Unit 5(b) Lecture Notes for MAT224

§1 5.2 Vector Spaces over a Field

Give an example of a complex vector space V.

Let V be the set of all complex-valued functions of a real-variable, $f : \mathbb{R} \to \mathbb{C}$. That is, f assigns to each real number t a complex number $f(t)$.

Your vector space V from above is also a real vector space. Give an example of a basis for V as a real vector space (with scalars from $\mathbb{R}$) and another basis for V as a complex vector space (with scalars from $\mathbb{C}$).

Let W be the subset of V consisting of differentiable functions with scaled by $a \in \mathbb{R}$ and Q be another subset of V consisting of differentiable functions scaled by $b \in \mathbb{C}$.

In general, is there a relationship between $\dim_{\mathbb{R}}(V)$ and $\dim_{\mathbb{C}}(V)$? These are the size of a basis for V when linear combinations can use real scalars and complex scalars respectively.

In general, $\dim_{\mathbb{C}}(V) = 2\dim_{\mathbb{R}}(V)$. This result arises from the fact that there are n real parts and n complex parts.

§2 5.3 Geometry of a Complex Vector Space

What is the definition of a Hermitian Inner Product?

Let V be a complex vector space. A Hermitian Inner Product on V is a complex valued function on pairs of vectors in V, denoted by $\langle u, v\rangle \in \mathbb{C}$ for $u, v \in V$, which satisfies the following properties:

a) For all u, v and $w \in V$ and $a, b \in \mathbb{C}$, $\langle au + bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$

b) For all $u, v \in V$, $\langle u, v\rangle = \overline{\langle u, v\rangle}$, and

c) For all $v \in V$, $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0$ implies $v = 0$.

How can you use a Hermitian inner product to calculate the length of a vector v? Choose an Hermitian inner product and use it to calculate the length of $\begin{pmatrix} 1 \\ i \\ 2+3i \end{pmatrix}$.

We can find the length of a vector v using the Hermitian inner product by first calculating $\langle v, v\rangle$ and taking the square root of the result. Using the described method, we have the length of $\begin{pmatrix} 1 \\ i \\ 2+3i \end{pmatrix}$ to be $\sqrt{15}$.

What is another name for the conjugate transpose of a matrix?

Another name for the conjugate transpose of a matrix is the adjoint.

What is the adjoint M^* of matrix M if

$$M^* = \begin{bmatrix} 2-2i & 4-i & 0 \\ 7+2i & i & 8i \\ 8 & -5 & -4i \end{bmatrix}$$

We obtain

$$M = \begin{bmatrix} 2+2i & 7-2i & 8 \\ 4+4i & -i & -5 \\ 0 & -8i & 4i \end{bmatrix}$$

Proposition 5.3.8 says: Let V be a finite dimensional Hermitian inner product space. The adjoint of $T : V \to V$ satisfies $< T(v), w >=< v, T^*(w) >$ for all $v, w \in V$.

What does this proposition tell us about the matrix $[T]^\alpha_\alpha$ of a self-adjoint (Hermitian) linear transformation $T : V \to V$ with α any basis of V?

The proposition tells us that the matrix $[T]^\alpha_\alpha$ of a self-adjoint linear transformation $T : V \to V$ is the same for any α of V.

It turns out that, like symmetric transformations, self-adjoint transformations are diagonalizable (Theorem 5.3.12) and have real eigenvalues (Theorem 5.3.13). What does that tell you about the eigenvectors of a self-adjoint transformation?

By Theorem 5.3.12 and Theorem 5.3.13, we know that there is an orthonormal basis of V consisting of vectors for T and that each eigenvector are linearly independent and orthogonal to each other.

§3 6.1 Triangular Form

What is a T-invariant subspace? Give an example of a transformation T and a subspace W which is T-invariant.

Let $T : V \to V$ be a linear mapping. A subspace $W \subset V$ is said to be invariant T if $T(W) \subset W$. For example, let $T : V \to V$. As such we have that $W = \{0\}$ or V is invariant under T.

How is the upper-triangular form of a matrix helpful in identifying T-invariant subspaces? You can illustrate with an example, rather than explain in words.

We know that $[T]_\beta^\beta$ is upper-triangular if and only if $T(W_i) \subset W_i$ for each of the subspaces W_i. Furthermore, if $T(W_i) \subset W_i$ for each of the subspaces W_i, we have W_i is a T-invariant subspace.

How can you tell from the characteristic polynomial of linear transformation $T : V \to V$, whether or not T is triangularizable?

We have that T is triangularizable if and only if the characteristic polynomial of T has $\dim(V)$ roots (counted with multiplicities) in the field F.

The Cayley-Hamilton Theorem (6.1.12) says: Let $T : V \to V$ be a linear mapping on a finite-dimensional vector space V, and let $p(t) = det(T - tI)$ be its characteristic polynomial. Assume that $p(t)$ has $\dim(V)$ roots in the field F over which V is defined. Then $p(T) = 0$ (the zero mapping on V).

What does the theorem say about transformation T, which is defined by

$$T\vec{v} = \begin{bmatrix} 2i & 1 \\ 0 & -2i \end{bmatrix} \vec{v}$$

We have $p(t) = \det(T - tI = t^2 + 4$. Using this, we have

$$A^{-1} = \frac{1}{4} \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

§4 6.2 A Canonical Form for Nilpotent Mappings

Recall: A **nilpotent** linear transformation N is a transformation such that N^k is the zero transformation for some $k \in \mathbb{Z}$, $k \geq 1$.

What do we know about the span of a cycle (see 6.2.1) defined by a nilpotent transformation (as opposed to a transformation that is not nilpotent)?

Let N be a nilpotent mapping, let $C(x)$ be the cyclic subspace generated by some $x \in V$, and let α be the cycle generated by x, viewed as a basis for $C(x)$. Since $N(N^i(x)) = N^{i+1}(x)$ for all i, we see that in $[N|_{C(x)}]_\alpha^\alpha$ all the entries except the immediately above the diagonal are 0, and the entries above the diagonal are all 1's. Thus, the matrix of N restricted to a cyclic subspace is not only upper- triangular, but an upper-triangular matrix of a very special form.

Let $N = (T - \lambda I)$ for some eigenvalue λ of linear transformation T. How do you create a tableau to depict the cycles of T. Include an example. Section 6.2 contains many of them. You can use one of those here.

Note: This section merely states that a canonical basis exists without showing you how to find one. For now, you do not have to concern yourself with finding x to generate the cycles. You simply need to know that x exists.

If the cycle tableau corresponding to a canonical basis for N is (see 6.2.10), then $\dim(\mathrm{Ker}(N)) = 3, \dim(\mathrm{Ker}(N^2)) = 5, \dim(\mathrm{Ker}(N^3)) = 7$, and $\dim(\mathrm{Ker}(N^4)) = 8$.