Unit 2(b) Lecture Notes for MAT224

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§1 2.2 Linear Transformations Between Finite Dimensional Vector Spaces Continued...

We know that the product $[T]_\alpha^\beta e_i = [T(\alpha_i)]_\beta$ where α_i is the i^{th} basis element in α and the subscript β means "as a linear combination of elements in basis β".

Note that e_i is the vector with 1 in the i^{th} entry and zero everywhere else.

Now write $\vec{v}$ as a linear combination of standard basis elements $e_1, e_2, \ldots, e_n$.

Explain why the product $[T]_\alpha^\beta[\vec{v}]$ is equal to a linear combination of vectors in basis β. What do the e_i's represent here?

Proof. Let $v = x_1v_1 + \ldots + x_kv_k \in V$. Then if $T(v_j) = a_{1j}w_1 + \ldots + a_{lj}w_l$

$$T(v) = \sum_{i=1}^{l}\left(\sum_{j=1}^{k} x_j a_{ij}\right) w_j.$$

Thus, the ith coefficient of $T(v)$ in terms of β is $\sum_{j=1}^{k} x_j a_{ij}$ and

$$[T(v)]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

□

§2 2.3 Kernel and Image

Define the set Ker(T) for a linear transformation $T : V \to W$.
The kernel of T, denoted Ker(T), is the subset of V consisting of all vectors $v \in V$ such that $T(v) = 0$.

Prove that Ker(T) is a subspace of V. Notice that elements in Ker(T) are in V by definition.

Proof. Since Ker(T) $\subset V$ it suffices to prove that Ker(T) is closed under addition and scalar multiplication. Since T is linear, for all u and $v \in$ Ker(T) and $v \in$ Ker(T) and $a \in \mathbb{R}$. $T(u + av) = T(u) + aT(v) = 0 + a0 = 0$ that is $u + av \in$ Ker(T). □

Define the set Image(T) for a linear transformation $T : V \to W$.
The subset of W consisting of all vectors $w \in W$ for which there exists a $v \in V$ such that $T(v) = w$ is called the image of T and is denoted by $\mathrm{Im}(T)$.

Prove that Image(T) is a subspace of W. Notice that elements in Image(T) are in W by definition.

Proof. Let w_1 and $w_2 \in \mathrm{Im}(T)$ and let $a \in \mathbb{R}$. Since w_1 and $w_2 \in \mathrm{Im}(T)$, there exists vectors v_1 and $v_2 \in V$ with $T(v_1) = w_1$ and $T(v_2) = w_2$. Then we have $aw_1 + w_2 \in aT(v_1) + T(v_2) = T(av_1 + v_2)$ since T is linear. Therefore $aw_1 + w_2 \in \mathrm{Im}(T)$ and $\mathrm{Im}(T)$ is a subspace of W. □

Is it possible to determine the pre-image of a vector $\vec{w} \in W$, if you know its image? In other words, let $T : V \to W$ be a linear transformation and let $\vec{w} \in T(W)$. So $T(v) = w$ for some $v \in V$. Can we tell which $v \in V$ is sent to W by V.

It is possible to determine which $v \in V$ is sent to W by V.

What are some of the things you can do with the matrix form of T (in other words $[T]_\alpha^\beta$) to learn about the properties of T?

The kernel and image of the matrix form of T can be used to determine the properties of T such as the dimension of α or β.

For a linear transformation $T : V \to W$, how can you use two of the following quantities to determine the third:

$$dim(Ker(T)) + dim(Im(T)) = dim(V)$$

Given the dimension of the kernel of T and the dimension of the image of T, we can determine the dimension of the subspace V by adding the previous two dimensions.

§3 2.4 Application of the Dimension Theorem

Give examples of deciding if a linear transformation $T : V \to W$ is injective or not based on:

- its kernel
- its image
- the relative dimensions of V and W
- the relative dimensions of Im(T) and W

Some examples are

- A linear transformation $T : V \to W$ is injective if and only if $\dim(\mathrm{Ker}(T)) = 0$.
- A linear mapping $T : V \to W$ on a finite-dimensional vector space V is injective if and only if $\dim(\mathrm{Im}(T)) = \dim(V)$.
- If $\dim(W)$ ¡ $\dim(V)$ and $T : V \to W$ is a linear mapping, then T is not injective.
- If V and W are finite dimensional, then a linear mapping $T : V \to W$ can be injective only if $\dim(W) \geq \dim(V)$.
- If W is finite-dimensional, then a linear mapping $T : V \to W$ is surjective if and only if $\dim(\mathrm{Im}(T)) = \dim(W)$.

Give the definition of "surjective" and explain how you can decide if a linear transformation T is surjective based on the relative dimensions of V and W.

The function f is said to be surjective if for each $q \in S_2$ there is some $p \in S_1$ with $f(p) = q$. A linear mapping $T : V \to W$ can be surjective if $\dim(V) \geq \dim(W)$.

Let $T : V - - > W$ be a linear transformation where dim(V)=dim(W).

(i) If T is surjective, does it also have to be injective? Why or why not?
(ii) If T is injective, does it also have to be surjective? Why or why not?

Let $\dim(V) = \dim(W)$. A linear transformation $T : V \to W$ is injective if and only if it is surjective.

Consider Proposition 2.4.11. Let $\vec{w_0} \notin Ker(T)$ for some linear transformation $T : V \to W$. Then we know we can write every vector $\vec{v} \in V$ as $\vec{v} = \vec{w_0} + \vec{k}$ for some $\vec{k} \in Ker(T)$. If we replace w_0 with w_1 then does k stay the same? In other words, $\vec{v} = \vec{w_1} + \vec{k}$ for the same $\vec{k}$ as before or $\vec{v} = \vec{w_1} + \vec{k'}$ for a potentially different $\vec{k'} \in Ker(T)$?

It is not the same, $\vec{v} = \vec{w_1} + \vec{k'}$ for a potentially different $\vec{k'} \in Ker(T)$.