Unit 3(b) Lecture Notes for MAT224

14-16 February 2023

§1 3.1 Determinant as Area

Let $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$ be two linearly independent vectors. What is the area of a 2-dimensional parallelogram that has $\vec{a}$ and $\vec{b}$ as adjacent sides?

The area of a 2-dimensional parallelogram that has $\vec{a}$ and $\vec{b}$ as adjacent sides is $|a_1 b_2 - a_2 b_1|$ using the determinant function.

What happens to the above forumla if we use vectors $\vec{a}$ and $\vec{b}$ that are linearly dependent?

If use vectors $\vec{a}$ and $\vec{b}$ that are linearly dependent, $\vec{a}$ and $\vec{b}$ would be scalar multiples of each other and would thereby always produce a parralelogram of area 0.

The "area" function $A(\vec{x}, \vec{y})$ is the function that takes in two vectors $\vec{x}, \vec{y} \in \mathbb{R}^2$ and outputes the area of a parallelogram with adjacent sides equal to $\vec{x}$ and $\vec{y}$. What are three of its properties?

The three properties of the area function is multilinearity, alternating, and normalization.

Do any other functions have all three properties?
(Answer this question after reading up to and including Theorem 3.2.8.)

The determinant function has all three properties.

If you have another function that multilinear (Definition 3.2.1) and alternating (Definition 3.2.2), how can you tell if it is the area function or not?

You can verify it using the definition of the 2×2 determinant.

§2 3.2 Determinant of an nxn Matrix

What does it mean for a function to be multilinear? How is this different from the definition of linear that we have been using up until now?

A function f of the rows of a matrix A are called multilinear if f is a linear function of each of its rows when the remaining rows are held fixed.

What does alternating mean?

A function f of the rows of a matrix A is said to be alternating if whenever any two rows of A are interchanged f changes sign.

Use Definition 3.2.4 and Proposition 3.2.5 to write down step-by-step instructions for calculating the determinant of a 4x4 matrix. Include an example with a matrix that has at least 3 zeros in the row or column along which you are expanding.

We calculate the determinant of a 4×4 matrix using the formula

$$\det A = \sum_{j=1}^{4} (-1)^{i-j} a_{ij} \det (A_i j)$$

. When we combine Proposition 3.1.6 and Theorems 3.2.10, 3.2.13 and 3.2.14, what do we know about invertibility and determinants?

A matrix is said to be inveritible if its determinant is non-zero.

§3 3.3 Further Properties of the Determinant

List four properties of determinants of matrices.

If A and B are $n \times n$ matrices.

- $\det(AB) = \det(A)\det(B)$
- If A is invertible, then $\det(A^-1) = 1/\det(A)$.
- Multilinearity
- Alternating

Let T be a linear transformation. Which theorem allows us to talk about "the determinant of a linear transformation" $det(T)$ instead of $det([T]_\alpha^\alpha)$ for different bases α? Explain.

Theorem (3.3.8) allows to talk about "the determinant of a linear transformation" $det(T)$ instead of $det([T]_\alpha^\alpha)$ for different bases α since there exists exactly one alternating multilinear function that allows for such simplification.

How can the determinant function $det(T)$ help us decide if a linear transformation T is an isomorphism? Explain why your answer works.

Since we can use the determinant function $det(T)$ to decide if it a linear transformation, such results can be used with conjunction with results from Chapter 2 to conclude if T is an isomorphism.

Give one example for each of part (i) and part (ii) of Proposition 3.3.12 that demonstrates those properties are true. A matrix $S = T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies the conditions.

What calculation are we doing if we can we use Cramer's Rule to do it? We are determining if a function is invertible in the form $Ax = b$.