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Quantum $\mathcal{N}=3$, $d=3$ Chern-Simons matter theories in harmonic superspace

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Abstract: We develop the background field method for studying classical and quantum aspects of $\mathcal{N}=3$, $d=3$ Chern-Simons and matter theories in $\mathcal{N}=3$ harmonic superspace. As one of the immediate consequences, we prove a nonrenormalization theorem implying the ultra-violet finiteness of the corresponding supergraph perturbation theory. We also derive the general hypermultiplet and gauge superfield propagators in a Chern-Simons background. The leading supergraphs with two and four external lines are evaluated. In contrast to the non-supersymmetric theory, the leading quantum correction to the massive charged hypermultiplet proves to be the super Yang-Mills action rather than the Chern-Simons one. The hypermultiplet mass is induced by a constant triplet of central charges in the $\mathcal{N}=3$, $d=3$ Poincare superalgebra.

Keywords: Extended Supersymmetry, Superspaces, Supersymmetric Gauge Theory, Chern-Simons Theories.

Contents

1. Introduction 1

2. Field models in $N=3$, $d=3$ harmonic superspace 3
   - 2.1 Gauge theory in standard $N=3$ superspace 3
   - 2.2 Gauge theory in $N=3$ harmonic superspace 3
   - 2.3 $N=3$ super Yang-Mills and Chern-Simons models 6
   - 2.4 $N=3$ hypermultiplets 7

3. Background field quantization 9
   - 3.1 The background field method for $N=3$ Chern-Simons theory 9
   - 3.2 Adding hypermultiplets 13
   - 3.3 Gauge and hypermultiplet propagators 14
   - 3.4 $N=3$, $d=3$ nonrenormalization theorem 16
   - 3.5 General structure of the on-shell effective action 17
   - 3.6 Effective action in the one-loop approximation 19

4. Examples of supergraph computations 19
   - 4.1 Hypermultiplet two-point function 19
   - 4.2 Gauge and ghost superfield two-point functions 21
   - 4.3 Vanishing of tadpoles and hypermultiplet self-energy 23

5. $N=3$ supersymmetry with central charges 25
   - 5.1 Massive hypermultiplet model 25
   - 5.2 $N=3$ SYM as a quantum correction in the massive hypermultiplet model 27
   - 5.3 Hypermultiplet self-interaction induced by quantum corrections 28

6. Discussion 31

A. Appendix. $N=3$ harmonic superspace conventions 32

References 35

1. Introduction

Three-dimensional extended supersymmetric gauge theories attract much attention due to their remarkable relationships with string/M theory. In this paper we develop a generic procedure for constructing quantum effective actions of $N=3$, $d=3$ supergauge theories in terms of unconstrained harmonic superfields.

The harmonic superspace approach [1, 2, 3, 4] is a powerful tool for studying field theories with extended supersymmetry in diverse dimensions. In particular, the $\mathcal{N}=3$, d=3 harmonic superspace was introduced in [5]. Recently [6], we applied this approach to the three-dimensional $\mathcal{N}=3$ supersymmetric Chern-Simons and matter models which are building blocks of the $\mathcal{N}=6$ and $\mathcal{N}=8$ supersymmetric Aharony-Bergman-Jafferis-Maldacena (ABJM) [7] and Bagger-Lambert-Gustavsson (BLG) [8] theories. The manifestly $\mathcal{N}=3$ supersymmetric off-shell formulation of these theories was constructed for the first time. The ABJM and BLG models are currently of great interest, because they describe the world-volume dynamics of M2 branes in superstring theory and so open a way for studying the AdS_4/CFT_3 correspondence.

Superspace formulations are most advantageous for studying the quantum aspects of supersymmetric field theories because they make manifest one or another amount of the underlying supersymmetries. Based on this general feature, it is natural to expect that the $\mathcal{N}=3$, d=3 harmonic superspace approach may prove very fruitful for quantum computations in the $\mathcal{N}=6$ or $\mathcal{N}=8$ superconformal models including the ABJM and BLG ones. However, there is very limited experience in quantizing $\mathcal{N}=3$, d=3 Chern-Simons or matter models directly in harmonic superspace [5] (as opposed to e.g. $\mathcal{N}=2$, d=4 supersymmetric theories [2, 3, 4]). The aim of the present paper is to partially fill this gap by working out the basic steps of the appropriate quantization procedure.

We develop the background field method for a general $\mathcal{N}=3$ Chern-Simons matter theory and use it to prove a nonrenormalization theorem which guarantees the quantum finiteness of such a theory. We derive the superfield propagators in these models and use them for the calculation of leading supergraphs with two and four external gauge and matter legs. These diagrams bring to light some interesting facts about the quantum theory. First, the leading quantum correction in the massive charged hypermultiplet generates the $\mathcal{N}=3$ super Yang-Mills action rather than the Chern-Simons term. This result is rather unexpected in comparison with $\mathcal{N}=0$ three-dimensional electrodynamics in which a single massive fermion generates the Chern-Simons action as a leading contribution to the effective action [9]. Second, the four hypermultiplet one-loop diagram produces a quartic hypermultiplet self-interaction. Like in the four-dimensional case [10], such a contribution is possible only in a model of massive charged hypermultiplets, with the hypermultiplet mass being induced by the central charge of the $\mathcal{N}=3$; d=3 Poincare superalgebra.

The paper is organized as follows. In Section 2 we review the formulation of the $\mathcal{N}=3$ hypermultiplet and Chern-Simons models in $\mathcal{N}=3$ harmonic superspace. In Section 3 we develop the background field method for a general $\mathcal{N}=3$ Chern-Simons matter theory, prove the nonrenormalization theorem and discuss the general structure of the quantum effective action. In Section 4 we consider some one-loop quantum computations in the case of vanishing background field. In Section 5 we study the realization of $\mathcal{N}=3$ supersymmetry with a central charge. This leads us to the massive hypermultiplet model, whose quantum aspects we consider as well. The final Section 6 contains a discussion of our results as well as prospects of their further applications to three-dimensional models with extended supersymmetry. In the Appendix we collect technical details of the $\mathcal{N}=3$, d=3 harmonic superspace approach.

2. Field models in $\mathcal{N}=3$, d=3 harmonic superspace

The basic aspects of the $\mathcal{N}=3$, d=3 harmonic superspace were worked out for the first time in [5]. In this paper we follow the notations used in our recent paper [6]. They are collected in the Appendix.

2.1 Gauge theory in standard $\mathcal{N}=3$ superspace

To begin with, we consider the gauge theory in standard $\mathcal{N}=3$, d=3 superspace with coordinates $(x^{\alpha\beta}, \theta^{ij}_\alpha)$ and covariant spinor derivatives D^{ij}_α given by (A.10). Following the standard geometric approach to gauge theories in superspace, we start by defining the superfield connections for the space-time and spinor derivatives,

$$\nabla^{ij}_\alpha = D^{ij}_\alpha + V^{ij}_\alpha\,;\qquad \nabla_{\alpha\beta} = \partial_{\alpha\beta} + V_{\alpha\beta}\,. \tag{2.1}$$

The main superfield constraint for these superfield connections is given by [5]

$$\{\nabla^{ij}_\alpha, \nabla^{kl}_\beta\} = i\nabla_{\alpha\beta}(\varepsilon^{ik}\varepsilon^{jl}+\varepsilon^{il}\varepsilon^{jk}) - \frac{1}{2}\varepsilon_{\alpha\beta}(\varepsilon^{ik}W^{jl}+\varepsilon^{il}W^{jk}+\varepsilon^{jl}W^{ik}+\varepsilon^{jk}W^{il})\,, \tag{2.2}$$

where $W^{ij}=W^{(ij)}$ is a superfield strength for these gauge connections. Using the Bianchi identities one can check that the commutators of other covariant derivatives do not involve new tensors except W^{ij} and its derivatives,

$$[\nabla_{\alpha\beta}, \nabla^{ij}_\gamma] = \varepsilon_{\gamma\alpha}F^{ij}_\beta + \varepsilon_{\gamma\beta}F^{ij}_\alpha\,, \tag{2.3}$$

$$[\nabla_{\alpha\beta}, \nabla_{\gamma\delta}] = \varepsilon_{\alpha\gamma}F_{(\beta\delta)} + \varepsilon_{\beta\delta}F_{(\alpha\gamma)} + \varepsilon_{\alpha\delta}F_{(\beta\gamma)} + \varepsilon_{\beta\gamma}F_{(\alpha\delta)}\,, \tag{2.4}$$

where

$$F^{ij}_\alpha = \frac{i}{4}(\nabla^{ik}_\alpha W^j_k + \nabla^{jk}_\alpha W^i_k)\,;\qquad F_{(\alpha\beta)} = \frac{1}{24}(\nabla_{\alpha\, ij}\nabla^{ik}_\beta W^j_k + \nabla_{\beta\, ij}\nabla^{ik}_\alpha W^j_k)\,. \tag{2.5}$$

Moreover, the Bianchi identities lead to the following off-shell constraint for W^{ij},

$$\nabla^{(ij}_\alpha W^{kl)} = 0\,. \tag{2.6}$$

In the next subsection we will show how this constraint is resolved within the harmonic superspace approach.

2.2 Gauge theory in $\mathcal{N}=3$ harmonic superspace

The $\mathcal{N}=3$ harmonic superspace is parametrized by the following coordinates [1]

$$z = \{x^{\alpha\beta}, \theta^{++}_\alpha, \theta^{--}_\alpha, \theta^0_\alpha, u^\pm_i\}\,, \tag{2.7}$$

where $\theta^{\pm\pm}_\alpha = \theta^{ij}_\alpha u^\pm_i u^\pm_j$, $\theta^0_\alpha = \theta^{ij}_\alpha u^+_i u^-_j$ and $u^\pm_i$ are the $SU(2)/U(1)$ harmonic coordinates subjected to the constraints $u^{+i}u^-_i = 1$, $u^{+i}u^+_i = 0$, $u^{-i}u^-_i = 0$. The harmonic projections

[1] Note that in [5] the $\mathcal{N}=3$, d=3 harmonic superspace with $O(3)/O(2)$ harmonics was introduced.

of the covariant spinor derivatives ∇^{ij}_α and the superfield strengths W^{ij} are defined as follows

$$\nabla^{++}_\alpha = u^+_i u^+_j \nabla^{ij}_\alpha; \quad \nabla^{--}_\alpha = u^-_i u^-_j \nabla^{ij}_\alpha; \quad \nabla^{0}_\alpha = u^+_i u^-_j \nabla^{ij}_\alpha;$$
$$W^{++} = u^+_i u^+_j W^{ij}; \quad W^{--} = u^-_i u^-_j W^{ij}; \quad W^{0} = u^+_i u^-_j W^{ij}. \tag{2.8}$$

There are obvious relations between the harmonic superfield strengths,

$$\text{(a)}\ \partial^{++}W^{++} = 0; \quad \text{(b)}\ W^{0} = \frac{1}{2}\partial^{--}W^{++}; \quad W^{--} = \partial^{--}W^{0} = \frac{1}{2}(\partial^{--})^2 W^{++}; \tag{2.9}$$

where harmonic derivatives in the central basis $\partial^{\pm\pm}$ are defined in Appendix. In terms of the above harmonic projections, the anticommutation relations (2.2) can be rewritten as

$$\{\nabla^{++}_\alpha, \nabla^{--}_\beta\} = 2i\nabla_{\alpha\beta} + 2\varepsilon_{\alpha\beta}W^{0}; \qquad \{\nabla^{0}_\alpha, \nabla^{0}_\beta\} = -i\nabla_{\alpha\beta};$$
$$\{\nabla^{++}_\alpha, \nabla^{0}_\beta\} = \varepsilon_{\alpha\beta}W^{++}; \qquad \{\nabla^{--}_\alpha, \nabla^{0}_\beta\} = \varepsilon_{\alpha\beta}W^{--}; \tag{2.10}$$

while the harmonic projections of the constraint (2.6) are given by

$$\nabla^{++}_\alpha W^{++} = 0; \tag{2.11}$$
$$\nabla^{--}_\alpha W^{--} = 0; \quad \nabla^{--}_\alpha W^{++} + 4\nabla^{0}_\alpha W^{0} + \nabla^{++}_\alpha W^{--} = 0;$$
$$\nabla^{0}_\alpha W^{++} + \nabla^{++}_\alpha W^{0} = 0; \qquad \nabla^{0}_\alpha W^{--} + \nabla^{--}_\alpha W^{0} = 0. \tag{2.12}$$

The relations (2.11) and (2.12) are none other than the Bianchi identities for the superfield strengths W^{++}, W^{--}, W^{0}. It is important to realize that the whole set of the constraints (2.12) can be produced from the relation (2.11) by the successive action of the harmonic derivative ∂^{--}. Thus eq. (2.11) is the basic constraint. As will be clear soon, it is nothing else as the Grassmann analyticity condition, and it can be solved by passing to the analytic basis in $N=3, d=3$ harmonic superspace and to an analytic gauge frame.

An important feature of the $N=3$, $d=3$ harmonic superspace is the existence of an analytic subspace in it. This subspace is closed under the $N=3$ supersymmetry and is parametrized by the following coordinates

$$\zeta_A = (x^{\alpha\beta}_A, \theta^{++}_\alpha, \theta^{0}_\alpha, u^\pm_i); \tag{2.13}$$

where

$$x^{\alpha\beta}_A = (\gamma_m)^{\alpha\beta} x^m_A = x^{\alpha\beta} + i(\theta^{++\alpha}\theta^{--\beta} + \theta^{++\beta}\theta^{--\alpha}). \tag{2.14}$$

The analytic basis of $N=3$, $d=3$ harmonic superspace (as opposed to the original, central basis $(x^{\alpha\beta}, \theta^{ij}_\alpha)$) is defined as the coordinate set

$$z_A = \{\zeta_A, \theta^{--}_\alpha\}. \tag{2.15}$$

In the analytic basis the Grassmann derivative D^{++}_α becomes short, $D^{++}_\alpha = \frac{\partial}{\partial\theta^{--\alpha}}$. Other Grassmann and harmonic derivatives in this basis are given by expressions (A.17),(A.18). The existence of the analytic subspace is crucial for constructing superfield actions, as it allows one to define the analytic (short) superfields, which are independent of the coordinate θ^{-}_{α},

$$D^{++}\Phi_A = 0 \quad \Rightarrow \quad \Phi_A = \Phi_A(\zeta_A): \tag{2.16}$$

As soon as the harmonic variables $u^{\pm}_i$ appear on equal footing with the other superspace coordinates, there is a set of the harmonic derivatives ∂^{++}, ∂^{--}, ∂^{0} given by (A.13). Clearly, these derivatives do not receive any gauge connections in the original gauge frame ("τ frame"), since the gauge transformations in it are associated with the harmonic-independent, gauge algebra valued superfield parameter $\tau = \tau(z)$, e.g.,

$$W^{ij} \to e^{i\tau}W^{ij}e^{-i\tau}\ ; \qquad \delta\partial^{++} = \delta\partial^{--} = 0: \tag{2.17}$$

However, in order to be able to deal with the manifestly analytic superfields (2.16) in non-trivial representations of the gauge group, one should define another gauge frame ("λ frame") in which the gauge group is represented by the analytic superfield transformations,

$$\Phi_A \to \Phi'_A = e^{i\lambda}\Phi_A\ ; \qquad D^{++}\lambda = 0: \tag{2.18}$$

Having two different representations of the same gauge group, one with the harmonic-independent gauge parameter τ and another with the analytic gauge parameter λ, one can define the invertible "bridge" e^{iv}, $v = v(z;u)$, which transforms as

$$e^{iv'} = e^{i\lambda}e^{iv}e^{-i\tau}\ ; \tag{2.19}$$

and thus relates the τ and λ frames [1]:

$$\Phi_{A(\tau)} = e^{-iv}\Phi_A\ ; \qquad \Phi_{A(\tau)} \to \Phi'_{A(\tau)} = e^{i\tau}\Phi_{A(\tau)}: \tag{2.20}$$

Respectively, the Grassmann and harmonic gauge covariant derivatives in the τ and λ frames are related as

$$\nabla^{++}_{(\lambda)} = e^{iv}\nabla^{++}e^{-iv} = D^{++}\ ; \ \nabla^{-}_{(\lambda)} = e^{iv}\nabla^{-}e^{-iv}\ ; \ \nabla^{0}_{(\lambda)} = e^{iv}\nabla^{0}e^{-iv}\ ; \tag{2.21}$$

$$\nabla^{\pm\pm}_{(\lambda)} = e^{iv}D^{\pm\pm}e^{-iv} = D^{\pm\pm} + iV^{\pm\pm}\ ; \qquad V^{\pm\pm} = -ie^{iv}(D^{\pm\pm}e^{-iv})\ ; \tag{2.22}$$

where $D^{\pm\pm}$ are the analytic-basis harmonic derivatives defined in (A.17).[2] Hereafter, we omit the subscript (λ), assuming that we will always make use of the λ frame.

It is crucial that the derivative ∇^{++} in the λ frame becomes short while the harmonic derivatives acquire gauge connections. Owing to the commutation relation $[D^{++}; \nabla^{++}] = 0$, the superfield V^{++} is analytic,

$$D^{++}V^{++} = 0: \tag{2.23}$$

The algebra of harmonic derivatives $[\nabla^{++}; \nabla^{--}] = D^{0}$ leads to the harmonic zero-curvature equation,

$$D^{++}V^{--} - D^{--}V^{++} + i[V^{++}; V^{--}] = 0\ ; \tag{2.24}$$

[2] We would equally choose the central-basis form of the harmonic derivatives in (2.21), because there is no direct correlation between the superspace bases and the gauge frames.

which defines the gauge prepotential V^{--} as a function of V^{++}. An explicit solution of this equation can be represented by the series [11],

$$V^{--}(z,u)=\sum_{n=1}^{\infty}(-1)^n\int du_1\dots du_n\frac{V^{++}(z,u_1)V^{++}(z,u_2)\dots V^{++}(z,u_n)}{(u^+u_1^+)(u_1^+u_2^+)\dots(u_n^+u^+)}\,. \tag{2.25}$$

It is important that not only the prepotential V^{--}, but the gauge connections for the Grassmann derivatives, as well as the superfield strengths in (2.10), can be expressed through V^{++}. In particular,

$$[\nabla^{--}_\alpha,D^{++}_\beta]=2\nabla^0_{\alpha\beta}\;\Rightarrow\;V^0_{\alpha\beta}=\frac{1}{2}D^{++}_{(\alpha}D^{--}_{\beta)}V^{--}\,; \tag{2.26}$$

$$\{D^{++}_\alpha,\nabla^0_\beta\}=\varepsilon_{\alpha\beta}W^{++}\;\Rightarrow\;W^{++}=-\frac{1}{4}D^{++\alpha}D^{++}_\alpha V^{--}\,, \tag{2.27}$$

where V^{--} is a function of V^{++} given by (2.25). The equations (2.9b), being rewritten in the λ frame, read

$$W^0=\frac{1}{2}\nabla^{--}W^{++}\,,\qquad W^{--}=\nabla^{--}W^0=\frac{1}{2}(\nabla^{--})^2W^{++}\,. \tag{2.28}$$

The equation (2.11), when written in the λ frame, just means the analyticity of the superfield strength W^{++},

$$D^{++}_\alpha W^{++}=0\;\Rightarrow\;W^{++}=W^{++}(\zeta_A)\,. \tag{2.29}$$

As a result, the superfield constraint (2.11) for W^{++} is solved by using the λ representation of the gauge group and the analytic basis (2.15) in the harmonic superspace. However, the relation $\partial^{++}W^{++}=0$, eq. (2.9a), which, in the τ frame, just states that W^{++} is homogeneous of degree 2 in u_i^+, becomes non-trivial in the λ frame:

$$\nabla^{++}W^{++}=0\,. \tag{2.30}$$

In particular, this constraint encodes the Bianchi identity for the gauge field component of the gauge superfield strength.

2.3 N = 3 super Yang-Mills and Chern-Simons models

As shown in the previous subsection, the N = 3, d = 3 gauge theory is described by the superfield strengths W^{++}, W^0, W^{--} which can be expressed through the single analytic gauge prepotential V^{++}. Since the superfield W^{++} is analytic, eq. (2.29), it can be used for constructing the actions directly in the analytic subspace. In particular, the super Yang-Mills (SYM) and Chern-Simons actions are given by [5]

$$S_{SYM}=\frac{1}{g^2}\mathrm{tr}\int d\zeta^{(-4)}W^{++}W^{++}\,, \tag{2.31}$$

$$S_{CS}=\frac{ik}{4\pi}\mathrm{tr}\sum_{n=2}^{\infty}\frac{(-1)^n}{n}\int d^3x d^6\theta\, du_1\dots du_n\frac{V^{++}(z,u_1)\dots V^{++}(z,u_n)}{(u_1^+u_2^+)\dots(u_n^+u_1^+)}\,. \tag{2.32}$$

Here g is the Yang-Mills coupling constant with the mass dimension $[g] = 1/2$ while k is the (integer) Chern-Simons level. The rules of integration over the analytic and full $\mathcal{N}=3$ superspaces are given in the Appendix. Both SYM and Chern-Simons actions are invariant under the following gauge transformations

$$V^{++} \to V^{++\prime} = e^{i\lambda}\nabla^{++}e^{-i\lambda}\,, \tag{2.33}$$

or, in the infinitesimal form,

$$\delta V^{++} = -\nabla^{++}\lambda = -D^{++}\lambda - i[V^{++},\lambda]\,, \tag{2.34}$$

where λ is an analytic gauge parameter.

One can partly fix the gauge freedom by passing to the Wess-Zumino gauge, in which

$$\begin{aligned} V^{++}_{WZ} = {}& 3(\theta^{++})^2 u^-_k u^-_l \phi^{kl}(x_A) + 2\theta^{++\alpha}\theta^{0\beta}A_{\alpha\beta}(x_A) + 2(\theta^0)^2\theta^{++\alpha}\lambda_\alpha(x_A) \\ & + 3(\theta^{++})^2\theta^{0\alpha}u^-_k u^-_l \lambda^{kl}_\alpha(x_A) + 3i(\theta^{++})^2(\theta^0)^2 u^-_k u^-_l X^{kl}(x_A)\,. \end{aligned} \tag{2.35}$$

Such a form of the gauge prepotential is most suitable for deriving the component structure of the SYM and Chern-Simons actions [3]

$$\begin{aligned} S_{SYM} = {}& \frac{1}{g^2}\mathrm{tr}\int d^3x\Big(\phi^{kl}\Box\phi_{kl} + \frac14 f^{\alpha\beta}f_{\alpha\beta} - i\lambda^\alpha\partial_{\alpha\beta}\lambda^\beta - \frac{i}{2}\lambda^{kl\alpha}\partial_{\alpha\beta}\lambda^\beta_{kl} - X^{kl}X_{kl} \\ & + \text{interaction}\Big)\,; \end{aligned} \tag{2.36}$$

$$\begin{aligned} S_{CS} = {}& \frac{k}{4\pi}\mathrm{tr}\int d^3x\Big(\phi^{kl}X_{kl} - \frac{2i}{3}\phi^i_j[\phi^k_i,\phi^j_k] + \frac{i}{2}\lambda^\alpha\lambda_\alpha - \frac{i}{4}\lambda^\alpha_{kl}\lambda^{kl}_\alpha \\ & - \frac12 A^{\alpha\beta}\partial_\alpha^\gamma A_{\beta\gamma} - \frac{i}{6}A^{\alpha\beta}[A_\alpha^\gamma,A_{\beta\gamma}]\Big)\,, \end{aligned} \tag{2.37}$$

where $f_{\alpha\beta} = \partial_\alpha^\gamma A_{\gamma\beta} + \partial_\beta^\gamma A_{\gamma\alpha}$.

Since the SYM model in three-dimensional space-time involves the dimensionful coupling constant, it is not superconformal. In contrast, the Chern-Simons theory has dimensionless coupling constant and therefore is superconformal. In this paper we will be basically interested in quantum aspects of superconformal models; so our main focus will be on the $\mathcal{N}=3$ Chern-Simons gauge theory, rather than on $\mathcal{N}=3$ SYM. One more class of $\mathcal{N}=3$ superconformal theories we shall study is those of matter hypermultiplets.

2.4 $\mathcal{N}=3$ hypermultiplets

There are two basic types of the hypermultiplet in four dimensions: the q hypermultiplet and the ω hypermultiplet. They describe the same physical degrees of freedom, though with different assignments with respect to the R symmetry SU(2) group. Quite analogously, both these types of hypermultiplets exist in three-dimensional space-time too. In particular, the q-hypermultiplet consists of a SU(2) doublet of complex scalars f^i and a doublet of complex spinors ψ^i_α on shell. These fields appear in the component expansions of the complex analytic superfield q^+ as

$$q^+ = u^+_i f^i + (\theta^{++\alpha}u^-_i - \theta^{0\alpha}u^+_i)\psi^i_\alpha - 2i(\theta^{++\alpha}\theta^{0\beta})\partial^A_{\alpha\beta}f^i u^-_i + \text{aux. fields}\,. \tag{2.38}$$

[3] In the component field formulation such actions were obtained in [12].

The free hypermultiplet action has the well known form,

$$S_q = -\int d\zeta^{(-4)} \tilde{q}^+ D^{++} q^+ \,; \quad \tilde{q}^+ = \widetilde{q^+} \,; \quad \tilde{\tilde{q}}^+ = -q^+ \,, \tag{2.39}$$

which yields the following free action for the physical components upon eliminating an infinite tower of the auxiliary fields:

$$S_{q,phys} = \int d^3x \left(\bar{f}_i \Box f^i + \frac{i}{2} \bar{\psi}_i \partial \psi^i \right). \tag{2.40}$$

The ω-hypermultiplet collects on shell a real scalar φ, a triplet of real scalars $\varphi^{(ij)}$, a real spinor ψ_α and a triplet of real spinors $\psi^{(ij)}_\alpha$ which appear in the component expansion of a real analytic superfield ω as

$$\begin{aligned} \omega = {} & \frac{1}{2}\varphi + \frac{1}{\sqrt{2}}\varphi^{ij} u^+_i u^-_j + \frac{i}{\sqrt{2}}\theta^{0\alpha}\psi_\alpha + \frac{i\sqrt{3}}{2}\theta^{++\alpha}\psi^{ij}_\alpha u^-_i u^-_j - i\sqrt{3}\,\theta^{0\alpha}\psi^{ij}_\alpha u^+_i u^-_j \\ & - i\sqrt{2}\,\theta^{++\alpha}\theta^{0\beta}\partial_{\alpha\beta}\varphi^{ij} u^-_i u^-_j + \text{aux. fields}. \end{aligned} \tag{2.41}$$

The free superfield action

$$S_\omega = \int d\zeta^{(-4)} D^{++}\omega D^{++}\omega \,; \qquad \tilde{\omega} = \omega \,, \tag{2.42}$$

gives the standard kinetic terms for the physical component fields,

$$S_{\omega,phys} = \frac{1}{2}\int d^3x \left(\varphi\Box\varphi + \varphi^{ij}\Box\varphi_{ij} + i\psi^\alpha\partial_{\alpha\beta}\psi^\beta + i\psi^{ij\,\alpha}\partial_{\alpha\beta}\psi^\beta_{ij} \right). \tag{2.43}$$

The minimal gauge interaction of hypermultiplets can be implemented by promoting the flat harmonic derivative D^{++} to the gauge covariant one $\nabla^{++} = D^{++} + V^{++}$:

$$S_q = -\int d\zeta^{(-4)} \tilde{q}^+ \nabla^{++} q^+ \,, \tag{2.44}$$

$$S_\omega = \int d\zeta^{(-4)} \nabla^{++}\omega \nabla^{++}\omega \,. \tag{2.45}$$

For the time being we do not specify the representation of gauge group on the matter fields. We only notice that there is a difference between the q and ω hypermultiplet models in this aspect: since the ω hypermultiplet is described by a real superfield, it can be naturally placed into a real representation, e.g. the adjoint representation, while the q hypermultiplet is well suited for putting it into a complex representation of the gauge group, e.g. the fundamental one. Actually, there is a duality-sort transformation between two types of the hypermultiplet [3], so this difference between them is, to some extent, conventional.

Apart from the minimal gauge interaction, one can consider the hypermultiplet self-interaction. For the model with a single q hypermultiplet there exists the unique possibility to construct a quartic $SU(2)_R$ invariant superfield potential [4]

$$S_4 = \lambda \int d\zeta^{(-4)} (\tilde{q}^+ q^+)^2 \,, \qquad [\lambda] = -1 \,. \tag{2.46}$$

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[4] To prevent a possible confusion, let us recall that such q superfield "potentials", after passing to the physical component fields, give rise to the sigma-model terms for the latter rather than to a scalar potential and Yukawa-type fermionic couplings [13]. However, these component potential terms can appear as an effect of presence of central charges in the supersymmetry algebra.

In Section 5.3 we will show that the self-interaction (2.46) emerges as a leading quantum correction in the model of massive charged hypermultiplet.

3. Background field quantization

The background field method is a powerful tool for studying the general structure of the quantum effective actions in gauge theories. The basic advantage of this method is that it gives an opportunity to evaluate the effective action with preserving the classical gauge invariance on all steps of quantum computations. The idea of the background field method consists in splitting the initial fields into the classical and quantum parts and fixing the gauge symmetry only for the quantum fields in the generating functional for the effective action. For supersymmetric field models the concrete realizations of such a splitting is a non-trivial task which requires a special study in every case. For $N=1$, $d=4$ supergauge theories the background field method is discussed in [14], [15]. For $N=2$, $d=4$ supersymmetric theories formulated in terms of (constrained) $N=2$ superfields such a method was worked out in [16]. For the $N=2$, $d=4$ supergauge theories in harmonic superspace this method was developed in [17], [18]. Subsequently, it was successfully applied for studying quantum aspects of these models.

In this section we formulate the background field method for the $N=3$, $d=3$ Chern-Simons matter theory with the following general action

$$S = S_{CS} + S_q\,, \qquad (3.1)$$

where the Chern-Simons and hypermultiplet actions are given by eqs. (2.32), (2.44).[5]

3.1 The background field method for N = 3 Chern-Simons theory

The background field method for the $N=3$, $d=3$ Chern-Simons theory is analogous in some points to the one for the $N=2$, $d=4$ SYM theory [17], because the harmonic superspace classical actions in both theories bear a close resemblance to each other.

The classical action in the $N=3$ Chern-Simons theory (2.32) is invariant under the gauge transformations (2.34). We split the gauge superfield V^{++} into the 'background' V^{++} and 'quantum' v^{++} parts,

$$V^{++} \to V^{++} + \kappa v^{++}\,, \qquad (3.2)$$

where

$$\frac{1}{\kappa^2} = \frac{ik}{4\pi}\,. \qquad (3.3)$$

Then, the infinitesimal gauge transformations (2.34) can be realized in two different ways:

(i) Background transformations

$$\delta V^{++} = -D^{++}\lambda - [V^{++},\lambda] = -\nabla^{++}\lambda\,, \qquad \delta v^{++} = [\lambda, v^{++}]\,; \qquad (3.4)$$

[5] Here we do not consider the $N=3$ SYM theory (2.31) since we concentrate on the conformally invariant models.

(ii) Quantum transformations

$$\delta V^{++} = 0\,; \qquad \delta v^{++} = \frac{1}{\kappa}\nabla^{++}\lambda - [v^{++},\lambda]\,. \tag{3.5}$$

Here the covariant harmonic derivative ∇^{++} involves the background superfield V^{++}. Upon the splitting (3.2), the Chern-Simons action (2.32) can be rewritten as (see [19] for details of such a derivation in the $N=2, d=4$ case)

$$S_{CS}[V^{++}+v^{++}] = S_{CS}[V^{++}] - \frac{1}{\kappa}\mathrm{tr}\int d\zeta^{(-4)}\, v^{++}W^{++}(V^{++}) + S_{CS}[V^{++},v^{++}]\,; \tag{3.6}$$

where $W^{++}(V^{++})$ is defined in (2.27) and

$$S_{CS}[V^{++},v^{++}] = \mathrm{tr}\sum_{n=2}^{\infty}\frac{(-\kappa)^{n-2}}{n}\int d^9z\,du_1\ldots du_n\,\frac{\tilde v^{++}(z,u_1)\ldots\tilde v^{++}(z,u_n)}{(u_1^+u_2^+)\ldots(u_n^+u_1^+)}\,. \tag{3.7}$$

We introduced $\tilde v^{++} = e^{-i\Omega}v^{++}e^{i\Omega}$, with the bridge superfield Ω being constructed from the background gauge superfield V^{++} by the rule (2.22). The action (3.7) implicitly depends on the background superfield V^{++} via the bridge superfield Ω which is a complicated function of V^{++}. Every term in (3.6) is manifestly invariant under the background gauge transformations (3.4). The second term in (3.6) is responsible for the Chern-Simons equation of motion for the background gauge superfield which is none other than $W^{++}(V^{++}) = 0$. This term is not essential while constructing the effective action.

Within the background field method, it is necessary to fix the gauge only with respect to the quantum gauge transformations (3.5). The corresponding gauge-fixing function is

$$F^{(4)}_{\tau} = \nabla^{++}v^{++}\,; \tag{3.8}$$

or, being rewritten in the λ frame,

$$F^{(4)}_{\lambda} = D^{++}\tilde v^{++} = e^{-i\Omega}(\nabla^{++}v^{++})e^{i\Omega} = e^{-i\Omega}F^{(4)}_{\tau}e^{i\Omega}\,. \tag{3.9}$$

Under the quantum gauge transformations (3.5) this function is transformed as

$$\delta F^{(4)}_{\lambda} = \frac{1}{\kappa}e^{-i\Omega}\{\nabla^{++}(\nabla^{++}\lambda + \kappa[v^{++},\lambda])\}e^{i\Omega}\,. \tag{3.10}$$

The corresponding Faddeev-Popov determinant

$$\Delta_{FP}[V^{++},v^{++}] = \mathrm{Det}\,\nabla^{++}(\nabla^{++} + \kappa v^{++}) \tag{3.11}$$

can be represented by a path integral $\int Db\,Dc\,\exp(iS_{FP})$ with two ghost superfields b, c in the adjoint representation of the gauge group and with the ghost-field action

$$S_{FP} = \mathrm{tr}\int d\zeta^{(-4)}\, b\nabla^{++}(\nabla^{++}c + \kappa[v^{++},c])\,. \tag{3.12}$$

Putting all these ingredients together, we obtain the following representation for the effective action,

$$e^{i\Gamma_{CS}[V^{++}]} = e^{iS_{CS}[V^{++}]}\int Dv^{++}\,Db\,Dc\,e^{iS_{CS}[V^{++},v^{++}] + iS_{FP}[b,c,V^{++},v^{++}]}\,\delta[F^{(4)} - f^{(4)}]\,; \tag{3.13}$$

where $f^{(4)}$ is an arbitrary Lie algebra valued analytic function and $\delta[F^{(4)} - f^{(4)}]$ is the proper functional delta-function which fixes the gauge.

To cast (3.13) in a more useful form, we average it with the following weight factor

$$\Delta[V^{++}]\exp\left\{-\frac{i\alpha}{2}\mathrm{tr}\int d^9z du_1 du_2\, f^{(4)}(z,u_1)\frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)^3} f^{(4)}(z,u_2)\right\}, \qquad (3.14)$$

where α is an arbitrary parameter. The functional $\Delta[V^{++}]$ can be found from the condition

$$1 = \Delta[V^{++}]\int \mathcal{D}f^{(4)}\exp\left\{-\frac{i\alpha}{2}\mathrm{tr}\int d^9z du_1 du_2\, f^{(4)}(z,u_1)\frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)^3} f^{(4)}(z,u_2)\right\}. \qquad (3.15)$$

Hence,

$$\Delta^{-1}[V^{++}] = \int \mathcal{D}f^{(4)}\exp\left\{-\frac{i\alpha}{2}\mathrm{tr}\int d\zeta_1^{(-4)} d\zeta_2^{(-4)} f^{(4)}(1)A(1,2)f^{(4)}(2)\right\}$$
$$= \mathrm{Det}^{-1/2}\hat{A}, \qquad (3.16)$$

where $\hat{A}$ is some analytic Lie algebra valued operator with the kernel $A(1,2)$. To compute $\mathrm{Det}\hat{A}$, we represent it by a functional integral over the analytic superfields,

$$\mathrm{Det}^{-1}\hat{A} = \int \mathcal{D}\chi^{(4)}\mathcal{D}\rho^{(4)}\exp\left\{i\,\mathrm{tr}\int d\zeta_1^{(-4)} d\zeta_2^{(-4)}\chi^{(4)}(1)A(1,2)\rho^{(4)}(2)\right\} \qquad (3.17)$$

and perform the following change of functional variables

$$\rho^{(4)} = (\nabla^{++})^2\sigma, \qquad \mathrm{Det}\frac{\delta\rho^{(4)}}{\delta\sigma} = \mathrm{Det}(\nabla^{++})^2. \qquad (3.18)$$

Then we obtain

$$\mathrm{tr}\int d\zeta_1^{(-4)} d\zeta_2^{(-4)}\chi^{(4)}(1)A(1,2)\rho^{(4)}(2)$$
$$= \mathrm{tr}\int d^9z du_1 du_2\,\chi^{(4)}(z,u_1)\frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)^3}D^{++}_{(2)}\sigma(z,u_2) = \frac{1}{2}\int d^9z du\,\chi^{(4)}(D^{--})^2\sigma$$
$$= -\mathrm{tr}\int d\zeta^{(-4)}\chi^{(4)}\hat{\Box}\sigma, \qquad (3.19)$$

where

$$\hat{\Box} = -\frac{1}{8}(D^{++})^2(\nabla^{--})^2. \qquad (3.20)$$

As a result, $\Delta[V^{++}]$ can be formally written as

$$\Delta[V^{++}] = \mathrm{Det}^{1/2}_{(0,0)}(\nabla^{++})^2\,\mathrm{Det}^{1/2}_{(4,0)}\hat{\Box}, \qquad (3.21)$$

where

$$\mathrm{Det}^{1/2}_{(0,0)}(\nabla^{++})^2 = \int \mathcal{D}\phi\, e^{iS_{NK}[\phi,V^{++}]},$$
$$S_{NK}[\phi,V^{++}] = -\frac{1}{2}\mathrm{tr}\int d\zeta^{(-4)}\nabla^{++}\phi\,\nabla^{++}\phi \qquad (3.22)$$

and

$$
\mathrm{Det}^{-1}_{(4,0)}\hat{\Box} = \int \mathcal{D}\chi^{(4)}\mathcal{D}\varphi \; e^{i\int d\zeta^{(-4)}\chi^{(4)}\hat{\Box}\varphi} \,. \tag{3.23}
$$

The analytic real bosonic superfield φ plays the role of Nielsen-Kallosh ghost in this theory. The classical action for this superfield coincides with the ω-hypermultiplet action (2.45).

Upon averaging (3.13) with the weight factor (3.14) we arrive at the following path-integral representation for the effective action

$$
e^{i\Gamma_{CS}[V^{++}]} = e^{iS_{CS}[V^{++}]}(\mathrm{Det}^{1/2}_{(4,0)}\hat{\Box}) \int \mathcal{D}v^{++}\mathcal{D}b\mathcal{D}c\mathcal{D}\varphi \; e^{iS_Q[v^{++},b,c,\varphi,V^{++}]} \,, \tag{3.24}
$$

where

$$
\begin{aligned}
S_Q[v^{++},b,c,\varphi,V^{++}] = \; & S_{CS}[V^{++}+v^{++}] + S_{GF}[V^{++},v^{++}] \\
& + S_{FP}[b,c,V^{++},v^{++}] + S_{NK}[\varphi,V^{++}] \,.
\end{aligned} \tag{3.25}
$$

Here $S_{GF}[v^{++},V^{++}]$ is the gauge-fixing contribution to the quantum action given by

$$
\begin{aligned}
S_{GF}[V^{++},v^{++}] &= \frac{1}{2\alpha}\mathrm{tr}\int d^9z du_1 du_2 (\mathcal{D}^{++}v^{++}(z,u_1))\frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)^3}(\mathcal{D}^{++}v^{++}(z,u_2)) \\
&= -\frac{1}{2\alpha}\mathrm{tr}\int d^9z du_1 du_2 \frac{v^{++}(z,u_1)v^{++}(z,u_2)}{(u_1^+ u_2^+)^2} \\
&\quad -\frac{1}{2\alpha}\mathrm{tr}\int d\zeta^{(-4)} v^{++}\hat{\Box}v^{++} \,.
\end{aligned} \tag{3.26}
$$

Let us consider the sum of quadratic in v^{++} parts of S_{CS} and S_{GF},

$$
\frac{1}{2}\left(1+\frac{1}{\alpha}\right)\mathrm{tr}\int d^9z du_1 du_2 \frac{v^{++}(z,u_1)v^{++}(z,u_2)}{(u_1^+ u_2^+)^2} - \frac{1}{2\alpha}\mathrm{tr}\int d\zeta^{(-4)} v^{++}\hat{\Box}v^{++} \,. \tag{3.27}
$$

The first term in (3.27) vanishes at $\alpha = -1$, and we will adopt this choice in what follows. As a result, we arrive at the following final representation for the effective action

$$
e^{i\Gamma_{CS}[V^{++}]} = e^{iS_{CS}[V^{++}]}(\mathrm{Det}^{1/2}_{(4,0)}\hat{\Box}) \int \mathcal{D}v^{++}\mathcal{D}b\mathcal{D}c\mathcal{D}\varphi \; e^{i(S_2[v^{++},b,c,\varphi,V^{++}]+S_{int}[v^{++},b,c,V^{++}])} \,, \tag{3.28}
$$

where

$$
\begin{aligned}
S_2[v^{++},b,c,\varphi,V^{++}] &= \frac{1}{2}\mathrm{tr}\int d\zeta^{(-4)} v^{++}\hat{\Box}v^{++} + \mathrm{tr}\int d\zeta^{(-4)} b(\nabla^{++})^2 c \\
&\quad + \frac{1}{2}\mathrm{tr}\int d\zeta^{(-4)} \varphi(\nabla^{++})^2\varphi \,,
\end{aligned} \tag{3.29}
$$

$$
\begin{aligned}
S_{int}[v^{++},b,c,V^{++}] &= \mathrm{tr}\sum_{n=3}^{\infty}\frac{(-1)^n \kappa^{n-2}}{n}\int d^9z du_1 \ldots du_n \frac{v^{++}(z,u_1)\ldots v^{++}(z,u_n)}{(u_1^+ u_2^+)\ldots(u_n^+ u_1^+)} \\
&\quad - \mathrm{tr}\int d\zeta^{(-4)} \nabla^{++} b[v^{++},c] \,.
\end{aligned} \tag{3.30}
$$

The equations (3.28)–(3.30) completely determine the structure of perturbative expansion for the effective action in the pure $N=3$ Chern-Simons theory in a manifestly supersymmetric and gauge invariant form.

3.2 Adding hypermultiplets

Now we include into considerations the q-hypermultiplet superfield with the following classical action

$$S_q = -\int d\zeta^{(-4)} \tilde{q}^+ (\nabla^{++} + i v^{++}) q^+ ; \tag{3.31}$$

where ∇^{++} is the covariant harmonic derivative with the background gauge superfield V^{++}. Here we do not specify the representation of the gauge group on the hypermultiplet. We split the hypermultiplet superfields into the background q^+, and quantum $\mathbf{q}^+$ parts,

$$q^+ \to q^+ + \mathbf{q}^+ ; \qquad \tilde{q}^+ \to \tilde{q}^+ + \tilde{\mathbf{q}}^+ : \tag{3.32}$$

Upon such a splitting, the classical action (3.31) can be rewritten as a sum of the following four pieces

$$S_q = S_q[\tilde{q}^+ ; q^+ ; V^{++}] + S_{lin} + S_2 + S_{int} ; \tag{3.33}$$

where $S_q[\tilde{q}^+ ; q^+ ; V^{++}]$ is given by (2.44) and is constructed solely from the classical fields, while the term S_{lin} is linear in the quantum fields,

$$S_{lin} = -\int d\zeta^{(-4)} (\tilde{\mathbf{q}}^+ \nabla^{++} q^+ + \tilde{q}^+ \nabla^{++} \mathbf{q}^+ + \tilde{q}^+ i v^{++} q^+) : \tag{3.34}$$

This term can be omitted since it does not contribute to the effective action. The pieces S_2 and S_{int} in (3.33) correspond, respectively, to that part of the action which is quadratic in the quantum superfields, and to the interaction term:

$$S_2 = -\int d\zeta^{(-4)} (\tilde{\mathbf{q}}^+ \nabla^{++} \mathbf{q}^+ + \tilde{\mathbf{q}}^+ i v^{++} q^+ + \tilde{q}^+ i v^{++} \mathbf{q}^+) ; \tag{3.35}$$

$$S_{int} = -\int d\zeta^{(-4)} \tilde{\mathbf{q}}^+ v^{++} \mathbf{q}^+ : \tag{3.36}$$

Now we can generalize the generating functional for the effective action (3.28) to the Chern-Simons matter theory,

$$e^{i\Gamma_{CS}[V^{++};\tilde{q}^+;q^+]} = e^{i(S_{CS}[V^{++}]+S_q[\tilde{q}^+;q^+;V^{++}])} (\mathrm{Det}^{1/2}_{(4,0)} \hat{\Box}) \int \mathcal{D}v^{++}\mathcal{D}b\mathcal{D}c\mathcal{D}\varphi\mathcal{D}\tilde{\mathbf{q}}^+\mathcal{D}\mathbf{q}^+ e^{i(S_2+S_{int})} ; \tag{3.37}$$

where

$$S_2 = \frac{1}{2}\mathrm{tr}\int d\zeta^{(-4)} v^{++} \hat{\Box} v^{++} + \mathrm{tr}\int d\zeta^{(-4)} b(\nabla^{++})^2 c + \frac{1}{2}\mathrm{tr}\int d\zeta^{(-4)} \varphi(\nabla^{++})^2\varphi$$
$$- \int d\zeta^{(-4)} (\tilde{\mathbf{q}}^+ \nabla^{++} \mathbf{q}^+ + \tilde{\mathbf{q}}^+ v^{++} q^+ + \tilde{q}^+ v^{++} \mathbf{q}^+) ; \tag{3.38}$$

$$S_{int} = \mathrm{tr}\sum_{n=3}^{\infty} \frac{(-1)^n \kappa^{n-2}}{n} \int d^9z du_1 \ldots du_n \frac{v^{++}(z,u_1)\ldots v^{++}(z,u_n)}{(u_1^+ u_2^+)\ldots(u_n^+ u_1^+)}$$
$$+ \int d\zeta^{(-4)} \tilde{\mathbf{q}}^+ v^{++} \mathbf{q}^+ - \mathrm{tr}\int d\zeta^{(-4)} \nabla^{++} b[v^{++}, c] : \tag{3.39}$$

The treatment of the ω hypermultiplet within the background field method is quite analogous to the above q hypermultiplet consideration.

3.3 Gauge and hypermultiplet propagators

It is seen from the action (3.38) that the gauge and hypermultiplet propagators are defined by the equations

$$\langle v^{++}(1)v^{++}(2)\rangle = G^{(2,2)}(1|2)\,. \qquad \hat{\Box} G^{(2,2)}(1|2) = \delta_A^{(2,2)}(1|2)\,; \tag{3.40}$$

$$\langle q^{+}(1)q^{+}(2)\rangle = G^{(1,1)}(1|2)\,. \qquad \nabla^{++} G^{(1,1)}(1|2) = \delta_A^{(3,1)}(1|2)\,; \tag{3.41}$$

$$\langle \omega(1)\omega(2)\rangle = G^{(0,0)}(1|2)\,. \qquad (\nabla^{++})^2 G^{(0,0)}(1|2) = \delta_A^{(4,0)}(1|2)\,; \tag{3.42}$$

where the analytic delta-function is given by

$$\delta_A^{(4-q,q)}(1|2) = \frac{1}{4} D^{++}_{(1)} \bar{D}^{++}_{(1)} \delta^9(z_1 - z_2)\,\delta^{(-q,q)}(u_1,u_2)\,. \tag{3.43}$$

Here $\delta^{(-q,q)}(u_1,u_2)$ is the standard harmonic delta-function [4].

The solutions of the equations (3.40), (3.41) and (3.42) are given by the following expressions

$$G^{(2,2)}(1|2) = \frac{1}{\hat{\Box}^2}\hat{\Box}\,\delta_A^{(2,2)}(1|2)\,; \tag{3.44}$$

$$G^{(1,1)}(1|2) = \frac{1}{12\,\hat{\Box}}(D^{++}_{(1)})^2(D^{++}_{(2)})^2 \bar{W}^0_{(1)} \frac{e^{\Omega(1)}e^{-\Omega(2)}\delta^9(z_1 - z_2)}{(u_1^+u_2^+)^3}\,; \tag{3.45}$$

$$G^{(0,0)}(1|2) = -\frac{1}{12\,\hat{\Box}}(D^{++}_{(1)})^2(D^{++}_{(2)})^2 \bar{W}^0_{(1)} \frac{e^{\Omega(1)}e^{-\Omega(2)}\delta^9(z_1 - z_2)(u_1^- u_2^-)}{(u_1^+u_2^+)^3}\,; \tag{3.46}$$

where

$$\bar{W}^0 = W^0 + \frac{1}{4}\hat{\Delta} + \frac{1}{2}\nabla^{\alpha 0}\nabla^0_\alpha \tag{3.47}$$

and the operators $\hat{\Box}$, $\hat{\Box}^2$, $\hat{\Delta}$ depending on the background gauge superfield V^{++} will be specified below.

The operator $\hat{\Delta}$ was introduced in (3.20). It has the following basic commutation relations with the Grassmann and harmonic derivatives

$$[D^{++}_\alpha, \hat{\Delta}] = 0\,; \qquad [\nabla^{++}, \hat{\Delta}]\,\Phi_A^{(q)} = (1-q)W^{++}\Phi_A^{(q)}\,. \tag{3.48}$$

When acting on the analytic superfields, the operators $\hat{\Box}$ and $\hat{\Box}^2$ can be represented as

$$\hat{\Box} = (\nabla^{\alpha 0})^2 - W^0\bar{W}^0 - W^{++}\nabla^{--}\,; \tag{3.49}$$

$$\begin{aligned}
\hat{\Box}^2 = {} & \nabla^m\nabla_m + 3W^{++}W^{--} + (W^0)^2 - ((\nabla^0)^2 W^0) + (D^{++}_\alpha W^{--})\nabla^{\alpha 0} - 2W^0(\nabla^0)^2 \\
& - 2(\nabla^{\alpha 0}W^{++})\nabla^0_\alpha\nabla^{--} - 2W^{++}(\nabla^0)^2\nabla^{--} - 2W^{++}\nabla^{\alpha 0}\nabla^{--}_\alpha + W^0 W^{++}\nabla^{--} \\
& + 3W^{++}W^0\nabla^{--} - ((\nabla^0)^2 W^{++})\nabla^{--} + W^{++}W^{++}(\nabla^{--})^2\,.
\end{aligned} \tag{3.50}$$

Since the expression (3.50) starts with the square $\nabla^m\nabla_m$, the operator $1/\hat{\Box}^2$ in (3.44) is well defined as a power series expansion around $\nabla^m\nabla_m$.

The hypermultiplet propagators (3.45) and (3.46) involve the operator

$$\hat{\Box} = \frac{1}{6}(D^{++})^2\bar{W}^0(\nabla^{--})^2\,; \tag{3.51}$$

where W^0 is defined in (3.47). This operator $\hat{\Box}$ reveals the following basic properties

$$[D^{++},\hat{\Box}] = 0;\qquad [\nabla^{++},\hat{\Box}]\Phi_A^{(q)} = (1-q)\left((\nabla^0)^2 W^{++} + [W^0, W^{++}]\right)\Phi_A^{(q)}; \tag{3.52}$$

where $\Phi_A^{(q)}$ is some analytic superfield. With making use of these properties, the operator $\hat{\Box}$ in application to the analytic superfields is reduced to

$$\hat{\Box} = \nabla^m\nabla_m + \frac{1}{4}\{W^{ij},W_{ij}\} + \frac{1}{2}[W^{++},W^{--}] + (D^{++}W^{--})\nabla^0 + (\nabla^0 W^{++})\nabla^{--}$$
$$-((\nabla^0)^2 W^{++})\nabla^{--} + [W^{++},W^0]\nabla^{--} - ((\nabla^0)^2 W^0). \tag{3.53}$$

The expression (3.53) starts with $\nabla^m\nabla_m$, hence the operator $\hat{\Box}^{-1}$ is well defined as a power series expansion.

For the vanishing background field the propagators (3.44), (3.45) and (3.46) take very simple form,

$$G_0^{(2,2)}(1|2) = -\frac{1}{\Box}(D^0)^2\,\delta_A^{(2,2)}(1|2); \tag{3.54}$$

$$G_0^{(1,1)}(1|2) = -\frac{1}{16\Box}(D^0_{(1)})^2(D^{++}_{(1)})^2(D^{++}_{(2)})^2\frac{\delta^9(z_1-z_2)}{(u_1^+u_2^+)^3}; \tag{3.55}$$

$$G_0^{(0,0)}(1|2) = -\frac{1}{16\Box}(D^0_{(1)})^2(D^{++}_{(1)})^2(D^{++}_{(2)})^2\,\delta^9(z_1-z_2)\frac{(u_1^-u_2^-)}{(u_1^+u_2^+)^3}. \tag{3.56}$$

Notice that the free q- and ω-hypermultiplet propagators are, respectively, antisymmetric and symmetric with respect to interchanging their arguments,

$$G_0^{(1,1)}(1|2) = -G_0^{(1,1)}(2|1);\qquad G_0^{(0,0)}(1|2) = G_0^{(0,0)}(2|1). \tag{3.57}$$

We will use these free propagators in the next Section where some examples of quantum computations within this approach will be presented.

An alternative representation for the free propagators (3.54), (3.55) and (3.56) is given by

$$G_0^{(2,2)}(1|2) = \frac{1}{2\pi i}\frac{1}{\sqrt{\rho^2}}[(\zeta_1^{++})^2 - 2(u_1^+u_2^-)^2(\zeta_1^{++}\zeta_2^{++})$$
$$+(u_1^+u_2^-)^4(\zeta_2^{++})^2]\,\delta^{(-2,2)}(u_1,u_2); \tag{3.58}$$

$$G_0^{(1,1)}(1|2) = \frac{1}{2\pi i}\frac{(u_1^+u_2^+)}{\sqrt{\rho^2}}; \tag{3.59}$$

$$G_0^{(0,0)}(1|2) = \frac{1}{2\pi i}\frac{(u_1^+u_2^+)(u_1^-u_2^-)}{\sqrt{\rho^2}}; \tag{3.60}$$

where

$$\rho^{\alpha\beta} = x_{A1}^{\alpha\beta} - x_{A2}^{\alpha\beta} - 2i\theta_1^{0(\alpha}\theta_2^{0\beta)} - \frac{2i}{(u_1^+u_2^+)}\Big((u_1^-u_2^-)\theta_1^{++(\alpha}\theta_2^{++\beta)} - (u_1^-u_2^+)\theta_1^{++(\alpha}\theta_2^{0\beta)}$$
$$-(u_1^+u_2^-)\theta_1^{0(\alpha}\theta_2^{++\beta)} + (u_1^-u_2^+)\theta_1^{++(\alpha}\theta_1^{0\beta)} + (u_1^+u_2^-)\theta_2^{0(\alpha}\theta_2^{++\beta)}\Big) \tag{3.61}$$

is manifestly analytic $N=3$ supersymmetric interval.

The quantization of the $N=3$, $d=3$ superfield theories was considered for the first time in the formalism with the $O(3)/O(2)$ harmonics in [5].

3.4 $\mathcal{N}=3$, $d=3$ nonrenormalization theorem

It is well known that the β-function for Chern-Simons coupling in an arbitrary Chern-Simons matter theory is trivial [20], the divergences may occur only in the sector of matter fields. As for supersymmetric Chern-Simons matter theory, one can hope that the supersymmetry may reduce the degree of such divergences or even ensure their full cancellation like in the $\mathcal{N}=4$, $d=4$ SYM theory. The nonrenormalization properties of some $\mathcal{N}=1$ and $\mathcal{N}=2$ Chern-Simons matter theories were discussed in [23]. It was shown that in the general case such $\mathcal{N}=1$ and $\mathcal{N}=2$ theories with scale-invariant superpotentials are not free of UV divergences, but for some particular superpotentials, when the supersymmetry is enhanced to $\mathcal{N}=6$ or $\mathcal{N}=8$, the cancellation of such divergences may occur [24].

Here we prove the nonrenormalization theorem in general $\mathcal{N}=3$ Chern-Simons matter theory. The general statement is as follows: The effective action in the $\mathcal{N}=3$ Chern-Simons model (2.32) with arbitrary number of q and ω hypermultiplets (2.44), (2.45) in an arbitrary representation of gauge group is completely finite, in the sense that superfield Feynman diagrams contributing to the effective action show up no any UV quantum divergences.

This statement is very similar to the nonrenormalization theorem for the $\mathcal{N}=2$, $d=4$ supergauge theory [18] which provides the finiteness of this theory beyond one loop. In fact, this analogy is even deeper: the form of the Chern-Simons action (2.32) is similar to the $\mathcal{N}=2$, $d=4$ SYM action (there is a dimensionless coupling constant in both cases), the only difference being in the fact that the integration is now performed over the three-dimensional space-time. The form of classical harmonic superfield Lagrangians for the q and ω hypermultiplets is completely the same as in four dimensions. The details of the background field method for the $\mathcal{N}=3$ Chern-Simons theory given in the previous section are analogous to those in the $\mathcal{N}=2; d=4$ case [17]. Therefore one can follow all the steps of proving the four-dimensional $\mathcal{N}=2$ nonrenormalization theorem in [18] to arrive at the same conclusion in the $\mathcal{N}=3$, $d=3$ Chern-Simons-matter theory. Of course, the gauge and matter propagators in $\mathcal{N}=3$, $d=3$ theory are slightly different from their four-dimensional counterparts, and this should be taken into account in the proof of the nonrenormalization theorem in the considered case. In what follows we compute the superficial degree of divergences in this theory and prove that the UV divergences are absent.

For calculating the superficial degree of divergence we need to know the structure of superfield propagators for the matter and gauge superfields. Within the background field method these propagators are given by the expressions (3.45), (3.46) and (3.44), respectively. However, for computing the superficial degree of divergence it is sufficient to know the free propagators (3.55), (3.56) and (3.54) since all terms which complement these propagators to the gauge covariant form are only able to diminish the degree of divergence of a diagram.

Let us consider some background-field dependent supergraph G with L loops, P propagators and N_{mat} external matter legs. In the process of computation of the contribution of such a graph N_D covariant spinor derivatives may hit the external legs as a result of integration by parts, thereby reducing the degree of divergence of the diagram. Like in the

$\mathcal{N}=2$, $d=4$ gauge theory, the superficial degree of divergence $\omega(G)$ of this graph is given by

$$\omega(G) = 3L - 2P + (2P - N_{mat} - 3L) - \frac{1}{2}N_D = -N_{mat} - \frac{1}{2}N_D\,. \tag{3.62}$$

Here $3L$ is the contribution of loop momenta, $-2P$ comes from the factors $\square^{-1}$ in the propagators while another $2P$ corresponds to the operators $(D^{++})^2(D^{0})^2$ standing in the numerators of the propagators. We point out that the number $2P$ in the term within round brackets in (3.62) is decreased by the number N_{mat} because each external matter leg effectively takes one $(D^{++})^2$ operator to restore full superspace measure by the rule (A.23). Another negative contribution $-3L$ in this term appears since for each loop we have to apply the identity (4.4) which reduces the number of the covariant spinor derivatives. Thus we see that any diagram with external matter legs is automatically finite. For the diagrams without external matter legs, the last contribution $-\frac{1}{2}N_D$ in (3.62) plays the crucial role. This contribution appears when N_D covariant spinor derivatives hit the background gauge superfield V^{++}. In full analogy with the $\mathcal{N}=2$, $d=4$ supergauge theory, one can argue that $N_D > 0$ as a result of using the background field method. Indeed, within the background field method the result of computing any diagram automatically comes out in a gauge covariant form. In other words, it is expressed in terms of the covariant superfield strengths W^{ij} given in (2.2) and their covariant spinor derivatives. These derivatives are expressed in terms of the gauge superfield V^{++} with some number of covariant spinor derivatives on it (see, e.g., (2.26), (2.27)). This means that these derivatives should be effectively taken off from the propagators, thereby decreasing the superficial degree of divergence of the resulting graph by the number N_D. As a result, we arrive at the inequality $\omega(G) < 0$, which proves the UV finiteness of all quantum diagrams in the model under consideration. Some examples of such one-loop quantum computations will be presented in Section 4, just to confirm the proof given here. It is worthwhile to forewarn that all calculations in Section 4 will be performed with massless propagators for both the matter and the gauge superfields, which may lead to infrared divergences like in (4.10). However, such divergences automatically disappear if one studies the contributions to the effective action within the background field method, when all the propagators are effectively massive. This completes our arguments towards the quantum finiteness of the $\mathcal{N}=3$ Chern-Simons matter theory.

3.5 General structure of the on-shell effective action

For simplicity, we discuss the general structure of low-energy on-shell effective action in the Abelian Chern-Simons theory interacting with q hypermultiplet,[6]

$$S = \int d\zeta^{(-4)}\left(\frac{1}{2}V^{++}W^{++} + \tilde{q}^+ D^{++} q^+ + \tilde{q}^+ V^{++} q^+\right). \tag{3.63}$$

The classical equations of motion are given by

$$\nabla^{++} q^+ = 0\,,\quad \nabla^{++}\tilde{q}^+ = 0\,,\quad W^{++} = \tilde{q}^+ q^+\,. \tag{3.64}$$

[6] Here we omit the Chern-Simons coupling constant for simplicity.

These equations mean that in the τ frame the hypermultiplet superfields are linear in harmonics,

$$q^+ = u^+_i q^i, \qquad \tilde{q}^+ = u^+_i \tilde{q}^i, \tag{3.65}$$

while the gauge superfield strength reads

$$W^{ij} = q^{(i}\tilde{q}^{j)}. \tag{3.66}$$

In general, the on-shell effective action can be written as a sum of two terms expressed as integrals over the analytic subspace and full superspace,

$$\Gamma = S + \bar{\Gamma} = S + \int d\zeta^{(-4)} L_{\rm analytic} + \int d^9z\, L_{\rm full}. \tag{3.67}$$

Here S is the classical action, while $\bar{\Gamma}$ corresponds to the quantum corrections. Since the model (3.63) is scale invariant and there is no room for the conformal anomaly, as soon as there are no any divergences, the effective action Γ should be scale-invariant as well. However, there exist no any other scale invariant analytic superspace invariants except for the terms of the classical action (3.63). Therefore the effective action should receive non-trivial contributions only in the form of integrals over the full superspace,

$$\bar{\Gamma} = \int d^9z\, L(q^i, \tilde{q}^i, W^{ij}, \ldots), \tag{3.68}$$

where dots stand for the terms with various gauge covariant derivatives of the superfields q^i, $\tilde{q}^i$ and W^{ij} while L is some scale-independent gauge invariant function of its arguments. The gauge invariance of the effective action (3.68) is ensured by the use of the background field method.

We should take into account that on shell all the superfield strengths W^{ij} in (3.68) are expressed through the hypermultiplet superfields by virtue of (3.66). Therefore, on shell the low-energy effective action can depend on the hypermultiplet superfields and their derivatives of arbitrary order,

$$\bar{\Gamma} = \int d^9z\, L(q^i, \tilde{q}^i, D^{ij}q^k, D^{ij}\tilde{q}^k, \partial_{\alpha\beta} q^i, \partial_{\alpha\beta} \tilde{q}^i, \ldots). \tag{3.69}$$

In principle, one can look for the pure potential-like terms in the effective action, i.e., terms containing no derivatives. However, such terms cannot appear in the effective action (3.69). Indeed, there is the unique $SU(2)$ invariant independent superfield combination $q^i\bar{q}_i$, but any Lagrangian depending only on $q^i\bar{q}_i$ would involve a scale, $L = L(q^i\bar{q}_i/\Lambda)$, $[\Lambda] = 1$. Therefore, the expansion of the effective action starts from the terms with derivatives. For instance, the following terms are admissible in the full superspace Lagrangian,

$$\frac{D^{ij}q^k D_{ij}\bar{q}_k}{(q^i\bar{q}_i)^2}, \quad \frac{D^{ij}\tilde{q}^k D_{ij}\bar{\tilde{q}}_k}{(q^i\bar{q}_i)^2}, \quad \frac{D^{ij}q^k D_{ij}\tilde{q}_k}{(q^i\bar{q}_i)^2}. \tag{3.70}$$

Further hints concerning the possible structure of the low-energy effective action can be gained from the explicit quantum superfield computations.

3.6 Effective action in the one-loop approximation

Let us turn back to the general non-Abelian Chern-Simons theory interacting with some number of q hypermultiplets with the action (3.1). Within the background field method the effective action is given by the generating functional (3.37). The one-loop contributions to the effective action are defined by the quadratic action (3.38),

$$\Gamma^{(1)} = \frac{i}{2}\mathrm{Tr}_{(4,0)}\ln\hat\Box - \frac{i}{2}\mathrm{Tr}_{Ad}\ln(\nabla^{++})^2 + \frac{i}{2}\mathrm{Tr}\ln\begin{pmatrix} 0 & \nabla^{++} & q^+ \\ -\nabla^{++} & 0 & q^+ \\ q^+ & q^+ & \hat\Box \end{pmatrix} : \tag{3.71}$$

The first term in (3.71) corresponds to $(\mathrm{Det}_{(4,0)}\hat\Box)$ in (3.37) while the second term $\frac{i}{2}\mathrm{Tr}_{Ad}\ln(\nabla^{++})^2$ is responsible for the contributions from the ghost superfields which are in the adjoint representation of the gauge group. The last matrix term in (3.71) appears from the second line of (3.38) and it takes into account both the hypermultiplet and gauge superfield contributions. Making the Cartan-Iwasawa decomposition of this matrix, we can rewrite the effective action in the following form

$$\Gamma^{(1)} = \frac{i}{2}\mathrm{Tr}_{(2,2)}\ln\hat\Box - \frac{i}{2}\mathrm{Tr}_{(4,0)}\ln\hat\Box - \frac{i}{2}\mathrm{Tr}_{Ad}\ln(\nabla^{++})^2 + \frac{i}{2}\mathrm{Tr}\ln H ; \tag{3.72}$$

where the operator H is given by

$$H = \begin{pmatrix} q^+\frac{1}{\hat\Box}q^+ & \nabla^{++} - q^+\frac{1}{\hat\Box}q^+ \\ -\nabla^{++} - q^+\frac{1}{\hat\Box}q^+ & q^+\frac{1}{\hat\Box}q^+ \end{pmatrix} : \tag{3.73}$$

The expression (3.72) is the starting point for the one-loop perturbation theory in the general $\mathcal{N}=3$ Chern-Simons matter theory.

4. Examples of supergraph computations

4.1 Hypermultiplet two-point function

Let us consider the q-hypermultiplet effective action in the case of Abelian gauge superfield

$$\Gamma_{hyp} = i\mathrm{Tr}\ln(D^{++}+V^{++}) = \sum_{n=2}^{\infty}\Gamma_{hyp,n} ; \qquad \Gamma_{hyp,n} = i\frac{(-1)^{n+1}}{n}\mathrm{Tr}\left(\frac{1}{D^{++}}V^{++}\right)^n : \tag{4.1}$$

Explicitly, the two-point function Γ_2 depicted in Fig. 1a, is given by

$$\Gamma_{hyp,2} = \frac{i}{2}\int d\zeta_1^{(-4)}d\zeta_2^{(-4)}G_0^{(1,1)}(1|2)V^{++}(2)G_0^{(1,1)}(2|1)V^{++}(1) : \tag{4.2}$$

Next, we apply the expression (3.55) for the propagator and use two $(D^{++})^2$ operators to restore the full $\mathcal{N}=3$ harmonic superspace measure by the rule (A.23),

$$\Gamma_{hyp,2} = -\frac{i}{32}\int d^3x_1 d^6\theta_1 d^3x_2 d^6\theta_2 du_1 du_2 V^{++}(1)V^{++}(2)\frac{1}{\Box}(D_1^0)^2\frac{\delta^9(z_1-z_2)}{(u_1^+u_2^+)^3}$$
$$\times\frac{1}{\Box}(D_1^{++})^2(D_2^{++})^2(D_2^0)^2\frac{\delta^9(z_2-z_1)}{(u_2^+u_1^+)^3} : \tag{4.3}$$

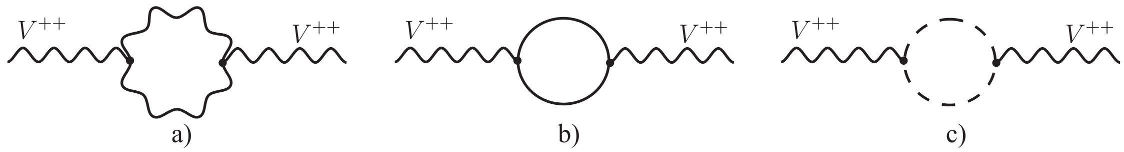

Figure 1: Hypermultiplet, gauge superfield and ghost contributions to Γ_2.

To shrink down the loop over Grassmann variables to a point we apply the identity

$$\delta^6(\theta_1-\theta_2)(D_1^{++})^2(D_2^{++})^2(D_2^0)^2\delta^9(z_2-z_1) = -16(u_1^+u_2^+)^4\delta^9(z_1-z_2)\,; \tag{4.4}$$

and pass to the momentum representation for the superfields,

$$\Gamma_{hyp,2} = \frac{1}{16}\int\frac{d^3p}{(2\pi)^3}d^6\theta\,\frac{du_1du_2}{(u_1^+u_2^+)^2}\frac{1}{\sqrt{p^2}}((D_1^0)^2V^{++}(p,\theta,u_1))V^{++}(-p,\theta,u_2)\,. \tag{4.5}$$

In the Abelian case the relation (2.25) between V^{++} and V^{--} becomes very simple,

$$V^{--}(z,u_1) = \int du_2\frac{V^{++}(z,u_2)}{(u_1^+u_2^+)^2}\,. \tag{4.6}$$

Using this relation, the expression (4.5) can be rewritten as

$$\begin{aligned}\Gamma_{hyp,2} &= \frac{i}{16}\int d^3x d^6\theta\, du\,((D^0)^2V^{++}(x,\theta,u))\frac{1}{\sqrt{\Box}}V^{--}(x,\theta,u)\\ &= \frac{i}{16}\int d\zeta^{(-4)}((D^0)^2V^{++})\frac{1}{\sqrt{\Box}}W^{++}\,.\end{aligned} \tag{4.7}$$

By $1/\sqrt{\Box}$ we denote a non-local operator which acts as the multiplication by $1/\sqrt{p^2}$ in the momentum representation. Finally, it is easy to see that (4.7) is non other than the Abelian SYM action with the insertion of the non-local operator $1/\sqrt{\Box}$,

$$\Gamma_{hyp,2} = \frac{i}{16}\int d\zeta^{(-4)}W^{++}\frac{1}{\sqrt{\Box}}W^{++}\,. \tag{4.8}$$

A similar result was obtained in the non-supersymmetric Chern-Simons matter theory [26], as well as in studying quantum corrections in BLG theory [24].

As a result, the leading contribution to the hypermultiplet effective action given by the two-point function reproduces the SYM action with the insertion of non-local operator $1/\sqrt{\Box}$. Of course, such a non-local operator appears because we do our computations in the massless theory in which the momentum integral $\int\frac{d^3k}{k^2(p+k)^2}$ is plagued by the infrared divergence at small p. At zero external momentum, $p=0$, one can regularize this integral by introducing the parameter ϵ as a cut-off at small k,

$$\int\frac{d^3k}{(2\pi)^3}\frac{1}{k^2(p+k)^2}\xrightarrow{\text{local limit}}\int\frac{d^3k}{(2\pi)^3}\frac{1}{k^4}\to\frac{i}{2\pi^2}\int_\epsilon^1\frac{dk}{k^2}=\frac{i}{2\pi^2}\frac{1}{\epsilon}\,. \tag{4.9}$$

Then, the action (4.8) in the local limit is given by

$$\Gamma_{hyp,2} = \frac{1}{4\pi^2}\int d\zeta^{(-4)} W^{++}W^{++} . \tag{4.10}$$

Alternatively, to avoid the regularization of the momentum integral (4.9), one can consider the model of massive hypermultiplet interacting with the Abelian gauge superfield. Such a model is studied in Section 5.

One can easily generalize the result (4.8) to the case of a hypermultiplet in some representation R of non-Abelian gauge group G,

$$\Gamma_{hyp,2} = -T(R)\frac{i}{16}\int d\zeta^{(-4)} W_0^{++a}\frac{1}{\sqrt{\Box}}W_0^{++a} . \tag{4.11}$$

Here, $\mathrm{tr}(T^a_R T^b_R) = T(R)\,\delta^{ab}$, $T(\mathrm{adjoint}) = 1$, and W_0^{++a} is the linear in V^{++} part of the full non-Abelian superfield strength W^{++a}.

4.2 Gauge and ghost superfield two-point functions

The next example of quantum computations is represented by the diagrams b) and c) at Fig. 1 which make the leading two-point contributions to the functional integral (3.28). To study the pure gauge superfield diagram a) it is sufficient to consider the Chern-Simons action (2.32) up to the cubic term,

$$\begin{aligned} S = S_2 + S_{gf} + S_3 = {} & \frac{ik}{8}\mathrm{tr}\int d\zeta^{(-4)} V^{++}(D^0)^2 V^{++} \\ & -\frac{ik}{24}\mathrm{tr}\int d^9z\, du_1 du_2 du_3 \frac{V^{++}(z,u_1)[V^{++}(z,u_2),V^{++}(z,u_3)]}{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)} . \end{aligned} \tag{4.12}$$

Let us expand the gauge superfields over the generators T^a of gauge group G, $V^{++} = V^{++a}T^a$, so that

$$[V^{++}(z,u_1), V^{++}(z,u_2)] = V^{++a}(z,u_1)V^{++b}(z,u_2) f^{abc} T^c , \tag{4.13}$$

where f^{abc} are the structure constants. As a result, the action (4.12) is rewritten as

$$\begin{aligned} S = {} & \frac{ik}{8}\int d\zeta^{(-4)} V^{++a}(D^0)^2 V^{++a} \\ & -\frac{ik}{24} f^{abc}\int d^9z\, du_1 du_2 du_3 \frac{V^{++a}(z,u_1)V^{++b}(z,u_2)V^{++c}(z,u_3)}{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)} . \end{aligned} \tag{4.14}$$

The contribution of this action to the one-loop effective action in the $\mathcal{N} = 3$ Chern-Simons theory is as follows

$$\begin{aligned} \Gamma_{CS} = \frac{i}{2}\mathrm{Tr}\ln\Big(\delta^{ab}\delta_A^{(2,2)}(1|2) - f^{abc}\frac{1}{16}(D^0_{(1)})^2(D^{++}_{(1)})^2(D^{++}_{(2)})^2\delta^9(z_1 - z_2) \\ \times\int du_3 \frac{V^{++c}(z,u_3)}{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)}\Big) . \end{aligned} \tag{4.15}$$

We single out in (4.15) the two-point contribution,

$$\Gamma_{CS,2} = \frac{i}{16\cdot 64} f^{abc} f^{bad} \int d\zeta_1^{(-4)} d\zeta_2^{(-4)} du_3 du_4 \times \frac{1}{\Box}(D^0_{(1)})^2 (D^{++}_{(1)})^2 (D^{++}_{(2)})^2 \delta^9(z_1-z_2) \frac{V^{++c}(z_1,u_3)}{(u_1^+u_2^+)(u_2^+u_3^+)(u_3^+u_1^+)} \times \frac{1}{\Box}(D^0_{(2)})^2 (D^{++}_{(2)})^2 (D^{++}_{(1)})^2 \delta^9(z_2-z_1) \frac{V^{++d}(z_2,u_4)}{(u_2^+u_1^+)(u_1^+u_4^+)(u_4^+u_2^+)}. \quad (4.16)$$

Further computations are analogous to those performed in the previous subsection: we restore the full superspace measure by the rule (A.23) and shrink down the loop over the Grassmann variables to a point using the identity (4.4),

$$\Gamma_{CS,2} = \frac{i}{4} f^{abc} f^{abd} \int d^3x_1 d^3x_2 d^6\theta\, du_1 du_2 du_3 du_4 \frac{1}{\Box}\delta^3(x_1-x_2)\frac{1}{\Box}\delta^3(x_2-x_1) \times \frac{V^{++d}(x_2,\theta,u_4)(D^0_{(1)})^2 V^{++c}(x_1,\theta,u_3)(u_1^+u_2^+)^2}{(u_2^+u_3^+)(u_3^+u_1^+)(u_1^+u_4^+)(u_4^+u_2^+)}. \quad (4.17)$$

To compute the harmonic integrals, we apply the following identity

$$\int du_1 du_2 \frac{(u_1^+u_2^+)^2}{(u_2^+u_3^+)(u_3^+u_1^+)(u_1^+u_4^+)(u_4^+u_2^+)} = -2\frac{(u_3^-u_4^-)}{(u_3^+u_4^+)}. \quad (4.18)$$

Passing to the momentum representation and computing the momentum integral, we find

$$\Gamma_{CS,2} = \frac{1}{16} f^{abc} f^{abd} \int \frac{d^3p}{(2\pi)^3} d^6\theta\, du_1 du_2 \frac{1}{\sqrt{-p^2}} (D^0_{(1)})^2 V^{++c}(p,\theta,u_1) V^{++d}(-p,\theta,u_2) \frac{(u_1^-u_2^-)}{(u_1^+u_2^+)}. \quad (4.19)$$

This expression is non-local only in the harmonic variables.

Finally, we consider the ghost field action (3.12) which can be rewritten for the vanishing background field as

$$S_{gh} = \int d\zeta^{(-4)} [b^a D^{++} D^{++} c^a + f^{abc} b^a D^{++} V^{++b} c^c]. \quad (4.20)$$

The one-loop effective action for the ghost superfields reads (minus sign is due to the odd statistics of ghost superfields)

$$\Gamma_{gh} = -i\,\mathrm{Tr}\ln\left[\delta^{ab}\delta_A^{(0,4)}(1|2) + f^{abc} V^{++c}(2) D^{++}_{(2)} G_0^{(0,0)}(1|2)\right]. \quad (4.21)$$

We need only the two-point contribution depicted in Fig. 1c,

$$\Gamma_{hg,2} = -\frac{i}{2}\int d\zeta^{(-4)}_{(1)} d\zeta^{(-4)}_{(2)} f^{abc} f^{bad} V^{++d}(1) V^{++c}(2) D^{++}_{(2)} G_0^{(0,0)}(1|2) D^{++}_{(1)} G_0^{(0,0)}(2|1). \quad (4.22)$$

Further we assume that the structure constants are normalized in such a way that $f^{abc}f^{bad} = -\delta^{cd}$, i.e., $T(\text{adjoint}) = 1$. Omitting the details of computations (which are analogous to those in the previous subsection), we obtain

$$\Gamma_{hg,2} = \frac{1}{16}\int \frac{d^3p}{(2\pi)^3} d^6\theta\, du_1 du_2 \frac{1}{\sqrt{-p^2}} (D^0_{(1)})^2 V^{++a}(p,\theta,u_1) V^{++a}(-p,\theta,u_2) \frac{(u_1^-u_2^+)(u_1^+u_2^-)}{(u_1^+u_2^+)^2}. \quad (4.23)$$

Now we sum up the gauge and ghost superfields two-point contributions (4.19), (4.23),

$$
\begin{aligned}
\Gamma_{gauge,2} &= \Gamma_{CS,2} + \Gamma_{gh,2} \\
&= -\frac{1}{16}\int \frac{d^3p}{(2\pi)^3} d^6\theta\, du_1 du_2 \frac{(D^0_{(1)})^2 V^{++a}(p,\theta;u_1) V^{++a}(-p,\theta;u_2)}{\sqrt{p^2}\,(u_1^+ u_2^+)^2}\,, \qquad (4.24)
\end{aligned}
$$

where the following identity has been used

$$
\frac{(u_1^- u_2^+)(u_1^+ u_2^-)}{(u_1^+ u_2^+)^2} - \frac{(u_1^- u_2^-)}{(u_1^+ u_2^+)} = -\frac{1}{(u_1^+ u_2^+)^2}\,. \qquad (4.25)
$$

The expression (4.24) can be rewritten in the analytic superspace,

$$
\Gamma_{gauge,2} = \frac{i}{16}\int d\zeta^{(-4)} W_0^{++a} \frac{1}{\sqrt{\square}} W_0^{++a}\,, \qquad (4.26)
$$

where W_0^{++a} is the linear in V^{++} part of the full non-Abelian superfield strength W^{++a}.

Note that the hypermultiplet two-point function (4.11) has exactly the same form, but opposite sign. Hence, these two contributions cancel out each other if one takes n hypermultiplet q_i^+ in representations R_i, providing that $\sum_{i=1}^n T(R_i) = 1$. For instance, one q-hypermultiplet in the adjoint representation is sufficient for these two contributions to cancel each other.

A similar cancellation between the hypermultiplet and gauge superfield two-point functions plays the important role in the $N=2$, $d=4$ gauge theory [3], where it is the manifestation of quantum UV finiteness of the $N=4$, $d=4$ SYM theory. However, in our case this cancellation is not of the same significance as for the four-dimensional models, because all quantum contributions are now divergenceless. The term (4.26) does not contribute to the Chern-Simons effective action since it vanishes on the classical equations of motion for the pure gauge superfields. Moreover, this term is gauge-variant. This was explained in [27] for non-supersymmetric Chern-Simons theory, but this is true in our case too. Recall that we work in the Fermi-Feynman gauge, $\xi = -1$, while the authors of [26, 24] used the Landau gauge $\xi = 0$ for which the contributions of the form (4.26) are absent in the pure Chern-Simons theory.

4.3 Vanishing of tadpoles and hypermultiplet self-energy

Now we shall consider the tadpole as well as hypermultiplet self-energy diagrams depicted in Fig. 2 and show that their contributions vanish as a consequence of the properties of Grassmann and harmonic distributions.

The vanishing of the pure gauge superfield diagram a) at Fig. 2 is obvious. Indeed, the gauge superfield propagator (3.54) involves four Grassmann derivatives acting on the delta function,

$$
G_0^{(2,2)}(1|2) = -\frac{1}{4\square}(D^0)^2 (D^{++})^2 \delta^9(z_1 - z_2)\delta^{(-2,2)}(u_1,u_2)\,. \qquad (4.27)
$$

Therefore it vanishes at the coincident points due to the deficit of Grassmann derivatives. By the same reason vanish similar tadpole diagrams with more vector legs outgoing from a single point.

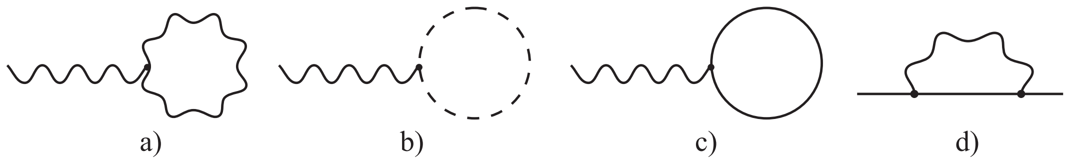

Figure 2: Tadpoles and hypermultiplet self-energy diagram.

The diagrams on Fig. 2b), 2c) vanish because of the properties of harmonics. Indeed, the hypermultiplet propagator (3.55) has six Grassmann derivatives which are necessary to kill all Grassmann variables of the Grassmann delta function,

$$(D^0_{(1)})^2 (D^{++}_{(1)})^2 (D^{++}_{(2)})^2 \frac{\delta^6(\theta_1-\theta_2)}{(u_1^+u_2^+)^3}\Big|_{\theta(1)=\theta(2)} = -16(u_1^+u_2^+)|_{u_1=u_2} = 0\,. \qquad (4.28)$$

The similar identity can be obtained for the ω-hypermultiplet propagator (3.56) which is responsible for the ghost field contribution depicted in Fig. 2b).

The hypermultiplet self-energy diagram requires a more careful consideration. Up to a numerical factor, it is given by

$$\Gamma_{qq} \propto \int d\zeta_1^{(-4)} d\zeta_2^{(-4)} q^+(1) q^+(2) \frac{1}{\Box}(D^0_{(1)})^2 (D^{++}_{(1)})^2 \delta^9(z_1-z_2)\,\delta^{(-2,2)}(u_1,u_2)$$
$$\times\frac{1}{\Box}(D^0_{(2)})^2 (D^{++}_{(1)})^2 (D^{++}_{(2)})^2 \frac{\delta^9(z_1-z_2)}{(u_1^+u_2^+)^3}\,. \qquad (4.29)$$

Now we restore the full superspace measure and shrink down the θ-loop using the identity

$$\delta^6(\theta_1-\theta_2)(D^0_{(1)})^2(D^0_{(2)})^2(D^{++}_{(1)})^2\delta^9(z_1-z_2) = -16(u_1^+u_2^+)^2(u_1^+u_2^-)^2\delta^9(z_1-z_2)\,. \qquad (4.30)$$

We have exactly six Grassmann derivatives for this identity. As a result,

$$\Gamma_{qq} \propto \int d^3x_1 d^3x_2 d^6\theta\, du_1 du_2\, q^+(x_1,\theta,u_1) q^+(x_2,\theta,u_2) \frac{(u_1^+u_2^-)}{(u_1^+u_2^+)}\delta^{(-2,2)}(u_1,u_2)$$
$$\times\frac{1}{\Box}\delta^3(x_1-x_2)\frac{1}{\Box}\delta^3(x_2-x_1)\,. \qquad (4.31)$$

In principle, the harmonic distribution $\frac{(u_1^+u_2^-)}{(u_1^+u_2^+)}\delta^{(-2,2)}(u_1,u_2)$ in (4.31) is potentially dangerous due to the problem of coincident harmonic singularities. But this problem is resolved here by passing to the analytic subspace and using the resulting $(D^{++})^2$ operator to produce extra harmonic factors,

$$(D^{++}_{(2)})^2 q^+(x,\theta,u_1) = (u_1^+u_2^+)^2[\Box_{(1)}(u_1^+u_2^+)^2 - 4D^{--}_{(1)}D^0_{(1)}(u_1^+u_2^+)(u_1^-u_2^+)$$
$$+4(D^0_{(1)})^2(u_1^-u_2^+)^2]q^+(x,\theta,u_1)\,. \qquad (4.32)$$

The factor $(u_1^+u_2^+)^2$ in the r.h.s. of (4.32) cancels the denominator of the harmonic distribution in (4.31) and gives zero due to the identity $(u_1^+u_2^+)\delta^{(-2,2)}(u_1,u_2) = 0$. As a result, the hypermultiplet self-energy contribution vanishes, $\Gamma_{qq} = 0$.

5. $N=3$ supersymmetry with central charges

The $N=3$ superalgebra without central charges is generated by the operators (A.10) with the anticommutation relations (A.12). In this section we shall study an extension of this superalgebra by the central charge operators Z^{ij}. The relations (A.12) are replaced by the following ones

$$\{Q^{ij}_\alpha, Q^{kl}_\beta\} = -i(\varepsilon^{ik}\varepsilon^{jl} + \varepsilon^{il}\varepsilon^{jk})\partial_{\alpha\beta} - \frac{1}{2}\varepsilon_{\alpha\beta}(\varepsilon^{ik}Z^{jl} + \varepsilon^{jk}Z^{il} + \varepsilon^{jl}Z^{ik} + \varepsilon^{jl}Z^{ik}) \,. \qquad (5.1)$$

The operators Z^{ij} commute with all other generators except those of the R-symmetry SU(2) algebra. We will show that just this modified $N=3$ superalgebra is inherent in the massive hypermultiplet model, in analogy with the four-dimensional case [10, 28, 29].

5.1 Massive hypermultiplet model

Let us consider the Abelian version of the q-hypermultiplet model (2.44)

$$S_m = -\int d^{(-4)}\zeta\, \tilde{q}^+ (D^{++} + V_0^{++}) q^+ \,, \qquad (5.2)$$

with the background gauge superfield given by

$$V_0^{++} = 3(\theta^{++})^2 u_i^- u_j^- Z^{ij} \,, \qquad Z^{ij} = Z^{ji} = \text{const}. \qquad (5.3)$$

One can easily find the relevant bridge superfield Ω_0,

$$\Omega_0 = 3\theta^{++}\theta^{0} u_k^- u_l^- Z^{kl} + \theta^{0} u_k^+ u_l^- Z^{kl} - \theta^{++} \theta^{--} u_k^+ u_l^- Z^{kl} - 2(\theta^{0})^2 u_k^+ u_l^- Z^{kl} \,, \qquad (5.4)$$

as a solution of the equation

$$D^{++} + V_0^{++} = e^{-\Omega_0} D^{++} e^{\Omega_0} \,. \qquad (5.5)$$

Now, using the relations (2.22), we obtain the connections for the covariant spinor derivatives

$$\mathcal{D}^{ij}_\alpha = D^{ij}_\alpha + V^{ij}_{0\alpha} \,, \qquad V^{ij}_{0\alpha} = \frac{1}{2}\theta^{ik}_\alpha Z^j_k + \frac{1}{2}\theta^{jk}_\alpha Z^i_k \,. \qquad (5.6)$$

These derivatives satisfy the following anticommutation relations

$$\{\mathcal{D}^{ij}_\alpha, \mathcal{D}^{kl}_\beta\} = i(\varepsilon^{ik}\varepsilon^{jl} + \varepsilon^{il}\varepsilon^{jk})\partial_{\alpha\beta} + \frac{1}{2}\varepsilon_{\alpha\beta}(\varepsilon^{ik}Z^{jl} + \varepsilon^{jk}Z^{il} + \varepsilon^{jl}Z^{ik} + \varepsilon^{jl}Z^{ik}) \,. \qquad (5.7)$$

The original supercharges (A.10) do not anticommute with (5.6). However, one can define the modified supercharges

$$\mathcal{Q}^{ij}_\alpha = Q^{ij}_\alpha - V^{ij}_{0\alpha} = Q^{ij}_\alpha - \frac{1}{2}\theta^{ik}_\alpha Z^j_k - \frac{1}{2}\theta^{jk}_\alpha Z^i_k \,, \qquad (5.8)$$

so that $\mathcal{Q}^{ij}_\alpha$ anticommute with (5.6), i.e. $\{\mathcal{Q}^{ij}_\alpha, \mathcal{D}^{kl}_\beta\} = 0$. One can easily check that the operators $\mathcal{Q}^{ij}_\alpha$ satisfy the anticommutation relations of $N=3$ superalgebra with central charges (5.1). The central charge operators are realized as the multiplication by the constants Z^{ij}.

It is worth noting that these constant central charges explicitly break the R-symmetry SU(2) down to $U(1) \times SU(2)$.

It is a non-trivial task to show that the equations of motion in the model (5.2) lead to the mass-shell condition for the hypermultiplet superfield q^+. To this end we introduce the following notation

$$Z^{++} = u^+_i u^+_j Z^{ij}\,; \quad Z^{--} = u^-_i u^-_j Z^{ij}\,; \quad Z^0 = u^+_i u^-_j Z^{ij} \tag{5.9}$$

and

$$\nabla^{++} = D^{++} + V_0^{++}\,; \qquad \nabla^{--} = D^{--} + V_0^{--}\,; \tag{5.10}$$

where

$$V_0^{--} = D^{--} v_0 = 2\theta^{++}\theta^{--} Z^{--} + 4(\theta^0)^2 Z^{--} - 4\theta^{--}\theta^0 Z^0 + (\theta^{--})^2 Z^{++}\,. \tag{5.11}$$

The equation of motion in the model (5.2) has the following important corollaries

$$\nabla^{++} q^+ = 0\,; \;\Rightarrow\; (\nabla^{--})^2 q^+ = 0\,; \;\Rightarrow\; (D^{++})^2(\nabla^{--})^2 q^+ = 0\,. \tag{5.12}$$

Hence, each operator from the set

$$(D^{++})^2(\mathcal{D}^0)^2(\nabla^{--})^2\,; \quad (D^{++})^2(\nabla^{--})^2(D^{++})^2(\nabla^{--})^2\,; \quad (D^{++})^2(\nabla^{--})^2 \tag{5.13}$$

annihilates the superfield q^+ on-shell. Here $\mathcal{D}^0 = D^0 + Z^0\theta^0\theta^0 - \frac{1}{2}Z^{--}\theta^{++}\theta^{0} - \frac{1}{2}Z^{++}\theta^{--}\theta^{0}$.
Based on the important identity for these operators

$$\frac{1}{12}(D^{++})^2(\mathcal{D}^0)^2(\nabla^{--})^2 + \frac{1}{192}(D^{++})^2(\nabla^{--})^2(D^{++})^2(\nabla^{--})^2 + \frac{1}{6}(D^{++})^2 Z^0(\nabla^{--})^2 = \Box + \frac{1}{2}Z^{ij}Z_{ij}\,, \tag{5.14}$$

which holds in application to the analytic superfields, we derive the mass-shell condition for the hypermultiplet superfield,

$$(\Box + m^2)q^+ = 0\,; \qquad m^2 = \frac{1}{2}Z^{ij}Z_{ij}\,. \tag{5.15}$$

Thus we have demonstrated that the model (5.2) does describe the massive hypermultiplet model with the mass squared being equal to the square of the central charge operators. All these considerations are analogous to those in the four-dimensional q-hypermultiplet model. Minor complications stem from the fact that in the three-dimensional case the central charge Z^{ij} has SU(2) indices. It is obvious that such a central charge indeed breaks the SU(2) R-symmetry of the $N=3$ superalgebra down to U(1).

The propagator of the massive hypermultiplet can be easily deduced from the full hypermultiplet propagator (3.45) by choosing the background superfield strengths to be constant, $W^{ij} = Z^{ij}$,

$$G_m^{(1,1)}(1|2) = -\frac{1}{48}\frac{1}{\Box + m^2}(D^{++}_{(1)})^2(D^{++}_{(2)})^2[3(\mathcal{D}^0_{(1)})^2 + 3Z^0 - Z^{++}\nabla^{--}_{(1)}]\frac{e^{\tilde v_0(2) - \tilde v_0(1)}\delta^9(z_1 - z_2)}{(u^+_1 u^+_2)^3}\,, \tag{5.16}$$

where the mass m is defined in (5.15).

5.2 $N=3$ SYM as a quantum correction in the massive hypermultiplet model

It is well known that the standard Chern-Simons action appears as a result of computation of the one-loop two-point diagram with a massive fermion inside and two vector fields on the external legs [9]. Naively, one could expect that the $N=3$ supersymmetric version of the Chern-Simons theory (2.32) can also be derived from the massive hypermultiplet two-point function of the form depicted in Fig. 1a. Surprisingly, such a computations in the $N=3$ supersymmetric theory yields the $N=3$ SYM action rather than the Chern-Simons one.

Indeed, consider the model of massive q-hypermultiplet interacting with the background Abelian gauge superfield V^{++},

$$S_{hyp,m} = -\int d\zeta^{(-4)} \tilde{q}^+ (D^{++} + V_0^{++} + V^{++}) q^+ \,; \tag{5.17}$$

where V_0^{++} is given by (5.3). The action (5.17) is invariant with respect to the following Abelian gauge transformations

$$\delta V^{++} = -D^{++}\lambda\,; \quad \delta\tilde{q}^+ = i\lambda\tilde{q}^+\,; \quad \delta q^+ = -i\lambda q^+\,; \tag{5.18}$$

λ being an analytic superfield gauge parameter. The formal expression for the massive hypermultiplet two-point function is given by

$$\Gamma_2 = \frac{i}{2}\int d\zeta^{(-4)}_{(1)} d\zeta^{(-4)}_{(2)} G_m^{(1,1)}(1|2)V^{++}(2)G_m^{(1,1)}(2|1)V^{++}(1)\,; \tag{5.19}$$

where the massive propagator is defined in (5.16). Subsequent computations are rather similar to those performed in subection 4.1 for the massless hypermultiplet, modulo complications related to the fact that the expression for the massive hypermultiplet propagator is more involved as compared to the massless one. As a result, we obtain

$$\Gamma_2 = \frac{i}{2}\frac{1}{(2\pi)^6}\int d^3p\, d^6\theta\, du\, V^{--}(p,\theta,u)(D^0)^2 V^{++}(-p,\theta,u)\int d^3k\frac{1}{k^2-m^2}\frac{1}{(k+p)^2-m^2}\,. \tag{5.20}$$

Computing the momentum integral at zero momenta p, we deduce the local part of the two-point function in the form

$$\Gamma_2 = \frac{1}{16\pi m}\int d^9z\, du\, V^{--}(D^0)^2V^{++} = -\frac{1}{16\pi m}\int d\zeta^{(-4)} W^{++}W^{++}\,. \tag{5.21}$$

As a result, we obtain the $N=3$ SYM action as a quantum correction in the massive hypermultiplet model.

The reason why the Chern-Simons term does not appear becomes clear in the component fields formulation. The hypermultiplet superfield q^+ contains the spinor ψ^i which is a doublet of the $SU(2)$ R-symmetry group. That part of the massive hypermultiplet action (5.17) which involves the spinor field ψ^i interacting with the vector field is given by

$$S_\psi = \frac{1}{2}\int d^3x\,(\bar\psi^i i\gamma^m D_m\psi_i + \bar\psi^i Z_{ij}\psi^j)\,; \tag{5.22}$$

where $D_{\alpha\beta} = \partial_{\alpha\beta} + iA_{\alpha\beta}$. One can choose the frame with respect to broken SU(2) rotations (acting on the doublet indices) in such a way that the central charge matrix takes the following form

$$\tilde{Z}^i_j = \begin{pmatrix} im & 0 \\ 0 & -im \end{pmatrix} . \tag{5.23}$$

The spinors ψ^1 and ψ^2 decouple from each other and the action (5.22) can now be rewritten as a sum of two standard actions of the massive 3D spinors with the opposite masses,

$$S = S[\psi^1; m] + S[\psi^2; -m] ; \tag{5.24}$$

$$S[\psi; m] = \frac{i}{2}\int d^3x\, (\bar\psi D \psi - m \bar\psi\psi) . \tag{5.25}$$

Each of the spinors in (5.24) makes the same contribution to the one-loop two-point function, modulo the sign (see, e.g., [9])

$$\Gamma_2[A; m] = \frac{1}{8}\frac{m}{|m|}\int d^3x\, \varepsilon_{mnp} A^m \partial^n A^p + \frac{1}{48|m|}\int d^3x\, F_{mn}F^{mn} ; \tag{5.26}$$

where $F_{mn} = \partial_m A_n - \partial_n A_m$. Therefore the Chern-Simons terms cancel each other in the full two-point function for the action (5.24) and so the leading contribution is given by the Maxwell term,

$$\Gamma_2[A; m] + \Gamma_2[A; -m] = \frac{1}{24|m|}\int d^3x\, F_{mn}F^{mn} . \tag{5.27}$$

The action (5.21) is none other than a supersymmetric generalization of (5.27).

The absence of the Chern-Simons term in the hypermultiplet low-energy effective action can be also understood from simple parity reasoning. Indeed, the hypermultiplet classical action (2.44) is even with respect to the P-reflection while the Chern-Simons one (2.32) is odd (see [6] for details). Since there are no any divergences in the one-loop computation (which, if existing, might produce an anomaly), the resulting hypermultiplet effective action should be also P-even. Hence, the Chern-Simons term cannot occur in the hypermultiplet effective action.

5.3 Hypermultiplet self-interaction induced by quantum corrections

It is known that the quartic hypermultiplet self-interaction (2.46) appears as a leading quantum correction in the model of $\mathcal{N}=2$, d=4 massive hypermultiplet interacting with the dynamical Abelian gauge superfield [10]. In this section we will show that a similar phenomenon takes place in the $\mathcal{N}=3$, d=3 gauge theory too.

The classical action of the model under consideration is given by

$$S_{CS,Ab} + S_{hyp,m} ; \tag{5.28}$$

where $S_{CS,Ab}$ is the Abelian Chern-Simons action,

$$S_{CS,Ab} = \frac{ik}{8}\int d\zeta^{(-4)} V^{++} W^{++} ; \tag{5.29}$$

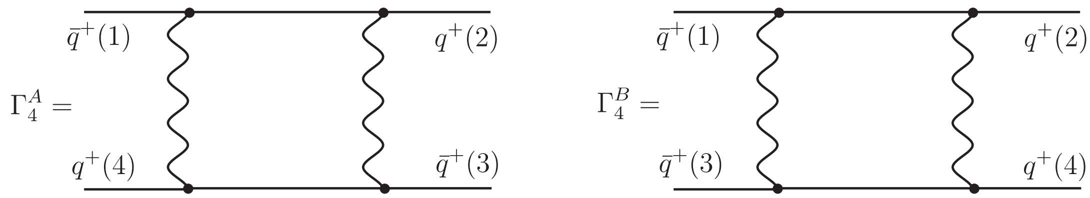

Figure 3: Four hypermultiplet contributions.

while $S_{hyp,m}$ is given by (5.17). The quartic hypermultiplet self-interaction (2.46) appears from the local parts of the diagrams depicted in Fig. 3. Given the propagators for the gauge superfield (3.54) and for the massive hypermultiplet (5.16), we represent these contributions as follows

$$
\begin{aligned}
\Gamma_4^A &= \frac{i}{4}\int d\zeta_1^{(-4)} d\zeta_2^{(-4)} d\zeta_3^{(-4)} d\zeta_4^{(-4)} \bar q^+(1) q^+(2) \bar q^+(3) q^+(4) \\
&\quad G_m^{(1,1)}(1|2) G_0^{(2,2)}(2|3) G_m^{(1,1)}(3|4) G_0^{(2,2)}(4|1)\,; \qquad (5.30)\\
\Gamma_4^B &= \frac{i}{4}\int d\zeta_1^{(-4)} d\zeta_2^{(-4)} d\zeta_3^{(-4)} d\zeta_4^{(-4)} \bar q^+(1) q^+(2) \bar q^+(3) q^+(4) \\
&\quad G_m^{(1,1)}(1|2) G_0^{(2,2)}(2|4) G_m^{(1,1)}(3|4) G_0^{(2,2)}(3|1)\,. \qquad (5.31)
\end{aligned}
$$

Further computations for Γ_4^A and Γ_4^B follow the same line. Therefore we consider in detail only computation of Γ_4^A.

First, we do the integration over $d\zeta_3^{(-4)}$ and $d\zeta_4^{(-4)}$ using the analytic delta-function in the gauge superfield propagator (3.54) and integrate by parts with respect to one of the $(D^0)^2$ operators,

$$
\Gamma_4^A = \frac{4\lambda^2 i}{k^2}\int d\zeta_1^{(-4)} d\zeta_2^{(-4)} \bar q^+(1) q^+(2) \frac{1}{\square} G_m^{(1,1)}(1|2)\,[(D^0_{(2)})^2 \bar q^+(2)\,(D^0_{(1)})^2 q^+(1) \frac{1}{\square} G_m^{(1,1)}(2|1)]\,. \qquad (5.32)
$$

Next, we have to substitute the massive hypermultiplet propagators (5.16) into this expression. It is important that for deriving the contribution of the form (2.46) it is sufficient to take into account only the following term in the massive hypermultiplet propagator (5.16)

$$
G_m^{(1,1)}(1|2) \to \frac{1}{16}\frac{1}{\square + m^2}(D^{++}_{(1)})^2 (D^{++}_{(2)})^2 (D^0_{(1)})^2 \frac{\delta^9(z_1 - z_2)}{(u_1^+ u_2^+)^3}\,. \qquad (5.33)
$$

All other terms in the propagator give rise to higher-order contributions involving derivatives. Substituting (5.33) into (5.32) and restoring the full superspace measure, we obtain

$$
\begin{aligned}
\Gamma_4^A &= \frac{i\lambda^2}{4k^2}\int d^9z_1 d^9z_2 du_1 du_2\, q^+(2) \frac{1}{\square(\square + m^2)} \frac{\delta^9(z_1 - z_2)}{(u_1^+ u_2^+)^3} \qquad (5.34)\\
&\quad (D^0_{(1)})^2 q^+(1)\,(D^0_{(2)})^2 q^+(2)\,(D^0_{(1)})^2 q^+(1)\,(D^0_{(2)})^2 (D^{++}_{(1)})^2 (D^{++}_{(2)})^2 \frac{1}{\square(\square + m^2)} \frac{\delta^9(z_2 - z_1)}{(u_2^+ u_1^+)^3}\,.
\end{aligned}
$$

In this expression, every derivative D^0 acts on everything to the right of it. Therefore, there is plenty of terms with the derivatives D^0 distributed in different ways among them.

However, a non-trivial result can be generated only by those terms where the operator $(D^0)^2$ is present as a whole. For these terms one can apply the identity (4.4) to end up with only one -integration. Two other such operators will produce the box operator by the rule (A.21). As a result, we are left with the following expression

$$
{}^A_4 = \frac{4\ {}^2 i}{k^2}\int d^3x_1 d^3x_2 d^6\ \frac{du_1 du_2}{(u_1^+ u_2^+)^2}\frac{1}{(\ +m^2)}\ {}^3(x_1\ \ x_2)\frac{1}{+m^2}\ {}^3(x_2\ \ x_1)
$$
$$
q^+(2)q^+(2)(D^0_{(1)})^2[q^+(1)q^+(1)]: \qquad (5.35)
$$

Here the term in the second line depends on different x's and u's, but on the same . Next, we pass to the momentum space and compute the momentum integral in the local limit,

$$
\int \frac{d^3p}{p^2(p^2\ \ m^2)^2} = \frac{i\ {}^2}{m^3}; \qquad (5.36)
$$

thus arriving at

$$
{}^A_4 = \frac{}{2m^3k^2}\int d^9z \frac{du_1du_2}{(u_1^+u_2^+)^2} q^+(z;u_2)q^+(z;u_2)(D^0_{(1)})^2[q^+(z;u_1)q^+(z;u_1)]: \qquad (5.37)
$$

The integrand in (5.37) contains a harmonic distribution. We need to single out a local part in this expression in order to get the contribution of the form (2.46). For this purpose we follow the same line as in [10]. We insert the operator $D^0 = [\nabla^{++};\nabla\ \ ]$ under the integral and consider only the contribution from the term $\nabla^{++}\nabla$ in this commutator,[7]

$$
\begin{aligned}
{}^A_4 &= \frac{}{2m^3k^2}\int d^9z \frac{du_1du_2}{(u_1^+u_2^+)^2} q^+(z;u_2)q^+(z;u_2)\frac{1}{2}D^0_{(1)}(D^0_{(1)})^2[q^+(z;u_1)q^+(z;u_1)] \\
&= \frac{}{4m^3k^2}\int d^9z \frac{du_1du_2}{(u_1^+u_2^+)^2} q^+(z;u_2)q^+(z;u_2) \\
&\qquad\qquad \nabla^{++}_{(1)}\nabla_{(1)}(D^0_{(1)})^2[q^+(z;u_1)q^+(z;u_1)]: \qquad (5.38)
\end{aligned}
$$

We integrate by parts with respect to ∇^{++} and use the standard equation for the harmonic distributions [4],

$$
D^{++}_{(1)}\frac{1}{(u_1^+u_2^+)^2} = D_{(1)}\ {}^{(2;\ 2)}(u_1;u_2); \qquad (5.39)
$$

which allows us to perform the u_2 integration using the harmonic delta-function,

$$
\begin{aligned}
{}^A_4 &= \frac{}{4m^3k^2}\int d^3x d^6\ du\, q^+q^+(\nabla\ \ )^2(D^0)^2[q^+q^+] \\
&= \frac{}{16m^3k^2}\int d^{(\ 4)}q^+q^+(D^{++})^2(D^0)^2(\nabla\ \ )^2[q^+q^+]: \qquad (5.40)
\end{aligned}
$$

From the operator $(\nabla\ \ )^2$ we need only the term $(V_0\ \ )^2$, where V_0 is given by (5.11). Such a term yields

$$
(D^{++})^2(D^0)^2(V_0\ \ )^2 = 32m^2: \qquad (5.41)
$$

[7] The second term $\nabla\ \ \nabla^{++}$ in the commutator contains the operator ∇^{++} which hits the hypermultiplet superfields, resulting in the free massive hypermultiplet equations of motion (5.12). Therefore, such terms do not contribute to the on-shell effective action. Moreover, such terms are non-local with respect to the harmonic variables while here we are interested in the local contributions to the effective action.

Finally, we find

$$\Gamma^A_4 = \frac{2}{mk^2}\int d\zeta^{(-4)} q^+ \tilde{q}^+ q^+ \tilde{q}^+ \, : \tag{5.42}$$

One can check that the computation of the second diagram Γ^B_4 on Fig. 3 yields the same result. Therefore, the final answer for Γ_4 is as follows

$$\Gamma_4 = \frac{4}{mk^2}\int d\zeta^{(-4)} q^+ \tilde{q}^+ q^+ \tilde{q}^+ \, : \tag{5.43}$$

It is known that in the $N = 2$, $d = 4$ hypermultiplet model such a quartic self-interaction results in a sigma model for the scalar fields with the target hyper-Kähler Taub-NUT metric [13, 4]. The self-interaction (2.46) gives rise to the same sigma model, but in the three-dimensional space-time.

6. Discussion

In this paper we laid down a basis for the systematic study of the quantum aspects of three-dimensional $N = 3$ supersymmetric gauge and matter models in harmonic superspace. We worked out the background field method for the general $N = 3$ Chern-Simons matter theory. It is a powerful tool for finding the quantum effective actions directly in $N = 3$, $d = 3$ harmonic superspace, preserving manifest gauge invariance and $N = 3$ supersymmetry at each step of the quantum calculations. The usefulness of this method was illustrated by a simple proof of the $N = 3$, $d = 3$ nonrenormalization theorem. Furthermore, we derived the propagators for the massless and massive hypermultiplets as well as for the Chern-Simons fields in harmonic superspace and employed them to compute the leading terms in the quantum two-point and four-point functions.

The derivation of propagators and the calculation of quantum diagrams in $N = 3$, $d = 3$ harmonic superspace closely mimic the analogous considerations in the four-dimensional $N = 2$ harmonic superspace approach [3]. However, in contrast to the four-dimensional case, there are no one-loop UV divergences in $N = 3$, $d = 3$ harmonic superspace, and all diagrams are finite. Only IR singularities may appear in the massless hypermultiplet theory, but they can be avoided either by using massive hypermultiplets or by doing all the calculations within the background field method, where all propagators are effectively massive.

The massive hypermultiplet model has some new features in comparison with the four-dimensional theory. As is well known [30], the massive hypermultiplet describes a BPS state, i.e. it respects supersymmetry with a central charge equal to the hypermultiplet mass. The $N = 2$, $d = 4$ superalgebra has a central charge (complex or real) which is a singlet with respect to the R-symmetry group. Therefore, it breaks the $U(2)$ R-symmetry group down to $SU(2)$. In three dimensions, this picture is slightly different. The $N = 3$, $d = 3$ superalgebra has a central charge which is a triplet, breaking the $SO(3) \simeq SU(2)$ R-symmetry group down to $SO(2) \simeq U(1)$. For this reason, the massive hypermultiplet propagator has a more complicated form (5.16) as compared to the four-dimensional case.

A new feature arises when considering quantum contributions in the massive charged hypermultiplet model. In the $N = 0$ analog of such a model, i.e. three-dimensional electrodynamics, a single massive spinor generates the Chern-Simons action in the one-loop two-point quantum diagram [9]. A similar feature is pertinent to the $\mathcal{N}=1$ and $\mathcal{N}=2$ models [31]. However, the one-loop two-point diagram in the $\mathcal{N}=3$ massive charged hypermultiplet theory produces the $\mathcal{N}=3$ super Yang-Mills action rather than the Chern-Simons one as the leading quantum correction. This may be explained by resorting to a parity argument: the $\mathcal{N}=3$ hypermultiplet is parity-even while the Chern-Simons term violates parity. Since no anomaly can appear, the Chern-Simons term is prohibited in the hypermultiplet low-energy effective action.

Another interesting feature of quantum computations is related to the one-loop four-point function with four external hypermultiplets in the model of a massive charged hypermultiplet interacting with a dynamical Chern-Simons field. We showed that these quantum diagrams produce, as the leading correction, a quartic hypermultiplet self-interaction which in components yields the Taub-NUT sigma model for the scalar fields. The same phenomenon was observed in the four-dimensional case [10].

Let us outline some further problems which can be studied and hopefully solved based on the results of the present work. Its natural continuation is the study of the $\mathcal{N}=3$ superfield low-energy effective action in the hypermultiplet and the Chern-Simons theory. So far, there have not been any attempts to constructing the effective actions in these theories. It is worthwhile to compare this situation with the $\mathcal{N}=2$, $d=4$ supersymmetric models, in which the hypermultiplet and gauge superfield effective actions have been studied to a large extent (see, e.g., [19, 32]). Even more tempting is the application of our quantum techniques to the $\mathcal{N}=6$ and $\mathcal{N}=8$ supersymmetric ABJM and BLG models, in order to describe the quantum-corrected low-energy dynamics of M2 branes in superstring theory. An important related question concerns the composite operators for the hypermultiplet superfields in the ABJM theory. Such operators are relevant for testing the AdS_4/CFT_3 version of the general "gravity/gauge" correspondence.

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A. Appendix. $\mathcal{N}=3$ harmonic superspace conventions

Three-dimensional notation. We use the Greek letters $\alpha,\beta,\ldots$ to label the spinorial indices corresponding to the $SO(1,2) \simeq SL(2,R)$ Lorentz group. The corresponding gamma-matrices can be chosen to be real, in particular,

$$(\gamma^0)_\alpha{}^\beta = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \quad (\gamma^1)_\alpha{}^\beta = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad (\gamma^2)_\alpha{}^\beta = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.1}$$

They satisfy the Clifford algebra

$$\{\gamma^m, \gamma^n\} = -2\eta^{mn}, \qquad \eta^{mn} = \mathrm{diag}(1,-1,-1), \tag{A.2}$$

and the following orthogonality and completeness relations

$$(\gamma^m)_\alpha{}^\beta(\gamma^n)_\beta{}^\alpha = 2\eta^{mn}, \qquad (\gamma^m)_{\alpha\beta}(\gamma_m)^{\gamma\delta} = -(\delta_\alpha^\gamma\delta_\beta^\delta + \delta_\alpha^\delta\delta_\beta^\gamma). \tag{A.3}$$

We raise and lower the spinor indices with the ε-tensor, e.g., $(\gamma_m)^{\alpha\beta} = \varepsilon^{\alpha\gamma}(\gamma_m)_\gamma{}^\beta$, $\varepsilon_{12} = 1$. The products of two and tree gamma-matrices are given by

$$(\gamma^m)_\alpha{}^\gamma(\gamma^n)_\gamma{}^\beta = -\eta^{mn}\delta_\alpha^\beta - \varepsilon^{mnp}(\gamma_p)_\alpha{}^\beta, \tag{A.4}$$

$$(\gamma^m)_\alpha{}^\beta(\gamma^n)_\beta{}^\gamma(\gamma^p)_\gamma{}^\delta = -\eta^{mn}(\gamma^p)_\alpha{}^\delta + \eta^{mp}(\gamma^n)_\alpha{}^\delta - \eta^{np}(\gamma^m)_\alpha{}^\delta + \varepsilon^{mnp}\delta_\alpha^\delta, \tag{A.5}$$

where $\varepsilon_{012} = \varepsilon^{012} = 1$.

The relations (A.3) are used to convert any vector index into a symmetric pair of space-time ones, e.g.,

$$x^{\alpha\beta} = (\gamma_m)^{\alpha\beta}x^m, \qquad x^m = \frac{1}{2}(\gamma^m)_{\alpha\beta}x^{\alpha\beta},$$
$$\partial_{\alpha\beta} = (\gamma^m)_{\alpha\beta}\partial_m, \qquad \partial_m = \frac{1}{2}(\gamma_m)^{\alpha\beta}\partial_{\alpha\beta}, \tag{A.6}$$

so that

$$\partial_m x^n = \delta_m^n, \qquad \partial_{\alpha\beta}x^{\gamma\delta} = \delta_\alpha^\gamma\delta_\beta^\delta + \delta_\alpha^\delta\delta_\beta^\gamma = 2\delta_{(\alpha}^\gamma\delta_{\beta)}^\delta. \tag{A.7}$$

Superspace and harmonic conventions. The R-symmetry of $\mathcal{N}=3$ superspace is $SO(3)_R \simeq SU(2)_R$. Therefore we label the three copies of Grassmann variables by a pair of symmetric $SU(2)$ indices i,j, i.e., $\theta^{ij}_\alpha = \theta^{ji}_\alpha$. Thus the $\mathcal{N}=3$ superspace is parametrized by the following real coordinates in the central basis

$$z = (x^m, \theta^{ij}_\alpha), \qquad \overline{x^m} = x^m, \qquad \overline{\theta^{ij}_\alpha} = \theta_{ij\alpha}. \tag{A.8}$$

The partial spinor derivatives are defined as follows

$$\frac{\partial}{\partial\theta^{ij}_\alpha}\theta^{kl}_\beta = \delta^\alpha_\beta\delta^k_{(i}\delta^l_{j)}. \tag{A.9}$$

The covariant spinor derivatives and supercharges read

$$D^{kj}_\alpha = \frac{\partial}{\partial\theta^\alpha_{kj}} + i\theta^{kj\beta}\partial_{\alpha\beta}, \qquad Q^{kj}_\alpha = \frac{\partial}{\partial\theta^\alpha_{kj}} - i\theta^{kj\beta}\partial_{\alpha\beta}. \tag{A.10}$$

They satisfy the following anticommutation relations

$$\{D^{ij}_\alpha, D^{kl}_\beta\} = i(\varepsilon^{ik}\varepsilon^{jl} + \varepsilon^{il}\varepsilon^{jk})\partial_{\alpha\beta}\,, \tag{A.11}$$
$$\{Q^{ij}_\alpha, Q^{kl}_\beta\} = -i(\varepsilon^{ik}\varepsilon^{jl} + \varepsilon^{il}\varepsilon^{jk})\partial_{\alpha\beta}\,. \tag{A.12}$$

We use the standard harmonic variables $u^\pm_i$ parametrizing the coset $SU(2)_R/U(1)_R$ [4]. In particular, the partial harmonic derivatives are

$$\partial^{++} = u^+_i \frac{\partial}{\partial u^-_i}\,,\quad \partial^{--} = u^-_i \frac{\partial}{\partial u^+_i}\,,\quad \partial^0 = [\partial^{++}, \partial^{--}] = u^+_i \frac{\partial}{\partial u^+_i} - u^-_i \frac{\partial}{\partial u^-_i}\,. \tag{A.13}$$

The harmonic projections of the Grassmann $N=3$ coordinates and spinor derivatives are defined as follows

$$\theta^{ij}_\alpha \to (\theta^{++}_\alpha, \theta^{--}_\alpha, \theta^0_\alpha) = (u^+_i u^+_j \theta^{ij}_\alpha, u^-_i u^-_j \theta^{ij}_\alpha, u^+_i u^-_j \theta^{ij}_\alpha)\,,$$
$$D^{ij}_\alpha \to (D^{++}_\alpha, D^{--}_\alpha, D^0_\alpha) = (u^+_i u^+_j D^{ij}_\alpha, u^-_i u^-_j D^{ij}_\alpha, u^+_i u^-_j D^{ij}_\alpha)\,. \tag{A.14}$$

The analytic subspace in the full $N=3$ superspace is parametrized by the following coordinates:

$$\zeta_A = (x^{\alpha\beta}_A, \theta^{++}_\alpha, \theta^0_\alpha, u^\pm_i)\,, \tag{A.15}$$

where

$$x^{\alpha\beta}_A = (\gamma_m)^{\alpha\beta} x^m_A = x^{\alpha\beta} + i(\theta^{++\alpha}\theta^{--\beta} + \theta^{++\beta}\theta^{--\alpha})\,. \tag{A.16}$$

The harmonic and Grassmann derivatives in the analytic coordinates are:

$$D^{++} = \partial^{++} + 2i\theta^{++\alpha}\theta^{0\beta}\partial^A_{\alpha\beta} + \theta^{++\alpha}\frac{\partial}{\partial\theta^{0\alpha}} + 2\theta^{0\alpha}\frac{\partial}{\partial\theta^{--\alpha}}\,,$$
$$D^{--} = \partial^{--} - 2i\theta^{--\alpha}\theta^{0\beta}\partial^A_{\alpha\beta} + \theta^{--\alpha}\frac{\partial}{\partial\theta^{0\alpha}} + 2\theta^{0\alpha}\frac{\partial}{\partial\theta^{++\alpha}}\,,$$
$$D^0 = \partial^0 + 2\theta^{++\alpha}\frac{\partial}{\partial\theta^{++\alpha}} - 2\theta^{--\alpha}\frac{\partial}{\partial\theta^{--\alpha}}\,,\quad [D^{++}, D^{--}] = D^0\,, \tag{A.17}$$

$$D^{++}_\alpha = \frac{\partial}{\partial\theta^{--\alpha}}\,,\quad D^{--}_\alpha = -\frac{\partial}{\partial\theta^{++\alpha}} + 2i\theta^{--\beta}\partial^A_{\alpha\beta}\,,\quad D^0_\alpha = -\frac{1}{2}\frac{\partial}{\partial\theta^{0\alpha}} + i\theta^{0\beta}\partial^A_{\alpha\beta}\,, \tag{A.18}$$

where $\partial^A_{\alpha\beta} = (\gamma^m)_{\alpha\beta}\frac{\partial}{\partial x^m_A}$. They satisfy the following relations:

$$\{D^{++}_\alpha, D^{--}_\beta\} = 2i\partial^A_{\alpha\beta}\,,\quad \{D^0_\alpha, D^0_\beta\} = -i\partial^A_{\alpha\beta}\,, \tag{A.19}$$

$$[D^{++}, D^{--}_\alpha] = 2D^0_\alpha\,,\quad [D^0, D^{--}_\alpha] = -2D^{--}_\alpha\,,\quad [D^{--}, D^0_\alpha] = D^{--}_\alpha\,. \tag{A.20}$$

Some useful identities involving these derivatives are as follows,

$$(D^0)^2 D^0_\alpha = -iD^{0\beta}\partial_{\alpha\beta}\,,\quad D^0_\alpha (D^0)^2 = iD^{0\beta}\partial_{\alpha\beta}\,,\quad (D^0)^4 = \Box\,. \tag{A.21}$$

The integration measures over the full and analytic harmonic superspaces are defined by

$$d^9z = -\frac{1}{16}d^3x\,(D^{++})^2(D^{--})^2(D^0)^2\,, \tag{A.22}$$
$$d\zeta^{(-4)} = \frac{1}{4}d^3x_A\,du\,(D^{--})^2(D^0)^2\,,\quad d^9z\,du = -\frac{1}{4}d\zeta^{(-4)}(D^{++})^2\,, \tag{A.23}$$

where $(D^{++})^2 = D^{++\alpha} D^{++}_{\alpha}$, etc. With such conventions, the superspace integration rules are most simple:

$$\int d^3x\, f(x) = \int d^9z\, (\theta^{++})^2 (\theta^{--})^2 (\theta^0)^2 f(x) = \int d\zeta^{(-4)} (\theta^{++})^2 (\theta^0)^2 f(x_A) \qquad (A.24)$$

for some field $f(x)$.

We denote the special conjugation in the $N=3$ harmonic superspace by $\sim$,

$$\widetilde{(u^{\pm}_i)} = u^{\pm i}\,; \quad \widetilde{(x^m_A)} = x^m_A\,; \quad \widehat{(\theta^{\pm\pm}_\alpha)} = \theta^{\pm\pm}_\alpha\,; \quad \widetilde{(\theta^0_\alpha)} = \theta^0_\alpha\,. \qquad (A.25)$$

It squares to -1 on the harmonics and to 1 on other superspace coordinates. All bilinear combinations of the Grassmann coordinates are imaginary

$$[\widehat{(\theta^{++}\theta^0)}] = -\theta^{++}\theta^0\,; \quad [\widehat{(\theta^{++})^2}] = -(\theta^{++})^2\,; \quad [\widehat{(\theta^0)^2}] = -(\theta^0)^2\,. \qquad (A.26)$$

The conjugation rules for the spinor and harmonic derivatives are

$$\widehat{(D^0_\alpha\Phi)} = -D^{0\alpha}\tilde\Phi\,; \quad [\widehat{(D^0)^2\Phi}] = -(D^0)^2\tilde\Phi\,; \quad \widehat{(D^{++}\Phi)} = D^{++}\tilde\Phi\,; \qquad (A.27)$$

where Φ and $\tilde\Phi$ are even superfields.

The analytic superspace measure is real, $\widehat{d\zeta^{(-4)}} = d\zeta^{(-4)}$, while the full superspace measure is imaginary, $\widetilde{d^9z} = -d^9z$.

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