CO 342

Graph Theory I

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1. Connectivity

1.1 Basic Graph Terminology

Def'n 1.1 **Graph**

A ***graph*** $G = (V, E)$ is an ordered pair such that

(a) V is a set; and *vertex set*

(b) E is a set of unordered pairs of elements of V. *edge set*

The elements of V are called the ***vertices*** of G and the elements of E are called the ***edges*** of G. Given $u, v \in V$, we denote uv to denote an edge ***joining*** u, v. Note that $uv = vu$ for every $u, v \in V$. For simplicity, we assume that

(a) G does not have any ***loop*** (i.e. $vv \notin E$ for every $v \in V$);[a]

(b) G has at least one vertex; and

(c) G is ***finite*** (i.e. V is a finite set).

[a] Note that, since we defined E to be a set of unordered pairs of elements of V, if $e, f \in E$ join the same vertices, then $e = f$. In other words, we do not allow ***multiple edges***.

Notation 1.2 $V(G), E(G)$

Let G be a graph. We write $V(G)$ to denote the vertex set and write $E(G)$ to denote the edge set of G.

Def'n 1.3 **Subgraph** of a Graph

Let G be a graph. We say H is a ***subgraph*** of G if $V(H) \subseteq V(G), E(H) \subseteq E(G)$, and all edges in $E(H)$ have both endpoints in $V(H)$. In particular,

(a) when $V(H) = V(G)$, we say H is a ***spanning*** subgraph; and

(b) if there exists $S \subseteq V(G)$ such that $V(H) = S$ and that

$$E(H) = \{e \in E(G) : \exists v, u \in S [e = vu]\},$$

then we say H is the subgraph ***induced*** by S, denoted as $G[S]$.

Def'n 1.4 **Degree** of a Vertex

Let G be a graph. The ***degree*** of $v \in V(G)$, denoted as $d_G(v)$, is the number of edges incident to v. Moreover, we write $\delta(G)$ to denote

$$\delta(G) = \min\{d_G(v) : v \in V(G)\}$$

and write $\Delta(G)$ to denote

$$\Delta(G) = \max\{d_G(v) : v \in V(G)\}.$$

Def'n 1.5 **Connected** Graph

Let G be a graph. We say G is ***connected*** if for every $u, v \in V(G)$, there is a u, v-path in G. A ***component*** of G is a maximally connected subgraph of G.

Proposition 1.1

Let G be a graph. The following are equivalent.

(a) G is connected.

(b) There exists $v \in V(G)$ such that for every $u \in V(G)$, there exists a u, v-path in G.

Notation 1.6

$G - F$, $G - e$, $G - U$, $G - v$

Let G be a graph.

(a) Given $F \subseteq E(G)$, we write $G - F$ to denote the graph such that $V(G - F) = V(G), E(G - F) = E(G) \setminus F$.

(b) In case $F = \{e\}$ for some $e \in E$, we simply write $G - e$ to denote $G - \{e\}$.

(c) Similarly, given $U \subseteq V(G)$, we write $G - U$ to denote the graph such that $V(G - U) = V(G) \setminus U$ and that

$$E(G - U) = \{e \in E(G) : \exists u, v \in V(G) \setminus U\, [e = uv]\}.$$

In other words, we remove every vertex in U from G and every edge in G that is incident to at least one vertex in U.

(d) In case $U = \{v\}$ for some $v \in E$, we simply write $G - v$ to denote $G - \{v\}$.

Def'n 1.7

Contraction of a Graph

Let G be a graph. For any $e \in E(G)$, the G ***contract*** e, denoted as G/e, is defined by *removing* e from G and *identifying* u, v as the same vertex.

1.2 Edge Connectivity

Def'n 1.8

***k*-edge-connected** Graph

We say a graph G is k-***edge-connected*** if $G - F$ is connected for every $F \subseteq E(G)$ such that $|F| < k$. The ***edge connectivity*** of G, denoted as $\kappa'(G)$, is the largest $k \in \mathbb{N}$ for which G is k-edge-connected.

Def'n 1.9

Cut-edge of a Graph

Let G be a graph. We say $e \in E(G)$ is a ***cut-edge*** if $G - \{e\}$ is disconnected.

(1.1) Let G be a graph.

(a) $\kappa'(G) = 1$ if and only if G has a cut-edge. Equivalently, G is 2-edge connected if and only if G is connected with no cut-edges.

(b) If G is disconnected, then we define $\kappa'(G) = 0$ and say G is 0-edge-connected.

(c) The ***trivial graph*** $(\{v\}, \varnothing)$ *cannot be disconneted*; we define its edge-connectivity to be 0.

Proposition 1.2

Let G be a graph. Then
$$\kappa'(G) \leq \delta(G).$$

Proof. If $\delta(G) = 0$, then there is an isolated vertex $v \in V(G)$, which means G is disconnected. So $\kappa'(G) = 0$, meaning $\delta(G) = 0 \leq 0 = \kappa'(G)$. Now assume $\delta(G) \geq 1$ and let $v \in V(G)$ be such that $d_G(v) = \delta(G)$. Let $F = \{e \in E(G) : \exists u \in V(G)[uv = e]\}$, the set of all edges incident with v. Then $G - F$ is disconnected, since v is not adjacent to any other vertices in G, which in turn exist since $\delta(G) \geq 1$. It follows that $\kappa'(G) \leq |F| = \delta(G)$. ∎

Def'n 1.10

Cut Induced by a Subset

Let G be a graph and let $S \subseteq V(G)$. We define the ***cut*** induced by S, denoted as $\delta_G(S)$, to be the set of all edges with exactly one end in S:
$$\delta_G(S) = \{e \in E(G) : \exists v \in V(G) \setminus S, u \in S[e = vu]\}.$$
We say $\delta_G(S)$ is ***nontrivial*** if $S \neq \varnothing$ and $S \neq V(G)$.

Proposition 1.3

Let G be a graph. Then the following are equivalent.

(a) G is connected.

(b) Every nontrivial cut of G is nonempty.

(1.2) Say we have a graph and a cut $C \subseteq E(G)$. Then it is easy to see that $G - C$ is disconnected when C is nontrivial. It follows that the size of any nontrivial cut is an upper bound on $\kappa'(G)$.

Proposition 1.4

Let G be a graph and let $F \subseteq E(G)$. If $G - F$ is disconnected, then F contains a nontrivial cut.

Proof. Suppose $G - F$ is disconnected and let H be a component of $G - F$. Since $G - F$ is disconnected, $V(H) \neq \varnothing$ and $V(H) \neq V(G)$. Consider $\delta_G(V(H))$. Since H is maximally connected in $G - F$, none of the edges in $\delta_G(V(h))$ can be in $G - F$. Hence, all edeges in $|delta_G(V(H))$ must be in F. Hence F contains a nontrivial cut $\delta_G(V(H))$, as required. ∎

Corollary 1.4.1

Let G be a graph. Then $\kappa'(G)$ is the size of a minimum nontrivial cut.

Def'n 1.11

Bond of a Graph

A ***bond*** is a minimal nonempty cut.

Proposition 1.5

Characterization of a Bond

Let $F = \delta_G(S)$ be a cut in a connected graph G, where $S \subseteq V(G)$. Then the following are equivalent.

(a) F is a bond.

(b) $G - F$ has exactly 2 components.

Proof.

- ($\Longrightarrow$) Suppose $G-F$ has at least 3 components. Assume, without loss of generality, that (at least) 2 of the components are contained in $G[S]$ and let H be one of them. Then $\delta_G(V(H)) \subset F$, so F is not minimal and hence not a bond.

- Suppose $G-F$ has exactly 2 components and suppose further F is not a bond, for the sake of contradiction. Then there is a nonempty cut $\delta_G(S)$ that is a proper subset of F. Then the two components of G_F are linked by some edge in F. It follows that $G-\delta_G(S)$ is connected, contradicting the fact that $\delta_G(S)$ is a cut. ∎

Proposition 1.6

Every cut is a disjoint union of bonds.

Def'n 1.12

Symmetric Difference of Two Sets

Let S,T be sets. Then the ***symmetric difference*** of S,T, denoted as $S\triangle T$, is defined as

$$S\triangle T = (S\cup T)\setminus(S\cap T).$$

Lemma 1.6.1

Let G be a graph and let $S,T \subseteq V(G)$. Then

$$\delta_G(S)\triangle\delta_G(T) = \delta_G(S\triangle T).$$

Proof of Proposition 1.6. We proceed inductively on the number of edges in F. If $|F|=0$, then is a disjoint union of no bonds, so we are done. Now assume t hat F is any nonempty cut (where the inductive hypothesis is that any cut with a smaller size than F is a disjoint union of bonds). We may assume F is not a bond. Then F contains a nonepmty cut F_1 by the definition of a bond. Now define

$$F_2 = F\setminus F_1,$$

where we note that $F\setminus F_1 = (F\cup F_1)\setminus(F\cap F_1) = F\triangle F_1$. Hence by Lemma 1.6.1, F_2 is a cut. Since F_1,F_2 are nonempty, both F_1,F_2 have fewer edges than F. It follows that F_1,F_2 are disjoint union of bonds by the inductive hypothesis. Thus $F = F_1\cup F_2$ is also a disjoint union of bonds, as required. ∎

1.3 Vertex Connectivity

Def'n 1.13

Separating Set, Cut-vertex of a Graph

Let G be a graph.

(a) If $S\subseteq V(G)$ is such that $G-S$ is disconnected, then we say S is a ***separating set***.

(b) If $v\in V(G)$ is such that $G-v$ has more components than G, then we say v is a ***cut-vertex***.

Def'n 1.14

Connectivity of a Graph

Let G be a graph.

(a) We say G is ***k-connected***, where $k \in \mathbb{N} \cup \{0\}$, if G has at least $k+1$ vertices and does not have a separating set of size at most $k-1$.

(b) The ***connectivity*** of G, denoted as $\kappa(G)$, is the largest $k \in \mathbb{N} \cup \{0\}$ for which G is k-connected.

(1.3)

(a) The complete graph K_n ($n \in \mathbb{N}$) has no separating sets. Hence $\kappa(K_n) = n-1$. In particular, K_n is the smallest $(n-1)$-connected graph: any $(n-1)$-connected graph has at least n vertices.

(b) Let G be a graph. If there exists a separating set of size $k \in \mathbb{K}$, then $\kappa(G) \leq k$.

Theorem 1.7
Expansion Lemma

Let G be a k-connected graph and let G' be obtained from G by adding a new vertex v and adding at least k edges joining v to at least k distinct vertices of G. Then G' is k-connected.

Proof. To show that G' is k-connected, we show that there is no separating set of size $k-1$. To do so, let $X \subseteq V(G')$ have $k-1$ vertices. We now have 2 cases.

- *Case 1. Suppose $v \in X$.* Then $G' - X = G - (X \setminus \{v\})$. Since G is k-connected and $|X \setminus \{v\}| = k-2$, so $G - (X \setminus \{v\})$ is connected. Hence $G' - X$ is connected, implying that X is not a separating set.
- *Case 2. Suppose $v \notin X$.* Then $X \subseteq V(G)$. Since $|X| = k-1$ and G is k-connected, $G - X$ is connected. Moreover, v has at least k neighbors by definition, so at least one of them, say w, is not in X. Then given any $u \in V(G-X)$, note that there is a u,w-path in $G-X$ by the connectedness of $G-X$, so by the fact that w, v are neighbors in $G'-X$, there is a u,v-path in $G'-X$. Hence $G'-X$ is connected and X is not a separating set.

Thus G' is k-connected, as required. ∎

Theorem 1.8
Whitney's Theorem

Let G be a graph. Then

$$\kappa(G) \leq \kappa'(G) \leq \delta(G).$$

Proof. We showed the second inequality in Proposition 1.2. We now prove $\kappa(G) \leq \kappa'(G)$. Note that we have the following two special cases.

- *Case 1. Suppose G is complete, say $G = K_n$ for some $n \in \mathbb{N}$.* Then $\kappa(G) = \kappa'(G) = n-1$.
- *Case 2. Suppose G is disconnected.* Then $\kappa(G) = \kappa'(G) = 0$.

Hence, we may assume G is connected and not complete. Let F be a minimum nontrivial cut in G, so that $|F| = \kappa'(G)$. Then F is a bond, since every minimum nontrivial cut is clearly minimal. So by Proposition 1.5, $G-F$ has exactly 2 components, say H_1, H_2. We break into two cases.

- *Case 3. Suppose there exists a vertex $v \in V(G)$ that is not incident to any edge in F.* Without loss of generality, say v is in H_1. Let S b e the set of all vertices in H_1 incident with some edge in F. Then $|S| \leq |F|$. Also, S is a separating set, since it separates v from all the vertices in H_2. So

$$\kappa(G) \leq |S| \leq |F| = \kappa'(G),$$

as required.

- *Case 4. Suppose all vertices are incident with some edge in F.* Since G is not complete, there exist two vertices u, v that are not adjacent. Without loss of generality, say $u \in H_1$. Let S be the set of all neighbors of u. Now observe that $v \notin S$, so S is a separating set as it separates u from v. We now count the size of S. To do so, we show that every vertex in S corresponds to a distinct edge in F.
  - *Case 4.1. Say $y \in H_1$.* Then y is incident to some edge in F, say e, (since we assumed that all vertices are incident with some edge in F), and clearly u is not incident to e, since both y, u are in H_1.
  - *Case 4.2. Say $y \in H_2$.* Then $uy \in F$.

  In other words, if $y \in H_1$, then y corresponds to an edge in F that is incident to y but not incident to u. If $y \in H_2$, then y corresponds to uy. And it is immediate from the definition that these correspondences are distinct.[1] It follows that $|S| \leq |F|$, which means

$$\kappa(G) \leq |S| \leq |F| = \kappa'(G).$$

This completes the proof. ∎

1.4 Menger's Theorem

(1.4) So far we have looked at the connectivity of graphs via *destructive* methods: that is, we remove, or *destroy*, few edges or vertices and see what happens. We now change our perspective to *constructive* methods: finding disjoint paths between vertices. We first do this in a *local* point view (i.e. given two vertices, we ask: how well connected are they?). Later, we extend this to a *global* description, using paths.

(1.5) Let us reiterate what we said in (1.4). Suppose that x, y are two nonadjacent vertices of a graph G. There are two ways to measure connectivity between x, y.

(a) How many vertices do we need to remove to disconnect x from y? *destructive*

(b) How many *internally disjoint* paths are there from x to y? *constructive*

Note that for (a) we want the *minimum* number of such vertices, whereas for (b) we want the *maximum* number of such paths. In fact, we can ask the following question: between (a), (b), which one should be larger?

Answer-ish. Suppose there are k paths from x to y. In order to disconnect x from y, we have to remove at least one vertex from each path. In other words, (a) is at least (b). ◁

Surprisingly, *Menger's theorem* shows that this inequality always holds with equality. Let us first define notions that we need.

Def'n 1.15 **X, Y-separating Set**
Let G be a graph and let $X, Y \subseteq V(G)$. We say $S \subseteq V(G) \setminus (X \cup Y)$ is an X, Y-***separating set*** if there is no path from any vertex in X to any vertex in Y in $G - S$. In case $X = \{x\}, Y = \{y\}$, we say S is an x, y-***separating set***.

[1] That is, we just informally described an injection from S to F.

Def'n 1.16 **Internally Disjoint x, y-paths**
Let G be a graph and let $x, y \in V(G)$. We say x, y-paths are ***internally disjoint*** (or ***i.d.***), if no two of them share any vertices other than x, y.

Note that, from our previous observations, if S is an x, y-separating set and P is a set of internally disjoint x, y-paths, then $|S| \geq |P|$.

Theorem 1.9
Menger's Theorem

Let x, y to nonadjacent distinct vertices of a graph G. Then the minimum size of an x, y-separating set is equal to the maximum size of a set of internally disjoint x, y-paths.

Notation 1.17 G/X
Let G be a graph and let $X \subseteq V(G)$. We denote G/X to be the graph obtained from G by removing all edges in $G[X]$ and identifying all vertices in X into one new vertex.

(1.6)
Wrong Approach to Manger's Theorem

Here is a *wrong*, but useful, approach to the Menger's theorem. We proceed inductively on the number of edges of the graph. Let S be a minimum x, y-separating set, and say $k = |S|$. Let X, Y be the components containing x, y in $G - S$, respectively. Then we *shrink* X by taking G/X, and get k i.d. x, y-paths by induction. Similarly, we *shrink* Y by taking G/Y, and get k i.d. x, y-paths by induction. Combining these paths results in k i.d. x, y-paths in G.

(1.7)
Proof Outline of Manger's Theorem

Here is why (1.6) is wrong: the induction works only when $G/X, G/Y$ have *fewer* edges than G. But if $X = \{x\}$ or $Y = \{y\}$, this is not the case, and the induction fails. Hence, much of the work of the upcoming proof is done to ensure that X, Y has at least 2 vertices, a vertex besides x, y, respectively. Here is an outline.

Proof Outline. We begin by assuming that a minimum x, y-separating set, say S, has size k. We desire to build k i.d. x, y-paths by using induction on $|E(G)|$. We break our proof into three cases.

- *Case 1. Suppose every edge is incident with x, y.* This case is relatively easy.

After we are done with Case 1, we may assume that there are vertices $u, v \in V(G) \setminus \{x, y\}$ that are adjacent. The vertices u, v will be the extra vertices we need in X, Y. Let $e = uv$.

- *Case 2. Suppose a minimum x, y-separating set has size k in $G - e$.* Then by applying induction we obtain k i.d. x, y-paths.

- *Case 3. Suppose a minimum x, y-separating set S has size at most $k - 1$ in $G - e$.* This is arguably the hardest case. For Case 3, we first show that any minimum x, y-separating set indeed has size $k - 1$ in $G - e$. Afterwards, we show that $S \cup \{u\}$ is a minimum x, y-separating set of size k in G/Y, and since G/Y has fewer edges than G, by applying induction we obtain k i.d. x, y-paths. Similarly, we obtain k i.d. x, y-paths in G/X. Combining these paths gives us k i.d. x, y-paths in G. $\triangleleft$

Proof of Menger's Theorem. We prove by inducting on the number of edges on G. When there are no edges, a minimum x, y-separating set has size 0, and there are 0 x, y-paths. We now assume G has at least one edge, and that the theorem holds for any graph with less than $|E(G)|$ edges. Let k be the size of a minimum x, y-separating set, where the goal is to find k internally disjoint x, y-paths. We have few cases.

- *Case 1. Suppose every edge of G is incident with x or y.* Let S be the set of all vertices adjacent to both x, y. Then the only x, y-paths are those of length 2 going through S, and they are internally disjoint. So there are $|S|$ internally disjoint x, y-paths. But S is a x, y-separating set, so $|S| = k$, as required.

We now assume there exists some edge uv that is not incident to x nor y. Let $H = G - uv$ and let S be a minimum x, y-separating set in H.

- *Case 2. Suppose $|S| = k$.* Since H has fewer edges than G, by induction there exists a set of k internally disjoint x,y-paths in H. These paths are also in G since $H \subseteq G$, and the result follows.

- *Case 3. Suppose $|S| < k$.*

  - *Claim 1. We claim that $S \cup \{u\}$ is an x,y-separating set in G.*

    Proof. Note $G - (S \cup \{u\}) = H - (S \cup \{u\})$. Also, $S \cup \{u\}$ is an x,y-separating set in H, since H is an x,y-separating set in H. It follows that $S \cup \{u\}$ is an x,y-separating set in G. ◁

  But any x,y-separating set in G in G has size at least k, so $|S \cup \{u\}| \geq k$. Therefore, $|S| \geq k-1$, which means $|S| = k-1$. Now write

  $$S = \{v_i\}_{i=1}^{k-1}$$

  for convenience. Let X,Y be the vertices of the components of $H - S$ containing x,y, respectively. Since $|S| = k-1 < k$, $G-S$ does not separate x,y. So there is an x,y-path in $G-S$. This path does not exist in $H-S$, so uv must be on this path. In fact, e must join a vertex in X with a vertex in Y. Without loss of generality, say $u \in X, v \in Y$. This means X,Y has at least two vertices, $X \supseteq \{x,u\}, Y \supseteq \{y,v\}$. We now consider G/Y, and let y^* be the new vertex.

  - *Claim 2. Any x,y^*-separating set in G/Y is an x,y separating set in G.*

    Proof. Let T be any x,y^*-separating set in G/Y. Then any x,y-path in G that does not use T corresponds to an x,y^*-path in G/Y that does not use T, which does not exist. Therefore, T is an x,y-separating set in G. ◁

  Claim 2 in particualr means any x,y^*-separating set in G/Y has size at least k. But $S \cup \{u\}$ is an x,y^*-separating set in G/Y, with size k. Therefore, a minimum x,y^*-separating set in G/Y has size exactly k. Since Y has at least 2 vertices, $G[Y]$ has att least one edge, so G/Y has fewer edges than G, so by induction there is a set of k internally disjoint x,y^*-paths in G/Y. In particular, each path must use a distinct vertex in $S \sup \{u\}$. Now for each $i \in \{1,\ldots,k-1\}$, let P_i be the parts of the paths from x to v_i, and let P_k be the part of the path from x to u. We can make the same argument for G/X, and find k paths: $Q_1,\ldots,Q_{k-1}$ from $v_1,\ldots,v_{k-1}$ to y, Q_k from v to y. Then,

  $$\{P_i + Q_i\}_{i=1}^{k-1} \cup \{P_k + uv + Q_k\}$$

  is a set of k internally disjoint x,y-paths, as required. ∎

(1.8) There are many variations and corollaries of Menger's theorem.

Corollary 1.9.1 Characterization of k-connectedness

Let G be a graph. The following are equivalent.

(a) G is k-connected.

(b) There exist k internally disjoint x,y-paths for any distinct $x,y \in V(G)$.

Proof.

- (a) $\Longrightarrow$ (b): Let $x,y \in V(G)$. We break into 2 cases.

  - *Case 1. Suppose x,y are not adjacent.* Since G is k-connected, any x,y-separating set has size at least k, so by Menger's theorem, there exist at least k internally disjoint x,y-paths in G.

- *Case 2. Suppose x, y are adjacent.* Then $G - xy$ is $(k-1)$-connected. So any x, y-separating set in $G - xy$ has size at least $k-1$. By Menger's theorem, there exist at least $k-1$ internally disjoint x, y-paths in $G - xy$. Together with the path $(\{x, y\}, \{xy\})$, we have at least k internally disjoint x, y-paths in G.

- (a) $\Longleftarrow$ (b): Let X be any set of $k-1$ vertices and let $x, y \in V(G) \setminus X$. By assumption, there exist k internally disjoint x, y-paths in G. So removing $k-1$ vertices leaves at least one such path: there exists an x, y-path in $G - X$ for any vertices x, y in $G - X$. Hence $G - X$ is connected, which means G is k-connected. ∎

Def'n 1.18

Fan

Let G be a graph. Let $v \in V(G)$ and let $X \subseteq V(G) \setminus \{v\}$. A k-***fan*** from v to X is a set of k internally disjoint paths that start with v and end with distinct vertices in X, and no internal vertices are in X.

Corollary 1.9.2
Fan Lemma

Let G be a k-connected graph. Let $v \in V(G)$ and let $X \subseteq V(G) \setminus \{v\}$ where $|X| \geq k$. Then there exists a k-fan from v to X.

Proof. Let H be the graph obtained from G from adding a vertex x^* and edges xx^* for all $x \in X$. Then observe that H is k-connected by the Expansion lemma (Theorem 1.7), so by Corollary 1.9.1, there are k internally disjoint v, x^*-paths, say $P_1, \ldots, P_k$. Now for each $i \in \{1, \ldots, k\}$, write P_i as

$$P_i = (\{v, u_1, \ldots, u_{l_i}, x^*\}, \{vu_1, u_1u_2, \ldots, u_{l-1}u_l, u_lx^*\})$$

and let $j_i \in \{1, \ldots, l_i\}$ be the smallest index such that $u_{j_i} \in X$. It follows that $\{Q_i\}_{i=1}^k$ by

$$Q_i = \left(\{v, u_1, \ldots, u_{j_i}\}, \{vu_1, u_1u_2, \ldots, u_{j_i-1}u_{j_i}\}\right)$$

for all $i \in \{1, \ldots, k\}$ is a k-fan from v to X. ∎

Corollary 1.9.3

Let G be a k-connected graph with $k \geq 2$ and let $X \subseteq V(G)$ be such that $|X| = k$. Then there exists a cycle in G with all vertices in X.

Proof. We proceed inductively on k. Suppose $k = 2$ and let $X = \{u, v\}$. Then by Corollary 1.9.1, there exist 2 internally disjoint u, v-paths in G; they form a cycle containing u, v. Now assume $k \geq 3$ and let $X \subseteq V(G)$ with $|X| = k$. Then G is also $(k-1)$-connected. Choose $v \in X$. Then $X \setminus \{v\}$ has $k-1$ vertices, so there is a cycle in G that has all vertices in $X \setminus \{v\}$. If C also has v, then we are done. So assume C does not have v. The $k-1$ vertices of $X \setminus \{v\}$ are on the cycle C, say they are in order $v_1, \ldots, v_{k-1}$. These $k-1$ vertices partitions C into $k-1$ internally disjoint paths, or segments. We now have 2 cases.

- *Case 1. Suppose C has at least k vertices.* Since G is k-connected and $|V(G)| \geq k$, there exists a k-fan from v to $V(C)$ by the Fan lemma. Since there are $k-1$ segment on C, by the pigeonhole principle, there is a segment S on C that contains the endpoints of 2 paths, say P_1, P_2, in the fan. Say P_1, P_2 end with vertices $w_1, w_2 \in V(C)$, respectively. Let P_3 be the w_1, w_2-path contained in S. Then no internal vertex of P_3 is in X, so $C - P_3 + P_1 + P_2$ is a cycle that contains X.

- *Case 2. Suppose C has exactly $k-1$ vertices.* Then there is a $(k-1)$-fan from v to $V(C)$, ending in all vertices in C. By removing any edge from C and adding 2 paths ending at the endpoints of the edge, we obtain a cycle containing X. ∎

Def'n 1.19

Disjoint X, Y-paths

Let G be a graph and let $X, Y \subseteq V(G)$. We say paths in G are ***disjoint*** X, Y-paths if

(a) each join a vertex in X with a vertex in Y;

(b) no oehter vertices are in X nor Y; and

(c) no two paths share any vertex, including the endpoints.

Note that we *allow* X, Y to have a nonempty intersection in Def'n 1.15 and 1.18, and an X, Y-separating set could include vertices in X or Y. In fact, X and Y are X, Y-separating sets.

Theorem 1.10
Menger's Theorem, Set Version

Let G be a graph and let $X, Y \subseteq V(G)$. Then the minimum size of an X, Y-separating set is equal to the maximum size of a set of disjoint X, Y-paths.

Proof. First observe that any size of X, Y-separating set must be at least the size of any set of disjoint X, Y-paths, since we need to remove at least one vertex from each path. It remains to show that there exist an X, Y-separating set and a set of disjoint X, Y-paths of the same size. To do so, we shall utilize Menger's theorem. Let H be the graph obrtained from G by adding 2 new vertices, say x^*, y^*, and adding edges x^*x, y^*y for all $x \in X, y \in Y$. Since x^*, y^* are not adjacent, by Menger's theorem, any minimum x^*, y^*-separating set S has the same size as a maximum set of internally disjoint x, y-paths $\mathcal{P}$.

- *Claim 1. S is an X, Y-separating set in G.*

  Proof. Suppose S is not an X, Y-separating set, for the sake of contradiction. Then there exists and X, Y-path Q from $x \in X$ to $y \in Y$ in $G - S$. Then $(V(Q) \cup \{x^*, y^*\}, E(Q) \cup \{x^*x, y^*y\})$ is an x^*, y^*-path in $H - S$, contradicting the fact that S is an x^*, y^*-separating set. ◁

We now obtain a set of disjoint X, Y-paths $\mathcal{P}'$ from $\mathcal{P}$ as follows. Say a path

$$(\{x^*, v_1, \ldots, v_l, y^*\}, \{x^*v_1, v_1v_2, \ldots, v_{l-1}v_l, v_ly^*\}) \in \mathcal{P}$$

is given. Let $i \in \{1, \ldots, l\}$ be the largest index such that $v_i \in X$, and let $j \in \{1, \ldots, l\}$ be the smallest index at least i such that $v_j \in Y$. We put the path $\left(\{v_i, \ldots, v_j\}, \{v_iv_{i+1}, \ldots, v_{j-1}v_j\}\right)$ in $\mathcal{P}'$. We see that each path in $\mathcal{P}'$ joins a vertex in X with a vertex in Y with no other vertices in X or Y, and paths in $\mathcal{P}'$ do not share any vertices since $\mathcal{P}$ is internally disjoint. Then $|S| = |\mathcal{P}| = |\mathcal{P}'|$, we have a set of disjoint X, Y-paths $\mathcal{P}'$ of the same size as an X, Y-separating set S. ∎

Def'n 1.20

Edge-disjoint x, y-paths

Let G be a graph and let $x, y \in V(G)$. We say x, y-paths are ***edge-disjoint*** if no 2 paths share an edge.

Theorem 1.11
Menger's Theorem, Edge Version

Let x, y be distinct vertices of a graph G. Then the minimum size of a cut that disconnects x from y is equal to the maximum size of a set of edge-disjoint x, y-paths.

Def'n 1.21

Line Graph of a Graph

Let G be a graph. Then the ***line graph*** of G, denoted as $\mathrm{L}(G)$, is the graph whose vertex set is $\{v_e : e \in E(G)\}$ and v_ev_f is an edge of $\mathrm{L}(G)$ if and only if e, f have a common endpoint in G.

Corollary 1.11.1
Characterization of k-edge-connectedness

Let G be a graph. The following are equivalent.

(a) G is k-edge-connected.

(b) There exist k edge-disjoint x,y-paths for any distinct $x,y \in V(G)$.

1.5 2-connected Graphs

Proposition 1.12
Cycle Characterizations of 2-connectedness

Let G be a graph with at least 3 vertices. The following are equivalent.

(a) G is 2-connected.

(b) For any distinct $u,v \in V(G)$, there is a cycle C such that $u,v \in V(C)$.

(c) $\delta(G) \geq 1$ and for any distinct $e,f \in E(G)$, there is a cycle C such that $e,f \in E(C)$.

Proof.

- (b) $\implies$ (a) This follows immediately from Corollary 1.9.1.
- (a) $\implies$ (c) Assume G is 2-connected. Since G is connected with at least 3 vertices, no isolated vertex exists, so $\delta(G) \geq 1$. Moreover, let $e,f \in E(G)$ be distinct and let $u,v,x,y \in V(G)$ be such that $e = uv, f = xy$. Let $A = \{u,v\}, B = \{x,y\}$. Since G is 2-connected, any A,B-separating set has size at least 2. Hence by the set version of Menger's theorem (Theorem 1.10), there exist 2 disjoint A,B-paths, P_1,P_2. In particular, both endpoints of e,f are on distinct paths. So
$$C = P_1 + f + P_2 + e$$
is a cycle which has e,f.
- (c) $\implies$ (b) Let $u,v \in V(G)$ be distinct. Since $\delta(G) \geq 1$, both u,v are incident to at least one edge.
  - *Claim 1. There exist distinct edges e,f that are incident to u,v, respectively.*

    Proof. For the sake of contradiction, suppose $e \in E(G)$ is the only edge incident to u,v. Then $e = uv$. But this means
    $$H = (\{u,v\}, \{uv\})$$
    is a component of G, since no edge other than $e = uv$ is incident to u,v. Since G has at least 3 vertices, there is $w \in V(G) \setminus \{v,u\}$. But then there is $f \in E(G)$ that is incident to w, and $f \neq e$. Since H is a component, it follows that there is no cycle of G that has e,f, contradicting (c). ◁

  By Claim 1, choose distinct e,f that are incident to u,v, respectively. Then by (c), there exists a cycle C which has e,f, which means C also has u,v. ∎

Def'n 1.22

Ear Decomposition of a Graph

Let F be a subgraph of G.

(a) An ***ear*** of F in G is a nontrivial path in G where only the two endpoints of the path are in F.

(b) An ***ear decomposition*** of G is a sequence of graphs $(G_0)_{i=0}^{k}$ where

(i) G_0 is a cycle in G; *starting cycle*

(ii) for each $r \in \{0,\ldots,k-1\}$,
$$G_{i+1} = G_i + P_i$$
where P_i is an ear of G_i; and *intermediate cycles*

(iii) $G_k = G$.

Proposition 1.13
Characterization of 2-connectedness by Ear Decompositions

Let G be a graph. The following are equivalent.

(a) G is 2-connected.

(b) G has an ear decomposition.

Proof.

- ($\Longrightarrow$) Let G be 2-connected and let G_0 be any cycle in G. For $i \in \mathbb{N} \cup \{0\}$, we inductively construct G_{i+1} from G_i as follows.

  (a) If $G_i = G$, then we are done.

  (b) Otherwise, there exists an edge $e \in E(G) \setminus E(G_i)$, say $e = uv$ for some $u, v \in V(G)$. Then by the set version of Menger's theorem, there exist 2 disjoint $\{u,v\}, G_i$-path in G, say Q_1, Q_2. Then observe that
  $$P_i = Q_1 + uv + Q_2$$
  is an ear of G_i. We then set $G_{i+1} = G_i + P_i$.

  Note that the above process terminates since we add at least one edge of G each time. At termination, say after k step, $(G_i)_{i=1}^k$ is an ear decomposition of G.

- ($\Longleftarrow$) Suppose $(G_i)_{i=0}^k$ is an ear decomposition of G. We proceed inductively to prove each G_i is 2-connected, thereby proving that G is also 2-connected. Since G_0 is a cycle, G_0 is 2-connected. Now suppose that G_i is 2-connected, where $i \in \mathbb{N} \cup \{0\}$ and let P_i be an ear of G_i such that
  $$G_{i+1} = G_i + P_i.$$
  We claim that every distinct vertices x, y in G_{i+1} are in a cycle, so that we can use Proposition 1.12. Suppose that the two endpoins of P_i are u, v and let $x, y \in V(G_{i+1})$. We have 3 cases.

  · *Case 1. Suppose $x, y \in V(G_i)$.* Then since G_i is 2-connected, there is a cycle C in G_i, hence in G_{i+1}, which has x, y.

  · *Case 2. Suppose $x, y \in V(P_i)$.* Since G_i is connected, there exists a u,v-path Q in G_i. It follows that $P_i + Q$ is a cycle which has x, y.

  · *Case 3. Suppose exactly one of x, y is in G_i.* Without loss of generality, say $x \in V(G_i), y \in V(P_i)$. Since G_i is 2-connected, there exist 2-fan from x to $\{u,v\}$ by the fan lemma. Then 2-fan combined with P_i is a cycle which has x, y.

  Hence every 2 distinct vertices in G_{i+1} are in a cycle, and by Proposition 1.12 G_{i+1} is 2-connected. Thus G_k is 2-connected, as required. ∎

Corollary 1.13.1

Let G be a graph.

(a) If G is 2-connected, then any cycle in G can be the starting cycle in the ear decomposition.

(b) If G admits an ear decomposition, then each intermediate graph in the ear deecomposition is 2-connected.

Proposition 1.14

Let G be a graph. If G admits an ear decomposition $(G_0)_{i=0}^{k}$, then $k = |E(G)| - |V(G)|$.

1.6 Blocks

Def'n 1.23 **Block** of a Graph

Let G be a graph. A ***block*** of G is a maximal connected subgraph with no cut-vertex.

(1.9)

(a) A connected graph with no cut-vertex is not necessarily 2-connected. We might have an isolated vertex, or a single edge (which must be a cut-edge). So a block is K_1, K_2, or 2-connected.

(b) We shall see results such as *this property holds for G if and only if it holds for every block of G*. In other words, we are breaking G into manageable pieces when doing such proofs.

Proposition 1.15

Let G be a graph. Then every block of G is an induced subgraph of G.

Proof. Observe that, given any connected subgraph of G, adding an edge to the subgraph does not create a new cut-vertex. This means, given any connected subgraph H of G without any cut-vertex, if there is $e \in E(G)$ such that $e = uv$ for some vertices u, v of H, then $H + e$ is connected without any cut-vertex. Hence by maximality, any block has every edge joining its vertices, so is induced by its vertex set in G. ∎

Proposition 1.16

Let G be a graph. If C is a cycle in G and B is the block containing C, then $V(C) \subseteq V(B)$.

Proof. Since C is a cycle, it does not have a cut-vertex. By maximality of B, the vertices of C are in B. ∎

Proposition 1.17

Let G be a graph and let B_1, B_2 be distinct blocks of G. Then B_1, B_2 share at most 1 vertex.

Proof. Suppose B_1, B_2 share at least 2 vertices, for the sake of contradiction. We may assume that there is at least 1 vertex in B_1 and 1 vertex in B_2 that are not shared by B_1, B_2.

- *Claim 1. $B_1 \cup B_2$ does not have a cut-vertex.*

  Proof. Let x be a vertex in $B_1 \cup B_2$. Since B_1, B_2 share at least 2 vertices, there is a vertex y in $B_1 \cup B_2$ that is not x. Since B_1, B_2 have no cut-vertices, $B_1 - x, B_2 - x$ are connected. Since y is in both blocks, there is a path from y to each vertex in $B_1 - x$ and to each vertex in $B_2 - x$. So $(B_1 \cup B_2) - x$ is connected, hence $B_1 \cup B_2$ has no cut-vertex. ◁

But by Claim 1, $B_1 \cup B_2$ is a block of G, which contradicts the maximality of B_1, B_2. ∎

Proposition 1.18

Let G be a graph. Every vertex of G that is in 2 or more blocks of G is a cut-vertex.

Proposition 1.19

Let G be a graph and let B be a block. If $v, u \in V(B)$, then every u, v-path is contained B.

Proposition 1.20

Let G be a graph. Then for any $e \in E(G)$, there exists unique block B of G that has e.

Corollary 1.20.1

Let G be a graph and let $B_1, \ldots, B_k$ be the distinct blocks of G. Then

$$\{E(B_i)\}_{i=1}^{k}$$

is a partition of $E(G)$.

Block Graph of a Graph

Def'n 1.24 Let G be a graph. The ***block graph*** of G is a graph with

- the vertex set $\mathcal{B} \cup C$; and
- the edge set $\{Bv : B \in \mathcal{B}, v \in V(G)\,[v \in V(B), v \text{ is a cut-vertex}]\}$;

where $\mathcal{B}$ is the collection of blocks of G and C is the set of the cut-vetices of G.

(1.10) Note that the block graph of G is bipartite with a bipartition $\{\mathcal{B}, C\}$, where

- $\mathcal{B}$ is the set of the blocks of G; and
- C is the set of the cut-vertices of G.

Proposition 1.21

Let G be a graph. Then the block graph of G is a forest.

1.7 Strong Orientation

Orientation of a Graph

Def'n 1.25 Let G be a graph.

(a) An ***orientation*** of G is an assignment a direction to each edge uv, either $u \to v$ or $v \to u$.

(b) For any vertices v, u in G, a u, v-path P is directed if $u, v_1, \ldots, v_k, v$ are the vertices of P and

$$u \to v_1, v_1 \to v_2, \ldots, v_k \to v$$

are the edges of P.

(c) An orientation of G is called ***strong*** if every distinct vertices u, v of G admits a directed u, v-path.

Proposition 1.22

If G is 2-connected, then G has a strong orientation.

Proof. Let $(G_i)_{i=0}^k$ be an ear decomposition of G. We proceed inductively to construct an orientation for each G_i, and prove that they are strong orientations. Since G_0 is a cycle, with vertices $v_1, \ldots, v_l$ and edges $v_1v_2, \ldots, v_{l-1}v_l, v_lv_1$, we can orient G_0 with

$$v_1 \to v_2, v_2 \to v_3, \ldots, v_l \to v_1.$$

The above results in a strong orientation. Now suppose G_i has a strong orientation, where $i \geq 0$. Let P_i be an ear of G_i, with vertices $w_1, \ldots, w_m$ and edges $w_1w_2, \ldots, w_{m-1}w_m$, that we add to G_i to obtain G_{i+1}. We keep the same orientation for G_i, and orient P_i as

$$w_1 \to w_2, w_2 \to w_3, \ldots, w_m \to w_1.$$

Let $x, y \in V(G_{i+1})$, where we desire to show there exists a directed x,y-path in the orientation of G_{i+1}. We break down into 5 cases.

- *Case 1. Suppose $x, y \in V(G_i)$.* Then by the inductive hypothesis, there exists a directed x,y-path in G_i so in G_{i+1}.
- *Case 2. Suppose $x \in V(G_i), y \in V(P_i)$.* Then by assumption there is a directed x_1,w_1-path in G_i, followed by a directed w_1,y-path in P_i.
- *Case 3. Suppose $x \in V(P_i), y \in V(G_i)$.* Then there is a directed x,w_m-path in P_i and directed w_m,y-path in G_i, combining which gives a directed x,y-path.
- *Case 4. Suppose $x, y \in V(P_i)$, with $x = w_\alpha, y = w_\beta$ for some $\alpha < \beta$.* Then there is a directed x,y-path in P_i.
- *Case 5. Suppose $x, y \in V(P_i)$, with $x = w_\alpha, y = w_\beta$ for some $\alpha > \beta$.* Then there is a directed x,w_m-path in P_i, directed w_m,w_1-path in G_i and directed w_1,y-path in P_i. ∎

Theorem 1.23

Let G be connected. The following are equivalent.

(a) G has a strong orientation.

(b) Every block of G has a strong orientation.

Proof.

- (a) $\Longrightarrow$ (b) Consider a strong orientation for G and let B be a block of G. Orient B by restricting the orientation of G onto B. To show that this is a strong orientation on B, observe that there is a directed x,y-path in G, and this path is in B by Proposition 1.19. Hence B admits a strong orientation.
- (b) $\Longrightarrow$ (a) Suppose each block has a strong orientation and keep the same orientation for G. Let $x, y \in V(G)$. If x, y are in the same block, then there exists a directed x,y-path in the block. Suppose x, y are in different blocks. In the block graph of G, consider the path between the blocks B_1, B_k, say with vertices
$$B_1v_1, \ldots, v_{k-1}, B_k,$$
where $x \in V(B_1), y \in V(B_k)$. Since each block is strongly oriented, we can find directed paths from x to v_1 in B_1, from v_1 to v_2 in B_2, …, from v_{k-2} to v_{k-1} in B_{k-1}, and from v_{k-1} to y in B_k. Combining these paths gives a directed x,y-path in G. ∎

Corollary 1.23.1

Let G be a graph. The following are equivalent.

(a) G has a strong orientation.

(b) G is 2-edge-connected.

1.8 3-connected Graphs

(1.11) We summarize 2 characterizations of 3-connected graphs without proof. These characterizations depend on contractions of edges.

Proposition 1.24

Let G be a 3-connected graph with at least 5 vertices. Then G has an edge e such that G/e is 3-connected.

Corollary 1.24.1
Tutte's Characterization of 3-connectedness

Let G be a graph. The following are equivalent.

(a) G is 3-connected.

(b) There are graphs $G_0, \dots, G_k$ such that

(i) $G_0 = K_4, G_k = G$; and

(ii) for each $\in \{1, \dots, k\}$, there exists an edge xy in G_i such that $\deg_{G_i}(x) \geq 3, \deg_{G_i}(y) \geq 3$ and $G_i = G_{i+1}/xy$.

Def'n 1.26

Wheel

A ***wheel*** is the graph that consists of a cycle with another vertex that is joined to all vertices on the cycle. We write W_n to denote the wheel whose cycle is of length $n \in \mathbb{N}$.

Theorem 1.25
Tutte's Wheel Theorem

Let G be a graph. The following are equivalent.

(a) G is 3-connected.

(b) There are 3-connected $G_0, \dots, G_k$ such that

(i) G_0 is a wheel, $G_k = G$; and

(ii) for every $i \in \{1, \dots, k\}$, there is an edge e in G_i such that $G_{i-1} = G_i - e$ or $G_{i-1} = G_i/e$.

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2.
Introductory Algebraic Graph Theory

2.1 Edge Spaces

Def'n 2.1 **Edge Space** of a Graph

Let G be a graph. We define the ***edge space*** of G, denoted as $\varepsilon(G)$, as

$$\varepsilon(G) = \mathbb{Z}_2^{E(G)}.$$

(2.1)

(a) Note that, if we index the vectors in order $E(G) = \{e_i\}_{i=1}^m$, then the vector $v \in \varepsilon(G)$ that corresponds to $\{e_{i_1}, \ldots, e_{i_k}\}$ would be such that

$$v_i = \begin{cases} 1 & \text{if } i = i_j \text{ for some } j \in \{1, \ldots, k\} \\ 0 & \text{otherwise} \end{cases}$$

for all $i \in \{1, \ldots, m\}$.

(b) We can use subsets of $E(G)$ as vectors. In fact, we shall use this representation most of the time.

(c) Addition is done over $\mathbb{Z}$, so

$$\begin{aligned} 0+0&=0 \\ 1+0&=1 \\ 0+1&=1 \\ 1+1&=0. \end{aligned}$$

This means, given $E_1, E_2 \subseteq E(G)$, the sum $E_1 + E_2$ represents the *symmetric difference* $E_1 \triangle E_2$.

(d) We do not have to worry about scalar multiplication, as we are working over $\mathbb{Z}_2 = \{0,1\}$. This in particular means it suffices for a subset of $\varepsilon(G)$ to be closed under addition to be a *subspace* of $\varepsilon(G)$.

(e) Suppose that $E_1, \ldots, E_k \in \varepsilon(G)$ are given. Then observe that

$$E_1, \ldots, E_k \text{ are linearly dependent} \iff \text{there are indices } i_1, \ldots, i_l \text{ such that } E_{i_1} \triangle \cdots \triangle E_{i_l} = \varnothing.$$

That is, linear dependence corresponds to an empty symmetric difference.

(f) Observe that $\{\{e_1\}, \ldots, \{e_m\}\}$ is a basis for $\varepsilon(G)$. This means $\dim(\varepsilon(G)) = |E(G)|$.

Also recall the following result from linear algebra.

Proposition 2.1

Let V be a vector space and let $W \subseteq V$ be a subspace of dimension $k \in \mathbb{N}$. Then for every $X \subseteq W$, X is a basis for W if two of the following conditions hold.

(a) X is linearly independent.

(b) $\operatorname{span}(X) = W$.

(c) $|X| = k$.

Moreover, if X is a basis for W, then all of the above hold.

Def'n 2.2 **Cut Space** of a Graph

Let G be a graph. The ***cut space*** of G, denoted as $\mathcal{C}^*(G)$, is the set of all cuts in G.

Proposition 2.2

Let G be a graph. Then $\mathcal{C}^*(G)$ is a subspace of $\varepsilon(G)$.

Proof. This is by Lemma 1.6.1. That is, the symmetric difference of any two cuts is a cut, so $\mathcal{C}^*(G)$ is closed under addition. ∎

Proposition 2.3

The set of all bonds in G spans $\mathcal{C}^*(G)$.

Proof. This follows from the fact that every cut is a disjoint union of bonds (Proposition 1.6), since the symmetric difference of two disjoint sets is the union of the sets. ∎

Def'n 2.3 **Fundamental Cut** of a Connected Graph

Let G be a connected graph and let T be a spanning tree of G. Given any edge e of T, we say the cut $\delta_G(S)$ induced by the set of vertices S of one of the components of $T-e$ is a ***fundamental cut*** with respect to T, denoted as D_e.

Proposition 2.4

Let G be connected and let T be a spanning tree. Then the set of all fundamental cuts with respect to T is a basis of $\mathcal{C}^*(G)$.

Proof. Note that, given any $e,f \in E(T)$,

$$e \in D_f \iff e = f.$$

This means the set of all fundamental cuts is linearly independent. Moreover, to show the spanning part, let F be a cut, where we show that F can be expressed as a linear combination of fundamental cuts. Consider the sum

$$M = F + \sum\nolimits_{e \in F \cap E(T)} D_e. \qquad [2.1]$$

The claim is that $M = \varnothing$. First note that $M \in \mathcal{C}^*(G)$ as a sum of cuts. Let $e \in E(T)$.

- *Case 1. Suppose $e \in F$.* Then observe that e precisely appears in F and in D_e in [2.1].
- *Case 2. Suppose $e \notin F$.* Then e does not appear in [2.1].

In either case, $e \notin M$. This means M is a cut in G that contains no edges from a spanning tree T, which means $M = \varnothing$. Thus

$$M + \sum\nolimits_{e \in F \cap E(T)} D_e = \varnothing,$$

which means

$$F = \sum\nolimits_{e \in F \cap E(T)} D_e.$$ ∎

Corollary 2.4.1

Let G be a graph with k components. Then

$$\dim\left(\mathcal{C}^*(G)\right) = |V(G)| - k.$$

Def'n 2.4

Cycle Space (Flow Space) of a Graph

Let G be a graph. The ***cycle space*** (or ***flow space***) of G, denoted as $\mathcal{C}(G)$, is the subspace of $\varepsilon(G)$ spanned by the (edge sets of the) cycles of G.

Def'n 2.5

Even Edge Set

Let G be a graph.

(a) Given $F \subseteq E(G)$, we say F is ***even*** if $\deg_F(v)$ is even for all $v \in V$.

(b) We say G is ***even*** if its edge set $E(G)$ is even.

Proposition 2.5

Let G be a graph and let $E_1, E_2 \subseteq E(G)$. If E_1, E_2 are even, then $E_1 \triangle E_2$ is even.

Proof. Let $v \in V(G)$ and let D_1, D_2 be the set of edges in E_1, E_2 incident with v. Edges in $D_1 \cap D_2$ do not appear in $E_1 \triangle E_2$. This means

$$\deg_{E_1 \triangle E_2}(v) = |D_1| + |D_2| - 2|D_1 \cap D_2|.$$

Since $|D_1|, |D_2|, 2|D_1 \cap D_2|$ are even, $\deg_{E_1 \triangle E_2}(v)$ is even. Thus $E_1 \triangle E_2$ is even. ∎

Proposition 2.6
Characterization of Even Space

Let G be a graph and let H be a subgraph of G. The following are equivalent.

(a) $H \in \mathcal{C}(G)$.

(b) H is even.

Proof. Let $F = E(H)$.

- (a) $\implies$ (b) Suppose $H \in \mathcal{C}(G)$. By definition,

$$F = C_1 \triangle C_2 \triangle \cdots C_k$$

  for some cycles $C_1, \ldots, C_k$.[1] Since every cycle is even, each C_i is even. It follows inductively that F is even from Proposition 2.5.

- (b) $\implies$ (a) Suppose F is even. We proceed inductively on $|F|$. If $|F| = 0$, then F is the empty sum of cycles.[2] Otherwise, each nontrivial component of H has minimum degree 2. From MATH 249, we know that there is a cycle C_k contained in H. This means $F \triangle C_k$ is even, with fewer edges than F. By induction, $F \triangle C_k$ is a sum of cycles of G, say

$$F \triangle C_k = C_1 \triangle C_2 \triangle \cdots C_{k-1}.$$

  It follows that

$$F = C_1 \triangle C_2 \triangle \cdots C_k,$$

  as required. ∎

[1] Precisely speaking, we are referring to the edge sets of $C_1, \ldots, C_k$.
[2] We are viewing $\triangle$ as the vector addition in $\varepsilon(G)$.

Def'n 2.6 **Fundamental Cycle** of a Connected Graph

Let G be a connected graph and let T be a spanning tree of G. For any $e \in E(G) \setminus E(T)$, the unique cycle in $T + e$, denoted as C_e, is called a ***fundamental cycle***.

Proposition 2.7

Let G be a connected graph and let T be a spanning tree of G. Then the set of all fundamental cycles with respect to T is a basis for $\mathcal{C}(G)$.

Proof. Each fundamental cycle is in $\mathcal{C}(G)$ and contains a unique edge $e \in E(G) \setminus E(T)$. This means the set of fundamental cycles is linearly independent. To show the spanning part, let $F \in \mathcal{C}(G)$ and let $F' = F \setminus E(T)$. Consider

$$M = F + \sum\nolimits_{e \in F'} C_e. \qquad [2.2]$$

So M is a sum of even subgraphs, which means M is even. Suppose $e \in E(G) \setminus E(T)$.

- *Case 1.* $e \in F$. Then $e \in F'$, so e appears exactly twice in [2.2].
- *Case 2.* $e \notin F$. Then $e \notin F'$, so e does not appear in [2.2].

This means $e \notin M$. This means the only edges in M are in T. But the only even edge set with the edges of a tree is the empty set, so $M = \varnothing$. This means

$$F = \sum\nolimits_{e \in F'} C_e.$$

Thus the set of fundamental cycles is a basis for $\mathcal{C}(G)$. ∎

Corollary 2.7.1

Let G be a graph. If G has k components, then $\dim(\mathcal{C}(G)) = |E(G)| - |V(G)| + k$.

Recall 2.7 **Orthogonal** Vectors

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. We say $v, w \in V$ are ***orthogonal*** if

$$\langle v, w \rangle = 0.$$

(2.2) Let G be a graph, with an edge set $E(G) = \{e_i\}_{i=1}^{k}$. Note that, since $\varepsilon(G)$ is over $\mathbb{Z}_2$, we cannot define an inner product over $\varepsilon(G)$. However, we can still define a *dot-product-like* bilinear form on $\varepsilon(G)$: define $\langle \cdot, \cdot \rangle : \varepsilon(G)^2 \to \mathbb{Z}_2$ by

$$\langle E_1, E_2 \rangle = \sum\nolimits_{i=1}^{k} (E_1)_i (E_2)_i,$$

where

$$(E_1)_i = \begin{cases} 1 & \text{if } e_i \in E_1 \\ 0 & \text{otherwise} \end{cases}$$

for all $i \in \{1, \dots, k\}$ and similarly for $(E_2)_i$. For convenience, we shall write

$$E_1 \cdot E_2$$

to denote $\langle E_1, E_2 \rangle$. Then Recall 2.7 motivates the following definition.

Def'n 2.8 **Orthogonal** Edge Sets

Let G be a graph. We say $E_1, E_2 \subseteq E(G)$ are ***orthogonal*** if $E_1 \cdot E_2 = 0$.

The following result is immediate: E_1, E_2 are orthogonal if and only if $|E_1 \cap E_2|$ is even.

Proposition 2.8

Let G be a graph. Then

$$\mathcal{C}^*(G)^{\perp} = \mathcal{C}(G).$$

Proof. We first show $\mathcal{C}^*(G)^{\perp} \subseteq \mathcal{C}(G)$. Let $F \in \mathcal{C}^*(G)^{\perp}$, where our goal is to prove that F is even. Let $v \in V(G)$. Then $\delta(\{v\}) \in \mathcal{C}^*(G)$. Since $F \in \mathcal{C}^*(G)^{\perp}$, F is orthogonal to v, which means $|F \cap \delta(\{v\})|$ is even. This means $\deg_F(v)$ is even, so F is even. Conversely, suppose $F \in \mathcal{C}(G)$ and let $D = \delta(X)$ be any cut, where $X \subseteq V(G)$ and we desire to show $|F \cap D|$ is even. Then $D \in \mathcal{C}^*(G)$. Consider

$$d = \sum\nolimits_{v \in X} d_F(v). \qquad [2.3]$$

Since F is even, d is even. Edges in F contribute to the sum in [2.3] in two ways.

(a) If both endpoints of an edge are in X, then it contributes 2 to the sum.

(b) If exactly one endpoint is in X, then it contributes 1 to the sum. Such an edge is in D, which means there are $|F \cap D|$ such edges.

But $|F \cap D|$ is even, so it follows that d is even. Thus F, D are orthogonal, as desired. ∎

(2.3) Note that, given a graph G with k, components,

$$\dim(\mathcal{C}^*(g)) + \dim(\mathcal{C}(G)) = (|V(G)| - k) + (|E(G)| - |V(G)| + k) = |E(G)|$$

by Corollary 2.4.1, 2.7.1. This is consistent with the result in linear algebra that $\dim(W) + \dim\left(W^{\perp}\right) = \dim(V)$ for any vector space V (equipped with a bilinear form) and a subspace $W \subseteq V$.

3.
Planar Graphs

3.1 Planar Graphs

Planar Graph

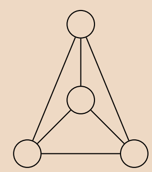
Def'n 3.1 Let G be a graph.

(a) A drawing of G on a plane whose edges only intersect at common endpoints is called a ***planar embedding*** of G.

(b) We say G is ***planar*** if there exists a planar embedding of G.

(EX 3.1) K_4 is a planar graph.

A Planar Embedding of K_4



Plane Graph

Def'n 3.2 A ***plane*** graph is a planar graph with a fixed planar embedding. We call vertices as ***points*** and edges as ***lines***.

(3.2) A rigorous discussion of planarity involves topology, which is out of the scope of this note. But we shall take many topological results for granted. Moreover, we will mainly focus on the combinatorial aspects of planar graphs.

Curve on a Topological Space

Recall 3.3 Let X be a topological space. We say $C \subseteq X$ is a ***curve*** on X if there exists continuous $f : I \to X$ such that $C = \text{image}(f)$, where $I \subseteq \mathbb{R}$ is an interval. We call f a ***parameterization*** of C.

Simple, Closed, Jordan Curve

Recall 3.4 Let C be a curve on a topological space X.

(a) We say C is ***simple*** if it does not intersect itself, except at the endpoints.[a]

(b) We say C is ***closed*** if it is a continuous image of a unit circle.

(c) We say C is ***Jordan*** if C is simple closed curve and $X = \mathbb{R}^2$.

[a] Rigorously speaking, given a curve C with a parameterization f on an interval I, we say C is simple if, for every $x, y \in I$, $f(x) = f(y)$ only if $x = y$ or x, y are the endpoints of I.

Connected Subset

Recall 3.5 Let X be a topological space.

(a) We say X is ***disconnected*** if X is a union of two disjoint nonempty open subsets of X.

(b) We say X is ***connected*** if X is not disconnected.

(c) A maximal connected subset of X is called a ***connected component*** of X.

Theorem 3.1
Jordan Curve Theorem

Let C be a Jordan curve. Then $\mathbb{R}^2 \setminus C$ contains exactly 2 connected components, one bounded and one unbounded.

Recall 3.6

Interior, Exterior of a Jordan Curve

Let C be a Jordan curve.

(a) The ***interior*** of C, denoted as $\operatorname{int}(C)$, is the bounded connected component of $\mathbb{R}^2 \setminus C$.

(b) The ***exterior*** of C, denoted as $\operatorname{ext}(C)$, is the unbounded connected component of $\mathbb{R}^2 \setminus C$.

Corollary 3.1.1

K_5 is not planar.

Proof. Suppose K_5 has a planar embedding for the sake of contradiction. Write $V(K_5) = \{v_1, v_2, v_3, v_4, v_5\}$. Then the lines

$$v_1v_2, v_2v_3, v_3v_1$$

form a Jordan curve C, since $(\{v_1, v_2, v_3\}, \{v_1v_2, v_2v_3, v_3v_1\})$ is a cycle in K_5. Then $v_4 \in \operatorname{int}(C)$ or $v_4 \in \operatorname{ext}(C)$. Without loss of generality, assume $v_4 \in \operatorname{int}(C)$. Then the lines v_4v_1, v_4v_2, v_4v_3 are contained in $\int(C)$, execpt at v_1, v_2, v_3, respectively. Let C_1, C_2, C_3 be the Jordan curves formed by the cycles

$$(\{v_2, v_3, v_4\}, \{v_2v_3, v_3v_4, v_4v_2\}), (\{v_1, v_3, v_4\}, \{v_1v_3, v_3v_4, v_4v_1\}), (\{v_1, v_2, v_4\}, \{v_1v_2, v_2v_4, v_4v_1\}),$$

respectively. Note that, for all $i \in \{1,2,3\}$, $\int(C_i) \subseteq \operatorname{int}(C)$ and $v_i \in \operatorname{ext}(C)$. Since there is a line joining v_5 to v_i for all $i \in \{1,2,3\}$, by the Jordan curve theorem $v_5 \in \operatorname{ext}(C_i)$ for all $i \in \{1,2,3\}$. So $v_5 \in \operatorname{ext}(C)$. But this contradicts the fact that $v_4 \in \int(C)$ and v_4, v_5 are joined by a line. Thus K_5 is not planar. ∎

Corollary 3.1.2

$K_{3,3}$ is not planar.

Def'n 3.7

Subdivision of a Graph

Let G be a graph. A ***subdivision*** of G is obtained from G by replacing each edge with a new path of length at least 1.

Proposition 3.2

Let G be a graph. The following are equivalent.

(a) G is planar.

(b) Every subdivision of G is planar.

Proof-ish. Note that an edge and a path are both represented by a simple curve. ∎

Corollary 3.2.1

Any graph containing a subdivision of K_5 or $K_{3,3}$ is not planar.