

LECTURE 1: Probability models and axioms

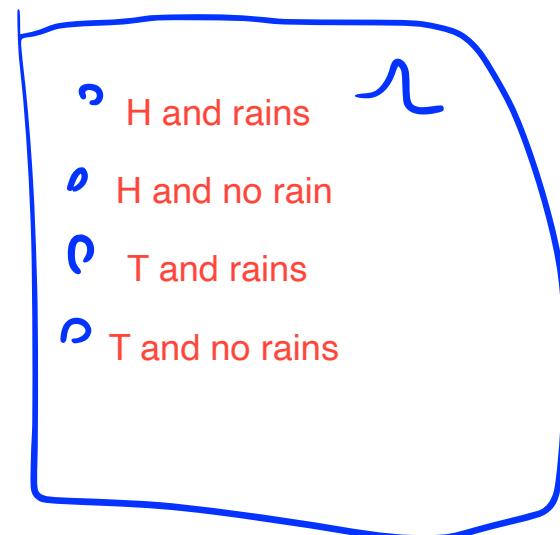
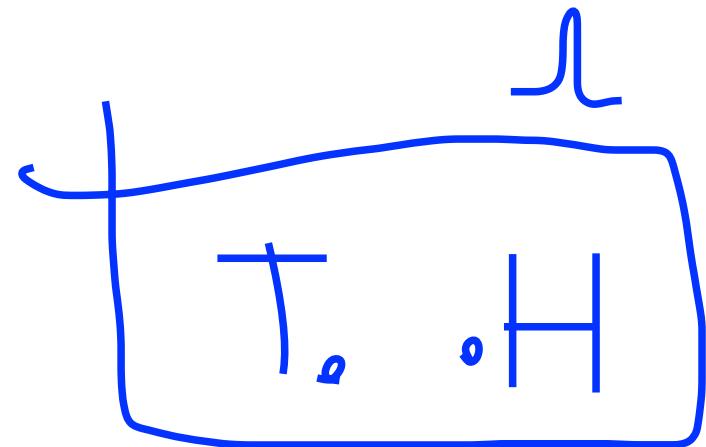
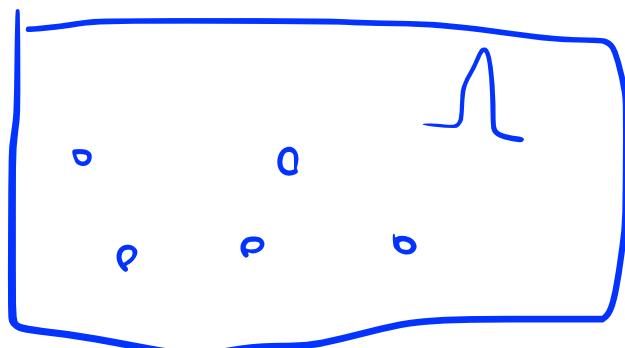
- Sample space
- Probability laws
 - Axioms
 - Properties that follow from the axioms
- Examples
 - Discrete
 - Continuous
- Discussion
 - Countable additivity
 - Mathematical subtleties
- Interpretations of probabilities

Sample space

- Two steps:
 - Describe possible outcomes
 - Describe beliefs about likelihood of outcomes

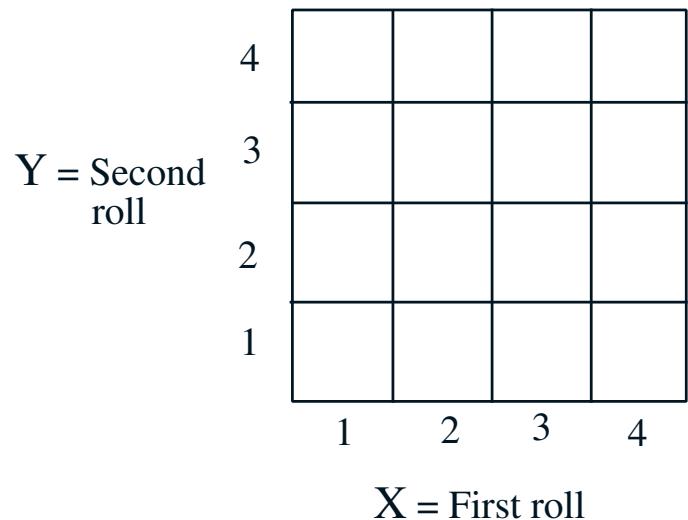
Sample space

- List (set) of possible outcomes, Ω
- List must be:
 - Mutually exclusive
 - Collectively exhaustive
 - At the “right” granularity

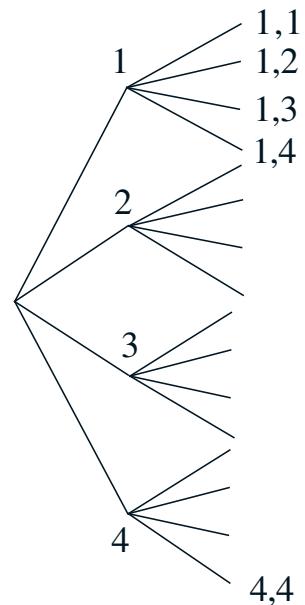


Sample space: discrete/finite example

- Two rolls of a tetrahedral die

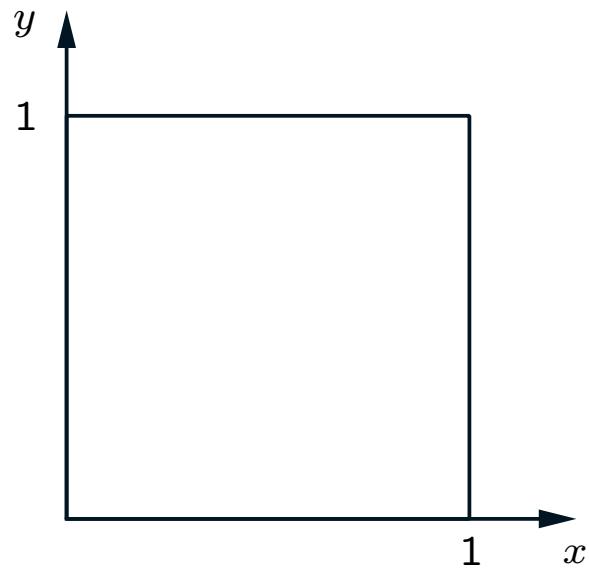


sequential description



Sample space: continuous example

- (x, y) such that $0 \leq x, y \leq 1$



Probability axioms

- **Event:** a subset of the sample space
 - Probability is assigned to events
- **Axioms:**
 - Nonnegativity: $P(A) \geq 0$
 - Normalization: $P(\Omega) = 1$
 - (Finite) additivity: (to be strengthened later)
If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$

Some simple consequences of the axioms

Axioms

$$P(A) \geq 0$$

$$P(\Omega) = 1$$

For disjoint events:

$$P(A \cup B) = P(A) + P(B)$$

Consequences

$$P(A) \leq 1$$

$$P(\emptyset) = 0$$

$$P(A) + P(A^c) = 1$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

and similarly for k disjoint events

$$P(\{s_1, s_2, \dots, s_k\}) = P(\{s_1\}) + \dots + P(\{s_k\})$$

$$= P(s_1) + \dots + P(s_k)$$

Some simple consequences of the axioms

Axioms

$$P(A) \geq 0$$

$$P(\Omega) = 1$$

For disjoint events:

$$P(A \cup B) = P(A) + P(B)$$

Some simple consequences of the axioms

- A, B, C disjoint: $\mathbf{P}(A \cup B \cup C) = \mathbf{P}(A) + \mathbf{P}(B) + \mathbf{P}(C)$
- $\mathbf{P}(\{s_1, s_2, \dots, s_k\}) =$

More consequences of the axioms

- If $A \subset B$, then $P(A) \leq P(B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cup B) \leq P(A) + P(B)$

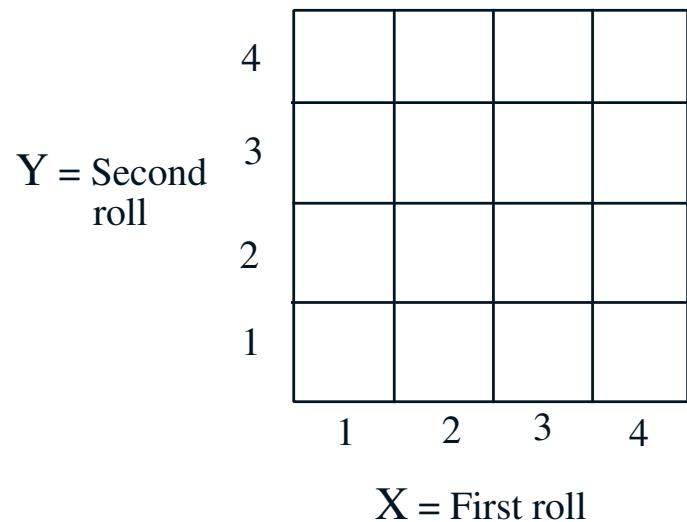
More consequences of the axioms

- $P(A \cup B \cup C) = P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C)$

Probability calculation: discrete/finite example

- Two rolls of a tetrahedral die
- Let every possible outcome have probability $1/16$

- $P(X = 1) =$



Let $Z = \min(X, Y)$

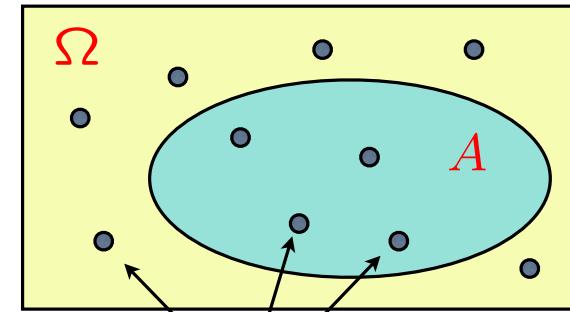
- $P(Z = 4) =$

- $P(Z = 2) =$

Discrete uniform law

- Assume Ω consists of n equally likely elements
- Assume A consists of k elements

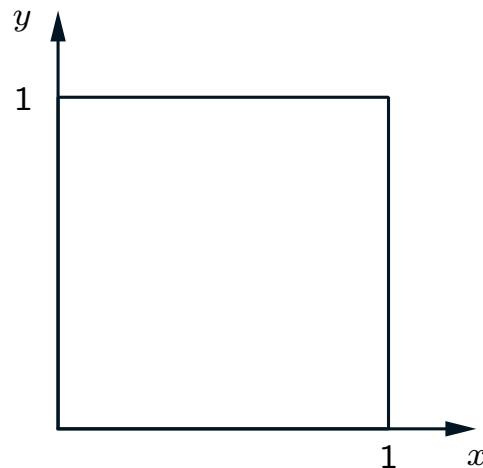
$$P(A) =$$



$$\text{prob} = \frac{1}{n}$$

Probability calculation: continuous example

- (x, y) such that $0 \leq x, y \leq 1$
- Uniform probability law: Probability = Area



$$P\left(\{(x, y) \mid x + y \leq 1/2\}\right) =$$

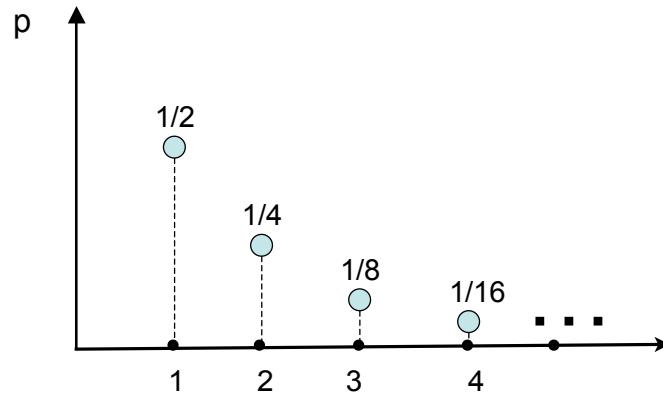
$$P\left(\{(0.5, 0.3)\}\right) =$$

Probability calculation steps

- Specify the sample space
- Specify a probability law
- Identify an event of interest
- Calculate...

Probability calculation: discrete but infinite sample space

- Sample space: $\{1, 2, \dots\}$
 - We are given $P(n) = \frac{1}{2^n}$, $n = 1, 2, \dots$



- $P(\text{outcome is even}) =$

Countable additivity axiom

- Strengthens the finite additivity axiom

Countable Additivity Axiom:

If A_1, A_2, A_3, \dots is an infinite **sequence** of **disjoint** events,
then $P(A_1 \cup A_2 \cup A_3 \cup \dots) = P(A_1) + P(A_2) + P(A_3) + \dots$

Mathematical subtleties

Countable Additivity Axiom:

If A_1, A_2, A_3, \dots is an infinite **sequence** of **disjoint** events,
then $\mathbf{P}(A_1 \cup A_2 \cup A_3 \cup \dots) = \mathbf{P}(A_1) + \mathbf{P}(A_2) + \mathbf{P}(A_3) + \dots$

- Additivity holds only for “countable” sequences of events
- The unit square (similarly, the real line, etc.) is **not countable** (its elements cannot be arranged in a sequence)
- “Area” is a legitimate probability law on the unit square, as long as we do not try to assign probabilities/areas to “very strange” sets

Interpretations of probability theory

- A narrow view: a branch of math
 - Axioms \Rightarrow theorems **“Thm:”** “Frequency” of event A “is” $P(A)$
- Are probabilities frequencies?
 - $P(\text{coin toss yields heads}) = 1/2$
 - $P(\text{the president of ... will be reelected}) = 0.7$
- Probabilities are often interpreted as:
 - Description of beliefs
 - Betting preferences

The role of probability theory

- A framework for analyzing phenomena with uncertain outcomes
 - Rules for consistent reasoning
 - Used for predictions and decisions

