Unit 1(a) Lecture Notes for MAT224

January 10-12, 2023

These notes are intended for study purposes. You should be able to fill in these blanks from the notes you take during lecture and/or the textbook. You are welcome to use them to work ahead. Your completed copy of these notes should be submitted to the Quercus assignment called "Unit 1(a) Lecture Notes" by April 7, 2023. You can scan your handwritten answers or you can type them out. See the Unit 1 Homework for details.

§1 Vector Spaces 1.1

What is a vector space?

V is a vector space if and only if it is a set V (whose elements are called vectors) together with

(a) an operation called vector addition, which for each pair of vectors $\mathbf{x}, \mathbf{y} \in V$ produces another vector V denoted $\mathbf{x} + \mathbf{y}$, and

(b) an operation called multiplication by a scalar (a real number), which for each vector $\mathbf{x} \in V$, and each scalar $c \in \mathbb{R}$ produces another vector in V denoted $c\mathbf{x}$.

Give three examples of vector spaces that are not $\mathbb{R}^2$, $\mathbb{R}^3$, or $\mathbb{R}^n$ for any $n \in \mathbb{N}$.

The set $P_n(\mathbb{R}) = \{p: \mathbb{R} \leftarrow \mathbb{R} \mid p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0,$ where the $a_i \in \mathbb{R}\}$ is a vector space with usual vector addition and scalar multiplication.

The set $M_{m\times n} = \{\text{set of } m \times n \text{ matrices}\}$ is a vector space with matrix addition and scalar multiplication.

The set $R^+ = \{x \in \mathbb{R} \mid x > 0\}$ is a vector space with specific vector addition and scalar multiplication.

§2 Subspaces 1.2

What is the subspace criterion? What does it say?

Let V be a vector space and $W \in V$ be a subset. W is a subspace of V if W itself is a vector space closed under vector addition and scalar multiplication.

What is another definition of a subspace that is NOT the subspace criterion?

A set W is a subspace of a vector space W if and only if $\forall x, y \in W$ and $\forall c \in \mathbb{R}$, $cx + y \in \mathbb{R}$.

Give an example of a subspace of $M_{2\times 2}(\mathbb{R})$. Prove it is a subspace.

An example of a subspace of $M_{2\times 2}(\mathbb{R})$ is $W = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} a + d = 0, b - 2c = 0\}$. To prove it is a subspace, we need to prove that it is closed under matrix addition and scalar multiplication, i.e. there exists $a \in \mathbb{R}$ such that $cx + y \in \mathbb{R}$ for $x, y \in W$.

Give an example of a subset S of a vector space V that is not a subspace. Explain why it is not a subspace.

Let $V \in \mathbb{R}^3$, and $W = \{(x, y, z) \in \mathbb{R}^3 \; x + y + z + 2 = 0\}$. Then W is not a subspace of V, since it does not contain $\vec{0}$.

§3 Linear Independence 1.3

Definition of Linear Independence: The set of vectors $\{v_1, v_2, \ldots v_n\}$ is linearly independent if and only if ...

Whenever, we have $a_i \in \mathbb{R}$ and $x_i \in S$ such that $a_1x_1 + a_2x_2 + \ldots + a_ix_i = 0$ for all i.

Definition of Linear Dependence: The set of vectors $\{v_1, v_2, \ldots v_n\}$ is linearly dependent if and only if ...

Whenever, we have $a_i \in \mathbb{R}$ and $x_i \in S$ such that $a_1x_1 + a_2x_2 + \ldots + a_ix_i = 0$ for all i except at least one i.

Can a set of only one vector be linearly dependent? Why or why not?

Yes, if $\vec{v} \in V = 0$ where $V = \{0\}$ then it may be dependent since there are infinite non-trivial solutions.

Can a set of 5 vectors in $\mathbb{R}^4$ be linearly independent? Let $B = \{v_1, v_2, v_3, v_4, v_5\}$, $v_i \in \mathbb{R}^4$, $i = 1, 2, \ldots 5$. Then B ...

No it will be dependant, since there will always be a redundant vector.

Give an example of a proof that a set is linearly independent.

Proof. Given the set $S = \{0, 1\}$, we can show that S is linearly independent since $a(0) + b(1) = 0$ and $b(1) = 0$. Thus there are no non-trivial solutions and S is linearly independent. □

Give an example of a proof that a set is linearly dependent.

Proof. Given the set $S = \{0\}$, $\alpha(0) = 0$ where $\alpha \in \mathbb{R}$. There are infinite non-trivial solutions and S is linearly independant. □

Which of the above two examples is also an example of disproving that a set is linearly independent? Why?

We can prove by contradiction a set is linearly independent by showing that it is not linearly dependent.