e_2 = (0,1)$. Let $T = R_\theta$ be rotation through an angle θ in the plane. Then, the matrix of R_θ is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

.
What is the process of getting from a linear transformation T to its matrix form $[T]_\alpha^\beta$? Is it possible to go backwards and determine T from its matrix?

Since the columns give the mapped basis vectors, we can always find a linear combination of the vectors to obtain the required image.