Unit 6a Lecture Notes for MAT224

§1 6.3 Jordan Canonical Form

Definition (6.3.2) Let $T : V \to V$ be a linear mapping on a finite-dimensional vector space V. Let λ be an eigenvalue of T with algebraic multiplicity m.

(a) The **λ-generalized eigenspace**, written K_λ, is the kernel of the mapping $(T - \lambda I)^m$ on V.

(b) The non-zero elements of K_λ are called **generalized eigenvectors** of T.

What is the dimension of the generalized eigenspace K_2 with respect to the standard basis $\mathcal{E}$, where $[T]_{\mathcal{E}}^{\mathcal{E}}$ is the following matrix?

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Since 2 is an eigenvalue of multiplicity 3, then $\dim(K_2) = 3$.

Consider a vector v with the property that $Tv = 2v$. This implies $Tv - 2v = 0$. So $(T - 2I)v =$ means that v is an eigenvector for $= 2$.
Now consider vector w such that $(T - 2I)^3 w = 0$. We can see that 3 is the multiplicity of $(2 - \lambda)$ in the characteristic polynomial of T. In other words, 3 is the algebraic multiplicity of $\lambda = 2$.

If $(T - 2I)^3 w = 0$, then $(T - 2I)(T - 2I)^2 w = 0$. So $(T - 2I)^2 w$ is an eigenvector.

Also, $(T - 2I)^2 (T - 2I) w = 0$. So $u = (T - 2I)w$ satisfies the equation:

$$(T - 2I)^2 u = 0$$

Is it possible to write a cycle of 3 vectors $\{v, (T - 2I)v, (T - 2I)^2 v\}$ as a basis for the K_2 from above?

It is not possible to write a cycle of 3 vectors $\{v, (T - 2I)v, (T - 2I)^2 v\}$ as a basis for the K_2 from above, since $\lambda = 2$ has a multiplicity of 3.

§2 6.4 Computing Jordan Form

What is Jordan Canonical Form and how can you figure out what it will look like for a transformation T without calculating a canonical basis?

We will look at Example 6.4.3 together. Then you will write the Jordan canonical form for the transformation in Example 6.4.4.

Example 6.4.3 Let $\beta = \{e^x, xe^x, x^2e^x, x^3e^x, e^{-x}, xe^{-x}\}$ and let $V = \{\beta\}$. Let $T : V \to V$ be defined by $T(f) = 2f + f''$. In order to find the Jordan canonical of T, we will first write $[T]_\beta^\beta$. For clarities sake, I will rename it M. So $M = [T]_\beta^\beta$. Then

$$M = \begin{bmatrix} 3 & 2 & 2 & 0 & 0 & 0 \\ 0 & 3 & 4 & 6 & 0 & 0 \\ 0 & 0 & 3 & 6 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Finding the Jordan canonical form (JCF) of a transformation requires similar calculations to diagonalizing a diagonalizable matrix. We take the characteristic polynomial and find that it is $(3-\lambda)^6$. If the dimension of the eigenspace for $\lambda = 3$ was also 6, it would be diagonalizable.

However, $\dim(E_3) = 2$. We can see this by looking at

$$(M - 3I)v = 0 \implies \begin{bmatrix} 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 6 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_3 \\ 4x_3 + 6x_4 \\ 6x_4 \\ 0 \\ -2x_6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So x_1 and x_5 can be anything. Therefore the matrix representation of the vectors spanning E_3 are $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. The actual basis elements are $\beta_1 = e^x$ and $\beta_5 = e^{-x}$.

This show us that the set of solutions v to the equation $(M - 3I)v = 0$ (the set of eigenvectors) is a 2-dimensional space.

Another way to say this is:

$\dim(\ker(M - 3I)) = 2$.

We can also calculate:
$\dim(\ker(M - 3I)^2)$
$\dim(\ker(M - 3I)^3)$
$\vdots$
$\dim(\ker(M - 3I)^6)$ because 6 is the algebraic multiplicity of $\lambda = 3$ (see the characteristic polynomial).

But it turns out we don't even have to look that far, since $\dim(\ker(M - 3I)^4) = 6$. So the kernel of $(M - 3I)^4$ is all of V.

Notice how $(M - 3I)v = 0$ implies that $((M - 3I)^2)v = (M - 3I)0 = 0$

Similarly, all solutions to $((M - 3I)^i)v = 0$ are solutions to $((M - 3I)^{i+1})v = 0$.

That is why we look at the values of $\dim(\ker(M - 3I)^{i+1}) - \dim(\ker(M - 3I)^i)$.
We are looking to see how many basis elements of $\ker((M - 3I)^{i+1})$ are not also elements of $\ker((M - 3I)^i)$.

Consider a vector that is in $\ker(M - 3I)^4)$, but not $\ker(M - 3I)^3$. Let's call it w. Then $(M - 3I)w$, $((M - 3I)^2)w$, and $((M - 3I)^3)w$ are all non-zero and linearly independent from each other (by theorem).

Let $S_1 = \{w, (M - 3I)w, ((M - 3I)^2)w, ((M - 3I)^3)w\}$. Then S_1 is a basis for a 4-dimensional $(M - 3I)$-invariant subspace. It is also a cycle of length 4.

But $\dim(\ker((M - 3I)^2)) = 4$. Compare that to the fact that $\dim(\ker(M - 3I) = 2$. So there are 4-2 = 2 extra basis elements of $\ker(M - 3I)^2$ than there are basis elements of E_3. This means we can create two non-overlapping cycles from basis elements of $\ker(M - 3I)^2$ that are not eigenvectors.

This information can be summarized with this tableau for eigenvalue $\lambda = 3$:

diag1.png

The number of squares in the first column is equal to the dimension of E_3, which is 2.

The number of squares in the second column is equal to

$\dim(\ker(M - 3I)^2) - \dim(\ker(M - 3I)) = 4 - 2 = 2$.

The number of squares in the third column is equal to
$\dim(\ker(M - 3I)^3) - \dim(\ker(M - 3I)^2) = 5 - 4 = 1$

The number of squares in the last column is equal to
$\dim(\ker(M - 3I)^4) - \dim(\ker(M - 3I)^3) = 6 - 5 = 1$

From this we can see that there are two non-overlapping cycles that general $(M - 3I)$-invariant subspaces, one of length four corresponding to the top row, and one of length two corresponding to the second (bottom) row.

So $M - 3I$ is a nilpotent matrix with zeros on the diagonal and two Jordan blocks, one that is 4×4 and the other is 2×2.

Instead of writing the matrix for $M - 3I$, we will add the 3's back to the main diagonal and write the Jordan canonical form of T:

$$\begin{bmatrix} 3 & 1 & 0 & 0 & & \\ 0 & 3 & 1 & 0 & & \\ 0 & 0 & 3 & 1 & & \\ 0 & 0 & 0 & 3 & & \\ & & & & 3 & 1 \\ & & & & 0 & 3 \end{bmatrix}$$

Now you try! Below is the matrix featured in Example 6.4.4. Find it's Jordan canoncial form by creating tableau diagrams for each of its eigenvalues. You can check your answer against the textbook's on pages 285 and 286.

$$A = \begin{bmatrix} 1-i & 1 & 0 & 0 & 0 \\ 0 & 1-i & 0 & 0 & 0 \\ i & -1 & 1 & 0 & 0 \\ 0 & i & 1 & 1 & 1 \\ -i & 1 & 0 & 0 & 1 \end{bmatrix}$$

Hint: The characteristic polynomial of A is $(1 - \lambda)^3((1 - i) - \lambda)^2$. Therefore, you should do two cycle tableaus: one for $\lambda = 1$ and the other for $\lambda = 1 - i$.

$$C = \begin{bmatrix} 1-i & 1 & & & \\ 0 & 1-i & & & \\ & & 1 & 1 & \\ & & 0 & 1 & \\ & & & & 1 \end{bmatrix}$$