Unit 3(a) Lecture Notes for MAT224

7-9 February 2023

§1 2.5 Composition of Linear Transformations

Let $S : U \to V$ and $T : V \to W$ be linear transformations. Prove that TS is also a linear transformation. What is its domain? What is its codomain?

Note that TS could also have been written $T \circ S$.
$TS\vec{v}$ and $(T \circ S)(\vec{v})$ have the same meaning.

Proof. Let a and $b \in \mathbb{R}$ and $u_1, u_2 \in U$. We wish to show that TS satisfies

$$\begin{aligned} TS(au_1 + bu_2) &= T(S(au_1 + bu_2)) \\ &= T(aS(u_1) + bS(u_2)) \\ &= aT(S(u_1))_b T(S(u_2)) \\ &= aTS(u_1) + bTS(u_2) \end{aligned}$$

□

Consider linear transformations $S : U \to V$ and $T : V \to W$.
We say that $A \subset B$ if $\forall a \in A$ we can show that $a \in B$ as well.

(Proposition 2.5.6) Prove that:
(i) $Ker(S) \subset Ker(TS)$

Proof. If $u \in$ Ker(S), $S(u) = 0$. Then $TS(u) = T(0) = 0$. Therefore $u \in \text{Ker}(TS)$ and we have proven (i). □

(ii) $Im(TS) \subset Im(T)$

Proof. If $w \in \text{Im}(TS)$ and $v \in V$, $TS(v) = w$. Then $T(v) = w$. Therefore $w \in \text{Im}(T)$ and we have proven (ii). □

Why does it not make sense to compare $Ker(T)$ and $Ker(TS)$?

It does it not make sense to compare $Ker(T)$ and $Ker(TS)$ since if $u \in Ker(S)$ but $S(u) \notin Ker(T)$ then it will not result to 0 as desired.

Similarly, why does it not make sense to compare $Im(S)$ and $Im(TS)$?

It does not make sense to compare $Im(S)$ and $Im(TS)$ since if the image of S is not in the domain of T, then it will not output any value.

If W is a subspace of V, what do we know about the dimensions of W and V? How does this relate to Corollary 2.5.7?

Corollary 2.5.7 says that:

(i) $dim(Ker(S)) \leq dim(Ker(TS))$

(ii) $dim(Im(TS)) \leq dim(Im(T))$

If W is a subspace of V, we know $\dim(W) \leq \dim(V)$. By (i), since $\mathrm{Ker}(T) \subset V$, we have $\dim \mathrm{Ker}(T) \leq \dim(V)$ and since $\mathrm{Ker}(TS)$ is a subspace of W which is a subspace of V, we have $\dim \mathrm{Ker}(T) \leq \dim \mathrm{Ker}.(TS)$. The reverse is true for the image.

Consider the matrices $[T]_\alpha^\beta$, $[S]_\gamma^\delta$, and $[TS]_\epsilon^\zeta$. Which of bases α, β, γ, δ, ϵ, and ζ have to be equal to each other in order for the following equation to be true:

$$[T]_\alpha^\beta [S]_\gamma^\delta = [TS]_\epsilon^\zeta$$

Explain your answer.

The bases β, γ, and ϵ must be equal and δ must be equal to ζ for the following equation to be true since it is the product of the matrices of the transformation.

In what ways are Propositions 2.5.4 and 2.5.14 similar? How are they different? Propositions 2.5.4 and 2.5.14 as both satisfy the following basic properties of associativity and distributivity. However they differ in the sense that one proposition discusses the properties in terms of matrix multiplication and the other linear transformations.

§2 2.6 Inverses of a Linear Transformation and Isomorphisms

Consider linear transformation $T : V \to W$, where V and W are finite dimensional vector spaces.

What does it mean for T to be:

(i) injective

(ii) surjective

(iii) invertible

Also, how are these concepts related?

A function between sets $f : S_1 \to S_2$ is said to be injective if whenever $f(p_1) = f(p_2)$ for $p_1, p_2 \in S_1$, we have $p_1 = p_2$. A function f is said to be surjective if for each $q \in S_2$ there is some $p \in S_1$, with $f(p) = q$. If $f : S_1 \to S_2$ is a function from one set to another, we say that $g : S_2 \to S_1$ is the inverse function of f if for every $x \in S_1$, $g(f(x)) = x$ for every $y \in S_2$, $f(g(y)) = y$. If $T : V \to W$ is injective and surjective, then the inverse function $S : W \to V$is a linear transformation.

Show that the inverse transformation of a bijection is also a linear transformation. In other words, if T is a bijective linear transformation, show that T^{-1} is also linear.

Proof. Let w_1 and $w_2 \in W$ and $S(w_1) = v_1$ and $S(w_2) = v_2$ are the unique vectors v and v_2 satisfying $T(v_1) = w_1$ and $T(v_2) = w_2$. By definition, $S(aw_1 + bw_2)$ is the unique vector v with $T(v) = aw_1 + bw_2$, but $v = av_1 + bv_2$ satisfies $T(av_1 + bv_2) = aT(v_1) + bT(v_2) = aw_1 + bw_2$. Thus, $S(aw_1 + bw_2) = av_1 + bv_2 = aS(w_1) + bS(w_2)$ as we desired. □

Two vector spaces V and W are said to be isomophic if we can define a bijection between them. In this situation, we call the bijection and "isomorphism". Define an isomorphism between the vector space of polynomials of degree at most 4, $P_3(\mathbb{R}$ and the vector space of 2×2 matrices $M_{2\times2}(\mathbb{R})$. An isomorphism between the vector space of polynomials of degree at most 4, $P_3(\mathbb{R}$ and the vector space of 2×2 matrices $M_{2\times2}(\mathbb{R})$ is $T(a + bx + cx^2 + dx^3) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

What must be true of the dimensions of two vectors space V and W if they are isomorphic to each other? It must be true that $\dim(V) = \dim(W)$.

Proposition 2.6.11 States that if $T : V \to W$ is an isomorphism between finite-dimensional vector spaces, then for any choice of bases α for V and β for W,

$$[T^{-1}]_{\beta}^{\alpha} = \left([T]_{\alpha}^{\beta}\right)^{-1}$$

Confirm that this is true when T is the derivative, $V = \{ax^2 + bx \mid a, b \in \mathbb{R}\}$ with basis $\{x, x^2\}$, and $W = \{mx + b \mid m, b \in \mathbb{R}\}$ with basis $\{1, x\}$.
Note that this was confirmed on paper, but was not transferred online.

Why is it okay to talk about T^{-1} here, even though the derivative in general does not have an inverse function?

It is okay to talk about T^{-1} here as the domain of V and W are restricted so that an inverse function could be found.

§3 2.7 Change of Basis

What is a change of basis matrix? Give an example, including the two bases.

The change of basis matrix $[T]_\alpha^\beta$ translates a vector v's representation in one basis α to another basis β.

Consider $[I]_\alpha^\beta$ and vector $[v]_\alpha$. Show why the product $[I]_\alpha^\beta[v]_\alpha$ produces $[v]_\beta$, the vector whose elements are the coefficients when we write vector v as a linear combination of elements in basis β.

By Proposition (2.2.15), $[I]_\alpha^\beta[v]_\alpha$ is the coordinate vector of $T(v)$ in the basis β. But since $I(v) = v$, the result follows immediately.

What has to be true of linear transformation T in order for $[T]_\alpha^\beta$ to equal the identity $n \times n$ matrix which looks like:

$$I_{n\times n}\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

For the linear transformation $[T]_\alpha^\beta$ to equal the identity $n \times n$ matrix, it must be true that T is a bijection and that α and β are related by the identity $n \times n$ matrix.

Let $T : V \to W$ be a linear transformation between finite dimensional vector spaces V with bases α and α' and W with bases β and β'. Let $I_V : V \to V$ and $I_W : W \to W$ be the respective identity transformations.

Theorem 2.7.5 tells us that

$$[T]_{\alpha'}^{\beta'} = [I_W]_\beta^{\beta'} \cdot [T]_\alpha^\beta \cdot [I_V]_{\alpha'}^\alpha$$

Explain which transformation each of those matrices represents. Also, explain why the result of

$$[I_W]_\beta^{\beta'}[T]_\alpha^\beta[I_V]_{\alpha'}^\alpha[v]_{\alpha'}$$

is written in terms of basis elements of β'.

(i) In the above, $I_v : V \to V$ and $I_w : W \to W$ are the respective identity transformations of V and W. Also, α and α' are bases of V while β and β' are bases of W.

(ii) The solution is written in terms of the result of the matrix product.

Give an example two bases α and α' for $P_2(\mathbb{R})$ and a polynomial $v \in P_2(\mathbb{R})$ so that $[v]_\alpha$ and $[v]_{\alpha'}$ have different coordinates.

Also find the change of basis matrix $[I]_\alpha^{\alpha'}$ so that

$$[I]_\alpha^{\alpha'}[v]_\alpha = [v]_{\alpha'}$$

Let $\alpha = \{1, x, x^2\}$ and $\alpha = \{1 + x, 1 - x, x^2\}$ be two bases of $P_2(\mathbb{R})$. As such we have

$$[I]_\alpha^{\alpha'} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & -0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$