CO 342

Graph Theory I

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1. Connectivity

1.1 Basic Graph Terminology

Def'n 1.1 **Graph**

A ***graph*** $G=(V,E)$ is an ordered pair such that

(a) V is a set; and *vertex set*

(b) E is a set of unordered pairs of elements of V. *edge set*

The elements of V are called the ***vertices*** of G and the elements of E are called the ***edges*** of G. Given $u,v\in V$, we denote uv to denote an edge ***joining*** u,v. Note that $uv=vu$ for every $u,v\in V$. For simplicity, we assume that

(a) G does not have any ***loop*** (i.e. $vv\notin E$ for every $v\in V$);[a]

(b) G has at least one vertex; and

(c) G is ***finite*** (i.e. V is a finite set).

[a] Note that, since we defined E to be a set of unordered pairs of elements of V, if $e,f\in E$ join the same vertices, then $e=f$. In other words, we do not allow ***multiple edges***.

Notation 1.2 $V(G), E(G)$

Let G be a graph. We write $V(G)$ to denote the vertex set and write $E(G)$ to denote the edge set of G.

Def'n 1.3 **Subgraph** of a Graph

Let G be a graph. We say H is a ***subgraph*** of G if $V(H)\subseteq V(G), E(H)\subseteq E(G)$, and all edges in $E(H)$ have both endpoints in $V(H)$. In particular,

(a) when $V(H)=V(G)$, we say H is a ***spanning*** subgraph; and

(b) if there exists $S\subseteq V(G)$ such that $V(H)=S$ and that

$$E(H)=\{e\in E(G): \exists v,u\in S\,[e=vu]\},$$

then we say H is the subgraph ***induced*** by S, denoted as $G[S]$.

Def'n 1.4 **Degree** of a Vertex

Let G be a graph. The ***degree*** of $v\in V(G)$, denoted as $d_G(v)$, is the number of edges incident to v. Moreover, we write $\delta(G)$ to denote

$$\delta(G)=\min\{d_G(v): v\in V(G)\}$$

and write $\Delta(G)$ to denote

$$\Delta(G)=\max\{d_G(v): v\in V(G)\}.$$

Def'n 1.5 **Connected** Graph

Let G be a graph. We say G is ***connected*** if for every $u,v\in V(G)$, there is a u,v-path in G. A ***component*** of G is a maximally connected subgraph of G.

Proposition 1.1

Let G be a graph. The following are equivalent.

(a) G is connected.

(b) There exists $v \in V(G)$ such that for every $u \in V(G)$, there exists a u, v-path in G.

Notation 1.6 — $G - F$, $G - e$, $G - U$, $G - v$

Let G be a graph.

(a) Given $F \subseteq E(G)$, we write $G - F$ to denote the graph such that $V(G - F) = V(G), E(G - F) = E(G) \setminus F$.

(b) In case $F = \{e\}$ for some $e \in E$, we simply write $G - e$ to denote $G - \{e\}$.

(c) Similarly, given $U \subseteq V(G)$, we write $G - U$ to denote the graph such that $V(G - U) = V(G) \setminus U$ and that
$$E(G - U) = \{e \in E(G) : \exists u, v \in V(G) \setminus U\, [e = uv]\}.$$
In other words, we remove every vertex in U from G and every edge in G that is incident to at least one vertex in U.

(d) In case $U = \{v\}$ for some $v \in E$, we simply write $G - v$ to denote $G - \{v\}$.

Def'n 1.7 — **Contraction** of a Graph

Let G be a graph. For any $e \in E(G)$, the G ***contract*** e, denoted as G/e, is defined by *removing* e from G and *identifying* u, v as the same vertex.

1.2 Edge Connectivity

Def'n 1.8 — ***k*-edge-connected** Graph

We say a graph G is ***k*-edge-connected** if $G - F$ is connected for every $F \subseteq E(G)$ such that $|F| < k$. The ***edge connectivity*** of G, denoted as $\kappa'(G)$, is the largest $k \in \mathbb{N}$ for which G is k-edge-connected.

Def'n 1.9 — **Cut-edge** of a Graph

Let G be a graph. We say $e \in E(G)$ is a ***cut-edge*** if $G - \{e\}$ is disconnected.

(1.1) Let G be a graph.

(a) $\kappa'(G) = 1$ if and only if G has a cut-edge. Equivalently, G is 2-edge connected if and only if G is connected with no cut-edges.

(b) If G is disconnected, then we define $\kappa'(G) = 0$ and say G is 0-edge-connected.

(c) The ***trivial graph*** $(\{v\}, \varnothing)$ *cannot be disconneted*; we define its edge-connectivity to be 0.

Proposition 1.2

Let G be a graph. Then

$$\kappa'(G) \leq \delta(G).$$

Proof. If $\delta(G) = 0$, then there is an isolated vertex $v \in V(G)$, which means G is disconnected. So $\kappa'(G) = 0$, meaning $\delta(G) = 0 \leq 0 = \kappa'(G)$. Now assume $\delta(G) \geq 1$ and let $v \in V(G)$ be such that $d_G(v) = \delta(G)$. Let $F = \{e \in E(G) : \exists u \in V(G)[uv = e]\}$, the set of all edges incident with v. Then $G - F$ is disconnected, since v is not adjacent to any other vertices in G, which in turn exist since $\delta(G) \geq 1$. It follows that $\kappa'(G) \leq |F| = \delta(G)$. ♦

Def'n 1.10

Cut Induced by a Subset

Let G be a graph and let $S \subseteq V(G)$. We define the ***cut*** induced by S, denoted as $\delta_G(S)$, to be the set of all edges with exactly one end in S:

$$\delta_G(S) = \{e \in E(G) : \exists v \in V(G) \setminus S, u \in S [e = vu]\}.$$

We say $\delta_G(S)$ is ***nontrivial*** if $S \neq \varnothing$ and $S \neq V(G)$.

Proposition 1.3

Let G be a graph. Then the following are equivalent.

(a) G is connected.

(b) Every nontrivial cut of G is nonempty.

(1.2) Say we have a graph and a cut $C \subseteq E(G)$. Then it is easy to see that $G - C$ is disconnected when C is nontrivial. It follows that the size of any nontrivial cut is an upper bound on $\kappa'(G)$.

Proposition 1.4

Let G be a graph and let $F \subseteq E(G)$. If $G - F$ is disconnected, then F contains a nontrivial cut.

Proof. Suppose $G - F$ is disconnected and let H be a component of $G - F$. Since $G - F$ is disconnected, $V(H) \neq \varnothing$ and $V(H) \neq V(G)$. Consider $\delta_G(V(H))$. Since H is maximally connected in $G - F$, none of the edges in $\delta_G(V(h))$ can be in $G - F$. Hence, all edeges in $|deltaa_G(V(H))$ must be in F. Hence F contains a nontrivial cut $\delta_G(V(H))$, as required. ♦

Corollary 1.4.1

Let G be a graph. Then $\kappa'(G)$ is the size of a minimum nontrivial cut.

Def'n 1.11

Bond of a Graph

A ***bond*** is a minimal nonempty cut.

Proposition 1.5
Characterization of a Bond

Let $F = \delta_G(S)$ be a cut in a connected graph G, where $S \subseteq V(G)$. Then the following are equivalent.

(a) F is a bond.

(b) $G - F$ has exactly 2 components.

Proof.

- ($\Longrightarrow$) Suppose $G - F$ has at least 3 components. Assume, without loss of generality, that (at least) 2 of the components are contained in $G[S]$ and let H be one of them. Then $\delta_G(V(H)) \subset F$, so F is not minimal and hence not a bond.
- Suppose $G - F$ has exactly 2 components and suppose further F is not a bond, for the sake of contradiction. Then there is a nonempty cut $\delta_G(S)$ that is a proper subset of F. Then the two components of G_F are linked by some edge in F. It follows that $G - \delta_G(S)$ is connected, contradicting the fact that $\delta_G(S)$ is a cut. ♦

Proposition 1.6

Every cut is a disjoint union of bonds.

Def'n 1.12

Symmetric Difference of Two Sets

Let S, T be sets. Then the ***symmetric difference*** of S, T, denoted as $S \triangle T$, is defined as

$$S \triangle T = (S \cup T) \setminus (S \cap T).$$

Lemma 1.6.1

Let G be a graph and let $S, T \subseteq V(G)$. Then

$$\delta_G(S) \triangle \delta_G(T) = \delta_G(S \triangle T).$$

Proof of Proposition 1.6. We proceed inductively on the number of edges in F. If $|F| = 0$, then is a disjoint union of no bonds, so we are done. Now assume t hat F is any nonempty cut (where the inductive hypothesis is that any cut with a smaller size than F is a disjoint union of bonds). We may assume F is not a bond. Then F contains a nonepmty cut F_1 by the definition of a bond. Now define

$$F_2 = F \setminus F_1,$$

where we note that $F \setminus F_1 = (F \cup F_1) \setminus (F \cap F_1) = F \triangle F_1$. Hence by Lemma 1.6.1, F_2 is a cut. Since F_1, F_2 are nonempty, both F_1, F_2 have fewer edges than F. It follows that F_1, F_2 are disjoint union of bonds by the inductive hypothesis. Thus $F = F_1 \cup F_2$ is also a disjoint union of bonds, as required. ♦

1.3 Vertex Connectivity

Def'n 1.13

Separating Set, Cut-vertex of a Graph

Let G be a graph.

(a) If $S \subseteq V(G)$ is such that $G - S$ is disconnected, then we say S is a ***separating set***.

(b) If $v \in V(G)$ is such that $G - v$ has more components than G, then we say v is a ***cut-vertex***.

Def'n 1.14

Connectivity of a Graph

Let G be a graph.

(a) We say G is k-***connected***, where $k \in \mathbb{N} \cup \{0\}$, if G has at least $k+1$ vertices and does not have a separating set of size at most $k-1$.

(b) The ***connectivity*** of G, denoted as $\kappa(G)$, is the largest $k \in \mathbb{N} \cup \{0\}$ for which G is k-connected.

(1.3)

(a) The complete graph K_n ($n \in \mathbb{N}$) has no separating sets. Hence $\kappa(K_n) = n-1$. In particular, K_n is the smallest $(n-1)$-connected graph: any $(n-1)$-connected graph has at least n vertices.

(b) Let G be a graph. If there exists a separating set of size $k \in \mathbb{K}$, then $\kappa(G) \leq k$.

Theorem 1.7
Expansion Lemma

Let G be a k-connected graph and let G' be obtained from G by adding a new vertex v and adding at least k edges joining v to at least k distinct vertices of G. Then G' is k-connected.

Proof. To show that G' is k-connected, we show that there is no separating set of size $k-1$. To do so, let $X \subseteq V(G')$ have $k-1$ vertices. We now have 2 cases.

- *Case 1. Suppose $v \in X$.* Then $G' - X = G - (X \setminus \{v\})$. Since G is k-connected and $|X \setminus \{v\}| = k-2$, so $G - (X \setminus \{v\})$ is connected. Hence $G' - X$ is connected, implying that X is not a separating set.
- *Case 2. Suppose $v \notin X$.* Then $X \subseteq V(G)$. Since $|X| = k-1$ and G is k-connected, $G - X$ is connected. Moreover, v has at least k neighbors by definition, so at least one of them, say w, is not in X. Then given any $u \in V(G-X)$, note that there is a u,w-path in $G-X$ by the connectedness of $G-X$, so by the fact that w,v are neighbors in $G'-X$, there is a u,v-path in $G'-X$. Hence $G'-X$ is connected and X is not a separating set.

Thus G' is k-connected, as required. ♦

Theorem 1.8
Whitney's Theorem

Let G be a graph. Then

$$\kappa(G) \leq \kappa'(G) \leq \delta(G).$$

Proof. We showed the second inequality in Proposition 1.2. We now prove $\kappa(G) \leq \kappa'(G)$. Note that we have the following two special cases.

- *Case 1. Suppose G is complete, say $G = K_n$ for some $n \in \mathbb{N}$.* Then $\kappa(G) = \kappa'(G) = n-1$.
- *Case 2. Suppose G is disconnected.* Then $\kappa(G) = \kappa'(G) = 0$.

Hence, we may assume G is connected and not complete. Let F be a minimum nontrivial cut in G, so that $|F| = \kappa'(G)$. Then F is a bond, since every minimum nontrivial cut is clearly minimal. So by Proposition 1.5, $G - F$ has exactly 2 components, say H_1, H_2. We break into two cases.

- *Case 3. Suppose there exists a vertex $v \in V(G)$ that is not incident to any edge in F.* Without loss of generality, say v is in H_1. Let S b e the set of all vertices in H_1 incident with some edge in F. Then $|S| \leq |F|$. Also, S is a separating set, since it separates v from all the vertices in H_2. So

  $$\kappa(G) \leq |S| \leq |F| = \kappa'(G),$$

  as required.

- *Case 4. Suppose all vertices are incident with some edge in F.* Since G is not complete, there exist two vertices u, v that are not adjacent. Without loss of generality, say $u \in H_1$. Let S be the set of all neighbors of u. Now observe that $v \notin S$, so S is a separating set as it separates u from v. We now count the size of S. To do so, we show that every vertex in S corresponds to a distinct edge in F.

  - *Case 4.1. Say $y \in H_1$.* Then y is incident to some edge in F, say e, (since we assumed that all vertices are incident with some edge in F), and clearly u is not incident to e, since both y, u are in H_1.
  - *Case 4.2. Say $y \in H_2$.* Then $uy \in F$.

  In other words, if $y \in H_1$, then y corresponds to an edge in F that is incident to y but not incident to u. If $y \in H_2$, then y corresponds to uy. And it is immediate from the definition that these correspondences are distinct.[1] It follows that $|S| \leq |F|$, which means

$$\kappa(G) \leq |S| \leq |F| = \kappa'(G).$$

This completes the proof. ♦

1.4 Menger's Theorem

(1.4) So far we have looked at the connectivity of graphs via *destructive* methods: that is, we remove, or *destroy*, few edges or vertices and see what happens. We now change our perspective to *constructive* methods: finding disjoint paths between vertices. We first do this in a *local* point view (i.e. given two vertices, we ask: how well connected are they?). Later, we extend this to a *global* description, using paths.

(1.5) Let us reiterate what we said in (1.4). Suppose that x, y are two nonadjacent vertices of a graph G. There are two ways to measure connectivity between x, y.

(a) How many vertices do we need to remove to disconnect x from y? *destructive*

(b) How many *internally disjoint* paths are there from x to y? *constructive*

Note that for (a) we want the *minimum* number of such vertices, whereas for (b) we want the *maximum* number of such paths. In fact, we can ask the following question: between (a), (b), which one should be larger?

Answer-ish. Suppose there are k paths from x to y. In order to disconnect x from y, we have to remove at least one vertex from each path. In other words, (a) is at least (b). ◁

Surprisingly, *Menger's theorem* shows that this inequality always holds with equality. Let us first define notions that we need.

X, Y-separating Set

Def'n 1.15 Let G be a graph and let $X, Y \subseteq V(G)$. We say $S \subseteq V(G) \setminus (X \cup Y)$ is an X, Y-***separating set*** if there is no path from any vertex in X to any vertex in Y in $G - S$. In case $X = \{x\}, Y = \{y\}$, we say S is an x, y-***separating set***.

[1] That is, we just informally described an injection from S to F.

Def'n 1.16 **Internally Disjoint x, y-paths**
Let G be a graph and let $x, y \in V(G)$. We say x, y-paths are ***internally disjoint*** (or ***i.d.***), if no two of them share any vertices other than x, y.

Note that, from our previous observations, if S is an x, y-separating set and P is a set of internally disjoint x, y-paths, then $|S| \geq |P|$.

Theorem 1.9
Menger's Theorem

Let x, y be nonadjacent distinct vertices of a graph G. Then the minimum size of an x, y-separating set is equal to the maximum size of a set of internally disjoint x, y-paths.

Notation 1.17 G/X
Let G be a graph and let $X \subseteq V(G)$. We denote G/X to be the graph obtained from G by removing all edges in $G[X]$ and identifying all vertices in X into one new vertex.

(1.6)
Wrong Approach to Manger's Theorem

Here is a *wrong*, but useful, approach to the Menger's theorem. We proceed inductively on the number of edges of the graph. Let S be a minimum x, y-separating set, and say $k = |S|$. Let X, Y be the components containing x, y in $G - S$, respectively. Then we *shrink* X by taking G/X, and get k i.d. x, y-paths by induction. Similarly, we *shrink* Y by taking G/Y, and get k i.d. x, y-paths by induction. Combining these paths results in k i.d. x, y-paths in G.

(1.7)
Proof Outline of Manger's Theorem

Here is why (1.6) is wrong: the induction works only when $G/X, G/Y$ have *fewer* edges than G. But if $X = \{x\}$ or $Y = \{y\}$, this is not the case, and the induction fails. Hence, much of the work of the upcoming proof is done to ensure that X, Y has at least 2 vertices, a vertex besides x, y, respectively. Here is an outline.

Proof Outline. We begin by assuming that a minimum x, y-separating set, say S, has size k. We desire to build k i.d. x, y-paths by using induction on $|E(G)|$. We break our proof into three cases.

- *Case 1. Suppose every edge is incident with x, y.* This case is relatively easy.

After we are done with Case 1, we may assume that there are vertices $u, v \in V(G) \setminus \{x, y\}$ that are adjacent. The vertices u, v will be the extra vertices we need in X, Y. Let $e = uv$.

- *Case 2. Suppose a minimum x, y-separating set has size k in $G - e$.* Then by applying induction we obtain k i.d. x, y-paths.

- *Case 3. Suppose a minimum x, y-separating set S has size at most $k - 1$ in $G - e$.* This is arguably the hardest case. For Case 3, we first show that any minimum x, y-separating set indeed has size $k - 1$ in $G - e$. Afterwards, we show that $S \cup \{u\}$ is a minimum x, y-separating set of size k in G/Y, and since G/Y has fewer edges than G, by applying induction we obtain k i.d. x, y-paths. Similarly, we obtain k i.d. x, y-paths in G/X. Combining these paths gives us k i.d. x, y-paths in G. $\triangleleft$

Proof of Menger's Theorem. We prove by inducting on the number of edges on G. When there are no edges, a minimum x, y-separating set has size 0, and there are 0 x, y-paths. We now assume G has at least one edge, and that the theorem holds for any graph with less than $|E(G)|$ edges. Let k be the size of a minimum x, y-separating set, where the goal is to find k internally disjoint x, y-paths. We have few cases.

- *Case 1. Suppose every edge of G is incident with x or y.* Let S be the set of all vertices adjacent to both x, y. Then the only x, y-paths are those of length 2 going through S, and they are internally disjoint. So there are $|S|$ internally disjoint x, y-paths. But S is a x, y-separating set, so $|S| = k$, as required.

We now assume there exists some edge uv that is not incident to x nor y. Let $H = G - uv$ and let S be a minimum x, y-separating set in H.

- *Case 2. Suppose* $|S| = k$. Since H has fewer edges than G, by induction there exists a set of k internally disjoint x,y-paths in H. These paths are also in G since $H \subseteq G$, and the result follows.

- *Case 3. Suppose* $|S| < k$.

  - *Claim 1. We claim that* $S \cup \{u\}$ *is an* x,y*-separating set in* G.

    Proof. Note $G - (S \cup \{u\}) = H - (S \cup \{u\})$. Also, $S \cup \{u\}$ is an x,y-separating set in H, since H is an x,y-separating set in H. It follows that $S \cup \{u\}$ is an x,y-separating set in G. ◁

  But any x,y-separating set in G in G has size at least k, so $|S \cup \{u\}| \geq k$. Therefore, $|S| \geq k-1$, which means $|S| = k-1$. Now write
  $$S = \{v_i\}_{i=1}^{k-1}$$
  for convenience. Let X, Y be the vertices of the components of $H - S$ containing x, y, respectively. Since $|S| = k-1 < k$, $G - S$ does not separate x, y. So there is an x,y-path in $G - S$. This path does not exist in $H - S$, so uv must be on this path. In fact, e must join a vertex in X with a vertex in Y. Without loss of generality, say $u \in X, v \in Y$. This means X, Y has at least two vertices, $X \supseteq \{x, u\}, Y \supseteq \{y, v\}$. We now consider G/Y, and let y^* be the new vertex.

  - *Claim 2. Any* x, y^**-separating set in* G/Y *is an* x, y *separating set in* G.

    Proof. Let T be any x, y^*-separating set in G/Y. Then any x,y-path in G that does not use T corresponds to an x, y^*-path in G/Y that does not use T, which does not exist. Therefore, T is an x,y-separating set in G. ◁

  Claim 2 in particualr means any x, y^*-separating set in G/Y has size at least k. But $S \cup \{u\}$ is an x, y^*-separating set in G/Y, with size k. Therefore, a minimum x, y^*-separating set in G/Y has size exactly k. Since Y has at least 2 vertices, $G[Y]$ has att least one edge, so G/Y has fewer edges than G, so by induction there is a set of k internally disjoint x, y^*-paths in G/Y. In particular, each path must use a distinct vertex in $S \sup \{u\}$. Now for each $i \in \{1, \ldots, k-1\}$, let P_i be the parts of the paths from x to v_i, and let P_k be the part of the path from x to u. We can make the same argument for G/X, and find k paths: $Q_1, \ldots, Q_{k-1}$ from $v_1, \ldots, v_{k-1}$ to y, Q_k from v to y. Then,
  $$\{P_i + Q_i\}_{i=1}^{k-1} \cup \{P_k + uv + Q_k\}$$
  is a set of k internally disjoint x,y-paths, as required. ♦

(1.8) There are many variations and corollaries of Menger's theorem.

Corollary 1.9.1
Characterization of k-connectedness

Let G be a graph. The following are equivalent.

(a) G is k-connected.

(b) There exist k internally disjoint x,y*-paths for any distinct* $x, y \in V(G)$.

Proof.

- (a) $\Longrightarrow$ (b): Let $x, y \in V(G)$. We break into 2 cases.

  - *Case 1. Suppose* x, y *are not adjacent.* Since G is k-connected, any x,y-separating set has size at least k, so by Menger's theorem, there exist at least k internally disjoint x,y-paths in G.

- *Case 2. Suppose x,y are adjacent.* Then $G-xy$ is $(k-1)$-connected. So any x,y-separating set in $G-xy$ has size at least $k-1$. By Menger's theorem, there exist at least $k-1$ internally disjoint x,y-paths in $G-xy$. Together with the path $(\{x,y\},\{xy\})$, we have at least k internally disjoint x,y-paths in G.

- (a) $\Longleftarrow$ (b): Let X be any set of $k-1$ vertices and let $x,y\in V(G)\setminus X$. By assumption, there exist k internally disjoint x,y-paths in G. So removing $k-1$ vertices leaves at least one such path: there exists an x,y-path in $G-X$ for any vertices x,y in $G-X$. Hence $G-X$ is connected, which means G is k-connected. ♦

Def'n 1.18

Fan

Let G be a graph. Let $v\in V(G)$ and let $X\subseteq V(G)\setminus\{v\}$. A k-***fan*** from v to X is a set of k internally disjoint paths that start with v and end with distinct vertices in X, and no internal vertices are in X.

Corollary 1.9.2
Fan Lemma

Let G be a k-connected graph. Let $v\in V(G)$ and let $X\subseteq V(G)\setminus\{v\}$ where $|X|\geq k$. Then there exists a k-fan from v to X.

Proof. Let H be the graph obtained from G from adding a vertex x^* and edges xx^* for all $x\in X$. Then observe that H is k-connected by the Expansion lemma (Theorem 1.7), so by Corollary 1.9.1, there are k internally disjoint v,x^*-paths, say $P_1,\dots,P_k$. Now for each $i\in\{1,\dots,k\}$, write P_i as

$$P_i=(\{v,u_1,\dots,u_{l_i},x^*\},\{vu_1,u_1u_2,\dots,u_{l-1}u_l,u_lx^*\})$$

and let $j_i\in\{1,\dots,l_i\}$ be the smallest index such that $u_{j_i}\in X$. It follows that $\{Q_i\}_{i=1}^k$ by

$$Q_i=\left(\{v,u_1,\dots,u_{j_i}\},\{vu_1,u_1u_2,\dots,u_{j_i-1}u_{j_i}\}\right)$$

for all $i\in\{1,\dots,k\}$ is a k-fan from v to X. ♦

Corollary 1.9.3

Let G be a k-connected graph with $k\geq 2$ and let $X\subseteq V(G)$ be such that $|X|=k$. Then there exists a cycle in G with all vertices in X.

Proof. We proceed inductively on k. Suppose $k=2$ and let $X=\{u,v\}$. Then by Corollary 1.9.1, there exist 2 internally disjoint u,v-paths in G; they form a cycle containing u,v. Now assume $k\geq 3$ and let $X\subseteq V(G)$ with $|X|=k$. Then G is also $(k-1)$-connected. Choose $v\in X$. Then $X\setminus\{v\}$ has $k-1$ vertices, so there is a cycle in G that has all vertices in $X\setminus\{v\}$. If C also has v, then we are done. So assume C does not have v. The $k-1$ vertices of $X\setminus\{v\}$ are on the cycle C, say they are in order $v_1,\dots,v_{k-1}$. These $k-1$ vertices partitions C into $k-1$ internally disjoint paths, or segments. We now have 2 cases.

- *Case 1. Suppose C has at least k vertices.* Since G is k-connected and $|V(G)|\geq k$, there exists a k-fan from v to $V(C)$ by the Fan lemma. Since there are $k-1$ segment on C, by the pigeonhole principle, there is a segment S on C that contains the endpoints of 2 paths, say P_1,P_2, in the fan. Say P_1,P_2 end with vertices $w_1,w_2\in V(C)$, respectively. Let P_3 be the w_1,w_2-path contained in S. Then no internal vertex of P_3 is in X, so $C-P_3+P_1+P_2$ is a cycle that contains X.

- *Case 2. Suppose C has exactly $k-1$ vertices.* Then there is a $(k-1)$-fan from v to $V(C)$, ending in all vertices in C. By removing any edge from C and adding 2 paths ending at the endpoints of the edge, we obtain a cycle containing X. ♦

Def'n 1.19 **Disjoint** X, Y-paths

Let G be a graph and let $X, Y \subseteq V(G)$. We say paths in G are ***disjoint*** X, Y-paths if

(a) each join a vertex in X with a vertex in Y;

(b) no oehter vertices are in X nor Y; and

(c) no two paths share any vertex, including the endpoints.

Note that we *allow* X, Y to have a nonempty intersection in Def'n 1.15 and 1.18, and an X, Y-separating set could include vertices in X or Y. In fact, X and Y are X, Y-separating sets.

Theorem 1.10
Menger's Theorem, Set Version

Let G be a graph and let $X, Y \subseteq V(G)$. Then the minimum size of an X, Y-separating set is equal to the maximum size of a set of disjoint X, Y-paths.

Proof. First observe that any size of X, Y-separating set must be at least the size of any set of disjoint X, Y-paths, since we need to remove at least one vertex from each path. It remains to show that there exist an X, Y-separating set and a set of disjoint X, Y-paths of the same size. To do so, we shall utilize Menger's theorem. Let H be the graph obrtained from G by adding 2 new vertices, say x^*, y^*, and adding edges x^*x, y^*y for all $x \in X, y \in Y$. Since x^*, y^* are not adjacent, by Menger's theorem, any minimum x^*, y^*-separating set S has the same size as a maximum set of internally disjoint x, y-paths $\mathcal{P}$.

- *Claim 1. S is an X, Y-separating set in G.*

  Proof. Suppose S is not an X, Y-separating set, for the sake of contradiction. Then there exists and X, Y-path Q from $x \in X$ to $y \in Y$ in $G - S$. Then $(V(Q) \cup \{x^*, y^*\}, E(Q) \cup \{x^*x, y^*y\})$ is an x^*, y^*-path in $H - S$, contradicting the fact that S is an x^*, y^*-separating set. ◁

We now obtain a set of disjoint X, Y-paths $\mathcal{P}'$ from $\mathcal{P}$ as follows. Say a path

$$(\{x^*, v_1, \ldots, v_l, y^*\}, \{x^*v_1, v_1v_2, \ldots, v_{l-1}v_l, v_ly^*\}) \in \mathcal{P}$$

is given. Let $i \in \{1, \ldots, l\}$ be the largest index such that $v_i \in X$, and let $j \in \{1, \ldots, l\}$ be the smallest index at least i such that $v_j \in Y$. We put the path $(\{v_i, \ldots, v_j\}, \{v_iv_{i+1}, \ldots, v_{j-1}v_j\})$ in $\mathcal{P}'$. We see that each path in $\mathcal{P}'$ joins a vertex in X with a vertex in Y with no other vertices in X or Y, and paths in $\mathcal{P}'$ do not share any vertices since $\mathcal{P}$ is internally disjoint. Then $|S| = |\mathcal{P}| = |\mathcal{P}'|$, we have a set of disjoint X, Y-paths $\mathcal{P}'$ of the same size as an X, Y-separating set S. ♦

Def'n 1.20 **Edge-disjoint** x, y-paths

Let G be a graph and let $x, y \in V(G)$. We say x, y-paths are ***edge-disjoint*** if no 2 paths share an edge.

Theorem 1.11
Menger's Theorem, Edge Version

Let x, y be distinct vertices of a graph G. Then the minimum size of a cut that disconnects x from y is equal to the maximum size of a set of edge-disjoint x, y-paths.

Def'n 1.21 **Line Graph** of a Graph

Let G be a graph. Then the ***line graph*** of G, denoted as $\mathrm{L}(G)$, is the graph whose vertex set is $\{v_e : e \in E(G)\}$ and v_ev_f is an edge of $\mathrm{L}(G)$ if and only if e, f have a common endpoint in G.

Corollary 1.11.1
Characterization of k-edge-connectedness

Let G be a graph. The following are equivalent.

(a) G is k-edge-connected.

(b) There exist k edge-disjoint x,y-paths for any distinct $x,y \in V(G)$.

1.5 2-connected Graphs

Proposition 1.12
Cycle Characterizations of 2-connectedness

Let G be a graph with at least 3 vertices. The following are equivalent.

(a) G is 2-connected.

(b) For any distinct $u,v \in V(G)$, there is a cycle C such that $u,v \in V(C)$.

(c) $\delta(G) \geq 1$ and for any distinct $e,f \in E(G)$, there is a cycle C such that $e,f \in E(C)$.

Proof.

- (b) $\Longrightarrow$ (a) This follows immediately from Corollary 1.9.1.
- (a) $\Longrightarrow$ (c) Assume G is 2-connected. Since G is connected with at least 3 vertices, no isolated vertex exists, so $\delta(G) \geq 1$. Moreover, let $e,f \in E(G)$ be distinct and let $u,v,x,y \in V(G)$ be such that $e = uv, f = xy$. Let $A = \{u,v\}, B = \{x,y\}$. Since G is 2-connected, any A,B-separating set has size at least 2. Hence by the set version of Menger's theorem (Theorem 1.10), there exist 2 disjoint A,B-paths, P_1, P_2. In particular, both endpoints of e,f are on distinct paths. So
$$C = P_1 + f + P_2 + e$$
is a cycle which has e,f.
- (c) $\Longrightarrow$ (b) Let $u,v \in V(G)$ be distinct. Since $\delta(G) \geq 1$, both u,v are incident to at least one edge.
  - *Claim 1. There exist distinct edges e,f that are incident to u,v, respectively.*

    Proof. For the sake of contradiction, suppose $e \in E(G)$ is the only edge incident to u,v. Then $e = uv$. But this means
    $$H = (\{u,v\}, \{uv\})$$
    is a component of G, since no edge other than $e = uv$ is incident to u,v. Since G has at least 3 vertices, there is $w \in V(G) \setminus \{v,u\}$. But then there is $f \in E(G)$ that is incident to w, and $f \neq e$. Since H is a component, it follows that there is no cycle of G that has e,f, contradicting (c). ◁

  By Claim 1, choose distinct e,f that are incident to u,v, respectively. Then by (c), there exists a cycle C which has e,f, which means C also has u,v. ♦

Def'n 1.22

Ear Decomposition of a Graph

Let F be a subgraph of G.

(a) An ***ear*** of F in G is a nontrivial path in G where only the two endpoints of the path are in F.

(b) An ***ear decomposition*** of G is a sequence of graphs $(G_0)_{i=0}^{k}$ where

(i) G_0 is a cycle in G; *starting cycle*

(ii) for each $r \in \{0, \ldots, k-1\}$,
$$G_{i+1} = G_i + P_i$$
where P_i is an ear of G_i; and *intermediate cycles*

(iii) $G_k = G$.

Proposition 1.13
Characterization of 2-connectedness by Ear Decompositions

Let G be a graph. The following are equivalent.

(a) G is 2-connected.

(b) G has an ear decomposition.

Proof.

- ($\Longrightarrow$) Let G be 2-connected and let G_0 be any cycle in G. For $i \in \mathbb{N} \cup \{0\}$, we inductively construct G_{i+1} from G_i as follows.

  (a) If $G_i = G$, then we are done.

  (b) Otherwise, there exists an edge $e \in E(G) \setminus E(G_i)$, say $e = uv$ for some $u, v \in V(G)$. Then by the set version of Menger's theorem, there exist 2 disjoint $\{u, v\}, G_i$-path in G, say Q_1, Q_2. Then observe that
  $$P_i = Q_1 + uv + Q_2$$
  is an ear of G_i. We then set $G_{i+1} = G_i + P_i$.

  Note that the above process terminates since we add at least one edge of G each time. At termination, say after k step, $(G_i)_{i=1}^{k}$ is an ear decomposition of G.

- ($\Longleftarrow$) Suppose $(G_i)_{i=0}^{k}$ is an ear decomposition of G. We proceed inductively to prove each G_i is 2-connected, thereby proving that G is also 2-connected. Since G_0 is a cycle, G_0 is 2-connected. Now suppose that G_i is 2-connected, where $i \in \mathbb{N} \cup \{0\}$ and let P_i be an ear of G_i such that
  $$G_{i+1} = G_i + P_i.$$
  We claim that every distinct vertices x, y in G_{i+1} are in a cycle, so that we can use Proposition 1.12. Suppose that the two endpoins of P_i are u, v and let $x, y \in V(G_{i+1})$. We have 3 cases.

  · *Case 1. Suppose* $x, y \in V(G_i)$. Then since G_i is 2-connected, there is a cycle C in G_i, hence in G_{i+1}, which has x, y.

  · *Case 2. Suppose* $x, y \in V(P_i)$. Since G_i is connected, there exists a u, v-path Q in G_i. It follows that $P_i + Q$ is a cycle which has x, y.

  · *Case 3. Suppose exactly one of x, y is in G_i.* Without loss of generality, say $x \in V(G_i), y \in V(P_i)$. Since G_i is 2-connected, there exist 2-fan from x to $\{u, v\}$ by the fan lemma. Then 2-fan combined with P_i is a cycle which has x, y.

  Hence every 2 distinct vertices in G_{i+1} are in a cycle, and by Proposition 1.12 G_{i+1} is 2-connected. Thus G_k is 2-connected, as required. ♦

Corollary 1.13.1

Let G be a graph.

(a) If G is 2-connected, then any cycle in G can be the starting cycle in the ear decomposition.

(b) If G admits an ear decomposition, then each intermediate graph in the ear deecomposition is 2-connected.

Proposition 1.14

Let G be a graph. If G admits an ear decomposition $(G_0)_{i=0}^{k}$, then $k = |E(G)| - |V(G)|$.

1.6 Blocks

Def'n 1.23 **Block** of a Graph

Let G be a graph. A ***block*** of G is a maximal connected subgraph with no cut-vertex.

(1.9)

(a) A connected graph with no cut-vertex is not necessarily 2-connected. We might have an isolated vertex, or a single edge (which must be a cut-edge). So a block is K_1, K_2, or 2-connected.

(b) We shall see results such as *this property holds for G if and only if it holds for every block of G.* In other words, we are breaking G into manageable pieces when doing such proofs.

Proposition 1.15

Let G be a graph. Then every block of G is an induced subgraph of G.

Proof. Observe that, given any connected subgraph of G, adding an edge to the subgraph does not create a new cut-vertex. This means, given any connected subgraph H of G without any cut-vertex, if there is $e \in E(G)$ such that $e = uv$ for some vertices u, v of H, then $H + e$ is connected without any cut-vertex. Hence by maximality, any block has every edge joining its vertices, so is induced by its vertex set in G. ♦

Proposition 1.16

Let G be a graph. If C is a cycle in G and B is the block containing C, then $V(C) \subseteq V(B)$.

Proof. Since C is a cycle, it does not have a cut-vertex. By maximality of B, the vertices of C are in B. ♦

Proposition 1.17

Let G be a graph and let B_1, B_2 be distinct blocks of G. Then B_1, B_2 share at most 1 vertex.

Proof. Suppose B_1, B_2 share at least 2 vertices, for the sake of contradiction. We may assume that there is at least 1 vertex in B_1 and 1 vertex in B_2 that are not shared by B_1, B_2.

- *Claim 1. $B_1 \cup B_2$ does not have a cut-vertex.*

  Proof. Let x be a vertex in $B_1 \cup B_2$. Since B_1, B_2 share at least 2 vertices, there is a vertex y in $B_1 \cup B_2$ that is not x. Since B_1, B_2 have no cut-vertices, $B_1 - x, B_2 - x$ are connected. Since y is in both blocks, there is a path from y to each vertex in $B_1 - x$ and to each vertex in $B_2 - x$. So $(B_1 \cup B_2) - x$ is connected, hence $B_1 \cup B_2$ has no cut-vertex. ◁

But by Claim 1, $B_1 \cup B_2$ is a block of G, which contradicts the maximality of B_1, B_2. ♦

Proposition 1.18

Let G be a graph. Every vertex of G that is in 2 or more blocks of G is a cut-vertex.

Proposition 1.19

Let G be a graph and let B be a block. If $v, u \in V(B)$, then every u, v-path is contained B.

Proposition 1.20

Let G be a graph. Then for any $e \in E(G)$, there exists unique block B of G that has e.

Corollary 1.20.1

Let G be a graph and let $B_1, \ldots, B_k$ be the distinct blocks of G. Then

$$\{E(B_i)\}_{i=1}^{k}$$

is a partition of $E(G)$.

Block Graph of a Graph

Def'n 1.24 Let G be a graph. The ***block graph*** of G is a graph with

- the vertex set $\mathcal{B} \cup C$; and
- the edge set $\{Bv : B \in \mathcal{B}, v \in V(G) [v \in V(B), v \text{ is a cut-vertex}]\}$;

where $\mathcal{B}$ is the collection of blocks of G and C is the set of the cut-vetices of G.

(1.10) Note that the block graph of G is bipartite with a bipartition $\{\mathcal{B}, C\}$, where

- $\mathcal{B}$ is the set of the blocks of G; and
- C is the set of the cut-vertices of G.

Proposition 1.21

Let G be a graph. Then the block graph of G is a forest.

1.7 Strong Orientation

Orientation of a Graph

Def'n 1.25 Let G be a graph.

(a) An ***orientation*** of G is an assignment a direction to each edge uv, either $u \to v$ or $v \to u$.

(b) For any vertices v, u in G, a u, v-path P is directed if $u, v_1, \ldots, v_k, v$ are the vertices of P and

$$u \to v_1, v_1 \to v_2, \ldots, v_k \to v$$

are the edges of P.

(c) An orientation of G is called ***strong*** if every distinct vertices u, v of G admits a directed u, v-path.

Proposition 1.22

If G is 2-connected, then G has a strong orientation.

Proof. Let $(G_i)_{i=0}^{k}$ be an ear decomposition of G. We proceed inductively to construct an orientation for each G_i, and prove that they are strong orientations. Since G_0 is a cycle, with vertices $v_1, \ldots, v_l$ and edges $v_1v_2, \ldots, v_{l-1}v_l, v_lv_1$, we can orient G_0 with

$$v_1 \to v_2, v_2 \to v_3, \ldots, v_l \to v_1.$$

The above results in a strong orientation. Now suppose G_i has a strong orientation, where $i \geq 0$. Let P_i be an ear of G_i, with vertices $w_1, \ldots, w_m$ and edges $w_1w_2, \ldots, w_{m-1}w_m$, that we add to G_i to obtain G_{i+1}. We keep the same orientation for G_i, and orient P_i as

$$w_1 \to w_2, w_2 \to w_3, \ldots, w_m \to w_1.$$

Let $x, y \in V(G_{i+1})$, where we desire to show there exists a directed x,y-path in the orientation of G_{i+1}. We break down into 5 cases.

- *Case 1. Suppose $x, y \in V(G_i)$.* Then by the inductive hypothesis, there exists a directed x,y-path in G_i so in G_{i+1}.
- *Case 2. Suppose $x \in V(G_i), y \in V(P_i)$.* Then by assumption there is a directed x_1, w_1-path in G_i, followed by a directed w_1, y-path in P_i.
- *Case 3. Suppose $x \in V(P_i), y \in V(G_i)$.* Then there is a directed x, w_m-path in P_i and directed w_m, y-path in G_i, combining which gives a directed x,y-path.
- *Case 4. Suppose $x, y \in V(P_i)$, with $x = w_\alpha, y = w_\beta$ for some $\alpha < \beta$.* Then there is a directed x,y-path in P_i.
- *Case 5. Suppose $x, y \in V(P_i)$, with $x = w_\alpha, y = w_\beta$ for some $\alpha > \beta$.* Then there is a directed x, w_m-path in P_i, directed w_m, w_1-path in G_i and directed w_1, y-path in P_i. ♦

Theorem 1.23

Let G be connected. The following are equivalent.

(a) G has a strong orientation.

(b) Every block of G has a strong orientation.

Proof.

- (a) $\Longrightarrow$ (b) Consider a strong orientation for G and let B be a block of G. Orient B by restricting the orientation of G onto B. To show that this is a strong orientation on B, observe that there is a directed x,y-path in G, and this path is in B by Proposition 1.19. Hence B admits a strong orientation.
- (b) $\Longrightarrow$ (a) Suppose each block has a strong orientation and keep the same orientation for G. Let $x, y \in V(G)$. If x, y are in the same block, then there exists a directed x,y-path in the block. Suppose x, y are in different blocks. In the block graph of G, consider the path between the blocks B_1, B_k, say with vertices
$$B_1v_1, \ldots, v_{k-1}, B_k,$$
where $x \in V(B_1), y \in V(B_k)$. Since each block is strongly oriented, we can find directed paths from x to v_1 in B_1, from v_1 to v_2 in B_2, …, from v_{k-2} to v_{k-1} in B_{k-1}, and from v_{k-1} to y in B_k. Combining these paths gives a directed x,y-path in G. ♦

Corollary 1.23.1

Let G be a graph. The following are equivalent.

(a) G has a strong orientation.

(b) G is 2-edge-connected.

1.8 3-connected Graphs

(1.11) We summarize 2 characterizations of 3-connected graphs without proof. These characterizations depend on contractions of edges.

Proposition 1.24

Let G be a 3-connected graph with at least 5 vertices. Then G has an edge e such that G/e is 3-connected.

Corollary 1.24.1
Tutte's Characterization of 3-connectedness

Let G be a graph. The following are equivalent.

(a) G is 3-connected.

(b) There are graphs $G_0, \dots, G_k$ such that

(i) $G_0 = K_4, G_k = G$; and

(ii) for each $\in \{1, \dots, k\}$, there exists an edge xy in G_i such that $\deg_{G_i}(x) \geq 3, \deg_{G_i}(y) \geq 3$ and $G_i = G_{i+1}/xy$.

Def'n 1.26

Wheel

A ***wheel*** is the graph that consists of a cycle with another vertex that is joined to all vertices on the cycle. We write W_n to denote the wheel whose cycle is of length $n \in \mathbb{N}$.

Theorem 1.25
Tutte's Wheel Theorem

Let G be a graph. The following are equivalent.

(a) G is 3-connected.

(b) There are 3-connected $G_0, \dots, G_k$ such that

(i) G_0 is a wheel, $G_k = G$; and

(ii) for every $i \in \{1, \dots, k\}$, there is an edge e in G_i such that $G_{i-1} = G_i - e$ or $G_{i-1} = G_i/e$.

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2.
Introductory Algebraic Graph Theory

2.1 Edge Spaces

Def'n 2.1 **Edge Space** of a Graph

Let G be a graph. We define the ***edge space*** of G, denoted as $\varepsilon(G)$, as

$$\varepsilon(G) = \mathbb{Z}_2^{E(G)}.$$

(2.1)

(a) Note that, if we index the vectors in order $E(G) = \{e_i\}_{i=1}^m$, then the vector $v \in \varepsilon(G)$ that corresponds to $\{e_{i_1}, \ldots, e_{i_k}\}$ would be such that

$$v_i = \begin{cases} 1 & \text{if } i = i_j \text{ for some } j \in \{1, \ldots, k\} \\ 0 & \text{otherwise} \end{cases}$$

for all $i \in \{1, \ldots, m\}$.

(b) We can use subsets of $E(G)$ as vectors. In fact, we shall use this representation most of the time.

(c) Addition is done over $\mathbb{Z}$, so

$$\begin{aligned} 0+0&=0 \\ 1+0&=1 \\ 0+1&=1 \\ 1+1&=0. \end{aligned}$$

This means, given $E_1, E_2 \subseteq E(G)$, the sum $E_1 + E_2$ represents the *symmetric difference* $E_1 \triangle E_2$.

(d) We do not have to worry about scalar multiplication, as we are working over $\mathbb{Z}_2 = \{0,1\}$. This in particular means it suffices for a subset of $\varepsilon(G)$ to be closed under addition to be a *subspace* of $\varepsilon(G)$.

(e) Suppose that $E_1, \ldots, E_k \in \varepsilon(G)$ are given. Then observe that

$$E_1, \ldots, E_k \text{ are linearly dependent} \iff \text{there are indices } i_1, \ldots, i_l \text{ such that } E_{i_1} \triangle \cdots \triangle E_{i_l} = \varnothing.$$

That is, linear dependence corresponds to an empty symmetric difference.

(f) Observe that $\{\{e_1\}, \ldots, \{e_m\}\}$ is a basis for $\varepsilon(G)$. This means $\dim(\varepsilon(G)) = |E(G)|$.

Also recall the following result from linear algebra.

Proposition 2.1

Let V be a vector space and let $W \subseteq V$ be a subspace of dimension $k \in \mathbb{N}$. Then for every $X \subseteq W$, X is a basis for W if two of the following conditions hold.

(a) X is linearly independent.

(b) $\operatorname{span}(X) = W$.

(c) $|X| = k$.

Moreover, if X is a basis for W, then all of the above hold.

Def'n 2.2 **Cut Space** of a Graph

Let G be a graph. The ***cut space*** of G, denoted as $\mathcal{C}^*(G)$, is the set of all cuts in G.

Proposition 2.2

Let G be a graph. Then $\mathcal{C}^*(G)$ is a subspace of $\varepsilon(G)$.

Proof. This is by Lemma 1.6.1. That is, the symmetric difference of any two cuts is a cut, so $\mathcal{C}^*(G)$ is closed under addition. ♦

Proposition 2.3

The set of all bonds in G spans $\mathcal{C}^*(G)$.

Proof. This follows from the fact that every cut is a disjoint union of bonds (Proposition 1.6), since the symmetric difference of two disjoint sets is the union of the sets. ♦

Def'n 2.3 **Fundamental Cut** of a Connected Graph

Let G be a connected graph and let T be a spanning tree of G. Given any edge e of T, we say the cut $\delta_G(S)$ induced by the set of vertices S of one of the components of $T-e$ is a ***fundamental cut*** with respect to T, denoted as D_e.

Proposition 2.4

Let G be connected and let T be a spanning tree. Then the set of all fundamental cuts with respect to T is a basis of $\mathcal{C}^*(G)$.

Proof. Note that, given any $e, f \in E(T)$,

$$e \in D_f \iff e = f.$$

This means the set of all fundamental cuts is linearly independent. Moreover, to show the spanning part, let F be a cut, where we show that F can be expressed as a linear combination of fundamental cuts. Consider the sum

$$M = F + \sum_{e \in F \cap E(T)} D_e. \tag{2.1}$$

The claim is that $M = \varnothing$. First note that $M \in \mathcal{C}^*(G)$ as a sum of cuts. Let $e \in E(T)$.

- *Case 1. Suppose $e \in F$.* Then observe that e precisely appears in F and in D_e in [2.1].
- *Case 2. Suppose $e \notin F$.* Then e does not appear in [2.1].

In either case, $e \notin M$. This means M is a cut in G that contains no edges from a spanning tree T, which means $M = \varnothing$. Thus

$$M + \sum_{e \in F \cap E(T)} D_e = \varnothing,$$

which means

$$F = \sum_{e \in F \cap E(T)} D_e.$$ ♦

Corollary 2.4.1

Let G be a graph with k components. Then

$$\dim(\mathcal{C}^*(G)) = |V(G)| - k.$$

Def'n 2.4

Cycle Space (Flow Space) of a Graph

Let G be a graph. The ***cycle space*** (or ***flow space***) of G, denoted as $\mathcal{C}(G)$, is the subspace of $\varepsilon(G)$ spanned by the (edge sets of the) cycles of G.

Def'n 2.5

Even Edge Set

Let G be a graph.

(a) Given $F \subseteq E(G)$, we say F is ***even*** if $\deg_F(v)$ is even for all $v \in V$.

(b) We say G is ***even*** if its edge set $E(G)$ is even.

Proposition 2.5

Let G be a graph and let $E_1, E_2 \subseteq E(G)$. If E_1, E_2 are even, then $E_1 \triangle E_2$ is even.

Proof. Let $v \in V(G)$ and let D_1, D_2 be the set of edges in E_1, E_2 incident with v. Edges in $D_1 \cap D_2$ do not appear in $E_1 \triangle E_2$. This means

$$\deg_{E_1 \triangle E_2}(v) = |D_1| + |D_2| - 2|D_1 \cap D_2|.$$

Since $|D_1|, |D_2|, 2|D_1 \cap D_2|$ are even, $\deg_{E_1 \triangle E_2}(v)$ is even. Thus $E_1 \triangle E_2$ is even. ♦

Proposition 2.6
Characterization of Even Space

Let G be a graph and let H be a subgraph of G. The following are equivalent.

(a) $H \in \mathcal{C}(G)$.

(b) H is even.

Proof. Let $F = E(H)$.

- (a) $\Longrightarrow$ (b) Suppose $H \in \mathcal{C}(G)$. By definition,

$$F = C_1 \triangle C_2 \triangle \cdots C_k$$

for some cycles $C_1, \ldots, C_k$.[1] Since every cycle is even, each C_i is even. It follows inductively that F is even from Proposition 2.5.

- (b) $\Longrightarrow$ (a) Suppose F is even. We proceed inductively on $|F|$. If $|F| = 0$, then F is the empty sum of cycles.[2] Otherwise, each nontrivial component of H has minimum degree 2. From MATH 249, we know that there is a cycle C_k contained in H. This means $F \triangle C_k$ is even, with fewer edges than F. By induction, $F \triangle C_k$ is a sum of cycles of G, say

$$F \triangle C_k = C_1 \triangle C_2 \triangle \cdots C_{k-1}.$$

It follows that

$$F = C_1 \triangle C_2 \triangle \cdots C_k,$$

as required. ♦

[1] Precisely speaking, we are referring to the edge sets of $C_1, \ldots, C_k$.
[2] We are viewing $\triangle$ as the vector addition in $\varepsilon(G)$.

Def'n 2.6 **Fundamental Cycle** of a Connected Graph

Let G be a connected graph and let T be a spanning tree of G. For any $e \in E(G) \setminus E(T)$, the unique cycle in $T + e$, denoted as C_e, is called a ***fundamental cycle***.

Proposition 2.7

Let G be a connected graph and let T be a spanning tree of G. Then the set of all fundamental cycles with respect to T is a basis for $\mathcal{C}(G)$.

Proof. Each fundamental cycle is in $\mathcal{C}(G)$ and contains a unique edge $e \in E(G) \setminus E(T)$. This means the set of fundamental cycles is linearly independent. To show the spanning part, let $F \in \mathcal{C}(G)$ and let $F' = F \setminus E(T)$. Consider

$$M = F + \sum\nolimits_{e \in F'} C_e. \qquad [2.2]$$

So M is a sum of even subgraphs, which means M is even. Suppose $e \in E(G) \setminus E(T)$.

- *Case 1.* $e \in F$. Then $e \in F'$, so e appears exactly twice in [2.2].
- *Case 2.* $e \notin F$. Then $e \notin F'$, so e does not appear in [2.2].

This means $e \notin M$. This means the only edges in M are in T. But the only even edge set with the edges of a tree is the empty set, so $M = \varnothing$. This means

$$F = \sum\nolimits_{e \in F'} C_e.$$

Thus the set of fundamental cycles is a basis for $\mathcal{C}(G)$. ♦

Corollary 2.7.1

Let G be a graph. If G has k components, then $\dim(\mathcal{C}(G)) = |E(G)| - |V(G)| + k$.

Recall 2.7 **Orthogonal** Vectors

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. We say $v, w \in V$ are ***orthogonal*** if

$$\langle v, w \rangle = 0.$$

(2.2) Let G be a graph, with an edge set $E(G) = \{e_i\}_{i=1}^k$. Note that, since $\varepsilon(G)$ is over $\mathbb{Z}_2$, we cannot define an inner product over $\varepsilon(G)$. However, we can still define a *dot-product-like* bilinear form on $\varepsilon(G)$: define $\langle \cdot, \cdot \rangle : \varepsilon(G)^2 \to \mathbb{Z}_2$ by

$$\langle E_1, E_2 \rangle = \sum\nolimits_{i=1}^k (E_1)_i (E_2)_i,$$

where

$$(E_1)_i = \begin{cases} 1 & \text{if } e_i \in E_1 \\ 0 & \text{otherwise} \end{cases}$$

for all $i \in \{1, \ldots, k\}$ and similarly for $(E_2)_i$. For convenience, we shall write

$$E_1 \cdot E_2$$

to denote $\langle E_1, E_2 \rangle$. Then Recall 2.7 motivates the following definition.

Def'n 2.8 **Orthogonal** Edge Sets

Let G be a graph. We say $E_1, E_2 \subseteq E(G)$ are ***orthogonal*** if $E_1 \cdot E_2 = 0$.

The following result is immediate: E_1, E_2 are orthogonal if and only if $|E_1 \cap E_2|$ is even.

Proposition 2.8

Let G be a graph. Then

$$\mathcal{C}^*(G)^{\perp} = \mathcal{C}(G).$$

Proof. We first show $\mathcal{C}^*(G)^{\perp} \subseteq \mathcal{C}(G)$. Let $F \in \mathcal{C}^*(G)^{\perp}$, where our goal is to prove that F is even. Let $v \in V(G)$. Then $\delta(\{v\}) \in \mathcal{C}^*(G)$. Since $F \in \mathcal{C}^*(G)^{\perp}$, F is orthogonal to v, which means $|F \cap \delta(\{v\})|$ is even. This means $\deg_F(v)$ is even, so F is even. Conversely, suppose $F \in \mathcal{C}(G)$ and let $D = \delta(X)$ be any cut, where $X \subseteq V(G)$ and we desire to show $|F \cap D|$ is even. Then $D \in \mathcal{C}^*(G)$. Consider

$$d = \sum\nolimits_{v \in X} d_F(v). \qquad [2.3]$$

Since F is even, d is even. Edges in F contribute to the sum in [2.3] in two ways.

(a) If both endpoints of an edge are in X, then it contributes 2 to the sum.

(b) If exactly one endpoint is in X, then it contributes 1 to the sum. Such an edge is in D, which means there are $|F \cap D|$ such edges.

But $|F \cap D|$ is even, so it follows that d is even. Thus F, D are orthogonal, as desired. ♦

(2.3) Note that, given a graph G with k, components,

$$\dim(\mathcal{C}^*(g)) + \dim(\mathcal{C}(G)) = (|V(G)| - k) + (|E(G)| - |V(G)| + k) = |E(G)|$$

by Corollary 2.4.1, 2.7.1. This is consistent with the result in linear algebra that $\dim(W) + \dim\left(W^{\perp}\right) = \dim(V)$ for any vector space V (equipped with a bilinear form) and a subspace $W \subseteq V$.

3.
Planar Graphs

3.1 Planar Graphs

Planar Graph

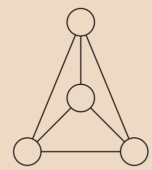
Def'n 3.1 Let G be a graph.

(a) A drawing of G on a plane whose edges only intersect at common endpoints is called a ***planar embedding*** of G.

(b) We say G is ***planar*** if there exists a planar embedding of G.

(EX 3.1) K_4 is a planar graph.

A Planar Embedding of K_4



Plane Graph

Def'n 3.2 A ***plane*** graph is a planar graph with a fixed planar embedding. We call vertices as ***points*** and edges as ***lines***.

(3.2) A rigorous discussion of planarity involves topology, which is out of the scope of this note. But we shall take many topological results for granted. Moreover, we will mainly focus on the combinatorial aspects of planar graphs.

Curve on a Topological Space

Recall 3.3 Let X be a topological space. We say $C \subseteq X$ is a ***curve*** on X if there exists continuous $f : I \to X$ such that $C = \text{image}(f)$, where $I \subseteq \mathbb{R}$ is an interval. We call f a ***parameterization*** of C.

Simple, Closed, Jordan Curve

Recall 3.4 Let C be a curve on a topological space X.

(a) We say C is ***simple*** if it does not intersect itself, except at the endpoints.[a]

(b) We say C is ***closed*** if it is a continuous image of a unit circle.

(c) We say C is ***Jordan*** if C is simple closed curve and $X = \mathbb{R}^2$.

[a] Rigorously speaking, given a curve C with a parameterization f on an interval I, we say C is simple if, for every $x, y \in I$, $f(x) = f(y)$ only if $x = y$ or x, y are the endpoints of I.

Connected Subset

Recall 3.5 Let X be a topological space.

(a) We say X is ***disconnected*** if X is a union of two disjoint nonempty open subsets of X.

(b) We say X is ***connected*** if X is not disconnected.

(c) A maximal connected subset of X is called a ***connected component*** of X.

Theorem 3.1
Jordan Curve Theorem

Let C be a Jordan curve. Then $\mathbb{R}^2 \setminus C$ contains exactly 2 connected components, one bounded and one unbounded.

Recall 3.6

Interior, Exterior of a Jordan Curve

Let C be a Jordan curve.

(a) The ***interior*** of C, denoted as $\operatorname{int}(C)$, is the bounded connected component of $\mathbb{R}^2 \setminus C$.

(b) The ***exterior*** of C, denoted as $\operatorname{ext}(C)$, is the unbounded connected component of $\mathbb{R}^2 \setminus C$.

Corollary 3.1.1

K_5 is not planar.

Proof. Suppose K_5 has a planar embedding for the sake of contradiction. Write $V(K_5) = \{v_1, v_2, v_3, v_4, v_5\}$. Then the lines

$$v_1v_2, v_2v_3, v_3v_1$$

form a Jordan curve C, since $(\{v_1, v_2, v_3\}, \{v_1v_2, v_2v_3, v_3v_1\})$ is a cycle in K_5. Then $v_4 \in \operatorname{int}(C)$ or $v_4 \in \operatorname{ext}(C)$. Without loss of generality, assume $v_4 \in \operatorname{int}(C)$. Then the lines v_4v_1, v_4v_2, v_4v_3 are contained in $\operatorname{int}(C)$, execpt at v_1, v_2, v_3, respectively. Let C_1, C_2, C_3 be the Jordan curves formed by the cycles

$$(\{v_2, v_3, v_4\}, \{v_2v_3, v_3v_4, v_4v_2\}), (\{v_1, v_3, v_4\}, \{v_1v_3, v_3v_4, v_4v_1\}), (\{v_1, v_2, v_4\}, \{v_1v_2, v_2v_4, v_4v_1\}),$$

respectively. Note that, for all $i \in \{1,2,3\}$, $\operatorname{int}(C_i) \subseteq \operatorname{int}(C)$ and $v_i \in \operatorname{ext}(C)$. Since there is a line joining v_5 to v_i for all $i \in \{1,2,3\}$, by the Jordan curve theorem $v_5 \in \operatorname{ext}(C_i)$ for all $i \in \{1,2,3\}$. So $v_5 \in \operatorname{ext}(C)$. But this contradicts the fact that $v_4 \in \operatorname{int}(C)$ and v_4, v_5 are joined by a line. Thus K_5 is not planar. ♦

Corollary 3.1.2

$K_{3,3}$ is not planar.

Def'n 3.7

Subdivision of a Graph

Let G be a graph. A ***subdivision*** of G is obtained from G by replacing each edge with a new path of length at least 1.

Proposition 3.2

Let G be a graph. The following are equivalent.

(a) G is planar.

(b) Every subdivision of G is planar.

Proof-ish. Note that an edge and a path are both represented by a simple curve. ♦

Corollary 3.2.1

Any graph containing a subdivision of K_5 or $K_{3,3}$ is not planar.

3.2 Faces

Face of a Planar Embedding

Def'n 3.8

A ***face*** of a planar embedding is a maximal subset of points on the plane that are path-connected, and do not include any part of the embedding. The ***outer*** face of an embedding is the unbounded face of the embedding. Each face is ***incident*** to the vertices and edges on its boundary. We say two faces are ***adjacent*** if they share at least one edge (or line) at their boundaries.

Proposition 3.3

Let f be a face of a plane graph G. Then there is an embedding of G with the boundary of f as the boundary of the outer face.

Proposition 3.4

A graph can be embedded on a plane if and only if the graph can be embedded on a sphere.

Proposition 3.5

Let G be a plane graph. If G is 2-connected, then each face of G is bounded by a cycle.[a]

[a] We say a face is ***bounded*** by a cycle if its boundary is a cycle.

Proof. There is an ear decomposition $(G_i)_{i=0}^{k}$ for G. We will treat each G_i as the plane graph whose drawing is a part of the plane graph G. We will inductively prove that the result holds for each G_i. We see that G_0 is a cycle, so there are 2 faces, each bounded by G_0. Suppose each face of G_i is bounded by a cycle and further suppose P_i is the ear such that $G_{i+1} = G_i + P_i$. Since G_{i+1} is a plane graph, the curve representing P_i must be entirely within a face f of G_i by the Jordan curve theorem. By induction, f is bounded by a cycle, say C. The two endpoints of P_i divides C into 2 paths Q_1, Q_2. Now f is divided into 2 faces, one bounded by $P_i + Q_1$ and the other bounded by $P_i + Q_2$, both of which are cycles. The remaining faces of G_i are still faces of G_{i+1}, so they remain bounded by cycles. So the result holds for G_{i+1}. By induction, the result holds for $G_k = G$. ♦

Corollary 3.5.1

In a 3-connected plane graph, all neighbors of a vertex lie on a common cycle.

Proof. Let G be a 3-connected plane graph, and let $v \in V(G)$. Then $G - v$ is a 2-connected plane graph. Moreover, the face f of $G - v$ that contains v is bounded by a cycle C by Proposition 3.5. Since all neighbors of v in G lie on the boundary of f, they are on C. ♦

Proposition 3.6

The cycle space of a 2-connected plane graph G is spanned by the set of all facial cycles of G.[a]

[a] A cycle is called ***facial*** if it is the boundary of a face.

Proof. Any element of $\mathcal{C}(G)$ is spanned by the cycles of G, so it suffices to show that each cycle is the sum of facial cycles. Let C be any cycle, and let S be the sum of all facial cycles of the faces inside C. Edges in the interior of C are counted twice in S, once on each side. So they do not appear in S. On the other hand, edges on the cycle C are counted once in S, as precisely one side is incide C. So $S = C$, as required. ♦

Def'n 3.9 **2-basis**

Let G be a graph. A 2-***basis*** of a subspace of $\varepsilon(G)$ is a basis β where each edge of G is in at most 2 elements in β.

(3.3) It follows from Proposition 3.6 that every 2-connected plane graph has a 2-basis. We shall prove later that a graph is planar if and only if its cycle space has a 2-basis, the result known as *MacLane's theorem*.

3.3 Bridges

Def'n 3.10 **Bridge** of a Subgraph

Let H be a subgraph of a connected graph G. A ***bridge*** of H is

(a) a component J of $G - V(H)$ together with the edges in $\delta_G(V(J))$ and their endpoints; or

(b) an edge not in H that joins two vertices in H (called a ***trivial*** bridge).

Def'n 3.11 **Vertex of Attachment (VOA)**

Let H be a subgraph of a connected graph G and let B be a bridge of H. A vertex v of B is called

(a) a ***vertex of attachment*** (***VOA***) if v is in H; and

(b) an ***internal*** vertex if v is not in H.

We call a bridge with $k \in \mathbb{N}$ VOA a k-***bridge***. We say two bridges are ***equivalent*** if they have the same VOA.

Def'n 3.12 **Segment**

Let C be a cycle. The k VOA of a k-bridge partitions C into k ***segments***.

(a) Two bridges of C ***avoid*** each other if the VOA of one bridge are entirely within one segment of the other bridge.

(b) Otherwise, we say they ***overlap***.

Def'n 3.13 **Skew** Bridges

Let C be a cycle. We say two bridges of C are ***skew*** if there exists a sequence of distinct vertices $(v_i)_{i=1}^4$ on C in cyclic order such that v_1, v_3 are VOA of one bridge and v_2, v_4 are VOA of the other bridge.

Proposition 3.7

Overlapping bridges of a cycle are

(a) skew; or

(b) equivalent 3-bridges.

Proof. Let B_1, B_2 be two overlapping bridges of C. Then they both have at least 2 VOA. Suppose B_1, B_2 are not equivalent. Then there must a VOA u in B_1 but not in B_2. Suppose u is in the segment S for B_2, say from x to y. Note that x, u, y are distinct. Since B_1, B_2 overlap, there exists a VOA v of B_1 that is not in S. Then x, u, y, v are in cyclic order where x, y are in B_2 and u, v are in B_1. So B_1, B_2 are skew. On the other hand, suppose B_1, B_2 are are equivalent k-bridges. But equivalent 2-bridges avoid each other, so $k \neq 2$. If $k = 3$, then they are equivalent 3-bridges. If $k \geq 4$, then they are skew by picking any 4 VOA. ♦

(3.4) Given a plane graph G and a cycle C in G, each bridge of C is connected. By the Jordan curve theorem, each bridge is either intirely in $\overline{\operatorname{int}(C)}$ or in $\overline{\operatorname{ext}(C)}$.

Inner, Outer Bridge

Def'n 3.14 Consider the setting of (3.4) and let B be a bridge of C. We say B is

(a) ***inner*** if $B \subseteq \overline{\operatorname{int}(C)}$; and

(b) ***outer*** if $B \subseteq \overline{\operatorname{ext}(C)}$.

Proposition 3.8 *Let G be a plane graph. Then for every cycle C of G,*

(a) inner bridges of C avoid one another; and

(b) outer bridges of C avoid one another.

To prove Proposition 3.8, we introduce some notions and results.

Branch Vertices of a Subdivision

Def'n 3.15 A vertex in a subdivision is called ***branch*** if its degree is at least 3.

We shall accept the following results without proof.

Fact 3.9 *Let C be a simple closed curve and let $p_1, \ldots, p_k \in C$ be distinct. Then for every $x \in \operatorname{int}(C)$ ($y \in \operatorname{ext}(C)$), there are k curves from x to p_i's such that*

(a) the curves are entirely in $\operatorname{int}(C)$ ($\operatorname{ext}(C)$) except at $p_1, \ldots, p_k$; and

(b) they do not intersect each other except at x (y).

Fact 3.10 *Let G be a connected graph and let H be obtained from G by adding 3 distinct vertices x_1, x_2, x_3 and joining each x_i to a vertex v_i in $V(G)$. Then there exists $v \in V(G)$ such that there is a 3-fan from v to $\{x_1, x_2, x_3\}$ in H.*

Proof-ish. First find a v_1, v_2-path and v_1, v_3-path in G. Let v be the *last* point of intersection and find a 3-fan from there. ♦

Proof of Proposition 3.8. We shall prove (a) only, since a similar proof holds for (b). We first remove all outer bridges of C, so C is the boundar of the outer face. Suppose, for the sake of contradiction, that there are overlapping inner bridges of C, say B_1, B_2. By Proposition 3.7, they are skew or equivalent 3-bridges.

- *Case 1. Suppose B_1, B_2 are skew.* Then there exist VOA v_1, v_3 in B_1 and v_2, v_4 in B_2 such that v_1, v_2, v_3, v_4 are distinct and in cyclic order on C. Since bridges are connected, there exists a v_1, v_3-path p_1 in B_1 and a v_2, v_4-path P_2 in B_2. Add a point x in the $\operatorname{ext}(C)$, and use Fact 3.9 to draw curves from x to $v_1, \ldots, v_4$ without intersecting. Treating x as a vertex and each curve as an edge, we have a 4-fan from x to $\{v_1, \ldots, v_4\}$ in $\overline{\operatorname{ext}(C)}$, and the resulting graph is plane. However, $\{x, v_1, \ldots, v_4\}$ forms the branch vertices of a subdivision of K_5, by using the fan, the cycle C, and the two paths P_1, P_2. Hence the resulting graph is not planar, which is a contradiction.

- *Case 2. Suppose B_1, B_2 are equivalent 3-bridges.* Let v_1, v_2, v_3 be the VOA for both bridges. Then by Fact 3.10, there are internal vertices x_1, x_2 in B_1, B_2, respectively, such that 3-fans ffrom x_1, x_2 to $\{v_1, v_2, v_3\}$ exist in B_1, B_2, respectively. Add a point y in the $\text{ext}(C)$, and use Fact 3.9 to draw a 3-fan from y to $\{v_1, v_2, v_3\}$ without intersecting. The resulting graph is plane. However, the 3-fans from x_1, x_2, y to v_1, v_2, v_3 form a $K_{3,3}$ subdivision, using $v_1, v_2, v_3, x_1, x_2, y$ as branch vertices and the bipartition $\{v_1, v_2, v_3\}, \{x_1, x_2, y\}$. But this means that the graph is not planar, a contradiction.

Thus all inner bridges of C avoid each other. ♦

3.4 Kuratowski's Theorem

Def'n 3.16 **Kuratowski Subgraph (KS)**
We call any subdivision of K_5 or $K_{3,3}$ a ***Kuratowski subgraph*** (***KS***).

Theorem 3.11
Kuratowski's Theorem

A graph is planar if and only if it does not contain a Kuratowski subgraph.

(3.5)
Proof Outline

The forward direction is provided by Corollary 3.2.1. Here is an outline for the reverse direction. We induct on the number of edges. Check base cases up to 6 edges. Assume G is not planar. The aim is to either find a KS or reach a contradiction (i.e. showing that G is planar).

(a) G is not planar if and only if a component of G is not planar. So we may assume G is connected.

(b) The case when G has a cut-vertex is easy. Apply induction to each *component* to either find a KS or find an embedding around the cut-vertex.

(c) The case when G has a separating set of size 2 is not hard. Apply induction to each *component*.

(d) Assume G is 3-connected. Then there exists an edge where G/e is 3-connected.

   (i) If G/e is not planar, t hen it has a KS by induction. *Uncontract e* to find a KS for G.

   (ii) If G/e is planar, then there is a planar embedding. We look at the cycle around the neighbors of the contracted vertex. When we *uncontract e*, we look at how the neighbors of the two ends of e land on the cycle. If they do not overlap, then we can find a planar embedding for G. If they do overlap, then we can find a KS.

Before we move on to the proof of Kuratowski's theorem, here are some results that we are going to utilize for the proof of Kuratowski's theorem.

Lemma 3.11.1

Suppose G is a plane graph and C is a simple closed curve. Then

(a) we can embed G into $\text{int}(C)$;

(b) if we pick a point v in the outer face of G and a point x on C, then we can embed G into $\overline{\text{int}(C)}$ where $v = x$; and

(c) if we pick a line uw on the outer face of G and a curve joining y, z on C, then we can embed G into $\overline{\text{int}(C)}$ where uw is the curve from y to z.

We also have a result from connectivity regarding minimal separating sets.

Lemma 3.11.2

If X is a minimal separating set of G, then every vertex in X is adjacent to at least one vertex in each component of $G - X$.

Proof of Kuratowski's Theorem. We prove the backward direction only, as Corollary 3.2.1 provides the forward direction.

We proceed by the induction on the number of edges. We can manually check that any graph with at most 6 edges is planar.

Now assume that G is a nonplanar graph with at least 7 edges, where we desire to find a Kuratowski subgraph in G. We break down the proof into few cases. We may assume that G is connected, since at least one component of G is nonplanar.

- *Case 1. Suppose G has a cut vertex, say v.* Let $H_1, \dots, H_k$ be the bridges of v. That is, $G - v$ has many components, and we add v (and adjacent edges) to each component to form $H_1, \dots, H_k$. Since there are at least 2 components in $G - v$, each bridge H_i has fewer edges than G.

  We now claim that at least one of $H_1, \dots, H_k$ is nonplanar. For the sake of contradiction, assume every H_i is planar. For each H_i, find an embedding where v on the outer face (i.e. find a face containing v, and use Proposition 3.3). We now find an embedding of G as follows. Place v anywhere on the plane. Partition a neighborhood of v into k *slices*, and embed each H_i to a distinct slice, making sure that v is the same point for each embedding. But this means G is planar, a contradiction.

  Hence we can find nonplanar H_i, and by induction H_i has a Kuratowski subgraph. Thus G contains a Kuratowski subgraph.

- *Case 2. Suppose G is 2-connected.* Let S be a separating set of G of size 2 and let $H_1, \dots, H_k$ be the bridges of S together with edge xy (even if xy is not an edge in G). Since S is a minimal separating set, both x, y are adjacent to some vertices in each component of $G - S$.

  Suppose H_i is not planar for some i. Note H_i has fewer edges than G. By induction, H_i has a Kuratowski subgraph, say J. If J does not contain xy, then we are done. Otherwise, there is an x, y-path P through a different bridge. We can replace xy with P in J to get a Kuratowski subgraph in G.

  On the other hand, assume every H_i is planar for the sake of contradiction. Then there exists an embedding of each H_i with xy on the outer face. We now construct an embedding of G inductively. Start with an embedding of H_1. Suppose we have embedded $H_1, \dots, H_i$. We look for a face containing xy, and embed H_{i+1} in this face. Once we embed H_k, remove xy if it is not in G. This means G is planar, a contradiction.

- *Case 3. Suppose G is 3-connected.* There exists $e = xy$ in G such that G/e is 3-conencted. Let z be the contracted vertex. We break into 2 cases.

  - *Case 3.1. Suppose G/e is not planar.* By induction, G/e contains a Kuratowski subgraph H. If H does not contain z, then H is a Kuratowski subgraph of G. We now suppose z is in H.

    Suppose z is not a branch vertex (i.e. $\deg_H(z) = 2$). Let u, v be the two neighbors of z in H. When we *uncontract* e, both u, v are adjacent to at least one of x, y. If both u, v are adjacent to the same vertex, say without loss of generality x, then we can replace the path (u, z, v) by (u, x, v). If u, v are adjacent to different vertices, say ux, vy are edges, then we can replace the path (u, z, v) by (u, x, y, v). In either case, we have found a Kuratowski subgraph in G.

    Suppose z is a branch vertex. Let S be the set of all neighbors of z in H. WHen we *uncontract* e, each vertex in S is adjacent to at least one of x, y. If x is adjacent to all vertices in S, then we can replace z with x to obtain a Kuratowski subgraph for G. Similar result holds when y is adjacent to all vertices in S. Suppose x is adjacent to all but one vertex in S, say w. Then y is adjacent to w. So we can replace z with x as the new branch vertex, and replace zw with the path (x, y, w). This gives a Kuratowski subgraph in G. Finally, when both x, y are adjacent to 2 vertices in S, and they are distinct. Then H is a subdivision of K_5. Let $w_1,\ldots,w_4$ be the other 4 branch vertices. Suppose without loss of generality the two neighbors of x are on the z,w_1-path, z,w_2-path in H, and the two neighbors of y are in z,w_3-path, z,w_4-path. Then there is a subdivision $K3,3$ with $\{x,w_3,w_4\},\{y,w_1,w_2\}$ as the bipartition of the branch vertices. This is a Kuratowski subgraph in G.

- *Case 3.2. Suppose G/e is planar.* Since G/e is 3-connected, the neighbors of z are in a cycle C. When we *uncontract* e, each such vertex is adjacent to at least one of x,y. Let H_x,H_y be the bridges of C in $G-xy$ containing x,y, respectively. Note that everything outside C is still planar.

  Suppose H_x,H_y avoid each other. First draw H_x in C: place x in $\operatorname{int}(C)$ and add nonintersecting lines to the neighbors of x. Then the VOA of H_y are within one segment P of H_x, say it is from u to v. Then $P+vx,xu$ form the boundary of a face. We then embed H_y and xy inside this face. This is a planar embedding of G, which is a contradiction. Hence H_x,H_y overlap.

  Since H_x,H_y overlap, they are skew or equivalent 3-bridges.

  Suppose H_x,H_y are skew. Then there exist vertices x_1,y_1,x_2,y_2 in cyclic order on C such that x_1,x_2 are in H_x, y_1,y_2 are in H_y. Then there is a $K_{3,3}$ subdivision in $C\cup\{xx_1,xx_2,yy_1,yy_2,xy\}$ with $\{x_1,y_1,y_2\},\{y,x_1,x_2\}$ as the bipartition of the brach vvertices. This is a Kuratowski subgraph in G.

  Suppose H_x,H_y are equivalent 3-bridges, with w_1,w_2,w_3 as their common VOA. Then $C\cup H_x\cup H_y\cup\{xy\}$ is a subdivision of K_5 with $\{x,y,w_1,w_2,w_3\}$ as the branch vertices. This is a Kuratowski subgraph in G. ♦

3.5 Wagner's Theorem

Def'n 3.17 **Minor** of a Graph

Let G be a graph. We say a graph H is a ***minor*** of G if H can be obtained from G by a series of vertex and edge deletions, and edge contractions.

(3.6) Each vertex in the minor is a result of a series of edge contractoins (possibly none), and the vertices involved must be connected. Another way to think about a minor is the following. First partition the vertices of G into $V_0,\ldots,V_k$ where $G[V_i]$ is connected for each $i\in\{1,\ldots,k\}$. We then delete k_0, and shrink $V_1,\ldots,V_k$ into k vertices.

Proposition 3.12 *Let G be a graph. If G has a subdivision of H, then H is a minor of G.*

Theorem 3.13 (Wagner's Theorem) *A graph G is planar if and only if G does not have K_5 or $K_{3,3}$ as a minor.*

Proposition 3.14 *A graph G contains a subdivision of K_5 or $K_{3,3}$ if and only if G has K_5 or $K_{3,3}$ as a minor.*

Proof. The forward direction is provided by Proposition 3.12. Suppose that G has K_5 or $K_{3,3}$ as a minor.

- *Case 1. Suppose G has $K_{3,3}$ as a minor.* So there is a subgraph of G where the vertices can be partitioned into $A_1, A_2, A_3, B_1, B_2, B_3$ such that each induces a connected subgraph, and shrinking all 6 sets results in a $K_{3,3}$. In G, between each pair A_i, B_j, there is an edge joining a vertex of A_i with a vertex of B_j. Since each A_i, B_j induces a connected subgraph, there exists a 3-fan from some vertex in each A_i, B_j to the 3 endpoints. These fans form a subdivision of $K_{3,3}$.
- *Case 2. Suppose G has K_5 as a minor.* We sketch the proof only. The 5 branch vertices correspond to 5 sets of vertices that induce connected subgraphs. There are edges between each set. If an appropriate 4-fan exists in all 5 sets, then we have a K_5 subdivision. Otherwise, we find some 3-fan in one set. Find an internally disjoint path from the fan to the 4th vertex. We can find appropiarte 3-fans from the other 4 sets, and this gives a $K_{3,3}$ subdivision. ♦

Theorem 3.15
Graph Minor Theorem

Any infinite sequence $(G_i)_{i=1}^{\infty}$ of finite graphs includes two graphs i, j with $i < j$ such that G_i is a minor of G_j.

Proof. Tl;dr. ♦

Def'n 3.18 **Minor-closed** Collection of Graphs

We say a collection $\mathcal{G}$ of graphs is ***minor-closed*** if any minor of any graph in $\mathcal{G}$ is also in $\mathcal{G}$.

(EX 3.7)

(a) The collection of planar graphs is minor-closed.

(b) More generally, any collection of all graphs that can be embedded on a particular surface is minor-closed.

(c) The collection of all forests is minor closed.

Def'n 3.19 **Minor-minimal** Graph

A graph G is ***minor-minimal*** in a minor-closed collection of graphs $\mathcal{G}$ if $G \notin \mathcal{G}$ but any proper minor of G is in $\mathcal{G}$.

(EX 3.8) By Wagner's theorem, the only minor-minimal graphs for the collection of planar graphs are $K_5, K_{3,3}$. We sometimes call $K_5, K_{3,3}$ the *forbidden minors*.

Proposition 3.16

For any minor-closed collection of graphs $\mathcal{G}$, there is a finite number of minor-minimal graphs.

Proof. Suppose there are infinitely many minor-minimal graphs, for the sake of contradiction. By the graph minor theorme, there exists two such graphs G, H such that G is a minor of H, contradicting the fact that H is minor-minimal. ♦

(3.9) Proposition 3.16 implies that, given any surface S, there exists a finite collection of graphs $\mathcal{G}$ where *G can be embedded in S if and only if G does not contain any graph in H as a minor.* For the torus, we do not exactly know what $\mathcal{G}$ is, but we at least know 16000 forbidden minors. One should also note that we cannot generalize Kuratowski's theorem in an analogous way; if we replace *minor* with *subdivision* above, then it is false in general.

3.6 MacLane's Theorem

Recall 3.20

2-basis

Let G be a graph. A 2-***basis*** of a subspace of $\varepsilon(G)$ is a basis β where each edge of G is in at most 2 elements in β.

Theorem 3.17
MacLane's Theorem

A graph G is planar if and only if the cycle space of G has a 2-basis.

(3.10)
Main Idea for the Backward Direction

If G is not planar, then we can use deletions and contractions to get K_5 or $K_{3,3}$. We show that if $\mathcal{C}(G)$ has a 2-basis, then any deletion or contraction results in a graph whose cycle space has a 2-basis. Finally, we show that $\mathcal{C}(K_5), \mathcal{C}(K_{3,3})$ do not have 2-bases. We prove each step as a separate proposition.

Proposition 3.18

If H is a minor of G and $\mathcal{C}(G)$ has a 2-basis, then $\mathcal{C}(H)$ has a 2-basis.

Proof Sketch. Suppose $G_0, \ldots, G_m$ is a sequence of graphs where $G_0 = G, G_m = H$, and G_{i+1} can be obtained from G_i by removing an isolated vertex, removing an edge, or contracting an edge. Note that deleting a vertex is equivalent to removing all edges incident with it and then removing the isolated vertex. We prove by induction that $\mathcal{C}(G_i)$ has a 2-basis for each i. By assumption, $G_0 = G$ has a 2-basis.

We now assume $i \geq 1$. Suppose G_i has c components, and $\{E_1, .., E_k\}$ is a 2-basis of $\mathcal{C}(G_i)$, where

$$k = |E(G_i)| - |V(G_i)| + c.$$

We now break into 3 cases.

- *Case 1. Suppose G_{i+1} is obtianed from G_i by removing an isolated vertex.* Then $E(G_{i+1}) = E(G_i)$, so all edges in $\{E_1, \ldots, E_k\}$ are still in G_{i+1}. Also, G_{i+1} has one fewer vertex and component than G, so

  $$\dim(\mathcal{C}(G_{i=1})) = |E(G_{i+1})| - (|V(G_i)| - 1) + (c-1) = |E(G_i)| - |V(G_i)| + c = k.$$

  Since $\{E_1, \ldots, E_k\}$ is independent with $\dim(\mathcal{C}(G_{i+1}))$ elements, it is a basis of $\mathcal{C}(G_{i+1})$. Hence $\{E_1, \ldots, E_k\}$ is a 2-basis for $\mathcal{C}(G_{i+1})$.

- *Case 2. Suppose G_{i+1} is obtained from G_i by removing an edge e.* If e is a cut-edge, then e is not in any cycle. This means $\{E_1, \ldots, E_k\}$ is still a basis for $\mathcal{C}(G_{i+1})$ and the result follows.

  On the other hand, suppose e is not a cut-edge. Then e is in 1 or 2 elements of $\{E_1, \ldots, E_k\}$. Suppose e is in exactly 1 element, say without loss of generality E_k. Then $\{E_1, \ldots, E_{k-1}\}$ is a 2-basis for $\mathcal{C}(G_{i+1})$ and the resultt follows.

  Now assume e is in two elements, say without loss of generality E_{k-1}, E_k. Then $\{E_1, \ldots, E_{k-2}, E_{k-1} + E_k\}$ is a 2-basis for $\mathcal{C}(G_{i+1})$.

- *Case 3. Suppose G_{i+1} is obtained from G_i by contracting an edge e.* Consider $\{E_1 - e, \ldots, E_k - e\}$. Note that if z is the contracted vertex and $e = uv$, then

  $$\deg_{E_j - e}(z) = \deg_{E_j}(u) + \deg_{E_j}(v) - 2,$$

  which is still even. So $E_j - e$ is even. Then $\{E_1 - e, \ldots, E_k - e\}$ is a 2 basis for $\mathcal{C}(G_{i+1})$. ♦

Proposition 3.19

$\mathcal{C}(K_5), \mathcal{C}(K_{3,3})$ do not have 2-bases.

Proof. Suppose $\mathcal{C}(K_5)$ has a 2-basis for the sake of contradiction. We see that $\dim(\mathcal{C}(K_5)) = 10-5+1 = 6$. So suppose $\{E_1, \ldots, E_6\}$ is a 2-basis for $\mathcal{C}(K_5)$. Let

$$E_7 = \sum_{i=1}^{6} E_i.$$

Since $\{E_1, \ldots, E_6\}$ is independent, E_7 is not empty. In fact, E_7 contains all the edges that appear in $E_1, \ldots, E_6$ exactly once. So $\{E_1, \ldots, E_7\}$ contains each edge exactly twice. Since K_5 has 10 edges, they appear in $\{E_1, \ldots, E_7\}$ 20 times in total. By pigenhole principle, at least one element E_i has at most 2 edges. However, each element is nonempty, and must contain at least one cycle, which has at least 3 edges. This is a contradiction.

The case for $\mathcal{C}(K_{3,3})$ is left as an exercise. ♦

Proof of MacLane's Theorem.

- ($\Longrightarrow$) Suppose G is planar. Then each block of G is planar. Then we can show that $\mathcal{C}(G)$ has a 2-basis if and only if the cycle space of each of its blocks has a 2-basis. If a block consists of only 1 edge, then its cycle space is the trivial subspace, so it has a 2-basis. If a block B is 2-connected, then in an embedding of B, its facial cycles span $\mathcal{C}(B)$. Every edge is in 2 facial cycles, so a subset of the facial cycles is a 2-basis for $\mathcal{C}(B)$.
- ($\Longleftarrow$) Suppose G is not planar. By Wagner's theorem, G has K_5 or $K_{3,3}$ as a minor. it follows from Proposition 3.18, 3.19 that $\mathcal{C}(G)$ does not have a 2-basis. ♦

3.7 Dual Graphs

(3.11) We discuss dual graphs and their results without proof.

Def'n 3.21 **Dual** of a Plane Graph

Let G be a graph. We define the ***dual*** of G, denoted as G^*, as follows. Each face f of G corresponds to a vertex f^* in G^*; for each edge e in G with faces f, g on its two sides, there is a corresponding edge $e^* = f^*g^*$.

Note that by definition, even if G is simple, G^* can have multiple edges and loops.

(a) Multiple edges appaer when several edges have the same faces on both sides.

(b) Loops appear when the two sides of an edge are in the same face.

Proposition 3.20

If G is a plane graph, then G^* is a plane graph.

(3.12) Proof Idea for Proposition 3.20

Given a plane graph, place a point in the interior of each face to represent the dual vertices. Then find *fans* to the midpoints of the edges on its boundary. For each edge, join the lines on both sides to form corresponding dual edges.

Proposition 3.21

For any plane graph G, G^* is connected.

(3.13)
Proof Idea for Proposition 3.21

Given a plane graph, take any two face vertices. Join them with any arbitrary curve. Trace the sequence of faces visited in the curve to find a walk in the dual.

Corollary 3.21.1

If G is a connected plane graph, then $(G^*)^* = G$.

Proposition 3.22

Let G be a conected plane graph and let e be an edge.

(a) If e is not a cut-edge, then $(G-e)^* = G^*/e^*$.

(b) If e is not a loop, then $(G/e)^* = G^* - e^*$.

Proposition 3.23

Let G be a connected plane graph.

(a) If C is a cycle in G, then $C^* = \{e^* : e \in C\}$ is a bond in G^*.

(b) If B is a bond in G, then $B^* = \{e^* : e \in B\}$ is a bond in G^*.

Corollary 3.23.1

For a connected plane graph G, $\mathcal{C}(G) = \mathcal{C}^*(G^*)$ and $\mathcal{C}^*(G) = \mathcal{C}(G^*)$.[a]

[a] We are *treating* e, e^* as the *same* element.

(3.14)

Proposition 3.23 motivates the following definition.

Def'n 3.22

Abstract Dual

Let G, G^* be two graphs with the same edge sets. We say G, G^* are ***abstract duals*** if the set of all cycles in G is the same as the set of all bonds in G^*, and the set of all bonds in G is the same as the set of all cycles in G^*.

(3.15)

If G is a connected plane graph, then G, G^* are abstract duals.

Proposition 3.24

If G, G^* are connected abstract duals, then $\mathcal{C}(G) = \mathcal{C}^*(G^*), \mathcal{C}^*(G) = \mathcal{C}(G^*)$.

Theorem 3.25
Whitney's Theorem

A connected graph G is planar if and only if G has an abstract dual.

Proof. If G is planar, then its plane dual G^* is its abstract dual. Conversely, suppose G has an abstract dual G^*. Then $\mathcal{C}(G) = \mathcal{C}^*(G^*)$. Now we can show that $\{\delta(\{v\})\}_{v \in V(G^*)}$ spans $\mathcal{C}^*(G^*)$. Moreover, each edge $e = uv$ is in two elements in the set: $\delta(\{u\}), \delta(\{v\})$. Hence a subset of $\{\delta(\{v\})\}_{v \in V(G^*)}$ is a 2-basis for $\mathcal{C}^*(G^*)$. But $\mathcal{C}^*(G^*) = \mathcal{C}(G)$, so we have a 2-basis for $\mathcal{C}(G)$. Thus by MacLane's theorem, G is planar. ♦

4. Matchings

4.1 Matchings

Def'n 4.1 **Matching** of a Graph

A ***matching*** M of a graph G is a subset of edges in G that do not share any vertices. A vertex incident with an edge in M is M-***covered***. Oterwise it is M-***exposed***.[a]

[a] We drop the suffix M- when M is understood.

(4.1) Even though M is a set of edges, we may sometimes consider M as a spanning subgraph of G. So a matching can be considered as a subgraph with maximum degree at most 1.

Def'n 4.2 **Maximum** Matching

A ***maximum*** matching in a graph G is a matching with the most number of edges. We write $\alpha'(G)$ to be the size of a maximum matching in G. The ***deficiency*** in G, denoted as $\operatorname{def}(G)$, is the number of vertices exposed by a maximum matching.

(4.2) By definition,

$$\operatorname{def}(G) = |V(G)| - 2\alpha'(G).$$

Def'n 4.3 **Perfect** Matching

A matching of a graph G is called ***perfect*** if it covers all vertices in G.[a]

[a] We also call a perfect matching a 1-***factor***.

(4.3) Let G be a bipartite graph with bipartition $\{A, B\}$. All edges join a vertex in A with a vertex in B. So a maximum matching has size equal to the numer of vertices in A that it covers. What prevents us from covering all of A is that, if there is some $S \subseteq A$ where its neighbor set is smaller than S, then there are not enough vertices to match all of S. If no such set exists, then Hall's theorem (from MATH 249) ensures that there is a matching covering all of A.

Def'n 4.4 **Neighbor Set** of a Set of Vertices

Let S be a vertices of a graph G. The ***neighbor set*** of S, denoted as $N_G(S)$, is the set of all vertices in $V(G) \setminus S$ that are adjacent to a vertex in S.[a] In case $S = \{v\}$, we write $N(v)$ instead.

[a] We write $N(S)$ instead when G is understood.

Theorem 4.1 Hall's Theorem

Let G be a bipartite graph with bipartition $\{A, B\}$. Then G has a matching that covers all vertices in A if and only if for all $S \subseteq A$, $|N_G(S)| \geq |S|$.

Proof.

- ($\Longrightarrow$) Let M be a matching that covers all of A. Then for any $S \subseteq A$, al ledges with one end in S have distinct vertices in $N(S)$ as the other end. So $|N(S)| \geq |S|$.
- ($\Longleftarrow$) Suppose $|N(S)| \geq |S|$ for all $S \subseteq A$. We prove that G has a matching that covers A by induction on $|A|$. When $|A| = 1$, the vertex in A must have at least 1 neighbor in B, since $|N(A)| \geq |A| = 1$. Any such edge is a matching that covers A.

  We now assume that $|A| \geq 2$. We break into 2 cases.

- *Case 1. Suppose $|N(S)| > |S|$ for all nonempty proper subsets S of A.* Consider any edge xy where $x \in A, y \in B$. Let $G' = G - \{x, y\}$. The goal is to show that the condition holds for any subset of $A \setminus \{x\}$, and use induction to find a matching that covers $A \setminus \{x\}$. Let $T \subseteq A \setminus \{x\}$. By assumption, in G, $|N_G(T)| > |T|$. But only y is removed from B to get G', so $N_{G'}(T) = N_G(T) \setminus \{y\} = N_G(T)$. In either case,

$$|N_{G'}(T)| \geq |N_G(T)| - 1 \geq |T|,$$

since $|N_G(T)| > |T|$. By induction, G' has a matching M' that covers $A \setminus \{x\}$. Then $M' \cup \{x, y\}$ is a matching that covers A in G.

- *Case 2. Suppose $|N(S)| = |S|$ for some nonempty proper subset S of A.* We break G into 2 parts. Let

$$G_1 = G[S \cup N_G(S)]$$
$$G_2 = G - (S \cup N_G(S)).$$

For G_1, let $T \subseteq S$. By assumption, $|N_G(T)| \leq |T|$. But all neighbors of T are in $N_G(S)$, so they are all in G_1. So

$$|N_{G_1}(T)| = |N_G(T)| \leq |T|.$$

Since S is a proper subset of A, by induction, G_1 has a matching M_1 that covers S in G_1.

For G_2, let $U \subseteq A \setminus S$. We claim that $|N_{G_2}(G)| \geq |U|$. Suppose otherwise for the sake of contradiction. Consider the set $S \cup U$ in G. We see that

$$N_G(S \cup U) = N_G(S) \cup N_{G_2}(U).$$

This implies

$$|N_G(S \cup U)| = |N_G(S)| + |N_{G_2}(U)| = |S| + |N_{G_2}(U)| < |S| + |U| = |S \cup U|,$$

a contradiction. Hence $|N_{G_2}(G)| \geq |U|$, as claimed.

Since S is nonempty, $|A \setminus S| < |A|$. Hence by induction, G_2 has a matching M_2 that covers $A \setminus S$. Then $M_1 \cup M_2$ is a matching in G that covers A. ♦

(4.4) What if we do not have a matching that covers A? What is the size of a maximum matching?

Idea. We must have some $S \subseteq A$ where $|N(S)| < |S|$. Then any matching must expose at least $|S| - |N(S)|$ vertices in A. ◁

This motivates the following definition.

Def'n 4.5

Deficiency of a Set of Vertices

For a set S of vertices of a graph G, the ***deficiency*** of S, denoted as $\operatorname{def}(S)$, is defined as

$$\operatorname{def}(S) = |S| - |N_G(S)|.$$

Note that we are using deficiency in a different way, applying to a set of vertices here. Also, if we have a set of maximum deficiency, we know that any matching must miss at least that many vertices.

Corollary 4.1.1

For a bipartite graph G with bipartition $\{A, B\}$,

$$\alpha'(G) = |A| - \max_{S \subseteq A} \operatorname{def}(S).$$

Proof. Note that $\operatorname{def}(\varnothing) = 0$, so $\max_{S \subseteq A} \operatorname{def}(S) \geq 0$. Let $S \subseteq A$ be a subset with maximum deficiency. From (4.4), any matching exposes at least $\operatorname{def}(S)$ vertices, so

$$\alpha'(G) \leq |A| - \operatorname{def}(S).$$

We now try to find a matching of size at least $|A| - \operatorname{def}(S)$ using Hall's theorem. We obtain G' from G by adding $\operatorname{def}(S)$ new vertices, and joint them to all vertices in A. Let $T \subseteq A$. The goal is to show that $|N_{G'}(T)| \geq |T|$. Then the neighbors of T consists of $N_G(T)$ and all the new vertices. So

$$|N_{G'}(T)| = |N_G(T)| - \operatorname{def}(S). \qquad [4.1]$$

Since S has maximum deficiency,

$$|T| - |N_G(T)| \leq \operatorname{def}(S). \qquad [4.2]$$

Combining [4.1], [4.2] gives

$$|N_{G'}(T)| \geq |T|,$$

so by Hall's theorem, G' has a matching M' that covers A. But at most $\operatorname{def}(S)$ edges in M' use new vertices, so removing these edges results in a matching of size at least $|A| - \operatorname{def}(S)$. Thus

$$\alpha'(G) = |A| - \operatorname{def}(S),$$

as required. ♦

Corollary 4.1.2
1-factorization

If G is a k-regular bipartite graph with $k \geq 1$, then G has a 1-factor. Moreover, the edges of G can be partitioned into k 1-factors.

Proof Sketch. Let $\{A, B\}$ be a bipartition. Since G is regular,

$$|A| = |B|.$$

Take $S \subseteq A$. There are $k|S|$ edges incident with S, whose other ends are in $N(S)$. Each vertex in $N(S)$ gets at most k of these edges, so

$$k|N(S)| \geq k|S|.$$

By Hall's theorem, there is a matching that covers A, which is a perfect matching (i.e. 1-factor) since $|A| = |B|$. By induction, we find k disjoint perfect matchings. ♦

Def'n 4.6

2-factor of a Graph

A 2-***factor*** of G is a 2-regular spanning subgraph of G.[a]

[a] Often times we think a 2-factor as a set of cycles that cover all vertices.

Proposition 4.2
2-factorization

Let G be a $2k$-regular graph with $k \geq 1$. Then G has a 2-factor. Moreover, the edges of G can be partitioned into 2-factors.

Recall 4.7

Eulerian Circuit

Let G be a graph. We say a walk on G is an ***Eulerian circuit*** if it is a closed walk that uses every edge exactly once.

Proposition 4.3

Let G be a connected graph. If every vertex has even degree, then G admits an Eulerian circuit.

Proof of Proposition 4.2. We may assume that G is connected. Since every vertex has even degree, it has an Eulerian circuit, say $(v_i)_{i=0}^{k}$. We create a new graph H as follows: for each vertex v in G, we add two vertices v^-, v^+ in H; for each edge $e_i = v_i v_{i+1}$, we add an edge $v_i^- v_{i+1}^+$ to H. Then H is bipartite with bipartition $\left\{\{v^-\}_{v\in V(G)}, \{v^+\}_{v\in V(G)}\right\}$. When the Eulerian circuit visits v_i, it uses two edges e_{i-1}, e_i incident with v_i. They correspond to edges in H that contributes to the degrees of v_i^+, v_i^- respectively. Since G is $2k$-regular, the Eulerian circuit visits each vertex k times. Hence H is k-regular, so has a perfect matching. By merging v^-, v^+ into v, we obtain a 2-factor for G.

Removing a 2-factor form a $2k$-regular graph resutls in a $(2k-2)$-regular graph. By induction, we can partition G into k 2-factors. ♦

4.2 Tutte's Theorem

(4.5) We now turn our attention to matchings for general graphs. That is, we desire to find a sufficient condition for the existence of a perfect matching.

Let G be a graph and let $S \subseteq V(G)$. Consider the components of $G - S$. Some have odd number of vertices, others have even number of vertices. Suppose G has a perfect matching. For an odd component, at least 1 vertex must be matched outside the component, at it must be matched with S. This means

$$|S| \geq \text{number of odd components of } G - S. \tag{4.3}$$

Hence [4.3] is a necessary condition for having a perfect matching. But if this condition holds for all $S \subseteq V(G)$, then G has a perfect matching. This result is known as *Tutte's theorem*.

Def'n 4.8

Odd Component

We say a component of a graph is ***odd*** if it has odd number of vertices. We denote the number of odd components of G by $o(G)$.

Theorem 4.4
Tutte's Theorem

A graph G has a perfect matching if and only if for every $S \subseteq V(G)$, $|S| \geq o(G-S)$.

Lemma 4.4.1
Parity Lemma

Let $S \subseteq V(G)$. Then

$$o(G-S) - |S| \equiv |V(G)| \bmod 2.$$

In particular, if $|V(G)|$ is even, then

$$|S| \equiv o(G-S) \bmod 2.$$

Proof of Tutte's Theorem.

- ($\implies$) If G has a perfect matching M, then any odd component of $G-S$ has one edge in M joining a vertex in the component with a distinct vertex in S. So $|S| \geq o(G-S)$.
- Assume $|S| \geq o(G-S)$ for all $S \subseteq V(G)$. By picking $S = \varnothing$, we see that

$$0 \geq o(G),$$

so G has no odd components. This means $|V(G)|$ is even, say $|V(G)| = 2n$. We proceed inductively on n.

When $n = 1$, there are 2 vertices, and they must be joined by an edge, since G has no component. So G admits a perfect matching.

We now assume $n > 1$. We break into 2 cases.

  - *Case 1. Suppose $|S| > o(G-S)$ for all $S \subseteq V(G)$ where $2 \leq |S| \leq 2n$.* By parity lemma,

$$|S| \geq o(G-S) + 2.$$

Let xy be an edge in G and let $G' = G - \{x, y\}$. The goal is to apply induction on G'. Let $T \subseteq V(G')$. Then $G' - T = G - (T \cup \{x, y\})$. By assumption,

$$|T \cup |x, y|| \geq o(G - (T \cup \{x, y\})) + 2 = o\left(G' - T\right) + 2$$

by parity lemma. But $|T \cup \{x, y\}| = |T| + 2$, so this gives

$$|T| \geq o\left(G' - T\right).$$

Hence by induction G' has a perfect matching M, and $M \cup \{xy\}$ is a perfect matching in G.

  - *Case 2. Suppose $|S| = o(G-S)$ for some $S \subseteq V(G)$ with $|S| \geq 2$.* Among all such sets, pick S to be a maximal one.

We break our proof into three claims.

    - *Claim 1. All components of $G-S$ are odd.*

Proof. Suppose otherwise, for the sake of contradiction. Let C be an even component of $G-S$. Let $x \in V(C)$. The goal is to show that $S \cup \{x\}$ creates a contradiction. We see that $C - x$ has odd number of vertices, so it contains an ood component. Odd components of $G-S$ are still odd components of $G - (S \cup \{x\})$. So

$$o(G - (S \cup \{x\})) \geq o(G-S) + 1 = |S| + 1$$

by the definition of S. But

$$o(G - (S \cup \{x\})) \leq |S \cup \{x\}| = |S| + 1$$

by the assumption of the reverse direction. So

$$o(G - (S \cup \{x\})) = |S \cup \{x\}|.$$

This contradicts the maximality of S. Thus every component of $G-S$ is odd. $\triangleleft$

⋆ *Claim 2. There is a matching between the odd components of $G-S$ with S.*

Proof. We construct a new graph H from G as follows. Shrink each component of $G-S$ into one vertex. Remove any edges in $G[S]$. Let T be the set of all vertices from shrinking the components. Then H is a bipartite graph with bipartition $\{S,T\}$. Moreover, since $|S| = o(G-S)$, $|S| = |T|$. Now we desire to show that there is a perfect matching using Hall's theorem.
We claim that $|N_H(X)| \geq |X|$ for all $X \subseteq T$. Suppose otherwise, for the sake of contradiction, so that there exists $X \subseteq T$ such that $|N_H(X)| < X$. Let $Y = N_H(X)$. In $G - N_H(X)$, the components represented by X are still odd components. So

$$o(G - N_H(X)) \geq |X| > |N_H(X)|,$$

which contradicts the assumption of the reverse direction. Hence we conclude

$$|N_H(X)| \geq |X|.$$

Thus by Hall's theorem, there is a perfect matching in H, which corresponds to a matching in G that covers S, each one matched to a distinct odd component of $G-S$. ◁

⋆ *Claim 3. There are matchings for the rest of the components.* So fix an odd component C of $G-S$. Suppose $a \in V(C)$ is matched to a vertex in S from the matching. The goal is to show that $C-a$ has a perfect matching.

Proof. We claim that for every $Z \subseteq V(C-a)$,

$$o((C-a)-Z) \leq |Z|.$$

Suppose not, so there exists $S \subseteq V(C-a)$ such that $o((C-a)-Z) > |Z|$. Since $C-a$ has even number of vertices, by parity lemma,

$$o((C-a)-Z) \geq |Z|+2.$$

Now consider

$$W = S \cup \{a\} \cup Z.$$

Then the odd components of $G-W$ are the odd components of $(C-a)-Z$ and odd components of $G-S$ except C. Hence

$$o(G-W) = o((C-a)-Z) + o(G-S) - 1.$$

But note that

$$o((C-a)-Z) + o(G-S) - 1 \geq |Z| + 2 + |S| - 1 = |Z| + |S| + 1 = |W|.$$

By the assumption of the reverse direction, it follows that

$$o(G-W) = |W|.$$

This contradicts the maximality of S, and the (sub)claim holds.
By induction, there exists a perfect matching in $C-a$. We can apply this to all components of $G-S$. ◁

By Claim 2, 3, we have a perfect matching in G

This concludes the proof of Tutte's theorem. ♦

4.3 Corollaries of Tutte's Theorem

(4.6) We discuss some of the corollaries of Tutte's theorem in this section.

Suppose in a graph G we find a set S such that $o(G-S) > |S|$. Then any matching must expose at least $o(G-S)-|S|$ vertices. This motivates the following definition.

Def'n 4.9 **Deficiency** of a Set of Vertices

Let G be a graph. For any $S \subseteq V(G)$, we define the ***deficiency*** of S, denoted as $\operatorname{def}(S)$, as

$$\operatorname{def}(S) = o(G-S) - |S|.$$

It follows from the above discussion that any matching exposes at least $\max_{S\subseteq V(G)} \operatorname{def}(S)$ vertices. Intuitively, a maximum matching would expose precisely $\max_{S\subseteq V(G)} \operatorname{def}(S)$ vertices. This result is known as *Tutte-Berge formula*.

Corollary 4.4.2
Tutte-Berge Formula

For any graph G,

$$\alpha'(G) = \frac{1}{2}\left(|V(G)| - \max_{S\subseteq V(G)} \operatorname{def}(S)\right).$$

Def'n 4.10 **Tutte Set**

A ***Tutte set*** is a set of vertices with maximum deficiency; that is,

$$\operatorname*{argmax}_{S\subseteq V(G)} \operatorname{def}(S).$$

(4.7)
Tutte Set

If S is a Tutte set, then $\operatorname{def}(S) = \operatorname{def}(G)$, the number of vertices exposed by a maximum matching in G. It should be noted that a graph can have multiple Tutte sets.

Let S be a Tutte set and let M be a maximum matching. Any matching exposes at least $\operatorname{def}(S)$ vertices, due to the structure of the odd components and S. But M exposes exactly $\operatorname{def}(S)$ vertices by the Tutte-Berge formula. So it is the case that each vertex in S is matched to a distinct odd component, a vertex from the remaining odd components is exposed, and the rest is covered. In particular, it follows that any maximum matching covers S and all even components of $G-S$.

This motivates the following definition.

Def'n 4.11 **Essential, Avoidable** Vertex

Let G be a graph and let v be a vertex.

(a) We say v is ***essential*** in G if every maximum matching in G covers v. Equivalently,

$$\alpha'(G-v) = \alpha'(G) - 1.$$

(b) We say v is ***avoidable*** if some maximum matching exposes v. Equivalently,

$$\alpha'(G-v) = \alpha'(G).$$

Lemma 4.4.3

Every vertex in a Tutte set is essential.

Proof. Let $S \subseteq V(G)$ be a Tutte set in a graph G and let $x \in S$. Consider the graph $G' = G - x$ and the set $S' = S \setminus \{x\}$. Then

$$G - S = G' - S',$$

so

$$o(G - S) = o(G' - S').$$

So

$$\operatorname{def}_G(G') \geq \operatorname{def}_{G'}(S') = o(G' - S') - |S'| = o(G - S) - (|S| - 1) = o(G - S) - |S| + 1 = \operatorname{def}(G) + 1,$$

since S is a Tutte set. It follows $\alpha'(G') < \alpha'(G)$, so x is essential. ♦

Corollary 4.4.4

Let S be a Tutte set in a graph G. Then

(a) every vertex in S is essential, and is matched to distinct odd components of $G - S$ in any maximum matching;

(b) each even component C of $G - S$ has a perfect matching, and each vertex of C is essential; and

(c) each odd component of $G - S$ has a near-perfect matching.[a]

[a] We say a matching is ***near-perfect*** if it covers all but one vertex.

(4.8) We now use Tutte's theorem to prove a sufficient condition for having a perfect matching.

Corollary 4.4.5
Petersen's Theorem

A 3-regular graph with no cut-edges has a perfect matching.

Proof. Let G be a 3-regular graph with no cut-edges. We verify the condition of the Tutte's theorem. So let $S \subseteq V(G)$ and let $k = o(G - S)$.

If $k = 0$, then $o(G - S) \leq |S|$.

We now assume $k \geq 1$, and let $C_1, \ldots, C_k$ be the odd components of $G - S$. Let F_i be the set of edges with one end in C_i and one end in S.

- *Claim 1. $|F_i|$ is odd.*

  Proof. Consider the sum

  $$\sum_{v \in V(C_i)} \deg_G(v).$$

  This is odd since $|V(C_i)|$ and $\deg_G(v) = 3$ are both odd. Each edge in C_i contributes 2 to the sum, and each edge in F_i contributes 1 to the sum. So $2|E(C_i)| + |F_i|$ is odd, implying that $|F_i|$ is odd. ◁

But note that $|F_i| \neq 1$, since there are no cut-edges. So $|F_i| \geq 3$ for each i. Over all k components, there are at least $3k$ edges incident with vertices in S. But each vertex in S has degree 3, so

$$3|S| \geq 3k.$$

Since $k \geq 1$, it follows that

$$|S| \geq k = o(G - S).$$

By Tutte's theorem, G has a perfect matching. ♦

(4.9) We can weaken the assumption to any 3-regular graph with at most 2 cut-edges.

4.4 Factor-critical Graphs

Factor-critical Graph

Def'n 4.12 Let G be a graph. We say G is ***factor-critical*** if $G-v$ has a 1-factor for all $v \in V(G)$.

(4.10) A factor-critical graph has odd number of vertices and is connected.

(EX 4.11) Here are some examples of factor-critical graphs:

(a) odd cycles; and

(b) odd complete graphs.

Proposition 4.5

Characterization of Factor-critical Graphs

Let G be a connected graph. The following are equivalent.

(a) G is factor-critical.

(b) Every vertex of G is avoidable.

Proof.

- (a) $\Longrightarrow$ (b) If G is factor-critical, then $G-v$ has a 1-factor, which is a maximum matching in G that exposes v. Hence v is avoidable.
- (b) $\Longrightarrow$ (a) Suppose every vertex is avoidable. Then $\varnothing$ is the only Tutte set by Lemma 4.4.3. So

$$\operatorname{def}(G) = \operatorname{def}(\varnothing) = o(G-\varnothing) - |\varnothing| = o(G).$$

Since G is connected, $o(G)$ is 0 or 1. But if $o(G) = 0$, then G has a 1-factor, so every vertex in G is essential, which is a contradiction. Hence $\operatorname{def}(G) = 1$, and by the Tutte-Berge formula,

$$\alpha'(G) = \frac{1}{2}(|V(G)| - 1).$$

But every $v \in V(G)$ is avoidable, so

$$\alpha'(G-v) = \alpha'(G) = \frac{1}{2}(|V(G)| - 1) = \frac{1}{2}|G-v|,$$

and $G-v$ has a 1-factor. Thus G is factor-critical. ♦

Proposition 4.6

If T is a maximal Tutte set in a graph G, then each component of $G-T$ is factor-critical.

Proof Idea. Proof of Proposition 4.6 is similar to the last part of the proof of Tutte's theorem. Also recall that, if T is a maximal Tutte set, then $G-T$ consists of only odd components. Each of them is factor-critical. ◁

Odd Closed-ear Decomposition of a Graph

Def'n 4.13 Let G be a graph. We say a sequence $(G_i)_{i=0}^{k}$ of G is an ***odd closed ear decomposition*** of G if

(a) G_0 is a vertex in G;

(b) for each $0 \le i \le k-1$,

$$G_{i+1} = G_i + P_i$$

for some ear P_i of G_i or an odd cycle P_i that intersects G_i at exactly one vertex; and

(c) $G_k = G$.

Theorem 4.7
Perfect Graph Theorem

Let G be a graph. The following are equivalent.

(a) G is factor-critical.

(b) G has an odd closed-ear decomposition.

Proposition 4.8

Let G be a graph and let M_1, M_2 be two matchings of G. Then $M_1 \triangle M_2$ has maximum degree 2.

Corollary 4.8.1

Consider the setting of Proposition 4.8. Then nontrivial components of $M_1 \triangle M_2$ are paths and even cycles, with edges alternate between M_1, M_2.

Proof of Perfect Graph Theorem.

- (b) $\implies$ (a) We provide a sketch only. We use induction with a case analysis.
  - *Case 1. Add an odd ear, find an exposed vertex x, and cover everything else.*
  - *Case 2. Add an odd cycle, find an exposed vertex x, and cover everything else.*
- (a) $\implies$ (b) Let G be factor-critical and let $G_0 = \{x\}$ for some vertex x. Let M_x be a 1-factor of $G - x$.

  We inductively build $G_{i=1}$ from G_i by adding odd ears or odd cycles while always maintaining an *invariant*: no edge of M_x is in the cut induced by any $V(G_i)$. This is true for G_0.

  Assume this invariant holds for G_i.

  - *Case 1. Suppose $G_i = G$.* We are done.
  - *Case 2. Suppose there exists an edge $uv \in E(G) \setminus E(G-i)$ where $u, v \in V(G_i)$.* Set $G_{i+1} = G_i + uv$. The invariant still holds since we did not add any edge from $\delta(V(G_i))$ to obtain G_{i+1}.
  - *Case 3. Suppose there exists an edge $uv \in E(G) \setminus E(G_i)$, where $u \in V(G_i), v \notin V(G_i)$.* By the invariant, $uv \notin M_x$. Let M_v be a 1-factor of $G - v$. Consider

    $$M_x \triangle M_v.$$

    We see that x is covered by M_v but not M_x. On the other hand, v is covered by M_x but not M_v. So the only vertices of degree 1 in $M_x \triangle M_v$ are x, v. Then by Corollary 4.8.1, v, x are the endpoints of a path in $M_x \triangle M_v$, say $(v_0, \ldots, v_m)$ with $v_0 = v, v_m = x$.

    Let j be the smallest index such that $v_j \in V(G_i)$. This exists since $v_m = x \in V(G_i)$. Then $v_{j-1}v_J$ is in the cut induced by $V(G_i)$, so by the invariant, $v_{j-1}v_j \in M_v$. From this, we conclude that j is even, so $(u, v_0, \ldots, v_j)$ has odd length, and we add this to G_i to obtain G_{i+1}.

    But note that every new vertices on P_i are covered by M_x, and their incident edges in M_x are on P_i. So no edge of M_x can be in the cut induced by $V(G_{i+1})$, so the invariant holds for G_{i+1}. This completes the construction, and we obtain an odd closed-ear decomposition of G. ♦

4.5 Gallai-Edmonds Structure Theorem

Theorem 4.9
Gallai-Edmonds Structure Theorem

Given a graph G, let D be the set of all avoidable vertices in G, let A be the set of all vertices in $V(G) \setminus D$ that are adjacent to some vertex in D, let $C = V(G) \setminus (D \cup A)$. Then

(a) *$G[D]$ consists of odd components that are factor-critical;*

(b) *$G[C]$ consists of even components that have perfect matchings;*

(c) *A is a Tutte set;*

(d) *Each nonempty $S \subseteq A$ is adjacent to at least $|S|+1$ odd components of $G[D]$; and*

(e) *If M is a maximum matching of G, then M contains a near-perfect matching of each component of $G[D]$, a perfect matching for $G[C]$, and matches each vertex in A to a distinct odd component of $G[D]$.*

Proof. Let T be a maximal Tutte set. Then $G-T$ contains only factor-critical odd components. Let H be a graph obtained from G by shrinking each component of $G-T$ and removing edges in $G[T]$. Then H is bipartitie. Moreover, any maximum matching matches vertices in T to distinct odd components. This corresponds to a matching in H that covers T. So for any $U \subseteq T$, $|N_H(U)| \geq |u|$.

Among all subsets of T, pick a maximal subset R such that $|N_H(R)| = |R|$. This exists since $|N_H(\varnothing)| = |\varnothing|$. Let R' be the odd components represented by $N_H(R)$. Now let

$$C' = R \cup R', A' = T \setminus R, D' = V(G) \setminus \left(C' \cup A'\right).$$

We now verify the 5 properties for D', A', C'.

(a) $G[D']$ consists of factor-critical components by our choice of T, a maximal Tutte set.

(b) We see that $|R|$ is equal to the number of components of R'. Hence a maximum matching matches R to all components of R'. Since the components of R' are factor-critical, there exist perfect matchings for the rest of R'. So $C' = R \cup R'$ has a perfect patching.

(c) There are no edges joining R to D'. So the odd components of $G-A'$ are precisely the components of D'. So

$$\operatorname{def}\left(A'\right) = o\left(G-A'\right) - \left|A'\right| = \left(o(G-T) - o\left(G\left[R'\right]\right)\right) - (|T|-|R|) = o(G-T) - |T| = \operatorname{def}(T).$$

Since T is a Tutte set, so is A'.

(d) For any $S \subseteq A'$, we know that $|N_H(S)| \geq |S|$. Suppose there exists $S \subseteq A'$ where $|N_H(S)| = |S|$ for contradiction. Then $|N_H(R \cup S)| = |R| + |S| = |R \cup S|$, which contradicts the maximality of R; recall that R is a maximal subset whose neighborhood has the same size.

(e) This follows from (a), …, (d).

We now claim that $D = D', A = A', C = C'$.

One wey to describe C is that it is the set of essential vertices not adjacent to any avoidable vertex. We see that C' only contains essential vertices as every maximum matching includes a perfect matching of C'. These vertices can be adjacent to vertices in A, but not D'. We see that A' is a Tutte set, so vertices in A' are essential. So $C' \subseteq C'$.

We now show that every vertex in D is avoidable. Let J be a component of D and let $v \in V(J)$. For each $S \subseteq A'$, it is adjacent to at least $|S|+1$ odd components of $G[D']$. Then S is adjacent to at least $|S|$ odd components other than J. Using Hall's theorem, there is a matching from A' to distinct odd components of $G[D'] - J$. We can obtain a near-perfect matching for each component of $G[D'] - J$. Since J is factor-critical, there is a near-perfect matching of J that exposes v. We also have a perfect matching for C'. This gives a maximum matching that exposes v. So v is avoidable. Hence all vertices in D' are avoidable while other vertices (A', C') are essential. So $D' = D$. Each vertex in A' is adjacent to at least one vertex in D' while no vertices in C' do. So $A' = A$, and $C' = C$. ♦