1.
Connectivity

1.1 Basic Graph Terminology

Def'n 1.1 **Graph**

A ***graph*** $G = (V, E)$ is an ordered pair such that

(a) V is a set; and *vertex set*

(b) E is a set of unordered pairs of elements of V. *edge set*

The elements of V are called the ***vertices*** of G and the elements of E are called the ***edges*** of G. Given $u, v \in V$, we denote uv to denote an edge ***joining*** u, v. Note that $uv = vu$ for every $u, v \in V$. For simplicity, we assume that

(a) G does not have any ***loop*** (i.e. $vv \notin E$ for every $v \in V$);[a]

(b) G has at least one vertex; and

(c) G is ***finite*** (i.e. V is a finite set).

[a] Note that, since we defined E to be a set of unordered pairs of elements of V, if $e, f \in E$ join the same vertices, then $e = f$. In other words, we do not allow ***multiple edges***.

Notation 1.2 $V(G), E(G)$

Let G be a graph. We write $V(G)$ to denote the vertex set and write $E(G)$ to denote the edge set of G.

Def'n 1.3 **Subgraph** of a Graph

Let G be a graph. We say H is a ***subgraph*** of G if $V(H) \subseteq V(G), E(H) \subseteq E(G)$, and all edges in $E(H)$ have both endpoints in $V(H)$. In particular,

(a) when $V(H) = V(G)$, we say H is a ***spanning*** subgraph; and

(b) if there exists $S \subseteq V(G)$ such that $V(H) = S$ and that

$$E(H) = \{e \in E(G) : \exists v, u \in S [e = vu]\},$$

then we say H is the subgraph ***induced*** by S, denoted as $G[S]$.

Def'n 1.4 **Degree** of a Vertex

Let G be a graph. The ***degree*** of $v \in V(G)$, denoted as $d_G(v)$, is the number of edges incident to v. Moreover, we write $\delta(G)$ to denote

$$\delta(G) = \min\{d_G(v) : v \in V(G)\}$$

and write $\Delta(G)$ to denote

$$\Delta(G) = \max\{d_G(v) : v \in V(G)\}.$$

Def'n 1.5 **Connected** Graph

Let G be a graph. We say G is ***connected*** if for every $u, v \in V(G)$, there is a u, v-path in G. A ***component*** of G is a maximally connected subgraph of G.

Proposition 1.1

Let G be a graph. The following are equivalent.

(a) G is connected.

(b) There exists $v \in V(G)$ such that for every $u \in V(G)$, there exists a u, v-path in G.

Notation 1.6 — $G - F$, $G - e$, $G - U$, $G - v$

Let G be a graph.

(a) Given $F \subseteq E(G)$, we write $G - F$ to denote the graph such that $V(G - F) = V(G), E(G - F) = E(G) \setminus F$.

(b) In case $F = \{e\}$ for some $e \in E$, we simply write $G - e$ to denote $G - \{e\}$.

(c) Similarly, given $U \subseteq V(G)$, we write $G - U$ to denote the graph such that $V(G - U) = V(G) \setminus U$ and that

$$E(G - U) = \{e \in E(G) : \exists u, v \in V(G) \setminus U [e = uv]\}.$$

In other words, we remove every vertex in U from G and every edge in G that is incident to at least one vertex in U.

(d) In case $U = \{v\}$ for some $v \in E$, we simply write $G - v$ to denote $G - \{v\}$.

Def'n 1.7 — **Contraction** of a Graph

Let G be a graph. For any $e \in E(G)$, the G ***contract*** e, denoted as G/e, is defined by *removing* e from G and *identifying* u, v as the same vertex.

1.2 Edge Connectivity

Def'n 1.8 — ***k*-edge-connected** Graph

We say a graph G is ***k*-edge-connected** if $G - F$ is connected for every $F \subseteq E(G)$ such that $|F| < k$. The ***edge connectivity*** of G, denoted as $\kappa'(G)$, is the largest $k \in \mathbb{N}$ for which G is k-edge-connected.

Def'n 1.9 — **Cut-edge** of a Graph

Let G be a graph. We say $e \in E(G)$ is a ***cut-edge*** if $G - \{e\}$ is disconnected.

(1.1) Let G be a graph.

(a) $\kappa'(G) = 1$ if and only if G has a cut-edge. Equivalently, G is 2-edge connected if and only if G is connected with no cut-edges.

(b) If G is disconnected, then we define $\kappa'(G) = 0$ and say G is 0-edge-connected.

(c) The ***trivial graph*** $(\{v\}, \varnothing)$ *cannot be disconneted*; we define its edge-connectivity to be 0.

Proposition 1.2

Let G be a graph. Then
$$\kappa'(G) \leq \delta(G).$$

Proof. If $\delta(G) = 0$, then there is an isolated vertex $v \in V(G)$, which means G is disconnected. So $\kappa'(G) = 0$, meaning $\delta(G) = 0 \leq 0 = \kappa'(G)$. Now assume $\delta(G) \geq 1$ and let $v \in V(G)$ be such that $d_G(v) = \delta(G)$. Let $F = \{e \in E(G) : \exists u \in V(G)[uv = e]\}$, the set of all edges incident with v. Then $G - F$ is disconnected, since v is not adjacent to any other vertices in G, which in turn exist since $\delta(G) \geq 1$. It follows that $\kappa'(G) \leq |F| = \delta(G)$. ∎

Def'n 1.10

Cut Induced by a Subset

Let G be a graph and let $S \subseteq V(G)$. We define the ***cut*** induced by S, denoted as $\delta_G(S)$, to be the set of all edges with exactly one end in S:
$$\delta_G(S) = \{e \in E(G) : \exists v \in V(G) \setminus S, u \in S[e = vu]\}.$$
We say $\delta_G(S)$ is ***nontrivial*** if $S \neq \varnothing$ and $S \neq V(G)$.

Proposition 1.3

Let G be a graph. Then the following are equivalent.

(a) G is connected.

(b) Every nontrivial cut of G is nonempty.

(1.2) Say we have a graph and a cut $C \subseteq E(G)$. Then it is easy to see that $G - C$ is disconnected when C is nontrivial. It follows that the size of any nontrivial cut is an upper bound on $\kappa'(G)$.

Proposition 1.4

Let G be a graph and let $F \subseteq E(G)$. If $G - F$ is disconnected, then F contains a nontrivial cut.

Proof. Suppose $G - F$ is disconnected and let H be a component of $G - F$. Since $G - F$ is disconnected, $V(H) \neq \varnothing$ and $V(H) \neq V(G)$. Consider $\delta_G(V(H))$. Since H is maximally connected in $G - F$, none of the edges in $\delta_G(V(h))$ can be in $G - F$. Hence, all edeges in $|deltaG(V(H))$ must be in F. Hence F contains a nontrivial cut $\delta_G(V(H))$, as required. ∎

Corollary 1.4.1

Let G be a graph. Then $\kappa'(G)$ is the size of a minimum nontrivial cut.

Def'n 1.11

Bond of a Graph

A ***bond*** is a minimal nonempty cut.

Proposition 1.5
Characterization of a Bond

Let $F = \delta_G(S)$ be a cut in a connected graph G, where $S \subseteq V(G)$. Then the following are equivalent.

(a) F is a bond.

(b) $G - F$ has exactly 2 components.

Proof.

- ($\Longrightarrow$) Suppose $G-F$ has at least 3 components. Assume, without loss of generality, that (at least) 2 of the components are contained in $G[S]$ and let H be one of them. Then $\delta_G(V(H)) \subset F$, so F is not minimal and hence not a bond.

- Suppose $G-F$ has exactly 2 components and suppose further F is not a bond, for the sake of contradiction. Then there is a nonempty cut $\delta_G(S)$ that is a proper subset of F. Then the two components of G_F are linked by some edge in F. It follows that $G-\delta_G(S)$ is connected, contradicting the fact that $\delta_G(S)$ is a cut. ∎

Proposition 1.6

Every cut is a disjoint union of bonds.

Def'n 1.12

Symmetric Difference of Two Sets

Let S, T be sets. Then the ***symmetric difference*** of S, T, denoted as $S \triangle T$, is defined as

$$S \triangle T = (S \cup T) \setminus (S \cap T).$$

Lemma 1.6.1

Let G be a graph and let $S, T \subseteq V(G)$. Then

$$\delta_G(S) \triangle \delta_G(T) = \delta_G(S \triangle T).$$

Proof of Proposition 1.6. We proceed inductively on the number of edges in F. If $|F| = 0$, then is a disjoint union of no bonds, so we are done. Now assume t hat F is any nonempty cut (where the inductive hypothesis is that any cut with a smaller size than F is a disjoint union of bonds). We may assume F is not a bond. Then F contains a nonepmty cut F_1 by the definition of a bond. Now define

$$F_2 = F \setminus F_1,$$

where we note that $F \setminus F_1 = (F \cup F_1) \setminus (F \cap F_1) = F \triangle F_1$. Hence by Lemma 1.6.1, F_2 is a cut. Since F_1, F_2 are nonempty, both F_1, F_2 have fewer edges than F. It follows that F_1, F_2 are disjoint union of bonds by the inductive hypothesis. Thus $F = F_1 \cup F_2$ is also a disjoint union of bonds, as required. ∎

1.3 Vertex Connectivity

Def'n 1.13

Separating Set, Cut-vertex of a Graph

Let G be a graph.

(a) If $S \subseteq V(G)$ is such that $G-S$ is disconnected, then we say S is a ***separating set***.

(b) If $v \in V(G)$ is such that $G-v$ has more components than G, then we say v is a ***cut-vertex***.

Def'n 1.14 **Connectivity** of a Graph

Let G be a graph.

(a) We say G is k-***connected***, where $k \in \mathbb{N} \cup \{0\}$, if G has at least $k+1$ vertices and does not have a separating set of size at most $k-1$.

(b) The ***connectivity*** of G, denoted as $\kappa(G)$, is the largest $k \in \mathbb{N} \cup \{0\}$ for which G is k-connected.

(1.3)

(a) The complete graph K_n ($n \in \mathbb{N}$) has no separating sets. Hence $\kappa(K_n) = n-1$. In particular, K_n is the smallest $(n-1)$-connected graph: any $(n-1)$-connected graph has at least n vertices.

(b) Let G be a graph. If there exists a separating set of size $k \in \mathbb{K}$, then $\kappa(G) \leq k$.

Theorem 1.7

Expansion Lemma

Let G be a k-connected graph and let G' be obtained from G by adding a new vertex v and adding at least k edges joining v to at least k distinct vertices of G. Then G' is k-connected.

Proof. To show that G' is k-connected, we show that there is no separating set of size $k-1$. To do so, let $X \subseteq V(G')$ have $k-1$ vertices. We now have 2 cases.

- *Case 1. Suppose $v \in X$.* Then $G' - X = G - (X \setminus \{v\})$. Since G is k-connected and $|X \setminus \{v\}| = k-2$, so $G - (X \setminus \{v\})$ is connected. Hence $G' - X$ is connected, implying that X is not a separating set.
- *Case 2. Suppose $v \notin X$.* Then $X \subseteq V(G)$. Since $|X| = k-1$ and G is k-connected, $G - X$ is connected. Moreover, v has at least k neighbors by definition, so at least one of them, say w, is not in X. Then given any $u \in V(G-X)$, note that there is a u,w-path in $G-X$ by the connectedness of $G-X$, so by the fact that w,v are neighbors in $G'-X$, there is a u,v-path in $G'-X$. Hence $G'-X$ is connected and X is not a separating set.

Thus G' is k-connected, as required. ∎

Theorem 1.8

Whitney's Theorem

Let G be a graph. Then

$$\kappa(G) \leq \kappa'(G) \leq \delta(G).$$

Proof. We showed the second inequality in Proposition 1.2. We now prove $\kappa(G) \leq \kappa'(G)$. Note that we have the following two special cases.

- *Case 1. Suppose G is complete, say $G = K_n$ for some $n \in \mathbb{N}$.* Then $\kappa(G) = \kappa'(G) = n-1$.
- *Case 2. Suppose G is disconnected.* Then $\kappa(G) = \kappa'(G) = 0$.

Hence, we may assume G is connected and not complete. Let F be a minimum nontrivial cut in G, so that $|F| = \kappa'(G)$. Then F is a bond, since every minimum nontrivial cut is clearly minimal. So by Proposition 1.5, $G - F$ has exactly 2 components, say H_1, H_2. We break into two cases.

- *Case 3. Suppose there exists a vertex $v \in V(G)$ that is not incident to any edge in F.* Without loss of generality, say v is in H_1. Let S b e the set of all vertices in H_1 incident with some edge in F. Then $|S| \leq |F|$. Also, S is a separating set, since it separates v from all the vertices in H_2. So

  $$\kappa(G) \leq |S| \leq |F| = \kappa'(G),$$

  as required.

- *Case 4. Suppose all vertices are incident with some edge in F.* Since G is not complete, there exist two vertices u, v that are not adjacent. Without loss of generality, say $u \in H_1$. Let S be the set of all neighbors of u. Now observe that $v \notin S$, so S is a separating set as it separates u from v. We now count the size of S. To do so, we show that every vertex in S corresponds to a distinct edge in F.
  - *Case 4.1. Say $y \in H_1$.* Then y is incident to some edge in F, say e, (since we assumed that all vertices are incident with some edge in F), and clearly u is not incident to e, since both y, u are in H_1.
  - *Case 4.2. Say $y \in H_2$.* Then $uy \in F$.

  In other words, if $y \in H_1$, then y corresponds to an edge in F that is incident to y but not incident to u. If $y \in H_2$, then y corresponds to uy. And it is immediate from the definition that these correspondences are distinct.[1] It follows that $|S| \leq |F|$, which means

$$\kappa(G) \leq |S| \leq |F| = \kappa'(G).$$

This completes the proof. ∎

1.4 Menger's Theorem

(1.4) So far we have looked at the connectivity of graphs via *destructive* methods: that is, we remove, or *destroy*, few edges or vertices and see what happens. We now change our perspective to *constructive* methods: finding disjoint paths between vertices. We first do this in a *local* point view (i.e. given two vertices, we ask: how well connected are they?). Later, we extend this to a *global* description, using paths.

(1.5) Let us reiterate what we said in (1.4). Suppose that x, y are two nonadjacent vertices of a graph G. There are two ways to measure connectivity between x, y.

(a) How many vertices do we need to remove to disconnect x from y? *destructive*

(b) How many *internally disjoint* paths are there from x to y? *constructive*

Note that for (a) we want the *minimum* number of such vertices, whereas for (b) we want the *maximum* number of such paths. In fact, we can ask the following question: between (a), (b), which one should be larger?

Answer-ish. Suppose there are k paths from x to y. In order to disconnect x from y, we have to remove at least one vertex from each path. In other words, (a) is at least (b). ◁

Surprisingly, *Menger's theorem* shows that this inequality always holds with equality. Let us first define notions that we need.

Def'n 1.15 **X, Y-separating Set**
Let G be a graph and let $X, Y \subseteq V(G)$. We say $S \subseteq V(G) \setminus (X \cup Y)$ is an X, Y-***separating set*** if there is no path from any vertex in X to any vertex in Y in $G - S$. In case $X = \{x\}, Y = \{y\}$, we say S is an x, y-***separating set***.

[1] That is, we just informally described an injection from S to F.

Def'n 1.16

Internally Disjoint x, y-paths

Let G be a graph and let $x, y \in V(G)$. We say x, y-paths are ***internally disjoint*** (or ***i.d.***), if no two of them share any vertices other than x, y.

Note that, from our previous observations, if S is an x, y-separating set and P is a set of internally disjoint x, y-paths, then $|S| \geq |P|$.

Theorem 1.9
Menger's Theorem

Let x, y to nonadjacent distinct vertices of a graph G. Then the minimum size of an x, y-separating set is equal to the maximum size of a set of internally disjoint x, y-paths.

Notation 1.17

G/X

Let G be a graph and let $X \subseteq V(G)$. We denote G/X to be the graph obtained from G by removing all edges in $G[X]$ and identifying all vertices in X into one new vertex.

(1.6)
Wrong Approach to Manger's Theorem

Here is a *wrong*, but useful, approach to the Menger's theorem. We proceed inductively on the number of edges of the graph. Let S be a minimum x, y-separating set, and say $k = |S|$. Let X, Y be the components containing x, y in $G - S$, respectively. Then we *shrink* X by taking G/X, and get k i.d. x, y-paths by induction. Similarly, we *shrink* Y by taking G/Y, and get k i.d. x, y-paths by induction. Combining these paths results in k i.d. x, y-paths in G.

(1.7)
Proof Outline of Manger's Theorem

Here is why (1.6) is wrong: the induction works only when $G/X, G/Y$ have *fewer* edges than G. But if $X = \{x\}$ or $Y = \{y\}$, this is not the case, and the induction fails. Hence, much of the work of the upcoming proof is done to ensure that X, Y has at least 2 vertices, a vertex besides x, y, respectively. Here is an outline.

Proof Outline. We begin by assuming that a minimum x, y-separating set, say S, has size k. We desire to build k i.d. x, y-paths by using induction on $|E(G)|$. We break our proof into three cases.

- *Case 1. Suppose every edge is incident with x, y.* This case is relatively easy.

After we are done with Case 1, we may assume that there are vertices $u, v \in V(G) \setminus \{x, y\}$ that are adjacent. The vertices u, v will be the extra vertices we need in X, Y. Let $e = uv$.

- *Case 2. Suppose a minimum x, y-separating set has size k in $G - e$.* Then by applying induction we obtain k i.d. x, y-paths.

- *Case 3. Suppose a minimum x, y-separating set S has size at most $k - 1$ in $G - e$.* This is arguably the hardest case. For Case 3, we first show that any minimum x, y-separating set indeed has size $k - 1$ in $G - e$. Afterwards, we show that $S \cup \{u\}$ is a minimum x, y-separating set of size k in G/Y, and since G/Y has fewer edges than G, by applying induction we obtain k i.d. x, y-paths. Similarly, we obtain k i.d. x, y-paths in G/X. Combining these paths gives us k i.d. x, y-paths in G. ◁

Proof of Menger's Theorem. We prove by inducting on the number of edges on G. When there are no edges, a minimum x, y-separating set has size 0, and there are 0 x, y-paths. We now assume G has at least one edge, and that the theorem holds for any graph with less than $|E(G)|$ edges. Let k be the size of a minimum x, y-separating set, where the goal is to find k internally disjoint x, y-paths. We have few cases.

- *Case 1. Suppose every edge of G is incident with x or y.* Let S be the set of all vertices adjacent to both x, y. Then the only x, y-paths are those of length 2 going through S, and they are internally disjoint. So there are $|S|$ internally disjoint x, y-paths. But S is a x, y-separating set, so $|S| = k$, as required.

We now assume there exists some edge uv that is not incident to x nor y. Let $H = G - uv$ and let S be a minimum x, y-separating set in H.

- *Case 2. Suppose $|S| = k$.* Since H has fewer edges than G, by induction there exists a set of k internally disjoint x,y-paths in H. These paths are also in G since $H \subseteq G$, and the result follows.

- *Case 3. Suppose $|S| < k$.*

  - *Claim 1. We claim that $S \cup \{u\}$ is an x,y-separating set in G.*

    Proof. Note $G - (S \cup \{u\}) = H - (S \cup \{u\})$. Also, $S \cup \{u\}$ is an x,y-separating set in H, since H is an x,y-separating set in H. It follows that $S \cup \{u\}$ is an x,y-separating set in G. ◁

  But any x,y-separating set in G in G has size at least k, so $|S \cup \{u\}| \geq k$. Therefore, $|S| \geq k-1$, which means $|S| = k-1$. Now write

  $$S = \{v_i\}_{i=1}^{k-1}$$

  for convenience. Let X,Y be the vertices of the components of $H - S$ containing x,y, respectively. Since $|S| = k-1 < k$, $G-S$ does not separate x,y. So there is an x,y-path in $G-S$. This path does not exist in $H-S$, so uv must be on this path. In fact, e must join a vertex in X with a vertex in Y. Without loss of generality, say $u \in X, v \in Y$. This means X,Y has at least two vertices, $X \supseteq \{x,u\}, Y \supseteq \{y,v\}$. We now consider G/Y, and let y^* be the new vertex.

  - *Claim 2. Any x,y^*-separating set in G/Y is an x,y separating set in G.*

    Proof. Let T be any x,y^*-separating set in G/Y. Then any x,y-path in G that does not use T corresponds to an x,y^*-path in G/Y that does not use T, which does not exist. Therefore, T is an x,y-separating set in G. ◁

  Claim 2 in particualr means any x,y^*-separating set in G/Y has size at least k. But $S \cup \{u\}$ is an x,y^*-separating set in G/Y, with size k. Therefore, a minimum x,y^*-separating set in G/Y has size exactly k. Since Y has at least 2 vertices, $G[Y]$ has att least one edge, so G/Y has fewer edges than G, so by induction there is a set of k internally disjoint x,y^*-paths in G/Y. In particular, each path must use a distinct vertex in $S \sup \{u\}$. Now for each $i \in \{1,\ldots,k-1\}$, let P_i be the parts of the paths from x to v_i, and let P_k be the part of the path from x to u. We can make the same argument for G/X, and find k paths: $Q_1,\ldots,Q_{k-1}$ from $v_1,\ldots,v_{k-1}$ to y, Q_k from v to y. Then,

  $$\{P_i + Q_i\}_{i=1}^{k-1} \cup \{P_k + uv + Q_k\}$$

  is a set of k internally disjoint x,y-paths, as required. ∎

(1.8) There are many variations and corollaries of Menger's theorem.

Corollary 1.9.1 Characterization of k-connectedness

Let G be a graph. The following are equivalent.

(a) G is k-connected.

(b) There exist k internally disjoint x,y-paths for any distinct $x,y \in V(G)$.

Proof.

- (a) $\Longrightarrow$ (b): Let $x,y \in V(G)$. We break into 2 cases.

  - *Case 1. Suppose x,y are not adjacent.* Since G is k-connected, any x,y-separating set has size at least k, so by Menger's theorem, there exist at least k internally disjoint x,y-paths in G.

- *Case 2. Suppose x, y are adjacent.* Then $G - xy$ is $(k-1)$-connected. So any x, y-separating set in $G - xy$ has size at least $k-1$. By Menger's theorem, there exist at least $k-1$ internally disjoint x, y-paths in $G - xy$. Together with the path $(\{x, y\}, \{xy\})$, we have at least k internally disjoint x, y-paths in G.

- (a) $\Longleftarrow$ (b): Let X be any set of $k-1$ vertices and let $x, y \in V(G) \setminus X$. By assumption, there exist k internally disjoint x, y-paths in G. So removing $k-1$ vertices leaves at least one such path: there exists an x, y-path in $G - X$ for any vertices x, y in $G - X$. Hence $G - X$ is connected, which means G is k-connected. ∎

Fan

Def'n 1.18 Let G be a graph. Let $v \in V(G)$ and let $X \subseteq V(G) \setminus \{v\}$. A k-***fan*** from v to X is a set of k internally disjoint paths that start with v and end with distinct vertices in X, and no internal vertices are in X.

Corollary 1.9.2
Fan Lemma

Let G be a k-connected graph. Let $v \in V(G)$ and let $X \subseteq V(G) \setminus \{v\}$ where $|X| \geq k$. Then there exists a k-fan from v to X.

Proof. Let H be the graph obtained from G from adding a vertex x^* and edges xx^* for all $x \in X$. Then observe that H is k-connected by the Expansion lemma (Theorem 1.7), so by Corollary 1.9.1, there are k internally disjoint v, x^*-paths, say $P_1, \ldots, P_k$. Now for each $i \in \{1, \ldots, k\}$, write P_i as

$$P_i = (\{v, u_1, \ldots, u_{l_i}, x^*\}, \{vu_1, u_1u_2, \ldots, u_{l-1}u_l, u_lx^*\})$$

and let $j_i \in \{1, \ldots, l_i\}$ be the smallest index such that $u_{j_i} \in X$. It follows that $\{Q_i\}_{i=1}^k$ by

$$Q_i = \left(\left\{v, u_1, \ldots, u_{j_i}\right\}, \left\{vu_1, u_1u_2, \ldots, u_{j_i-1}u_{j_i}\right\}\right)$$

for all $i \in \{1, \ldots, k\}$ is a k-fan from v to X. ∎

Corollary 1.9.3

Let G be a k-connected graph with $k \geq 2$ and let $X \subseteq V(G)$ be such that $|X| = k$. Then there exists a cycle in G with all vertices in X.

Proof. We proceed inductively on k. Suppose $k = 2$ and let $X = \{u, v\}$. Then by Corollary 1.9.1, there exist 2 internally disjoint u, v-paths in G; they form a cycle containing u, v. Now assume $k \geq 3$ and let $X \subseteq V(G)$ with $|X| = k$. Then G is also $(k-1)$-connected. Choose $v \in X$. Then $X \setminus \{v\}$ has $k-1$ vertices, so there is a cycle in G that has all vertices in $X \setminus \{v\}$. If C also has v, then we are done. So assume C does not have v. The $k-1$ vertices of $X \setminus \{v\}$ are on the cycle C, say they are in order $v_1, \ldots, v_{k-1}$. These $k-1$ vertices partitions C into $k-1$ internally disjoint paths, or segments. We now have 2 cases.

- *Case 1. Suppose C has at least k vertices.* Since G is k-connected and $|V(G)| \geq k$, there exists a k-fan from v to $V(C)$ by the Fan lemma. Since there are $k-1$ segment on C, by the pigeonhole principle, there is a segment S on C that contains the endpoints of 2 paths, say P_1, P_2, in the fan. Say P_1, P_2 end with vertices $w_1, w_2 \in V(C)$, respectively. Let P_3 be the w_1, w_2-path contained in S. Then no internal vertex of P_3 is in X, so $C - P_3 + P_1 + P_2$ is a cycle that contains X.

- *Case 2. Suppose C has exactly $k-1$ vertices.* Then there is a $(k-1)$-fan from v to $V(C)$, ending in all vertices in C. By removing any edge from C and adding 2 paths ending at the endpoints of the edge, we obtain a cycle containing X. ∎

Def'n 1.19
Disjoint X,Y-paths
Let G be a graph and let $X,Y \subseteq V(G)$. We say paths in G are ***disjoint*** X,Y-paths if

(a) each join a vertex in X with a vertex in Y;

(b) no oehter vertices are in X nor Y; and

(c) no two paths share any vertex, including the endpoints.

Note that we *allow* X,Y to have a nonempty intersection in Def'n 1.15 and 1.18, and an X,Y-separating set could include vertices in X or Y. In fact, X and Y are X,Y-separating sets.

Theorem 1.10
Menger's Theorem, Set Version

Let G be a graph and let $X,Y \subseteq V(G)$. Then the minimum size of an X,Y-separating set is equal to the maximum size of a set of disjoint X,Y-paths.

Proof. First observe that any size of X,Y-separating set must be at least the size of any set of disjoint X,Y-paths, since we need to remove at least one vertex from each path. It remains to show that there exist an X,Y-separating set and a set of disjoint X,Y-paths of the same size. To do so, we shall utilize Menger's theorem. Let H be the graph obrtained from G by adding 2 new vertices, say x^*, y^*, and adding edges x^*x, y^*y for all $x \in X, y \in Y$. Since x^*, y^* are not adjacent, by Menger's theorem, any minimum x^*, y^*-separating set S has the same size as a maximum set of internally disjoint x,y-paths $\mathcal{P}$.

- *Claim 1. S is an X,Y-separating set in G.*

  Proof. Suppose S is not an X,Y-separating set, for the sake of contradiction. Then there exists and X,Y-path Q from $x \in X$ to $y \in Y$ in $G - S$. Then $(V(Q) \cup \{x^*, y^*\}, E(Q) \cup \{x^*x, y^*y\})$ is an x^*, y^*-path in $H - S$, contradicting the fact that S is an x^*, y^*-separating set. ◁

We now obtain a set of disjoint X,Y-paths $\mathcal{P}'$ from $\mathcal{P}$ as follows. Say a path

$$(\{x^*, v_1, \ldots, v_l, y^*\}, \{x^*v_1, v_1v_2, \ldots, v_{l-1}v_l, v_ly^*\}) \in \mathcal{P}$$

is given. Let $i \in \{1, \ldots, l\}$ be the largest index such that $v_i \in X$, and let $j \in \{1, \ldots, l\}$ be the smallest index at least i such that $v_j \in Y$. We put the path $(\{v_i, \ldots, v_j\}, \{v_iv_{i+1}, \ldots, v_{j-1}v_j\})$ in $\mathcal{P}'$. We see that each path in $\mathcal{P}'$ joins a vertex in X with a vertex in Y with no other vertices in X or Y, and paths in $\mathcal{P}'$ do not share any vertices since $\mathcal{P}$ is internally disjoint. Then $|S| = |\mathcal{P}| = |\mathcal{P}'|$, we have a set of disjoint X,Y-paths $\mathcal{P}'$ of the same size as an X,Y-separating set S. ∎

Def'n 1.20
Edge-disjoint x,y-paths
Let G be a graph and let $x,y \in V(G)$. We say x,y-paths are ***edge-disjoint*** if no 2 paths share an edge.

Theorem 1.11
Menger's Theorem, Edge Version

Let x,y be distinct vertices of a graph G. Then the minimum size of a cut that disconnects x from y is equal to the maximum size of a set of edge-disjoint x,y-paths.

Def'n 1.21
Line Graph of a Graph
Let G be a graph. Then the ***line graph*** of G, denoted as $\mathrm{L}(G)$, is the graph whose vertex set is $\{v_e : e \in E(G)\}$ and v_ev_f is an edge of $\mathrm{L}(G)$ if and only if e, f have a common endpoint in G.

Corollary 1.11.1
Characterization of k-edge-connectedness

Let G be a graph. The following are equivalent.

(a) G is k-edge-connected.

(b) There exist k edge-disjoint x,y-paths for any distinct $x,y \in V(G)$.

1.5 2-connected Graphs

Proposition 1.12
Cycle Characterizations of 2-connectedness

Let G be a graph with at least 3 vertices. The following are equivalent.

(a) G is 2-connected.

(b) For any distinct $u,v \in V(G)$, there is a cycle C such that $u,v \in V(C)$.

(c) $\delta(G) \geq 1$ and for any distinct $e,f \in E(G)$, there is a cycle C such that $e,f \in E(C)$.

Proof.

- (b) $\implies$ (a) This follows immediately from Corollary 1.9.1.
- (a) $\implies$ (c) Assume G is 2-connected. Since G is connected with at least 3 vertices, no isolated vertex exists, so $\delta(G) \geq 1$. Moreover, let $e,f \in E(G)$ be distinct and let $u,v,x,y \in V(G)$ be such that $e = uv, f = xy$. Let $A = \{u,v\}, B = \{x,y\}$. Since G is 2-connected, any A,B-separating set has size at least 2. Hence by the set version of Menger's theorem (Theorem 1.10), there exist 2 disjoint A,B-paths, P_1, P_2. In particular, both endpoints of e,f are on distinct paths. So
$$C = P_1 + f + P_2 + e$$
is a cycle which has e,f.
- (c) $\implies$ (b) Let $u,v \in V(G)$ be distinct. Since $\delta(G) \geq 1$, both u,v are incident to at least one edge.
  - *Claim 1. There exist distinct edges e,f that are incident to u,v, respectively.*

    Proof. For the sake of contradiction, suppose $e \in E(G)$ is the only edge incident to u,v. Then $e = uv$. But this means
$$H = (\{u,v\},\{uv\})$$
is a component of G, since no edge other than $e = uv$ is incident to u,v. Since G has at least 3 vertices, there is $w \in V(G) \setminus \{v,u\}$. But then there is $f \in E(G)$ that is incident to w, and $f \neq e$. Since H is a component, it follows that there is no cycle of G that has e,f, contradicting (c). ◁

  By Claim 1, choose distinct e,f that are incident to u,v, respectively. Then by (c), there exists a cycle C which has e,f, which means C also has u,v. ∎

Def'n 1.22

Ear Decomposition of a Graph

Let F be a subgraph of G.

(a) An ***ear*** of F in G is a nontrivial path in G where only the two endpoints of the path are in F.

(b) An ***ear decomposition*** of G is a sequence of graphs $(G_0)_{i=0}^{k}$ where

(i) G_0 is a cycle in G; *starting cycle*

(ii) for each $r \in \{0, \dots, k-1\}$,

$$G_{i+1} = G_i + P_i$$

where P_i is an ear of G_i; and *intermediate cycles*

(iii) $G_k = G$.

Proposition 1.13
Characterization of 2-connectedness by Ear Decompositions

Let G be a graph. The following are equivalent.

(a) G is 2-connected.

(b) G has an ear decomposition.

Proof.

- ($\Longrightarrow$) Let G be 2-connected and let G_0 be any cycle in G. For $i \in \mathbb{N} \cup \{0\}$, we inductively construct G_{i+1} from G_i as follows.

  (a) If $G_i = G$, then we are done.

  (b) Otherwise, there exists an edge $e \in E(G) \setminus E(G_i)$, say $e = uv$ for some $u, v \in V(G)$. Then by the set version of Menger's theorem, there exist 2 disjoint $\{u, v\}, G_i$-path in G, say Q_1, Q_2. Then observe that
  $$P_i = Q_1 + uv + Q_2$$
  is an ear of G_i. We then set $G_{i+1} = G_i + P_i$.

  Note that the above process terminates since we add at least one edge of G each time. At termination, say after k step, $(G_i)_{i=1}^{k}$ is an ear decomposition of G.

- ($\Longleftarrow$) Suppose $(G_i)_{i=0}^{k}$ is an ear decomposition of G. We proceed inductively to prove each G_i is 2-connected, thereby proving that G is also 2-connected. Since G_0 is a cycle, G_0 is 2-connected. Now suppose that G_i is 2-connected, where $i \in \mathbb{N} \cup \{0\}$ and let P_i be an ear of G_i such that
  $$G_{i+1} = G_i + P_i.$$
  We claim that every distinct vertices x, y in G_{i+1} are in a cycle, so that we can use Proposition 1.12. Suppose that the two endpoins of P_i are u, v and let $x, y \in V(G_{i+1})$. We have 3 cases.

  · *Case 1. Suppose* $x, y \in V(G_i)$. Then since G_i is 2-connected, there is a cycle C in G_i, hence in G_{i+1}, which has x, y.

  · *Case 2. Suppose* $x, y \in V(P_i)$. Since G_i is connected, there exists a u, v-path Q in G_i. It follows that $P_i + Q$ is a cycle which has x, y.

  · *Case 3. Suppose exactly one of* x, y *is in* G_i. Without loss of generality, say $x \in V(G_i), y \in V(P_i)$. Since G_i is 2-connected, there exist 2-fan from x to $\{u, v\}$ by the fan lemma. Then 2-fan combined with P_i is a cycle which has x, y.

  Hence every 2 distinct vertices in G_{i+1} are in a cycle, and by Proposition 1.12 G_{i+1} is 2-connected. Thus G_k is 2-connected, as required. ∎

Corollary 1.13.1

Let G be a graph.

(a) If G is 2-connected, then any cycle in G can be the starting cycle in the ear decomposition.

(b) If G admits an ear decomposition, then each intermediate graph in the ear deecomposition is 2-connected.

Proposition 1.14

Let G be a graph. If G admits an ear decomposition $(G_0)_{i=0}^{k}$, then $k = |E(G)| - |V(G)|$.