Unit 1(b) Lecture Notes for MAT224

January 17-19, 2023

These notes are intended for study purposes. You should be able to fill in these blanks from the notes you take during lecture and/or the textbook. You are welcome to use them to work ahead. Your completed copy of these notes should be submitted to the Quercus assignment called "Unit 1(b) Lecture Notes" by April 7, 2023. You can scan your handwritten answers or you can type them out. See the Unit 1 Homework for details.

§1 Linear Combinations 1.3 Continued...

What is the definition of a linear combination?

Let V be a vector space with scalars in $\mathbb{R}$. A linear combination of vectors $v_1, v_2, v_3, \ldots, v_k \in V$ is defined to be ...

Any sum $a_1v_1 + a_2v_2 + a_3v_3 + \ldots + a_kv_k$ where $a_i \in \mathbb{R}$ and $v_k \in V$.

Write the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$ as a linear combination of elements in $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}$.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \frac{5}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Define the set $W_1 + W_2$ for subspaces W_1 and W_2 of vector space V. Is it a subspace of V?

$W_1 + W_2 = \{x \in V\ x = x_1 + x_2, x_1 \in W_1 \text{ and } x_2 \in W_2\}$. It is a subspace of V.

If $W_1 = span(S_1)$ and $W_2 = span(S_2)$ where $S_1 \subset V$ and $S_2 \subset V$ and W_1 and W_2 subspaces of V, we can write $W_1 + W_2$ as the span of which set? Explain.

We write $W_1 + W_2 = span(S_1 \cup S_2)$. We prove that $span(S_1 \cup S_2) \subseteq W_1 + W_2$ and $W_1 + W_2 \subseteq span(S_1 \cup S_2)$ so they are equivalent.

Let V be a set of vectors. Under which condition (if any) is $span(V)$ not a vector space? Explain.

$span(V)$ is not a vector space if V is dimensionless.

Use arithmetic, explanation in words, and, if you want, matrices to show that any vector with three real coordinates, say $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ where $x_0, y_0, z_0 \in \mathbb{R}$, can be written as a linear combination of elements from the linearly independent set
$\left\{ \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right\} \subset \mathbb{R}$.
We know $x_0, y_0, z_0 \in \mathbb{R}$ and that vectors $v_n, u_n, w_n \in \mathbb{R}$, so these three vectors are basis vectors since they are 3 linearly independent vectors in $\mathbb{R}^3$ so they span $\mathbb{R}$ and can form a linear combination of all vectors in $\mathbb{R}$.

§2 Linear Independence 1.4

Mislabeled as 1.3 in last week's notes

Done last week.

§3 Bases and Dimension 1.6

What is a basis of vector space V?

A basis is a set of linearly independent vectors such that the span of the basis is the vector space itself.
Write the theorem that allows us to write every function as a linear combination of an odd function and an even function.

Note: Function f is odd if and only $f(-x) = -f(x)$.
Function g is even if and only if $g(-x) = g(x)$.

Let $f(X) = g(x) + h(x)$ where g is even and h is odd, then $f(-x) = g(x) - h(x)$ and $f(x) + f(-x) = 2g(x)$ where $g(x) = \frac{f(x)+f(-x)}{2}$ (even). Then we can manipulate to form odd functions as well.

Restate Theorem 1.6.6. in plane English: A linearly independent subset of a span...

Every linearly independent set may be extended to a basis by adjoining further vectors.

Read the proof for Theorem 1.6.10 and write it here. Then explain how it gives us a maximum number of distinct vectors we can have in a linearly independent subset of vector space V.

Let S be a spanning set for the vector space V which has n elements. No linearly independent set in V has more than n elements. Since, when we have a space in $\mathbb{R}^n$, we know n gives the spatial dimension which determines hw many bases vectors (in S) are available. Adding an $n+1$ vector will form a dependency in every dimension in the basis.

Relate the statement of Theorem 1.6.18 to a Venn Diagram of two overlapping circles. Also, explain the theorem in words.

We know $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2)$ where the redundant vectors are in $\dim(W_1 \cap W_2)$.

From the Unit 1 Homework:

Question: It is possible to define a finite basis of a vector space V as a subset $\{v_1,\ v_2, v_3, \ldots, v_n\}$ of V that is neither too big nor too small. Define "too big" and "too small" so that the above statement is accurate.

"Too big" means...

"Too small" means...

Unit 1 Homework Answer