1: Use the four point rectification method to rectify the image (a).

The projective transformation formula given in the textbook written in homogeneous form is:

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x+h_{12}y+h_{13}}{h_{31}x+h_{32}y+h_{33}}$$

$$y' = \frac{x'_3}{x'_3} = \frac{h_{21}x+h_{22}y+h_{23}}{h_{31}x+h_{32}y+h_{33}}$$

Here, the homogeneous points (x, y,1) are from the original image (image world) and homogeneous points(x',y', 1) are from the rectified image (reald world), if we multiply out the above equations, it leads to:

$$x'(h_{31}x + h_{32}y + h_{33}) = h_{11}x + h_{12}y + h_{13}$$
$$y'(h_{31}x + h_{32}y + h_{33}) = h_{21}x + h_{22}y + h_{23}$$

Since the transformation matrix only considers the ratio of components, we can set $h_3 = 1$, and rearrange the above equation

$$x' = h_{11}x + h_{12}y + h_{13} - h_{31}xx' - h_{32}yx'$$
$$y' = h_{21}x + h_{22}y + h_{23} - h_{31}xy' + h_{32}yy'$$

To vectorize it

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x\ y\ 1\ 0\ 0\ 0\ -xx'\ -yx' \\ 0\ 0\ 0\ x\ y\ 1 - xy'\ -yy' \end{bmatrix} [h_{11}\ h_{12}\ h_{13}\ h_{21}\ h_{22}\ h_{23}\ h_{31}\ h_{32}]^T$$

In this way, if we are able to provide four points information, it would be able to solve this linear algebraic equation.

In image a, the points are selected are labeled with 1, 2, 3, 4 and the points selected in image b is 5, 6, 7, 8; we can use a computer program to pick the coordinate of points one clicked by mouse as long as all the points are in the same coordinate system.
The coordinate for selected points in image a are 1, 2, 3, 4:

v1 = (274, 32, 1), v2 = (353, 41, 1), v3 =(277, 295, 1), v4 = (352, 269, 1)

And coordinates for points 5, 6, 7, 8 in image b

v5 = (81, 122, 1), v6 = (181, 112, 1), v7 = (84, 333, 1), v8 = (177, 379, 1)

Now we can set the mapped coordinate in the rectified image 1', 2', 3', 4' as (275, 30), (370, 30), (275, 300), (370, 300), and the mapped in the rectified image 5', 6', 7', 8' as (81, 122), (200,122),(81,375), (200,375),

Then we can calculate the H for image a is below, then the transformation can be made with H

$$H = [0.38163438, -0.041528057, 82.809883; -0.11005338, 0.65524989, 29.482882; -0.001166273, -0.00012187734, 1]$$

The rectified images are below:

Image a before

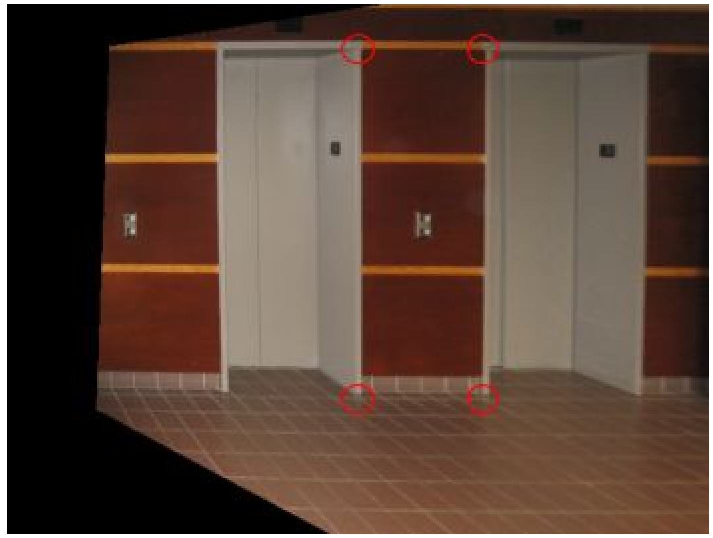


Image a after

Image b before

Image b after

Discussion: The four point method requires selection of four point in the original image and we have to specify four point in the target image, thus the coordinate of four points in the target image might not be so accurate and quite arbitrary.

Source Code Link: https://github.com/snmnmin12/ComputerVision

2. Using the line at infinity to rectify the image (a) to affinity

The projective distortion by taking the vanishing line present in the image plane back to infinity. The homograph used in this part is

$$H_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$$

Here, the line at infinity is set to be $L_i = [l_1 \;\; l_2 \;\; l_3]$, and we take the corners of a rectangular frame in the projective distorted image and form two sets of parallel lines in the world plane. Defining this corner points as x_1, x_2, x_3, x_4 we find the sets of parallel lines by taking their cross-product pairwise, in image a, the four points selected are

$$x_1 = (274, 32, 1),\; x_2 = (353, 41, 1),\; x_3 = (277, 295, 1),\; x_4 = (352, 269, 1)$$

so that

$$L_1 = x_1 \times x_2 = (-9, 79, -62)$$
$$L_2 = x_3 \times x_4 = (26, 75, -29327)$$
$$L_3 = x_1 \times x_3 = (-263, 3, 71966)$$
$$L_4 = x_2 \times x_4 = (-228, -1, 80525)$$
$$P = L_1 \times L_3 = (-2.31218e+06, -265555, -2729)$$
$$Q = L_2 \times L_4 = (313541, 4.76983e+06, 947)$$
$$L_i = P \times Q = (1.27654e+10, 1.33398e+09, -1.09455e+13)$$

Here, it set to be $l_1 \parallel l_2$ and $l_3 \parallel l_4$, the two pairs of parallel lines have intersection points at infinity points if it is in the world image. The intersections in the distorted images are defined as P, Q points, so the line at infinity is the cross product of P and Q; Once the line at infinity is obtained, the same transformation can be applied to all homogeneous points in the projective image world to the affinity image world with Hp for image a

Hp = [1, 0, 0;
0, 1, 0;
-0.0011662724, -0.00012187564, 1]

$$x' = H_p x$$

After the transformation of each coordinate x in original image and normalized the 3-d vector x' to make it homogeneous will remove the projective distortion for the selected parallel lines.

Discussion: this method requires the selection of two parallel lines and find the intersection points, it does not require the SVD computation to get the transformation function H, so it is quite simple.

The rectified images are attached below:

Image a before projective removal

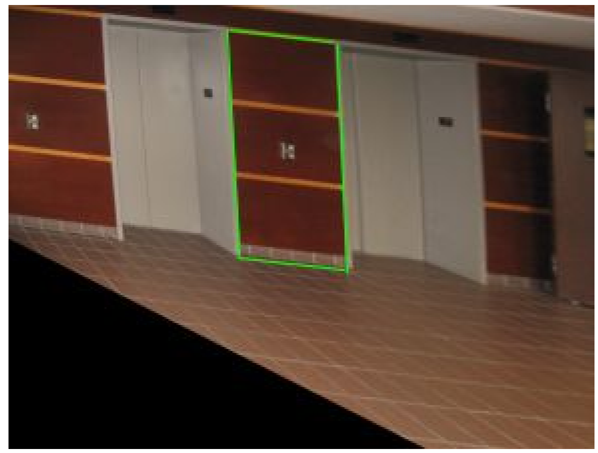


Image a after projective removal

Image b before

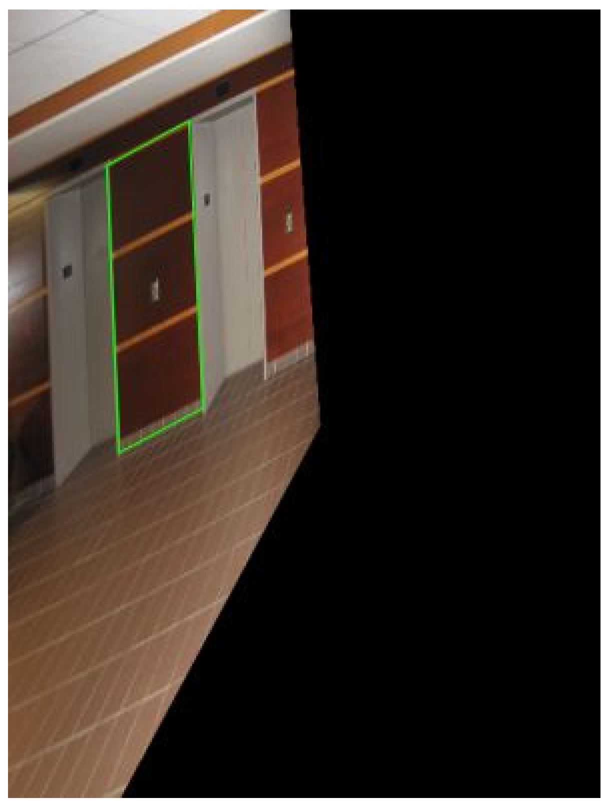


Image b after removal of projective

Source Code Link: https://github.com/snmnmin12/ComputerVision

3. Built on 2, using two step rectification using $C_\infty{}^*$

Now the image plane only contains affine distortion, which can be removed if knowing that the angle between two lines $l = [l_1\ l_2\ l_3]$ and $m = [m_1\ m_2\ m_3]$ in the similarity image plane

$$cos(\theta) = \frac{(l_1*m_1+l_2*m_2)}{\sqrt{(l_1^2+l_2^2)^2*(m_1^2+m_2^2)^2}} = \frac{(l^T C_\infty^* m)}{\sqrt{(l^T C_\infty^* l}*\sqrt{m^T C_\infty^* m}}$$

In which where C*$^\infty$ = [1 0 0; 0 1 0; 0 0 0]. Using the fact that C*' = HC*H^T we can rewrite the numerator as

$$l^T C_\infty^* m = l^T H^{-1} C_\infty{}^{*'} H^{-T}\ m\ =\ l'^T C_\infty{}^{*'} m'$$

If we can choose perpendicular lines l and m which makes the two lines l and m are orthogonal with each other which makes $l^T C_\infty^* m = l^T H^{-1} C^{*'} H^{-T} m\ =\ l'^T C^{*'} m'\ =\ 0$ which makes the below equations works

$$l'^T C_\infty^{*'} m' = l'^T H C_\infty^* H^T m'\ =\ 0$$

Here, we set H_a = [A 0;0 1] corresponding to the affine homography transformation, and

$$l'^T H_a C_\infty^* H_a m' = [l_1', l_2', l_3'] \begin{bmatrix} AA^T & 0 \\ 0^T & 0 \end{bmatrix} \begin{bmatrix} m_1' \\ m_2' \\ m_3' \end{bmatrix}$$

Now AA^T is symmetric matrix, so we can have it define AA^T as S

$$s = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}$$

Then we are able to derive the equations below:

$$l'_1 s_{11} m'_1\ +\ s_{12}(l'_1 m'_2 + l'_2 m'_1) + s_{22} l'_2 m'_2 = 0$$

In this equation, it has 3 variables to be solved and actually only the ratio of these variables is important, thus we can set one variable to be 1 and we can set s_{22} = 1, then the above equation can be formulized as

$$l'_1 s_{11} m'_1\ +\ s_{12}(l'_1 m'_2 + l'_2 m'_1)\ =\ -l'_2 m'_2$$

We need to perpendicular line group to solve the above equation, suppose line l perpendicular to line m, lien o perpendicular to line n, then the below linear system can be constructed:

$$\begin{bmatrix} l'_1 m'_1 & l'_1 m'_2 + l'_2 m'_1 \\ o'_1 n'_1 & o'_1 n'_2 + o'_2 n'_1 \end{bmatrix} \begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix} = \begin{bmatrix} -l'_2 m'_2 \\ -o'_2 n'_2 \end{bmatrix}$$

Solving this equations leads to the solution to

$$s = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}$$

To solve $S = AA^T$, SVD decomposition can be performed

$$A = UDU^r$$
$$S = AA^T = UDU^r UDU^T = UD^2U^T$$

The two pairs of orthogonal lines selected is marked in the images with below values as well as line l and m:

$$v_1 = (274,\ 32,\ 1),\ v_2 = (353,\ 41,\ 1)\quad l = v_1 \times v_2 = (-9\ ,79,\ -62)$$
$$v_3 = (353,\ 41,\ 1),\ v_4 = (352,\ 110, 1)\quad m = v_3 \times v_4 = (-6,\ -1, 24398\ )$$

As for the two perpendicular diagonal lines o and n:

$$v_5 = (274,\ 32,\ 1),\ v_6 = (352,\ 110,\ 1)\quad o = v_5 \times v_6 = (-9\ ,79,\ -62)$$
$$v_7 = (353,\ 41,\ 1),\ v_8 = (275,\ 111, 1)\quad n = v_7 \times v_8 = (-6,- 1, 24398\ )$$

Once we know the perpendicular line group l, m and o, n in the original image, the first thing to be done is to transform them into affine image word first with projective removal transformation H_p with below equations:

$$l' = H_p^{-T} l$$
$$m' = H_p^{-T} m$$
$$o' = H_p^{-T} o$$
$$n' = H_p^{-T} n$$

The linear equation constructed above above leads to the

s_{11} = 2.6605446,
s_{12} = 0.35226965
S = [2.6605446,   0.35226965
0.35226965, 1]

SVD is performed on the matrix S, it gives

U = [0.97994095, -0.19928782
0.19928791, 0.97994101]
D2 = [2.7321849, 0
0, 0.92835993]
D = [1.6529, 0
0, 0.9635]
A = [1.6256, 0.1346
0.1346, 0.9909]
$C_\infty^{*'}$ = [2.6605446, 0.35226965, 0
0.35226965, 1, 0
0, 0 , 1]
H_a = [1.6256, 0.1346, 0;
0.1346, 0.9909, 0;
0, 0, 1]

Thus the overall transformation from projective image to similarity image is a combination of inverse projective transformation and inverse affinity transformation, we already know that H_p will transform projective image to affinity image, and H_a^{-1} will transform affinity image to similarity image, so the overall transformation is $H_p H_a^{-1}$

Discussion:

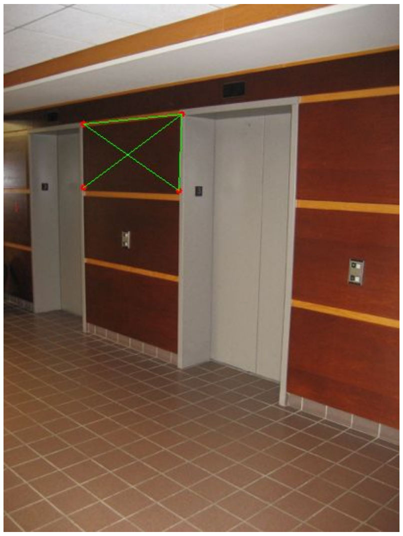
This method requires the selection of two parallel group on the same planes from the affinity images and perform one single SVD computation in order to obtain the transformation matrix H. This is method is also not difficult in terms of the line group selected.

The original image and rectified images are attached below for reference:

Image a- original form

Image a - projective removal

Image a - affine removal



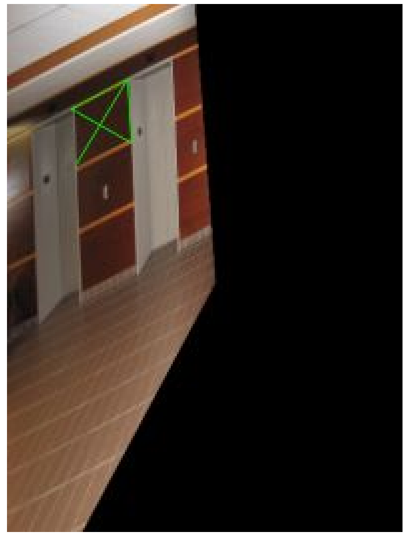


Image b - original

Image b projective removal

Image b affine removal

Source Code Link: https://github.com/snmnmin12/ComputerVision

4. Challenge: Single Step Method

Single step method can be used to get rid of both projective and affine distortions simultaneously by using dual $C_\infty{}^*$. The homography H can be expressed as

$$H = \begin{bmatrix} A & 0 \\ v^T & 1 \end{bmatrix}$$

$$C_\infty^{*\prime} = H C_\infty^* H^T = \begin{bmatrix} AA^T & Av \\ v^T A^T & v^T v \end{bmatrix}$$

The general form for conic $C_\infty^{*\prime}$ is

$$C_\infty^{*\prime} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

Remember that $l'^T C^{*\prime} m' = 0$

$$l_1 m_1 a + (l_2 m_1 + m_2 l_1) b/2 + l_2 m_2 c + (l_1 m_3 + l_3 m_1) d/2 + (l_2 m_3 + l_3 m_2) e/2 + l_3 m_3 f = 0$$

In matrix form:

$$\begin{bmatrix} l_1 m_1 \\ \frac{l_1 m_2 + l_2 m_1}{2} \\ l_2 m_2 \\ \frac{l_1 m_3 + l_3 m_1}{2} \\ \frac{l_2 m_3 + l_3 m_2}{2} \\ l_3 m_3 \end{bmatrix}^T \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = 0$$

Here, 5 orthogonal line pairs can be selected to construct a linear equations system $Mb = 0$ with M 5x6. Then, the null space can be calculated to find vector b to build the $C_\infty^{*'}$ and then perform the SVD for AA^t terms in the SVD matrix, we are able to solve for A and H.

The orthogonal lines paris selected from image a are marked in the below images with points coordinates below:

$$v_{11} = (274,\ 32,\ 1),\ v_{12} = (353,\ 41,\ 1),\ v_{13} = (353,\ 41,\ 1),\ v_{14} = (352,\ 110,\ 1)$$
$$v_{21} = (274,\ 32,\ 1),\ v_{22} = (352,\ 110,\ 1),\ v_{23} = (353,\ 41,\ 1),\ v_{24} = (275,\ 111, 1)$$
$$v_{31} = (33,\ 186,\ 1),\ v_{32} = (49,\ 184,\ 1),\ v_{33} = (49,\ 184,\ 1),\ v_{34} = (51,\ 208,\ 1)$$
$$v_{41} = (311,\ 154,\ 1),\ v_{42} = (313,\ 172,\ 1),\ v_{43} = (313,\ 172,\ 1),\ v_{44} = (321,\ 170,\ 1)$$
$$v_{51} = (277,\ 204,\ 1),\ v_{52} = (276,\ 295,\ 1),\ v_{53} = (276,\ 295,\ 1),\ v_{54} = (352,\ 268,\ 1)$$
$$l_1 = (-9,\ 79,\ -62),\ m_1 = (-69,\ -1,\ 24398)$$
$$l_2 = (-78,\ 78,\ 18876),\ m_2 = (-70,\ -78,\ 27908)$$
$$l_3 = (\ 2,\ 16,\ -3042),\ m_3 = (-24,\ 2,\ 808)$$
$$l_4 = (-18,\ 2,\ 5290),\ m_4 = (2,\ 8,\ -2002)$$
$$l_5 = (-91,\ -1,\ 25411),\ m_5 = (27,\ 76,\ -29872)$$

M = [621, -2721, -79, -107652, 963752, -1512676;
5460, 312, -6084, -1749072, 352248, 5.2679142e+08;
-48, -190, 32, 37312, 3422, -2457936;
-36, -70, 16, 23308, 19158, -10590580;
-2457, -3471.5, -76, 1702224.5, 980554, -7.5907738e+08]

$$b = (a,\ b, c, d, e, f) =$$
$$(0.91012394,\ 0.24278428,\ 0.3357459,\ 0.0021675078,\ 0.00037061362,\ 1.2495102e-06]$$

$C_\infty^{*'}$ = [0.91012394, 0.12139214, 0.0010837539;
0.12139214, 0.3357459, 0.00018530681;
0.0010837539, 0.00018530681, 1.2495102e-06]

After SVD decomposition on the $C_\infty^{*'}(1:2, 1:2)$ terms, we are able to find the matrix A

A = [
0.95067573, 0.079621486;
0.079621471, 0.57393932
]

Then we are able to find transformation matrix H

H = [
0.95067573, 0.079621486, 0;
0.079621471, 0.57393932, 0;
0.0011260248, 0.0001666571, 1
]

This is the transformation H from target image to original image, so we can apply H^{-1} to the original image to get the target image after removal.

Image a - orthogonal lines

Image a - after rectification

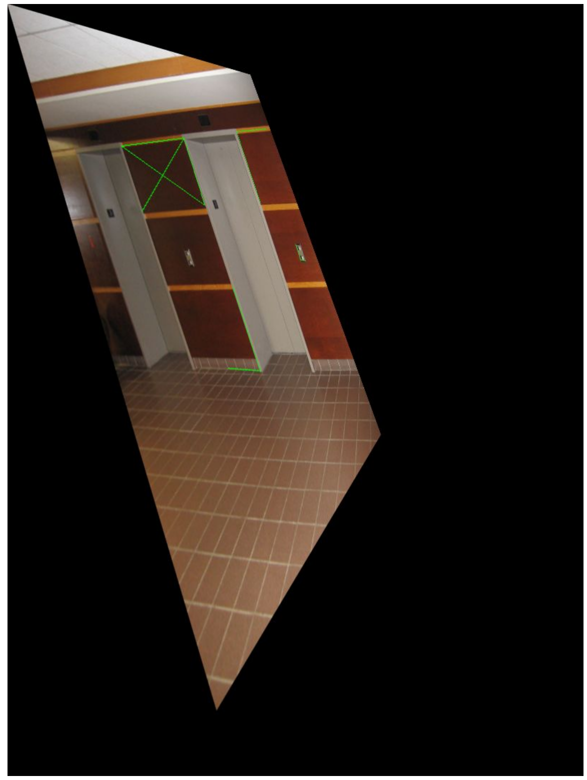


Image b before

Image b after

Discussion:
The single-step method is more straightforward than the two-step method, it requires several selection of the five pairs of orthogonal lines to provide an accurate results. The two-step method, however, requires less lines. On the other hand, the single-step algorithm is shorter, it requires at least two SVD decompositions to create the homography matrix, while in the two-step method only

Also the problem I encountered for single step is when I assume the last term f =1 in C, it gives undesirable output image. Then I realized that the actual value of f might be zero, so I started to solve the null space of the matrix M. In this way, I am able to obtain the correct transformation. To sum up, I found the two-step method less problematic for my testing.

Source Code Link: https://github.com/snmnmin12/ComputerVision