# I. Polynomial Rings and Ideals

## 1. Preliminaries

We are going to work with *fields* (e.g. $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F}_q$ where $q$ is a prime power, $\mathbb{Z}/p$ for some prime $p$, $K(t)$ where $K$ is a field).

Def'n 1.1. **Algebraically Closed** Field
We say a field $K$ is ***algebraicaly closed*** if every nonconstant $f \in K[x]$ has a root.

For instance, $x^2 + 1 \in \mathbb{R}[x]$ but it fails to have roots. Algebraically speaking, algebraically closed fields are *nice*. But computationally, we want to avoid them.

**Theorem 1.1.**
Every field $K$ has an *algebraic closure* $\overline{K}$, the unique smallest algebraically closed field containinig $K$.

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Given a polynomial ring $K[x_1, \ldots, x_n]$, we shall often write $K[\vec{x}]$ for convenience. $K[\vec{x}]$ is a vector space of *finite* $k$-linear sums of *monoids*. That is,

$$\vec{x}^{\vec{a}} = x^{a_1} \cdots x^{a_n}$$

and

$$f = \sum_{\vec{a} \in \mathbb{N}^n} \vec{x}^{\vec{a}}.$$

Def'n 1.2. **Degree**, **Support** of a Polynomial
Let $f$ be a polynomial. Then the **degree** of $f$ is

$$\deg(f) = \max\left\{\sum_i a_i : c_{\vec{a}} \neq 0\right\}.$$

The ***support*** of $f$ is

$$\operatorname{supp}(f) = \{\vec{a} : c_{\vec{a}} \neq 0\}.$$

Note that $K[\vec{x}]$ is ***graded***.

Def'n 1.3. **Homogeneous** Polynomial
We say $f$ is **homogeneous** of degree $i$ if it is only supported in degree $i$.

Let $S = K[\vec{x}]$ and write

$$S_i = \text{the subset of } S \text{ of degre } i \text{ homogeneous polynomials.}$$

As a vector space,

$$S = \bigoplus_{i=0}^{\infty} S_i.$$

General infinite dimensional vector spaces are *bad* in many sense (e.g. if $\dim(V) = \infty$, then we cannot tell if the dual of $V$ is nonempty!). But here the story is different, since every $S_i$ is finite dimensional, spanned by monomials of degree $i$. For instance, every $S_i$ has a natural dual, so it is possible to discribe the dual of $S$. Also, an easy combinatorial argument shows that

$$\dim(S_i) = \binom{i+n-1}{n-1}.$$

The argument is that there are $n + i$ dots and we are choosing $n - i$ dots to remove.

Def'n 1.4. **Hilbert Series** of a Graded Ring
A ***Hilbert series*** of a graded ring $S$ is

$$H(S;t) = \sum_{i\in\mathbb{N}} \dim(S_i)\, t^i.$$

For instance,

$$H(K[\vec{x}];t) = \sum_{i\in\mathbb{N}} \binom{i+n-1}{n-1} t^i = \frac{1}{(1-t)^n},$$

where the last equality follows from the negative binomial theorem.

Def'n 1.5. **Ideal** of a Polynomial Ring
An ***ideal*** is a nonempty subset $I \subseteq K[\vec{x}]$ such that

$$f, h \in I, g \in K[\vec{x}] \implies fg + h \in I.$$

Def'n 1.6. Ideal **Generated** by a Subset
Let $F \subseteq K[\vec{x}]$. We define the ideal ***generated*** by $F$, denoted as $\langle F\rangle$ (or $(F)$), to be

$$\langle F\rangle = \bigcap \{I \supseteq F : I \text{ is an ideal}\},$$

or the smallest ideal containing $F$.

**Proposition 1.2.**
For any $F \subseteq K[\vec{x}]$, let $I_F$ be the set of all finite $K[\vec{x}]$-linear combinations of elements of $F$. Then

$$I_F = \langle F\rangle.$$

**Proof Sketch.**

(a) $I_F$ is clearly an ideal containing $F$.

(b) Any ideal containing $F$ contains $I_F$.

**QED**

**Question 1.1.**
Given polynomials $f, g_1, \ldots, g_n \in K[\vec{x}]$, how do we tell if $f \in \langle g_1, \ldots, g_n\rangle$?
For instance, consider $\mathbb{Q}[x,y,z]$, $I = \langle xy - z, yz - x, xz - y\rangle$. Is $z^3 - z \in I$?

**Answer.** This is a hard question, and we are going to develop a systematic method for this.

**QED**

Def'n 1.7. **Homogeneous** Ideal
An ideal $I \subseteq K[\vec{x}]$ is ***homogeneous*** if there exist homogeneous polynomials that generate $I$.[1]

[1] Polynomials need not have the same degree.

**Proposition 1.3.** Characterization of Homogeneous Ideals
Let $S = K[\vec{x}]$ and let $I \subseteq S$ be an ideal. The following are equivalent.

(a) $I$ is homogeneous.

(b) For every $f \in I$, every homogeneous components are in $I$.[1]

(c) $I = \bigoplus_{j=0}^{\infty} I_j$, where $I_j = I \cap S_j$, the homogeneous subspace of $I$ of degree $j$.

[1] e.g. If $f(x, y) = xy - y$, then it must be the case that $xy \in I, y \in I$.

**Proof.** (a) $\implies$ (b) Let $G$ be a set of homogeneous polynomials generators[1] and let $f \in I$. Write $f$ as

$$f = \sum_{i=1}^{m} h_i g_i$$

for some $h_1, \ldots, h_m \in K[\vec{x}], g_1, \ldots, g_m \in G$. Let $d_i = \deg(g_i)$ for all $i$. Then

$$f_j = \sum_{i=1}^{m} h_{i,j} g_i,$$

where $h_{i,j}$ is the homogeneous degree $j - d_i$ part of $h_i$, so $f_j \in I$.

(b) $\implies$ (c) Given any $f \in I$, by assumption, $f_j \in I$, so $f_j \in I \cap S_j = I_j$. This means

$$I = \sum_{j=0}^{\infty} I_j = \bigoplus_{j=0}^{\infty} I_j.$$

(c) $\implies$ (b) This is by the definition of direct sum.

(b) $\implies$ (a) Let $G$ be a set of generators for $I$. Take all the homogeneous components of all polynomials of $G$:

$$F = \left\{ g_j : g \in G, j \in \mathbb{N} \cup \{0\} \right\},$$

which is a set of homogeneous generators for $I$.

[1] i.e. Every $g \in G$ is homogeneous and $\langle G \rangle = I$

QED

Def'n 1.8. **Quotient Ring**

Let $I \subseteq K[\vec{x}]$ be an ideal. Given any $f \in K[\vec{x}]$, the ***residue class*** of $f$ modulo $I$ is the set

$$f + I = \{f + i : i \in I\}.$$

Then there is an equivalence relation $\sim$ on $K[\vec{x}]$ by

$$f \sim g \iff f + I = g + I.$$

The ***quotient ring*** $S/I$ is the ring structure on the set of all residue classes $S/\sim$, with addition

$$(f + I) + (g + I) = (f + g) + I$$

and multiplication

$$(f + I)(g + I) = fg + I.$$

**Proposition 1.4.**

Let $I$ be an ideal of $K[\vec{x}]$ and let $f, g \in K[\vec{x}]$. Then

$$f + I = g + I \iff f - g \in I.$$

**Proof.** ($\Longleftarrow$) If $f-g \in I$, then $f = g+i$ for some $i \in I$, which means

$$f + I = (g+i) + I = g + I.$$

($\Longrightarrow$) If $f+I = g+I$, then for any $i \in I, f+i \in g+I$. In particular, there is $j \in I$ such that $f+j = g$, so that $f-g = j \in I$.

**QED**

Normally, when we give a definition like Def'n 1.8, we should check that $K[\vec{x}]/I$ satisfy all the ring axioms. Instead, we only check the following.

**Proposition 1.5.** Multiplication Is Well-defined on Quotient Rings

Let $I \subseteq K[\vec{x}]$ be an ideal. If $f+I = \hat{f}+I, g+I = \hat{g}+I$, then $fg + I = \hat{f}\hat{g} + I$.

**Proof.** We have $\hat{f} = f+i, \hat{g} = g+j$, so that

$$\hat{f}\hat{g} = (f+i)(g+j) = fg + \underbrace{ig}_{\in I} + \underbrace{fj}_{\in I} + \underbrace{ij}_{\in I} \in fg + I.$$

**QED**

**Proposition 1.6.** Operations on Ideals

Let $I, J \subseteq K[\vec{x}]$ be ideals. Then

(a) $I + J = \{f+g : f \in I, g \in J\}$; and

(b) $I \cap J$

are ideals.

It turns out that $\{fg : f \in I, g \in J\}$ is *not* an ideal. Hence we define the *product ideal* as follows.

Def'n 1.9. **Product Ideal**, **Colon Ideal** of Two Ideals

Let $I, J$ be ideals. We define the ***product ideal*** of $I, J$, denoted as $IJ$, to be

$$IJ = \langle fg : f \in I, g \in J \rangle .$$

We also define the ***colon ideal*** of $I, J$, denoted as $I : J$, as

$$I : J = \{f \in S : \forall j \in J\,[fj \in I]\} = \{f \in S : fJ \subseteq I\} .$$

**Proposition 1.7.**

Let $I, J$ be ideals of $K[\vec{x}]$. Then

$$IJ \subseteq I \cap J.$$

**Proof.** Given any $f \in I, g \in J$, since ideals are closed under multiplication, $fg \in I$. A symmetric argument shows $fg \in J$. This means $fg \in I \cap J$.

Since any $k \in IJ$ can be written as

$$\sum_{i=0}^{n} h_i \overbrace{\underbrace{f_i}_{\in I} \underbrace{g_i}_{\in J}}^{I \cap J},$$

it follows $k \in I \cap J$.

Thus $IJ \subseteq I \cap J$.

QED

**Example 1.2.**

Let $S = \mathbb{Q}[x]$, $I = \left\langle x^3 + 6x^2 + 12x + 8 \right\rangle, J = \left\langle x^2 + x - 2 \right\rangle$. By factoring,

$$I = \left\langle (x+2)^3 \right\rangle, J = \langle (x+2)(x-1) \rangle .$$

This means

(a) $I \cap J = \left\langle (x+2)^3 (x-1) \right\rangle$, the ideal generated by the *lcm* of the generators of $I, J$;

(b) $I + J = \langle (x+2) \rangle$, the ideal generated by the *gcd* of the generators;

(c) $IJ = \left\langle (x+2)^4 (x-1) \right\rangle$; and

(d) $I : J = \left\langle (x+2)^2 \right\rangle$.

Def'n 1.10. **Radical** of an Ideal

Let $I$ be an ideal. The ***radical*** of $I$ is

$$\sqrt{I} = \left\{ f \in S : \exists k \in \mathbb{N} \left[ f^k \in I \right] \right\} .$$

**Example 1.3.** Examples of Radicals of Ideals

Consider $I, J$ from Example 1.2. Then

$$\sqrt{\langle I \rangle} = \langle x + 2 \rangle , \sqrt{J} = \langle (x+2)(x-1) \rangle .$$

**Proposition 1.8.** Every Radical of an Ideal Is an Ideal

Let $I$ be an ideal. Then $\sqrt{I}$ is also an ideal.

**Proof.** Let $f, g \in \sqrt{I}$ and let $p \in S$. We have $f^k \in I$ and $g^l \in I$ for some $k, l \in \mathbb{N}$.

Now

$$(fp)^k = \underbrace{f^k}_{\in I} p^k \in I,$$

so that $fp \in \sqrt{I}$.

Also,

$$(f+g)^{k+l} = \sum_{i=0}^{k+l} \underbrace{\left( \right)}_{\text{some binomial coefficient}} f^i g^{k+l-i} \in I,$$

since given any $i \in \{0, \ldots, k+l\}$, $i \geq l$ so that $f^i \in I$ or $k + l - i \geq k$ so that $g^{k+l-i} \in I$. Thus $f + g \in \sqrt{I}$.

QED

Def'n 1.11. **Radical** Ideal

We say $I$ is an ***radical*** ideal if $I = \sqrt{I}$.

**Proposition 1.9.**

For all ideal $I$, $\sqrt{\sqrt{I}} = \sqrt{I}$.

**Proof.** Clearly $\sqrt{\sqrt{I}} \supseteq \sqrt{I}$.

Let $f \in \sqrt{\sqrt{I}}$. Then $f^k \in \sqrt{I}$ for some $k \in \mathbb{N}$. This implies there is $l \in \mathbb{N}$ such that $f^{kl} = \left(f^k\right)^l \in I$. Hence $f \in \sqrt{I}$.

QED

# II. Algebraic Geometry

Algebraic geometry allows us to turn equations into pictures which gives visual intuitions.

## 1. Introduction

Here is a general setting for algebriac geometry. We fix

$$S = K[\vec{x}]$$

Def'n 2.1. **Vanishing Locus** of a Subset of a Polynomial Ring

For any $F \subseteq S$, we define a ***variety*** $V(F)$ (or $V_K(F)$ when we want to specify the field $K$) by

$$V(F) = \{\vec{p} \in K^n : \forall f \in F\,[f(\vec{p}) = 0]\},$$

called the ***vanishing locus*** of $F$.

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**Example 2.1.**

Consider $K = \mathbb{R}$. Then

$$V\left(x^2 + y^2 - 1\right) = \left\{(p_1, p_2) \in \mathbb{R}^2 : p_1^2 + p_2^2 = 1\right\},$$

which is the set of points on the unit circle.

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**Example 2.2.**

Consider $K = \mathbb{R}$, $S = \mathbb{R}[x, y, z]$. Then by inspection,

$$(0,0,0), (1,1,1) \in V(xy - z, yz - x, xz - y).$$

Note that, when at least one of $x, y, z$ is 0, then the others must be also 0.

If two variables have absolute value larger than 1, say $|x|, |y| > 1$, then since $xy - z = 0$, we see that $z > |x|\,|y|$. But this makes it impossible to satisfy $yz - x = 0$.

Similar problem arises when $|x|, |y| < 1$ or $|x| > 1, |y| < 1$.

So the conclusion is that, other than $(0,0,0), (1,1,1)$, every point $(p_1, p_2, p_3)$ in the variety must have that: $|p_1| = |p_2| = |p_3| = 1$. By checking all 8 possibilities (or 7 if we discard $(1,1,1)$), we see that

$$(-1,-1,1), (-1,1,-1), (1,-1,-1) \in V(xy - z, yz - x, xz - y).$$

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**Example 2.3.**

Note that, when $K = \mathbb{R}$, $S = K[x, y]$,

$$V(1) = \emptyset.$$

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**Example 2.4.** Sadnesses

(a) Consider $K = \mathbb{R}$, $S = K[x, y]$. Then

$$V\left(x^2 + y^2 + 1\right) = \emptyset.$$

Hence, in some sense, *some information is lost.* *sadness 1*

(b) Consider $K = \mathbb{R}$, $S = K[x]$. Then $V(x) = \{0\} = V\left(x^2\right)$. So two different polynomials give the same variety. *sadness 2*

We are going to fix both *sadness* simultaneously.

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**Lemma 2.1.**
Let $I = \langle F \rangle$. Then $V(F) = V(I)$.

**Proof.** Suppose $p \in V(I)$. Then $g(p) = 0$ for all $g \in I$. In particular, $f(p) = 0$ for all $f \in F$, so $V(I) \subseteq V(F)$.
Conversely, suppose $p \in V(F)$. Then $f(p) = 0$ for all $f \in F$. Let $g \in I$. Then

$$g = \sum_{i=0}^{n} h_i f_i$$

for some $f_0, \ldots, f_n \in F, h_0, \ldots, h_n \in S$. So

$$g(p) = \sum_{i=0}^{n} (h_i f_i)(p) = \sum_{i_0}^{n} h_i(p) \underbrace{f_i(p)}_{=0} = 0.$$

Hence $V(F) \subseteq V(I)$.

QED

## 2. Schemes

**Lemma 2.2.**
Let $I$ be an ideal. Then

$$V(I) = V\left(\sqrt{I}\right).$$

**Proof.** Suppose $p \in V\left(\sqrt{I}\right)$. Then $f(p) = 0$ for all $p \in \sqrt{I}$. But $I \subseteq \sqrt{I}$, so $f(p) = 0$ for all $f \in I$. Hence $p \in V(I)$.
Conversely, suppose $p \in V(I)$. Then $f(p) = 0$ for all $f \in I$. Let $g \in \sqrt{I}$. Then $g^k \in I$ for some $k \in \mathbb{N}$, which means $g^k(p) = 0$. Since a field does not have a zero divisor, it follows $g(p) = 0$. This means $p \in V\left(\sqrt{I}\right)$.

QED

To fix this issue that *turning ideals in to variety* looses information, we are going to introduce the notion of *scheme*.

Def'n 2.2. **Radical Ideal** of a Variety
Let $X = V(F)$ be a variety. Then the ***radical ideal*** of $X$ is the set

$$I(X) = \{f \in K[\vec{x}] : \forall x \in X [f(x) = 0]\}.$$

**Lemma 2.3.**
Every radical ideal of a variety is a radical ideal.

**Proof Sketch.** Any $K[\vec{x}]$-linear combination of 0 is again 0 and any (positive integer) power of 0 is also 0.

QED

**Lemma 2.4.**
Given $F \subseteq K[\vec{x}]$,

$$\sqrt{\langle F \rangle} \subseteq I(V(F)).$$

Consequently, $\langle F \rangle \subseteq I(V(F))$.

**Theorem 2.5.** Nullstellensatz
Suppose $K$ is algebraically closed. Then for any $F \subseteq K[\vec{x}]$, $I(V(F)) = \sqrt{\langle F \rangle}$.

Let $F \subseteq K[\vec{x}]$. Then for any field extension $K' \supseteq K$, it makes sense to consider $V_{K'}(F)$. In fact, for any commutative ring $R \supseteq K$, we can still define

$$VparV_R(F) = \{\vec{p} \in R^n : \forall f \in F [f(\vec{p}) = 0]\}.$$

**Example 2.5.** $V_{\mathbb{R}}\left(x^2 + y^2 + 1\right)$

We have

$$V_{\mathbb{R}}\left(x^2 + y^2 + 1\right) = \emptyset = V_{\mathbb{R}}(1)$$

which is bad, but

$$V_{\mathbb{C}}\left(x^2 + y^2 + 1\right) = \text{a conic section}$$

and

$$V_{\mathbb{C}}(1) = \emptyset.$$

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**Example 2.6.**

We have

$$V_{\mathbb{R}}(x) = \{0\} = V_{\mathbb{R}}\left(x^2\right).$$

For this time, we cannot solve this problem by entering the world of complex numbers, as

$$V_{\mathbb{C}}(x) = \{0\} = V_{\mathbb{C}}\left(x^2\right).$$

But we can use commutative rings instead. Let $R = \mathbb{R}[\varepsilon] / \left\langle \varepsilon^2 \right\rangle = \left\{a + b\varepsilon + \left\langle \varepsilon^2 \right\rangle : a, b \in \mathbb{R}\right\}$.[1] This makes:

$$V_R(x) = \left\{0 + \left\langle \varepsilon^2 \right\rangle\right\} \neq \left\{b\varepsilon + \left\langle \varepsilon^2 \right\rangle\right\} = V_R\left(x^2\right).$$

[1] The intution behind quotienting with $\left\langle \varepsilon^2 \right\rangle$ is that, $\varepsilon$ is too small such that when it is squared, it vanishes. Of course such an element does not exist in $\mathbb{R}$, so we just throw in one.

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Def'n 2.3. **Scheme** of a Set of Polynomials

For every $F \subseteq K[\vec{x}]$, we define the ***scheme*** of $F$, denoted as $V_\infty(F)$, by

$$V_\infty(F) = \{V_R(F) : R \supseteq K \text{ is a ring extension of } K\}.$$

**Theorem 2.6.**

There is a bijection

$$\text{ideals of } K[\vec{x}] \leftrightarrow \text{schemes.}$$

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Let $I, J$ be ideals.

(a) $V_\infty(I + J) = V_\infty(I) \cap V_\infty(J)$.

(b) $V_\infty(I \cap J) = V_\infty(I) \cup V_\infty(J)$.

(c) $V_\infty(IJ) = V_\infty(I \cap J)$ with possibly extra infinitesimal fuzz (i.e. $\varepsilon$ in Example 2.6).

(d) $V_\infty(I : J) = V_\infty(I) \setminus V_\infty(J)$ and then patch the holes.

**Example 2.7.**

Let $I = \left\langle x^2 - y \right\rangle, J = \langle y - x - 2 \rangle$. Then

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Def'n 2.4. **Monomial Ideal**
A ***monomial ideal*** in $K[\vec{x}]$ is an ideal generated by monomials.

Monomial ideals are easy to understand.

**Example 2.8.** Monomials Are Easy
If $u = \vec{x}^{\vec{a}}$, $v = \vec{x}^{\vec{b}}$, then

$$u|v \iff \forall i\,[a_i \leq b_i]\,.$$

Moreover,

$$\gcd(u, v) = \vec{x}^{\min(\vec{a},\vec{b})},$$

and

$$\operatorname{lcm}(u, v) = \vec{x}^{\max(\vec{a},\vec{b})}.$$

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**Proposition 2.7.** Characterization of Monomial Ideals
Let $I \subseteq K[\vec{x}]$ be an ideal. The following are equivalent.

(a) $I$ is a monomial ideal.

(b) For every $f \in I$, every monomial components are in $I$. That is, $\operatorname{supp}(f) \subseteq I$.

**Proof.** (a) $\implies$ (b) Let $M$ be a set of monomial generators for $I$: $I = \langle M \rangle$. Let $f \in I$. Write

$$f = \sum_{m \in M} c_m m$$

for some $\{c_m\}_{m\in M} \subseteq K[\vec{x}]$. Hence

$$\operatorname{supp}(f) \subseteq \bigcup_{m \in M} \operatorname{supp}(c_m m)\,.$$

Suppose $u \in \operatorname{supp}(f)$. Then

$$u \in \operatorname{supp}(c_m m)$$

for some $m \in M$. But each $v \in \operatorname{supp}(c_m m)$ looks like

$$v = wm$$

for some monomial $w$. Hence $u = wm$ for some monomial $w$, so that $u \in \langle m \rangle \subseteq \langle M \rangle = I$.

(b) $\implies$ (a) Let $G$ be a generating set for $I$. Then by letting $M = \bigcup_{g\in G} \operatorname{supp}(g) \subseteq I$, we have $\langle M \rangle = I$. Hence $M$ is a generating set of monomials.

**QED**

**Corollary 2.7.1.**
Let $I$ be a monomial ideal and let $M \subseteq I$ be a set of monomials. Then

$$\langle M \rangle = I \iff \text{for every monomial } v \in I\text{, there exists } m \in M \text{ such that } m|v.$$

**Proof.** ($\implies$) Suppose $\langle M \rangle = I$ and let $v \in I$ be a monomial. Then

$$v = \sum_{m \in M} c_m m$$

for some $\{c_m\}_{m\in M}$, so

$$v \in \bigcup_{m\in M} \operatorname{supp}(c_m m).$$

This means

$$v \in \operatorname{supp}(c_m m)$$

for some $m \in M$. This means $v$ is a multiple of $m$. Thus $m|v$.

($\Longleftarrow$) Suppose that for every monomial $v \in I$, there exists $m \in M$ such that $m|v$. Let $f \in I$. Since $I$ is a monomial ideal, $\operatorname{supp}(f) \subseteq I$. Say

$$\operatorname{supp}(f) = \{v_1, \ldots, v_k\}$$

so that

$$f = \sum_{i=1}^{k} c_i v_i$$

for some $c_1, \ldots, c_k \in K$. By assumption, we have monomials $m_1, \ldots, m_k \in M$ and $w_1, \ldots, w_k \in K[\vec{x}]$ such that $v_i = m_i w_i$. Then

$$f = \sum_{i=1}^{k} c_i m_i w_i$$

so $f \in \langle M \rangle$. Thus $\langle M \rangle = I$.

**QED**

Def'n 2.5. **Hyperplane** of a Vector Space

A ***hyperplane*** is a codimension 1 vecor subspace.

We say a hyperplane is ***coordinate*** if it is spanned by axes.

**Example 2.9.**

Let $S = K[x, y, z, w]$. Then

$$\begin{aligned}
V_\infty(x) &= y, z, w\text{-coordinate hyperplane} = x^\perp\text{-coordinate hyperplane},\\
V_\infty(xy) &= V_\infty(x) \cup V_\infty(y) = x^\perp \cup y^\perp,\\
V_\infty(x) &= x^\perp \text{ (but fuzzy – we have a \textit{very small} thickness)},\\
V_\infty\left(x^2 y\right) &= \left(\text{fuzzy } x^\perp\right) \cup y^\perp,\\
V_\infty\left(x y^3\right) &= x^\perp \cup \left(\text{very fuzzy } y^\perp\right)
\end{aligned}$$

In general,

$$V_\infty(\text{monomial}) = \text{union of fuzzy coordinate hyperplanes}.$$

More precisely, if the monomial is $\vec{x}^{\vec{a}}$, then

$$\begin{aligned}
a_i > 0 &\iff x_i^\perp \text{ appears},\\
a_i &\longleftrightarrow \text{fuzziness of } x_i^\perp.
\end{aligned}$$

**Example 2.10.**

In $K[x, y, z, w]$,

$$V(xy, xz) = V(\langle xy\rangle + \langle xz\rangle) = V(xy) \cap V(xz) = \left(x^\perp \cup y^\perp\right) \cap \left(x^\perp \cup z^\perp\right) = x^\perp \cup \left(y^\perp \cap z^\perp\right),$$

which is the union of $y, z, w$-hyperplane and $x, w$-subspace. In general,

$$V(\text{monomial ideal}) = \text{fuzzy union of coordinate subspaces}.$$

Consider the partial order of $\mathbb{N}^n$ such that

$$(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n) \iff a_1 \leq b_1, \ldots, a_n \leq b_n.$$

Take $S \subseteq \mathbb{N}^n$. Say $a \in S$ is ***minimal*** if for every $b \in S$, $a \leq b$.

**Theorem 2.8.** Dickson's Lemma

Let $S \subseteq \mathbb{N}^n$. Then $S$ has finitely many minimal elements.

**Proof.** We proceed inductively on $n$.

If $n = 1$, then $S$ has at most 1 minimal element.

Suppose $n > 1$. Let $T = \left\{(a_1, \ldots, a_{n-1}) \in \mathbb{N}^{n-1} : (a_1, \ldots, a_n) \in S\right\}$. By inductive hypothesis, $G$ has a finite number of minimal elements $a^1, \ldots, a^k$. For each $a^i$, choose $a_n^i \in \mathbb{N}$ such that $\left(a^i, a_n^i\right) \in S$.

Let $b = \max_i \left\{a_n^i\right\}$. For each $c \in \{0, \ldots, b\}$, let

$$T^c = \left\{(a_1, \ldots, a_{n-1}) \in \mathbb{N}^{n-1} : (a_1, \ldots, a_{n-1}, c) \in S\right\}.$$

By inductive hypothesis, each $T^c$ has a finite set $T^c_{\min}$ of minimum elements. Let

$$\overline{T_{\min}} = \left\{\left(a^i, a_n^i\right) \in \mathbb{N}^n : i \in \{1, \ldots, k\}\right\} \subseteq S$$

and let

$$\overline{T^c_{\min}} = \{(a, c) \in \mathbb{N}^n : a \in T^c_{\min}\} \subseteq S.$$

Claim 1. *Let $S_{\min}$ be the set of minimal elelments of $S$. Then $S_{\min} \subseteq \overline{T_{\min}} \cup \bigcup_{c=0}^{b-1} \overline{T^c_{\min}}$.*

Consider $u = (u_1, \ldots, u_n) \in S$. If $u_n \geq b$, then $u \geq t$ for some $t \in \overline{T_{\min}}$. If $u_n < b$, then $u \geq t$ for some $t \in \overline{T^{u_n}_{\min}}$.

*(Done with Claim 1)*

Since the union $\overline{T_{\min}} \cup \bigcup_{c=0}^{b-1} \overline{T^c_{\min}}$ is finite, the result follows.

**QED**

**Corollary 2.8.1.**

Let $I$ be a monomial ideal and let $M$ be a generating set ofmonomials. Then there is a finite $M' \subseteq M$ such that $I = \langle M' \rangle$.

**Proof Sketch.** We consider monomials as elements of $\mathbb{N}^n$ and use Dickson's lemma.

**QED**

Def'n 2.6. **Minimal** Set of Monomials

Let $M$ be a set of monomials. We say $M$ is ***minimal*** if for every proper subset $N \subset M$, $\langle N \rangle \subset \langle M \rangle$.

**Proposition 2.9.**

Every monomial ideal $I$ has a unique minimal set of monomial generators.

**Proof.** By Dickson's lemma, minimal monomial generating sets exist and are finite. Suppose we have minimal sets of monomials generators $U = \{u_i\}_{i=1}^r$, $V = \left\{v_j\right\}_{j=1}^s$. Then

$$\langle u_i \rangle_{i=1}^r = \left\langle v_j \right\rangle_{j=1}^s.$$

For each $i$, there is $j$ such that $v_j | u_i$. But $u_k | v_j$ for some $k$, so that $u_k | u_i$. Since the set $\{u_i\}_{i=1}^r$ is minimal, $u_k = u_i = v_j$. Continuing this argument, $U \subseteq V$.

By symmetry, $V \subseteq U$ as well, which concludes the proof.

**QED**

Def'n 2.7. **Canonical Generating Set** of a Monomial Ideal

Let $I$ be a monomial ideal. The ***canonical generating set*** of $I$, denoted as $G(I)$, is the unique minimal set of monomial generators in Proposition 2.9.

**Proposition 2.10.**

Every ascending chain $(I_n)_{n=1}^\infty$ of monomial ideals stabilizes. That is, there is $N \in \mathbb{N}$ such that $I_{n+1} = I_n$ for all $n > N$.

**Proof.** Let
$$G = \bigcup_{n=1}^{\infty} G(I_n).$$
By Dickson's lemma, $G$ has a finite minimal set $G'$. Since $G'$ is finite, $G' \subseteq \bigcup_{n=1}^{\infty} G(I_n)$ for some $N \in \mathbb{N}$. Let $m > N$ and let $u \in I_m$ be a monomial. Then
$$\begin{aligned} u \text{ is divisible by an element of } G(I_m) &\implies u \text{ is divisible by an element of } G \\ &\implies u \text{ is divisible by an element of } G' \\ &\implies u \text{ is divisible by an element of } \bigcup_{n=1}^{N} G(I_n). \end{aligned}$$
Thus $u \in I_N$, so $I_m \subseteq I_N$, implying $I_m = I_N$.

**QED**

## 4. Operations on Monomial Ideals

Let $I, J$ be monomial ideals. The sum $I + J$ is a monomial ideal with
$$G(I+J) \subseteq G(I) \cup G(J).$$
The product $IJ$ is also monomial, with
$$G(IJ) \subseteq \{uv : u \in G(I), v \in G(J)\}.$$

**Example 2.11.**
Let $I = (xy, yz^2), J = (x^2y, yz) \subseteq K[x, y, z]$. Then
$$I = y^{\perp} \cup (y\text{-axis with some fuzz in } z\text{-direction})$$
and
$$J = y^{\perp} \cup (y\text{-axis with some fuzz in } x\text{-direction}).$$
It turns out that,
$$I + J = \langle xy, yz^2, x^2y, yz \rangle.$$
This means
$$I + J = y^{\perp} \cup (y\text{-axis}),$$
since the *fuzziness in $I, J$ are in orthogonal direction*. Thus
$$I + J = \langle xy, yz \rangle.$$

**Example 2.12.**
Consider $I, J$ from Example 2.11. Then
$$IJ = \langle x^3y^2, xy^2z, x^2y^2z^2, y^2z^3 \rangle.$$
Since $x^2y^2z^2$ is a multiple of $xy^2z$, we can get rid of it:
$$IJ = \langle x^3y^2, xy^2z, y^2z^3 \rangle.$$
This means we can make all three generators vanish by
(a) making $y^2$ vanish (i.e. $y^{\perp}$ with some fuzz); or
(b) making $x, z, x^3, z^3$ vanish (i.e. $y$-axis with lots of fuzz).
Hence
$$IJ = \left(y^{\perp} \text{ with some fuzz}\right) \cup (y\text{-axis with lots of fuzz}).$$

**Proposition 2.11.**

Let $I, J$ be monomial ideals. Then $I \cap J$ is monomial and

$$I \cap J = \langle \operatorname{lcm}(u, v) : u \in G(I), v \in G(J) \rangle .$$

**Proof.** Let $f \in I \cap J$. Then $f \in I$ and $f \in J$, where $I, J$ are monomial, so $\operatorname{supp}(f) \subseteq I$, $\operatorname{supp}(f) \subseteq J$. Hence $\operatorname{supp}(f) \subseteq I \cap J$. Thus $I \cap J$ is monomial.

Let $w \in I \cap J$ be monomial. Then there is $u \in G(I), v \in G(J)$ which divide $w$. Thus $\operatorname{lcm}(u, v) | w$, so

$$w \in \langle \operatorname{lcm}(u, v) : u \in G(I), v \in G(J) \rangle .$$

The converse containment is clear.

**QED**

**Example 2.13.**

Consider $I, J$ from Example 2.11. Then

$$I \cap J = \left\langle x^2y, xyz, x^2yz^2, yz^2 \right\rangle$$

and we can get rid of $x^2yz^2$ since it is a multiple of every other generator. Hence

$$I \cap J = \left\langle x^2y, xyz, yz^2 \right\rangle .$$

To make every generator vanish, we can make

(a) $y$ vanish ($y^{\perp}$); or

(b) $x, z, xz, x^2, z^2$ vanish ($y$-axis with fuzz in any other direction!).

**Proposition 2.12.**

Let $I, J$ be monomial ideals. Then $I : J$ is monomial with

$$I : J = \bigcap_{v \in G(J)} I : \langle v \rangle$$

and

$$I : \langle v \rangle = \left\langle \frac{u}{\gcd(u, v)} : u \in G(I) \right\rangle .$$

**Proof.** The first equality

$$I : J = \bigcap_{v \in G(J)} I : \langle v \rangle$$

is in Assignment 2.

Claim 1. *$I : \langle v \rangle$ is monomial for any $v \in G(J)$.*

Suppose $f \in I : \langle v \rangle$. Then $fv \in I$, so $\operatorname{supp}(fv) \subseteq I$. This means $\operatorname{supp}(f) \subseteq I : \langle v \rangle$. Thus $I : \langle v \rangle$ is monomial.

*(End of Claim 1)*

*(End of Claim 1)*

Since finite intersection of monomial ideals is monomial by Proposition 2.11, it follows that $I : J$ is monomial.

Let $u \in I : \langle v \rangle$ be a monomial. Then $uv \in I$ and so $i | uv$ for some $i \in G(I)$. By dividing $i$ by $\gcd(i, v)$, $\frac{i}{\gcd(i,v)} | u$, so $u \in \left\langle \frac{i}{\gcd(i,v)} : i \in G(I) \right\rangle$. Hence

$$I : \langle v \rangle \subseteq \left\langle \frac{u}{\gcd(u, v)} : u \in G(I) \right\rangle .$$

The reverse containment is clear.

**QED**

**Example 2.14.**

Consider $I, J$ from Example 2.11. Then

$$I : J = \left(I : \left\langle x^2 y\right\rangle\right) \cap \left(I : \langle yz\rangle\right) = \left\langle 1, z^2\right\rangle \cap \langle x, z\rangle = \langle x, z\rangle = (y\text{-axis}).$$

---

Def'n 2.8. **Squarefree** Monomial

We say a monomial is ***squarefree*** if it has no exponents greater than 1.

We say a monomial ideal $I$ is ***squarefree*** if $I$ can be generated by squarefree monomials.

We note that in order to check that a monomial ideal $I$ is squarefree, it suffices to check its canonical generating set $G(I)$.

Recall 2.9. **Prime** Ideal

Let $I$ be an ideal. We say $I$ is ***prime*** if $I$ is a proper ideal and for all $f, g \in \mathbb{K}[\vec{x}]$,

$$fg \in I \implies f \in I \text{ or } g \in I.$$

**Proposition 2.13.**

Let $I \subseteq S$ be an ideal. Then

$$I \text{ is prime} \iff S/I \text{ is a domain}.$$

---

**Proposition 2.14.**

Let $I$ be a squarefree monomial ideal. Then $I$ is a finite intersection of monomial prime ideals.

**Proof.** Write $G(I) = \{u_1, \ldots, u_r\}$. We proceed inductively on $r$.

When $r = 1$, then $I = \langle u_1\rangle = \langle x_{1_1} \cdots x_{1_d}\rangle = \bigcap_{j=1}^{d} \left\langle x_{1_j}\right\rangle$, where $u_1 = x_{1_1} \cdots x_{1_d}$. But every principle ideal generated by a single variable is prime, so the result follows.

Now suppose $r > 1$. For all $i$, write

$$u_1 = x_1 \cdots x_d.$$

Now

$$\bigcap_{j=1}^{d} \left\langle x_j, u_2, \ldots, u_r\right\rangle = \langle u_1, \ldots, u_r\rangle$$

by the intersection lemma for polynomial ideals. By induction, $\langle u_2, \ldots, u_r\rangle = \bigcap_{i=1}^{s} P_i$ for some prime ideals $P_1, .., P_s$. So

$$I = \bigcap_{j=1}^{d} \left\langle x_j, u_2, \ldots, u_r\right\rangle = \bigcap_{j=1}^{d} \left(\left\langle x_j\right\rangle + \bigcap_{i=1}^{s} P_i\right) = \bigcap_{j=1}^{d}\bigcap_{i=1}^{s} \left(\left(x_j\right) + P_i\right).$$

But we know that the sum of two monomial prime ideals is monomial prime. Thus $I$ is a finite intersection of monomial prime ideals.

**QED**

**Corollary 2.14.1.**

Let $I$ be a monomial ideal. Then

$$I \text{ is radical} \iff I \text{ is squarefree}.$$

**Proof.** ( $\Longleftarrow$ ) Suppose $I$ is squarefree. Then $I$ can be written as a finite intersection of monomial prime ideals. By Problem 1 on Assignment 3, it follows that $I$ is radical.

( $\Longrightarrow$ ) Suppose $I$ is not squarefree. Then $G(I)$ contains a monomial

$$f = \prod_{i=1}^{r} x_{j_i}^{a_i}.$$

Let

$$m = \max_{1 \le i \le r} a_i > 1.$$

Then

$$\left(x_{j_1} \cdots x_{j_r}\right)^m \in I,$$

so that

$$x_{j_1} \cdots x_{j_r} \in \sqrt{I}.$$

But $x_{j_1} \cdots x_{j_r} \notin I$, since it divides an element $f$ which is one of the minimal generators of $I$. Hence $I$ is not radical.

**QED**

**Theorem 2.15.**

Let $I$ be monomial. Then

$$\sqrt{I} = \left\langle \sqrt{u} : u \in G(I) \right\rangle,$$

where $\sqrt{u}$ is obtained by re-writing every nonzero exponent of $u$ to 1.

**Proof.** Let $J = \langle \sqrt{u} : u \in G(I) \rangle$. Then $G(I) \subseteq J$, which means $I \subseteq J$. Also $J$ is radical, so $\sqrt{I} \subseteq J$.

For each $u \in G(I)$, let $m_u$ be the highest exponent in $u$. Then $(\sqrt{u})^{m_u} \in I$, so $\sqrt{u} \in \sqrt{I}$. Thus $J \subseteq \sqrt{I}$.

**QED**

## 5. Grobner Bases

Algebraically, we are going to *flatly degenerate* ideals to monomials. Geometrically, we are going to flatly degenerate schemes to fuzzy union of coordiate subspaces.

There will be some properties that will be invariant under degeneration. This would be great, as we will be able to conveniently calculate things on monomials. For instance, Hilbert series is invariant. Consequently, *dimension*, *(multi-)degree*, *arithmetic genus* will be also invariant.

Other (non-invariant) properties can only get *worse* (e.g. *Betti numbers*, *Cohen-Macaulay-ness*,*Gorensteinness*, *Castelnuovo-Mumford regularity*, *primality*, *radicalness*). In some cases, we can tell that some didn't get worse.

Recall that we are going to identify monomials with elements in $\mathbb{N}^n$ with partial order

$$(a_1, \ldots, a_n) \le (b_1, \ldots, b_n) \iff \forall i \left[a_i \le b_i\right].$$

We extend this to a total order.

Def'n 2.10. **Monomial Order**

A ***monomial order*** is a total order on $\mathbb{N}^n$ such that

(a) if $a < b$ in $\mathbb{N}^n$, then $a + c < b + c$ for all $c \in \mathbb{N}^n$; and *shifting*

(b) for all $a \in \mathbb{N}^n$, $a \ge (0, \ldots, 0)$.

**Example 2.15.** Lexicographic Order (Lex)

The ***lexicographic order*** (***lex***) given by

$$a < b \iff \text{first nonzero entry of } b - a > 0$$

is a monomial order.

**Example 2.16.** Graded Lexicographic Order (Grlex)

The ***graded lexicographic order*** (***grlex***) given by

$$a < b \iff |a| < |b| \text{ or } (|a| = |b| \text{ and } a < b \text{ using the lexicographic order})$$

is a monomial order.

---

**Example 2.17.** Reverse Lexicographic Order (Grevlex)

The ***reverse lexicographic order*** (***grevlex***) given by

$$a < b \iff |a| < |b| \text{ or } (|a| = |b| \text{ and last nonzero entry of } b - a \text{ is negative})$$

is a monomial order.

---

**Example 2.18.**

Consider the order $x > y > z$ on three variables $x, y, z$ and consider monomials

$$\{x^2, xz^2, y^3\}.$$

According to lex:

$$x^2 > xz^2 > y^3.$$

According to grlex:

$$xz^2 > y^3 > x^2.$$

According to grevlex:

$$y^3 > xz^2 > x^2.$$

---

**Example 2.19.**

Again consider the order $x > y > z$ and consider monomials of degree 2. Then according to grlex:

$$x^2 > xy > xz > y^2 > yz > z^2.$$

According to grevlex:

$$x^2 > xy > y^2 > xz > yz > z^2.$$

Hence we see that it is *impossible* to permute the variables to obtain the same order on grlex and grevlex.

---

**Proposition 2.16.**

Let $<$ be a monomial order on $\mathbb{N}^n$. Then $<$ can be extended to a partial order $\leq$ such that

(a) if $u, v \in \mathbb{N}^n$ are monomials $u_j = u_{i_k} = u_{i_r}$ with $u|v$, then $u \leq v$; and

(b) if there is a decreasing sequence $(u_i)_{i=1}^{\infty} \in \mathbb{N}^n$, then there is $N \in \mathbb{N}$ such that $u_i = u_N$ for all $i \geq N$. *Well-ordering*

**Proof.**

(a) If $u|v$, then $v = uw$ for some $w$. But $1 \leq w$, so $u \leq uw = v$.

(b) Let $M = \{u_i\}_i$. By Dickson's lemma, there are finitely many minimal elements with respect to the partial order, say

$$u_{i_1}, \ldots, u_{i_r},$$

with $i_1 < i_2 < \cdots < i_r$. Let $j > i_r$. Then $u_{i_k}|u_j$ for some $k \in \{1, \ldots, r\}$. Therefore $u_{i_k}|u_j$. But then

$$u_{i_k} \geq u_{i_r} \geq u_j \geq u_{i_k},$$

so that $u_j = u_{i_k} = u_{i_r}$.

**QED**

Def'n 2.11. **Initial Monomial** of a Polynomial
Let $S = K[\vec{x}]$ and fix a monomial order $<$. For $f \in S$, if $f$ is nonzero, we define the ***initial monomial*** (or ***leading monomial***) of $f$, denoted as $\mathrm{in}_<(f)$, as

$$\mathrm{in}_<(f) = <\text{-greatest monomial of } \mathrm{supp}(f).$$

The coefficient $k$ of $\mathrm{in}_<$ is called the ***leading coefficient*** (or ***initial coefficient***), and $k\,\mathrm{in}_<(f)$ is called the ***leading term*** (or ***initial term***).

In case $f = 0$, we set $\mathrm{in}_<(f) = 0$.

Def'n 2.12. **Initial Ideal** of an Ideal
Let $I$ be an ideal. We define the ***initial ideal*** of $I$, denoted as $\mathrm{in}_<(I)$, by

$$\mathrm{in}_<(I) = \langle \mathrm{in}_<(f) : f \in I \rangle .$$

**Example 2.20.**
Let $S = K[x, y, z, w]$ ordered by grevlex with $x > y > z > w$. Let

$$I = \left\langle xy - zw, xz - y^2 \right\rangle .$$

Then the leading terms of the generators are $xy, y^2$, so

$$\left\langle xy, y^2 \right\rangle \subseteq \mathrm{in}(I) .$$

Moreover,

$$y(xy - zw) + x\left(xz - y^2\right) = x^2 z$$

which is not a multiple of $xy, y^2$, so

$$\left\langle xy, y^2, x^2 z \right\rangle \subseteq \mathrm{in}(I) .$$

But how are we supposed to know when we are done?

---

By Dickson's lemma, $\mathrm{in}(I)$ is generated by some finite set of monomials, which must be initial monomials of some elements of $I$. Thus there exist $g_1, \ldots, g_k \in I$ such that

$$\mathrm{in}(I) = \langle \mathrm{in}(g_i) \rangle_{i=1}^k .$$

Def'n 2.13. **Grobner Basis** for an Ideal
Consider the above setting. Such a set $G = \{g_i\}_{i=1}^k$ is called a **Grobner basis** for $I$.

**Theorem 2.17.** Macaulay (1927)
Let $I \subseteq S = K[\vec{x}]$ be an ideal. Then the residue classes of monomials in $S \setminus \mathrm{in}(I)$ form a $K$-basis for $S/I$.

**Proof.** Suppose the classes of the monomials are linearly dependent. Then there is some nonzero linear combination

$$f = \sum_{i=1}^{k} c_i m_i \in I,$$

with $c_1, \ldots, c_k \in K$ and monomials $m_1, \ldots, m_k$ from $S \setminus \mathrm{in}(I)$. But $\mathrm{in}(f) \in \mathrm{in}(I)$, which is a contradiction.

Let $f \in S$ and consider $f + I \in S/I$. We want to write $f + I$ as a $K$-linear combination of monomials in $S \setminus \mathrm{in}(I)$. Consider the set

$$\Omega = \{g \in S : g + I \text{ cannot be written as a } K\text{-linear combination of monomials in } S \setminus \mathrm{in}(I)\} .$$

Suppose $\Omega \neq \emptyset$ for contradiction. Let $g \in \Omega$ with smallest $\text{in}(g)$. Let $c \in K$ be the leading coefficient of $g$. Let $f' = f - c\,\text{in}(f)$. By minimality assumption, $f' \notin \Omega$, so that

$$f' + I = \sum_{i=1}^{k} c_i m_i$$

for some $c_1, \ldots, c_k \in K$ and monomials $m_1, \ldots, m_k \in S \setminus \text{in}(I)$.

Now

$$f + I = f' + c\,\text{in}(f) + I = c\,\text{in}(f) + c_1 m_1 + \cdots + c_k m_k + I,$$

so if $\text{in}(f) \in S \setminus \text{in}(I)$, then $f \notin \Omega$, which is a contradiction.

Hence $\text{in}(f) \in \text{in}(I)$, so there is $h \in I$ such that $f, h$ have the same leading term. Then $\text{in}(f-h) < \text{in}(f)$, so $f - h \notin \Omega$. This means

$$(f-h) + I = d_1 n_1 + \cdots + d_s n_s + I$$

for some $d_1, \ldots, d_s \in k$ and monomials $n_1, \ldots, n_s \in S \setminus \text{in}(I)$. But $h \in I$, so $f - h + I = f + I$, so $f + I$ is written as a $K$-linear combination of residue classes of monomial ideals, a contradiction.

**QED**

**Corollary 2.17.1.**

If $I$ is a homogeneous ideal, then

$$S/I = \bigoplus_{k=0}^{\infty} S_k/I_k$$

is graded and

$$\dim(S/I)_k = \dim(S/\text{in}(I))_k.$$

Recall that for a graded algebra $R = \bigoplus_{n=0}^{\infty} R_n$, the ***Hilbert function*** $H : \mathbb{N} \to \mathbb{N}$ is given by

$$H(n) = \dim(R_n), \qquad \forall n \in \mathbb{N}.$$

The ***Hilbert series*** is

$$H(R,t) = \sum_{n=1}^{\infty} \dim(R_n)\, t^n = \sum_{n=1}^{\infty} H(n)\, t^n.$$

**Corollary 2.17.2.**

If $I$ is a homogeneous ideal, then

$$H(S/I, t) = H(S/\text{in}(I), t).$$

**Theorem 2.18.**

An ideal $I$ has only finitely many initial ideals.

**Proof.** Let

$$F_0 = \{\text{in}_<(I) : < \text{ is a monomial order}\}.$$

Suppose that $F_0$ is infinite, for contradiction. Clearly $I$ is not the zero ideal, so we may fix nonzero $g_1 \in I$. For each monomial $m \in \text{supp}(g_1)$, let

$$F_0^m = \{\text{in}_<(I) \in F_0 : m \in \text{in}_<(I)\}.$$

Then

$$F_0 = \bigcup_{m \in \text{supp}(g_1)} F_0^m,$$

where supp $(g_1)$ is finite. Hence there exists $m_1 \in$ supp $(g_1)$ such that $F_0^{m_1}$ is infinite. Denote $F_1 = F_0^{m_1}$.

Note $F_1$ consists of infinintely many initial ideals, each containing $m_1$, so that $J \neq \langle m_1 \rangle$ for some $J \in F_1$ . By Macaulay's theorem, for any $J \in F_1$, the (residue classes $m + I$ of) monomials $m \in S \setminus J$ form a basis of the quotient ring $S/I$. Hence the monomials of $S \setminus \langle m_1 \rangle$ must be linearly dependent modulo $I$. Hence there is nonzero $g_2 \in I$ with

$$g_2 = \sum_{j=1}^{r} c_j u_j$$

with $c_1, \ldots, c_r \in K$ and $u_1, \ldots, u_r \in S \setminus \langle m_1 \rangle$ are monomials.

Now define

$$F_1^m = \{J \in F_1 : m \in J\}$$

for all $m \in$ supp $(g_2)$. Then we can choose $m_2 \in$ supp $(g_2)$ such that $F_2 = F_1^{m_2}$ is infinite. In particular, $m_2 \notin \langle m_1 \rangle$, so

$$\langle m_1 \rangle \subset \langle m_1, m_2 \rangle .$$

Since $F_2$ is infinite, there exists $J \in F_2$ with $J \neq \langle m_1, m_2 \rangle$, so the monomials of $S \setminus \langle m_1, m_2 \rangle$ are linearly dependent modulo $I$. So find nonzero $g_3 \in I$ which can be written as a $K$-linear combination of monomials in $S \setminus \langle m_1, m_2 \rangle$. Choose $m_3 \in$ supp $(g_3)$ such that $F_3 = \{J \in F_2 : m_3 \in J\}$ is infinite. This means

$$\langle m_1 \rangle \subset \langle m_1, m_2 \rangle \subset \langle m_1, m_2, m_3 \rangle .$$

Continuing this process, we have a strictly increasing chain

$$\langle m_1 \rangle \subset \langle m_1, m_2 \rangle \subset \langle m_1, m_2, m_3 \rangle \subset \cdots .$$

But this contradicts the ascending chain condition of monomial ideals.

Thus $F_0$ is finite, as required.

**QED**

**Theorem 2.19.** Hilbert Basis

Every ideal $I \subseteq K[\vec{x}]$ is finitely generated. Precisely, if $g_1, \ldots, g_m \in K[\vec{x}]$ form a Grobner basis of $I$, then $\langle g_1, \ldots, g_m \rangle = I$.

**Proof.** Let $\{g_1, \ldots, g_m\}$ be a Grobner basis of $I$ and let $f \in I$. We induct on in $(f)$.

Note in $(f) \in$ in $(I)$, so in $(f) =$ in $(g_i)\, w$ for some $i$ and monomial $w$. Let $c$ be the leading coefficient of $f$ and let $d$ be the leading coefficient of $g_i$. Define

$$h = f - \frac{c}{d} w g_i \in I.$$

If $h = 0$, then $f = \frac{c}{d} w g_i \in \langle g_1, \ldots, g_m \rangle$. So suppose $h \neq 0$. Then in $(h) <$ in $(f)$, so by induction, $h \in \langle g_1, \ldots, g_m \rangle$. But this means $f = h + \frac{c}{d} w g_i \in \langle g_1, \ldots, g_m \rangle$, as required.

**QED**

**Corollary 2.19.1.**

Let

$$I_1 \subseteq I_2 \subseteq \cdots$$

be an ascending chain of ideals in $S$. Then there is $N \in \mathbb{N}$ such that $I_k = I_N$ for all $k \geq N$.

**Proof.** Fix a monomial order $<$. Then

$$\text{in}\,(I_1) \subseteq \text{in}\,(I_2) \subseteq \cdots$$

is an ascending chain of monomial ideals, which stabilizes due to Dickson's lemma (Proposition 2.10). That is, there is $N \in \mathbb{N}$ such that in $(I_k) =$ in $(I_N)$ for all $k \geq N$.

But by *cheating lemma*, if $I \subseteq J$ and in $(I) =$ in $(J)$, then $I = J$. Thus $I_1 \subseteq I_2 \subseteq \cdots$ stabilizes.

**QED**

**Example 2.21.** Commuting Matrices Problem -- Still Open!

Let

$$V = \left\{(A, B) \in \left(\mathbb{R}^{n\times n}\right)^2 : AB = BA\right\} \subseteq K^{2n^2}.$$

What is $I = I(V)$?

Obviously, we can write down homogeneous quadratics concerning dot products of rows and columns. But no one knows if these quadratics are enough.

---

## 6. Division Algorithm

**Example 2.22.** Division Algorithm for Univariate Polynomials

Consider the case

$$f = x^3 + 4x^2 + 3x - 7, g = x - 1 \in K[x].$$

Then by doing long division, we obtain

$$f = g\left(x^2 + 5x + 8\right) + 1.$$

---

In multivariate case, the leading term depends on the monomial order that we use.

**Example 2.23.** Division Algorithm for Multivariate Polynomials

Let

$$f = xy^2 + 1, g_1 = xy + 1, g_2 = y + 1 \in K[x, y].$$

We want to write $f$ in terms of

$$f = q_1 g_1 + q_2 g_2 + r,$$

where $r$ is a *remainder*, which should be *small* (but what do we even mean by saying small?).

We are going to divide $f$ by $g_1$ first and then divide the remainder of $f/g_1$ by $g_2$.

In this case, the leading terms are clear, $xy^2$ for $f$, $xy$ for $g_1$, and $y$ for $g_2$. Note that the leading term $xy$ for $g_1$ goes into the leading term $xy^2$ for $f$ $y$ times, so we have

$$f - g_1 y = \left(xy^2 + 1\right) - (xy + 1)\, y = -y + 1.$$

Note that the leading term $xy$ does not divide $-y + 1$, so $-y + 1$ is the remainder of the division $f/g_1$. But note that the leading term $y$ for $g_2$ goes into $-y$ $-1$ times, so that

$$(f - g_1 y) - g_2(-1) = (-y + 1) - (y + 1)(-1) = 2.$$

Thus

$$f = y(xy + 1) - (y + 1) + 2.$$

---

In multivariate case, when the leading term is not divisible, we send it to remainders and remove it from the process.

**Example 2.24.**

Consider

$$f = x^2 y + xy^2 + y^2, g_1 = xy - 1, g_2 = y^2 - 1 \in K[x, y].$$

Then

$$f - xg_1 - yg_1 = \left(x^2 y + xy^2 + y^2\right) - x(xy - 1) - y(xy - 1) = y^2 + x + y.$$

Note that the leading term is $x$ now, which is not divisible by the leading terms $xy, y^2$ of $g_1, g_2$. So we record it as a part of the remainder, and proceed the division with $y^2 + y$. Now the leading term $y^2$ is divisible by the leading term $y^2$ of $g_2$, so that

$$\left(y^2 + y\right) - \left(y^2 - 1\right) = y + 1.$$

Now the terms $y, 1$ are not divisible by the leading terms $xy, y^2$, so we conclude that $x + y + 1$ is the remainder.

---

**Theorem 2.20.** Division Algorithm for Multivariate Polynomials

Let $f \in S = K[\vec{x}]$ and let $g_1, \ldots, g_m \in S$ be nonzero. The division algorithm produces polynomials $q_1, \ldots, q_m, r \in S$ such that

(a) $f = \left(\sum_{j=1}^{m} q_j g_j\right) + r$;

(b) $\operatorname{supp}(r) \cap \langle \operatorname{in}(g_1), \ldots, \operatorname{in}(g_m) \rangle$;[1] and

(c) $\operatorname{in}\left(q_j g_j\right) \leq \operatorname{in}(f)$.

[1] This makes sure that $r$ is small.

---

**Theorem 2.21.**

Fix a monomial order. Suppose $\{g_1, \ldots, g_m\}$ is a Grobner basis for a monomial ideal $I$. Then every $f \in S$ has a unique remainder aon division by $g_1, \ldots, g_m$. That is, no matter which $g_j$ we start dividing $f$ by, the quotients $q_j$'s in Theorem 2.20 may change, but $r$ stays the same.

**Proof.** Suppose we divide in 2 ways to get

$$f = r + \sum_{j=1}^{m} q_j g_j = s + \sum_{j=1}^{m} p_j g_j.$$

Then

$$(r - s) + \sum_{j=1}^{m} \left(q_j - p_j\right) g_j, = \left(r + \sum_{j=1}^{m} q_j g_j\right) - \left(s + \sum_{j=1}^{m} p_j g_j\right) = f - f = 0 \in I.$$

So $r - s \in I$, since the summation $\sum_{j=1}^{m} \left(q_j - p_j\right) g_j$ is a linear combination of basis elements $g_1, \ldots, g_m$ of $I$.

For contradiction, suppose $h = r - s \neq 0$. Then $\operatorname{in}(h) \in \operatorname{in}(I) = \langle \operatorname{in}(g_1), \ldots, \operatorname{in}(g_m) \rangle$. But $\operatorname{in}(h) \in \operatorname{supp}(r) \cup \operatorname{supp}(s)$, so either $r$ or $s$ has a monomial in $\langle \operatorname{in}(g_1), \ldots, \operatorname{in}(g_m) \rangle$, contradicting (b) of Theorem 2.20.

**QED**

---

**Corollary 2.21.1.** Algorithm for Ideal Membership

Let $\{g_1, \ldots, g_m\}$ be a Grobner basis for a monomial ideal $I$ and let $f \in S$. Then

$$f \in I \iff f \text{ has remainder 0 on division by } g_1, \ldots, g_m.$$

**Proof.** ($\Longleftarrow$) If we run division algorithm and get $f = r + \sum_{j=1}^{m} q_j g_j$ with $r = 0$, then clearly $f \in \langle g_1, \ldots, g_m \rangle = I$.

($\Longrightarrow$) Supose $f \in I$. Write $f = r + \sum_{j=1}^{m} q_j g_j$ by running division algorithm. Then $r \in I$. If $r \neq 0$, then $\operatorname{in}(r) \in \operatorname{in}(I) = \langle \operatorname{in}(g_1), \ldots, \operatorname{in}(g_m) \rangle$, contradicting the fact that $r$ is a remainder.

**QED**

## 7. Buchberger's Algorithm for Finding a Grobner Basis

Note that Corollary 2.21.1 is left as useless so far, since we know a Grobner basis exists for any monomial ideal, but we do not have a way of generating it.

Def'n 2.14. $S$**-polynomial**

Fix a monomial order. For all $f, g \in K[x]$, the $S$-***polynomial*** of $f$ by $g$, denoted as $S(f,g)$, is defined as

$$S(f,g) = \frac{\operatorname{lcm}(\operatorname{in}(f), \operatorname{in}(g))}{c \operatorname{in}(f)} f - \frac{\operatorname{lcm}(\operatorname{in}(f), \operatorname{in}(g))}{d \operatorname{in}(g)} g,$$

where $c, d \in K$ are the leading coefficients of $f, g$, respectively.

Note that if $f, g \in I$, where $I$ is an ideal, then so is $S(f,g)$.

**Algorithm 2.1.** Buchberger's Algorithm

```
INPUT:   monomial ideal I and a generating set F = {f_1, ..., f_k}
01.  for pair (f,f') ∈ F^2:
02.        compute S(f,f')
03.        for j ∈ {1, ..., k}:
04.              use division algorithm to divide S(f,f') by f_j to get a remainder r_j
05.              if r_j ≠ 0:
06.                    update F ← F ∪ {r_j}
07.                    restart from line 1
08.  return F, which is a Grobner basis
```

**Proposition 2.22.**

Buchberger's algorithm terminates in finite time.

**Proof.** Each time we enlarge the set of generators $F$, we strictly increase $\langle \operatorname{in}(f) : f \in F \rangle$. But this cannot get bigger forever by the ascending chain condition (Corollary 2.19.1).

**QED**

Def'n 2.15. **Reduces to** 0 Modulo $g_1, \ldots, g_m$

We say $f \in K[\vec{x}]$ ***reduces to*** 0 modulo $g_1, \ldots, g_m$ if

$$f = \sum_{i=1}^{m} q_i g_i$$

for some $q_1, \ldots, q_m$ with $\operatorname{in}(f) \geq \operatorname{in}(q_i g_i)$.

**Theorem 2.23.**

Fix a monomial order and let $I = \langle g_1, \ldots, g_m \rangle$ with each $g_i \neq 0$. The following are equivalent.

(a) $\{g_1, \ldots, g_m\}$ is a Grobner basis for $I$.

(b) Every $S(g_i, g_j)$ reduce to 0 modulo $g_1, \ldots, g_m$.

**Proof.** (a) $\Longrightarrow$ (b) If $\{g_1, \ldots, g_m\}$ is a Grobner basis for $I$, then every $f \in I$ reduces to 0 modulo $g_1, \ldots, g_m$.

(b) $\Longrightarrow$ (a) It suffices to show that

$$\operatorname{in}(I) = \langle \operatorname{in}(g_1), \ldots, \operatorname{in}(g_m) \rangle.$$

The $\supseteq$ containment is clear, so we only show $\subseteq$. Let $f \in I$ with $f \neq 0$. Write

$$f = \sum_{i=1}^{m} h_i g_i$$

for some $h_1, \ldots, h_m$. Then

$$\operatorname{in}(f) \leq \max_{1 \leq i \leq m} \operatorname{in}(h_i g_i).$$

Assume we have picked $h_i$'s so as to minimize the difference $\delta = \max_{1 \leq i \leq m} \operatorname{in}(h_i g_i) - \operatorname{in}(f)$.

If $\delta = 0$, then there is $i$ such that $\operatorname{in}(f) = \operatorname{in}(h_i g_i) = \operatorname{in}(h_i) \operatorname{in}(g_i)$, so $\operatorname{in}(f) \in \langle \operatorname{in}(g_1), \ldots, \operatorname{in}(g_m) \rangle$.

Otherwise,

$$\operatorname{in}(f) < \max_{1\le i\le m} \operatorname{in}(h_i g_i).$$

We are going to show that we can choose $h_i$'s better to make the gap $\delta$ smaller, which will bring the contradiction. Let

$$u = \max_{1\le i\le m} \operatorname{in}(h_i g_i) > \operatorname{in}(f)$$

and reorder the summation $\sum_{i=1}^{m} h_i g_i$ so that $\operatorname{in}(g_i h_i) = u$ for $i \le r$ and $\operatorname{in}(h_i g_i) < u$ for $i > r$. We can descale all the $g_i$ to have leading coefficient 1. Let $w_i = \operatorname{in}(h_i)$ and let $c_i$ be the leading coefficient of $h_i$. Since $\operatorname{in}(f) < u$, it follows that

$$\sum_{i=0}^{r} c_i = 0.$$

For $i \le r$, we have $u = w_i \operatorname{in}(g_i) = w_1 \operatorname{in}(g_1)$. This means $\operatorname{lcm}(\operatorname{in}(g_1), \operatorname{in}(g_i))$ divides $u$, so that

$$u = \operatorname{lcm}(\operatorname{in}(g_1), \operatorname{in}(g_i))\, v_i$$

for some monomial $v_i$.

Now

$$\begin{aligned} v_i S(g_1, g_i) &= v_i \left( \frac{\operatorname{lcm}(\operatorname{in}(g_1), \operatorname{in}(g_i))}{\operatorname{in}(g_i)} g_1 - \frac{\operatorname{lcm}(\operatorname{in}(g_1), \operatorname{in}(g_i))}{\operatorname{in}(g_i)} g_i \right) \\ &= \frac{u}{\operatorname{in}(g_1)} g_1 - \frac{u}{\operatorname{in}(g_i)} g_i = w_1 g_1 - w_i g_i. \end{aligned}$$

Both $w_1 g_1$, $w_i g_i$ have $u$ as the leading term, so that

$$\operatorname{in}(v_i S(g_1, g_i)) < u.$$

Now recall that the $S$-polynomials $S(g_i, g_j)$ reduce to 0 modulo $g_1, \ldots, g_m$, so write

$$S(g_1, g_i) = \sum_{j=1}^{m} q'_{i,j} g_j$$

for $q'_{i,1}, \ldots, q'_{i,m}$ with $\operatorname{in}\left(q'_{i,j} g_j\right) \le \operatorname{in}(S(g_1, g_i))$. Then

$$v_i S(g_1, g_i) = v_i \sum_{j=1}^{m} q'_{i,j} g_j = \sum_{j=1}^{m} q_{i,j} g_j$$

where

$$q_{i,j} = q'_{i,j} v_i.$$

So

$$u > \operatorname{in}(v_i S(g_1, g_i)) \ge \operatorname{in}(q_{i,j} g_j)$$

with

$$w_1 g_1 - w_i g_i - \sum_{j=1}^{m} q_{i,j} g_j = v_i S(g_1, g_i) - \sum_{j=1}^{m} q_{i,j} g_j.$$

Now

$$\sum_{i=2}^{r} c_i (w_1 g_1 - w_i g_i) = \sum_{i=1}^{r} c_i (w_1 g_1 - w_i g_i) = \sum_{i=1}^{r} c_i w_1 g_1 - \sum_{i=1}^{r} c_i w_i g_i = -\sum_{i=1}^{r} c_i w_i g_i.$$

since $c_1 = 0$ and $\sum_{i=1}^{r} c_i = 0$. Thus

$$\begin{aligned} f &= \sum_{i=1}^{r} h_i g_i + \sum_{i=r+1}^{m} h_i g_i = \sum_{i=1}^{r} h_i g_i + \sum_{i=2}^{r} c_i \left( w_1 g_1 - w_i g_i - \sum_{j=1}^{m} q_{i,j} g_j \right) + \sum_{i=r+1}^{m} h_i g_i \\ &= \sum_{i=1}^{r} h_i g_i - \sum_{i=1}^{r} c_i w_i g_i - \sum_{i=2}^{r} \sum_{j=1}^{m} c_i q_{i,j} g_j + \sum_{i=r+1}^{m} h_i g_i \\ &= \sum_{i=1}^{r} \left(h_i - c_i w_i\right) g_i + \sum_{i=r+1}^{m} h_i g_i - \sum_{j=1}^{m} \left( \sum_{i=2}^{r} c_i q_{i,j} \right) g_j = \sum_{i=1}^{m} h_i' g_i, \end{aligned}$$

where

$$h_i' = \begin{cases} h_i - c_i w_i - \sum_{k=2}^{r} c_k q_{k,j} & \text{if } i \leq r \\ h_i - \sum_{k=2}^{r} c_k q_{k,j} & \text{if } i > r \end{cases}.$$

Now compare $\operatorname{in}\left(h_i' g_i\right), \operatorname{in}\left(h_i g_i\right)$, we have

$$\operatorname{in}\left(h_i' g_i\right) \leq \operatorname{in}\left(h_i g_i\right)$$

and strict inequaity for $i \leq r$. Hence $\max_{1 \leq i \leq m} \operatorname{in}\left(h_i' g_i\right) < u$, a contradiction.

QED

**Lemma 2.24.**
Suppose $\gcd\left(\operatorname{in}(f), \operatorname{in}(g)\right) = 1$. Then $S(f, g)$ reduces to 0 modulo $f, g$.

**Proof.** Assume that the leading coefficients are 1. Write

$$f = \operatorname{in}(f) + f'$$

and

$$g = \operatorname{in}(g) + g'.$$

Since

$$\operatorname{lcm}(f, g) = fg$$

by $\gcd\left(\operatorname{in}(f), \operatorname{in}(g)\right) = 1$, we have

$$S(f, g) = \operatorname{in}(g) f - \operatorname{in}(f) g = \left(g - g'\right) f - \left(f - f'\right) g = f' g + g' f$$

It remains to check that

$$\operatorname{in}\left(f' g\right), \operatorname{in}\left(g' f\right) \leq \operatorname{in}\left(S(f, g)\right).$$

Suppose, for contradiction,

$$\operatorname{in}\left(f' g\right) = \operatorname{in}\left(g' f\right).$$

Then

$$\operatorname{in}\left(f'\right) \operatorname{in}(g) = \operatorname{in}\left(g'\right) \operatorname{in}(f).$$

Since $\operatorname{in}(f), \operatorname{in}(g)$ are coprime, so $\operatorname{in}(f) \mid \operatorname{in}\left(f'\right)$, which is absurd, since $\operatorname{in}\left(f'\right) < \operatorname{in}(f)$.

Hence $\operatorname{in}\left(f' g\right) < \operatorname{in}\left(g' f\right)$ or $\operatorname{in}\left(f' g\right) > \operatorname{in}\left(g' f\right)$. Assume without loss of generality that

$$\operatorname{in}\left(g' f\right) < \operatorname{in}\left(f' g\right).$$

This means

$$\operatorname{in}\left(f' g\right) = \operatorname{in}\left(S(f, g)\right),$$

which means $\operatorname{in}\left(g' f\right) < \operatorname{in}\left(S(f, g)\right)$ as well, which is what we required.

QED

**Example 2.25.**

Let $f = xy - zw, g = xz - y^2 \in K[x, y, z, w]$ and let

$$I = \langle f, g \rangle .$$

We compute a Grobner basis with respect to grevlex $x > y > z > w$.

We start with the generating set

$$F = \{f, g\} .$$

Observe that

$$\text{in}(f) = xy, \text{in}(g) = y^2,$$

which are not coprime. Hence Lemma 2.24 does not apply.

We have

$$S(f, g) = yf - (-x)g = x^2z - yzw.$$

We then run a division algorithm on $S(f, g)$ by $f, g$. Note that the leading terms $xy, y^2$ do not divide $x^2z$, so we throw the leading term $x^2z$ of $S(f, g)$ into remainder. But $xy, y^2$ do not divide the new lead term $yzw$ either, so it follows that the remainder is $x^2z - yzw$.

Hence define

$$h = x^2z - yzw$$

and update

$$F \leftarrow F \cup \{h\} = \{f, g, h\} .$$

For this time, we can skip $S(f, g)$, which was already computed.

For $f, h$, note that the leading terms $xy, x^2z$ are not coprime, and we have

$$S(f, h) = xzf - yh = \left(x^2yz - xz^2w\right) - \left(x^2yz - y^2zw\right) = y^2zw - xz^2w.$$

Now note that

$$-zwg = -zw\left(xz - y^2\right) = y^2zw - xz^2w = S(f, h) ,$$

so that $S(f, h)$ reduces to 0 modulo $f, g, h$.

But for $g, h$, note that

$$\text{in}(g) = y^2, \text{in}(h) = x^2z,$$

so that the leading terms are coprime. Then Lemma 2.24 applies and $S(g, h)$ reduces to 0 modulo $f, g, h$, so that we can skip computing $S(g, h)$.

Thus

$$F = \{f, g, h\}$$

is a Grobner basis for $I$.

---

**Example 2.26. Classical Determinantal Variety**

Let

$$D_r(m, n) = \left\{M \in K^{m\times n} : \text{rank}(M) \leq r\right\} .$$

Then observe that

$$D_r(m, n) = V_K\left(\left\{\underbrace{\text{all the } (r+1) \times (r+1) \text{ minors of } m \times n \text{ matrix of variables } x_{i,j}, 1 \leq i \leq m, 1 \leq j \leq n.}_{=J}\right\}\right)$$

We ask:

*is $J$ the set of all relations (i.e. $J = I(D_r(m, n))$)?*

Equivalently,

*is $\langle J \rangle$ radical?*

Consider the special case $D_1(2,n)$ and let

$$J = \left\{ 2 \times 2 \text{ minors of } \begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix} \right\}.$$

For convenience, define

$$P_{i,j} = x_i y_j - x_j y_i = \text{the minor of columns } i, j, \qquad \forall i < j.$$

We are going to understand this case by finding a Grobner basis. Fix a lexicographic order

$$x_1 > \cdots > x_n > y_1 > \cdots > y_n.$$

Then

$$\text{in}\left(P_{i,j}\right) = x_i y_j = \text{the diagonal term}, \qquad \forall i < j.$$

To find a Grobner basis, we observe that

$$i \neq k, j \neq l \implies \gcd\left(\text{in}\left(P_{i,j}\right), \text{in}\left(P_{k,l}\right)\right) = 1,$$

so that we can skip those $S$-polynomials.

Suppose $i = k$. Without loss of generality, say $j < l$. Then

$$S\left(P_{i,j}, P_{i,l}\right) = \left(x_i y_j y_l - x_j y_i y_l\right) - \left(x_i y_j y_l - x_l y_i y_j\right) = x_l y_i y_j - x_j y_i y_l = y_i\left(x_l y_j - x_j y_l\right) = -y_i P_{j,l}.$$

So these $S$-polynomials reduce to 0.

In case $j = l$ with $i < k$, an analogous calculation results in

$$x_j P_{i,k}$$

so that $S\left(P_{i,j}, P_{k,j}\right)$ reduces to 0.

Thus $J$ is a Grobner basis, so that

$$\text{in}(J) = \left\langle \text{in}\left(P_{i,j}\right)\right\rangle_{i<j} = \left\langle x_i y_j \right\rangle_{i<j}.$$

But then note that $\left\langle x_i y_j \right\rangle_{i<j}$ is radical. Hence it follows that

$$D_1(2,n) = \langle J \rangle \text{ is also radical.}$$

---

A more general result holds for Example 2.26.

**Theorem 2.25.** Hochster-Eagen

$D_r(m,n)$ is radical.

---

## 8. Reduced Grobner Bases

Def'n 2.16. **Reduced** Grobner Basis

Let $G = \{g_k\}_{j=1}^{m}$ be a Grobner basis for an ideal $I$. We say $G$ is ***reduced*** if

(a) all leading coefficients of $g_j$'s are monic; and

(b) for $i \neq j$, no $u \in \text{supp}(g_i)$ is divisible by $\text{in}(g_j)$.

**Theorem 2.26.**

Let $I$ be an ideal and fix a monomial order. Then there is a unique reduced Grobner basis for $I$.

**Proof.** Let $U = \{u_1, \ldots, u_m\}$ be the set of minimal generators for $\mathrm{in}(I)$. For each $u_i$, choose $g_i \in I$ with $\mathrm{in}(g_i) = u_i$.

Claim 1. $G = \{g_1, \ldots, g_m\}$ *is a Grobner basis.*

Run division algorithm on $g_1$ modulo $g_2, \ldots, g_m$ to write

$$g_1 = \sum_{i=2}^{m} q_i g_i + r_1.$$

Then no $u \in \mathrm{supp}(r_1)$ is divisible by any $\mathrm{in}(g_i)$ for $i \geq 2$. Also, $\mathrm{in}(g_1) \geq \mathrm{in}(q_i g_i)$ for $i \geq 2$.

If, for some $i \geq 2$,

$$\mathrm{in}(g_1) = \mathrm{in}(q_i g_i),$$

then

$$u_i \,\mathrm{in}(q_i) = \mathrm{in}(g_i)\,\mathrm{in}(q_i) = \mathrm{in}(q_i g_i) = \mathrm{in}(g_1) = u_1,$$

which is absurd.

Hence we have in fact

$$\mathrm{in}(g_1) > \mathrm{in}(q_i g_i), \qquad \forall i \geq 2.$$

This means

$$\mathrm{in}(r_1) = \mathrm{in}(g_1) = u_1,$$

so replace $g_1$ with $h_1 = \frac{r_1}{k_1}$, where $k_1$ is the leading coefficient of $r_1$.

Say we have *fixed* $g_1, \ldots, g_n$ to $h_1, \ldots, h_n$ to obtain a Grobner basis $\{h_1, \ldots, h_n, g_{n+1}, \ldots, g_m\}$. That is, each $\mathrm{in}(h_i) = u_i$, $h_i$ is monic, and no $u \in \mathrm{supp}(h_i)$ is divisible by any $u_j$ for $j \neq i$. Let $r_{n+1}$ be a remainder of $g_{n+1}$ modulo generators. Then

$$\mathrm{in}(r_{n+1}) = \mathrm{in}(g_{n+1}) = u_{n+1}$$

and no $u \in \mathrm{supp}(r_{n+1})$ is divisible by $u_j$ for $j \neq i$. We then replace $g_{n+1}$ by $h_{n+1} = \frac{r_{n+1}}{k_{n+1}}$, where $k_{n+1}$ is the leading coefficient for $r_{n+1}$.

By induction, $\{h_1, \ldots, h_n\}$ is a Grobner basis.

*(End of Claim 1)*

Now that we have shown existence, let us prove uniqueness. Suppose $\{g_i\}_{i=1}^{m}, \{h_j\}_{j=1}^{l}$ are reduced Grobner bases for $I$. Then the minimal generators for $\mathrm{in}(I)$ must be $\{\mathrm{in}(g_i)\}_{i=1}^{m}$. But so is $\{\mathrm{in}(h_j)\}_{j=1}^{l}$, so it follows $m = l$. Hence reorder elements so that each $\mathrm{in}(g_i) = \mathrm{in}(h_i)$.

Let $i \in \{1, \ldots, m\}$ and let

$$f = g_i - h_i \in I.$$

Suppose $f \neq 0$. Then $\mathrm{in}(f) \in \mathrm{supp}(g_i) \cup \mathrm{supp}(h_i)$. Without loss of generality, say $\mathrm{in}(f) \in \mathrm{supp}(g_i)$. Since $\mathrm{in}(f) \in \mathrm{in}(I)$, we know $\mathrm{in}(f)$ is a multiple of some $\mathrm{in}(g_j)$. But we know $\mathrm{in}(g_i) \notin \mathrm{supp}(f)$ since $f = g_i - h_i$ and $\mathrm{in}(g_i) = \mathrm{in}(h_i)$, which means $i \neq j$. But if $i \neq j$, then we have a contradiction against reducedness of $\{g_i\}_{i=1}^{m}$.

Thus $f = 0$, so that $g_i = h_i$, as required.

**QED**

# III. Geometry of Grobner Degeneration

## 1. Integral Weight Order

**Example 3.1.**

Let $I = \langle xy - 1 \rangle \subseteq \mathbb{R}[x, y]$. Then $V_\infty(I)$ is the hyperbola of $y = \frac{1}{x}$ whereas $V_\infty(\in(I))$ consists of precisely the $x, y$-axes. We are going to investigate how this *degeneration* happens in a continuous fasion.

Consider adding a dimension to the picture and the polynomial

$$xy - t \in \mathbb{R}[x, y, t].$$

Then observe that the $x, y$ cross section at $t = 0$ is the $x, y$-axes and the cross section at $t = 1$ is the $y = \frac{1}{x}$ hyperbola.

We call $V_\infty(xy - t)$ a *hyperbolic paraboloid*. We can think about hyperbolic paraboloid $P$ as a *family* of conics over a line. That is, there is a projection

$$\begin{aligned} \varphi : P &\to \mathbb{R} \\ (x, y, t) &\mapsto t \end{aligned}$$

and a ***fibre*** at $t = t_0$

$$\pi^{-1}(t_0) = \{(x, y, t) \in P : t = t_0\},$$

so that the family $\left\{\pi^{-1}(t)\right\}_{t \in \mathbb{R}}$ completely describes $P$. Each fibre is a hyperbola, except $\pi^{-1}(0)$ which is the union of two orthogonal lines.

We call this family $\left\{\varphi^{-1}(t)\right\}_{t \in \mathbb{R}}$ is ***flat*** if $\pi$ is flat.

---

Def'n 3.1. **Isomorphic** Scheme

Let $I \subseteq S, J \subseteq R$ be ideals in polynomial rings $S, R$. We say $V_\infty(I), V_\infty(J)$ are ***isomorphic*** and write $V_\infty(I) \cong V_\infty(J)$ if $S/I \cong R/J$ as rings.

**Example 3.2.**

For over $P$, all the fibres $\pi^{-1}(t)$ for $t \neq 0$ are isomorphic. That is,

$$\mathbb{R}[x, y] / \langle xy - t_0 \rangle \cong \mathbb{R}\left(x, x^{-1}\right), \qquad \forall t_0 \neq 0.$$

We call each $\mathbb{R}[x, y] / \langle xy - t_0 \rangle$ for $t_0 \neq 0$ a *general fibre* and $\mathbb{R}[x, y] / \langle xy \rangle$ a *special fibre*. Hence we have degenerated the general fibre to the special one.

---

In general, we chose an ***integral weight function***

$$\lambda : \mathbb{Z}^n \to \mathbb{Z},$$

which is $\mathbb{Z}$-linear.

This gives a partial order $\leq_\lambda$, called an ***integral weight order***, on monomials by comparing the weights:

$$m_1 \leq_\lambda m_2 \iff \lambda(m_1) \leq \lambda(m_2)$$

for all monomials $m_1, m_2$.

Def'n 3.2. **Initial Form** with respect to Integral Weight Order

Let $\leq_\lambda$ be an integral weight order. Given $g \in S$, we define the ***inital form***, denoted as, $\text{in}_\lambda(g)$, to be the sum of all the maximal terms with respect to $\leq_\lambda$.

The next theorem shows that integral weight orders are enough to capture all initial ideals.

**Theorem 3.1.**

Let $<$ be a monomial order and let $\{g_k\}_{k=1}^{n}$ be a Grobner basis for $I$ with respect to $<$. Then there is a finite set of pairs of monomials

$$m_1^1 < m_2^1, \ldots, m_1^r < m_2^r$$

such that for any integral weight order $<_\lambda$ with

$$m_1^1 <_\lambda m_2^1, \ldots, m_1^r <_\lambda m_2^r,$$

we have that

$$\mathrm{in}_\lambda (I) = \{\mathrm{in}_\lambda (g)\}_{g\in I}$$

is a monomial ideal with

$$\mathrm{in}_\lambda (I) = \mathrm{in}_< (I)$$

and $\{g_k\}_{k=1}^{n}$ is a $<_\lambda$-Grobner basis for $I$ (i.e. $\mathrm{in}_\lambda (I) = \langle \mathrm{in}_\lambda (g_k)\rangle_{k=1}^{n}$).

**Proof.** For each $g_k$ and each non-initial monomial $m \in \mathrm{supp}\,(g_k)$, put

$$m < \mathrm{in}_< (g_k)\,.$$

Let $\mathcal{F}$ be the set of such pairs. Now imagine running Buchberger's algorithm on $\{g_k\}_{k=1}^{n}$, to verify that it is a Grobner basis. This yields quadratically many $S$-polynomials. Run division algorithm on each. Each time we need to identify a leading term of some polynomial $f$, add $(m < \mathrm{in}_< (f))$ to $\mathcal{F}$ for each $m \neq \mathrm{in}_< (f)$ in $\mathrm{supp}\,(f)$.

Now take any integral weight order $<_\lambda$ which agrees on $\mathcal{F}$. That is, for all $(m < f) \in \mathcal{F}$, $m <_\lambda f$. If we run Buchburger's algorithm on $\{g_1, \ldots, g_n\}$ with respect to $<_\lambda$, the process is exactly the same; the division algorithm and Buchberger's criterion all work fine for partial orders provided that every initial form encountered is a single term.

Thus $\langle \mathrm{in}_\lambda (g_k)\rangle_{k=1}^{n} = \mathrm{in}_\lambda (I)$.

**QED**

Here is the general idea of the degeneration. Let $k = \mathbb{C}$ and let $\hat{t} \in \mathbb{C}$ be nonzero. Also fix an integral weight order $\leq_\lambda$. Then

$$\begin{aligned} \varphi_\lambda^j : S &\to S \\ x_j &\mapsto x_j \hat{t}^{-\lambda(x_j)} \end{aligned}$$

is an automorphism of rings.

Note that the composition of two $\varphi_\lambda^{j_1}, \varphi_\lambda^{j_2}$ is commutative. Define

$$\varphi_\lambda = \varphi_\lambda^1 \circ \cdots \circ \varphi_\lambda^n r$$

which is also an automorphism. Since $\varphi_\lambda$ is an automorphism, it follows that

$$S/I \cong S/\varphi_\lambda (I)$$

for all ideal $I \subseteq S$. But as $\hat{t} \to 0$,

$$\varphi_\lambda (I) \to \mathrm{in}_\lambda (I)\,,$$

because the leading form starts to dominate.

To do this better, let

$$T = S\,[t]$$

be a polynomial ring in 1 more variable. For $f \in S$, let $\tilde{f} \in T$ be defined by

$$\tilde{f} = t^M \varphi_\lambda (f)\,,$$

where $M = \max_{m \in \operatorname{supp}(f)} \lambda(m)$. In other words,

$$\tilde{f} = t^M f\left(t^{-\lambda(x_1)} x_1, \ldots, t^{-\lambda(x_n)} x_n\right).$$

Note that the leading form $\mathrm{in}_\lambda(f)$ remains unchanged, since we are multiplying by $t^M$. But all other terms have $t$.

For an ideal $I \subseteq S$, let

$$\tilde{I} = \langle \tilde{g} \rangle_{g \in I} \subseteq S[t].$$

Then $V_\infty(\tilde{I})$ is a ***flat family*** with general fibre isomorphic to $V_\infty(I)$ and special fibre isomorphic to $V_\infty(\mathrm{in}_\lambda(I))$.

There are few tricks that we can use to identify flatness.

(a) Projections $\pi : X \times Y \to X$ from a product are flat.

(b) Flatness is a local property on the base. For instance, any map that is a locally a projection is flat.

(c) If $X$ is a reduced and irreducible variety and $C$ is a smooth curve, then any map $\varphi : X \to C$ is flat.

(d) If all fibres come from homogeneous ideals, then flatness is equivalent to *every fibre has the same Hilbert function*.

## 2. Syzygies and Betti Numbers

Let $R$ be a commutative ring.

**Def'n 3.3. $R$-module**

An $R$-***module*** $M$ is an abelian group with a scalar multiplication by elements of $R$ such that

(a) $(r+s)m = rm + sm$ for all $r, s \in R, m \in M$;

(b) $r(m+n) = rm + rn$ for all $r \in R, n, m \in M$;

(c) $(rs)m = r(sm)$ for all $r, s \in R, m \in M$; and

(d) $1m = m$.

**Example 3.3.**

Let $M$ be an $R$-module.

(a) If $R$ is a field, then $M$ is a vector space.

(b) If $R = \mathbb{Z}$, then $M$ is an abelian group. In fact any (additive) abelian group $G$ is a $\mathbb{Z}$-module in a natural way:

$$r \cdot g = \underbrace{g + \cdots + g}_{r \text{ times}}.$$

(c) If $R$ is a polynomial ring, then any ideal $I \subseteq R$ is an $R$-module, and so is $R/I$.

---

**Def'n 3.4. Submodule Generated by a Subset**

Let $M$ be a module and let $G \subseteq M$. We define the submodule ***generated*** by $G$, denoted as $\langle G \rangle$, to be the set of all finite $R$-linear combinations of elements of $G$.

If $\langle G \rangle = M$, then we call $G$ a ***system of generators*** (or ***generating set***) for $M$.

If $M$ admits a finite generating set, then we say $M$ is ***finitely generated***.

**Def'n 3.5. Basis for a Module**

If every element of a module $M$ can be written uniquely as a $R$-linear combination of elements of $G \subseteq M$, then we say $G$ is a ***basis*** for $M$.

As the next example shows, not every module has a basis, unlike vector spaces.

**Example 3.4.** Module without Basis

Consider $I = \langle x, y \rangle \subseteq K[x, y]$.

Clearly $I$ does not have an 1-element basis.

Suppose $I$ has a basis $G \subseteq I$ with at least 2 elements and let $g_1, g_2 \in G$ be distinct. Then

$$g = g_1 g_2 \in I$$

but

$$g = \underbrace{g_1}_{\text{ring}} \overbrace{g_2}^{\text{module}} = \underbrace{g_2}_{\text{ring}} \overbrace{g_1}^{\text{module}}.$$

---

Def'n 3.6. **Free** Module

We say a module $F$ is ***free*** if it admits a basis.

**Example 3.5.**

Every vector space is a free module.

---

**Lemma 3.2.**

Let $F$ be a free module. Then every basis of $F$ has the same cardinality.

**Proof.** Let $I$ be a maximal ideal of $R$ (which exists due to AoC). Then a basis of $F$ induces a basis of the $R/I$-module $F/IF$. But a ring-modulo-its-maximal-ideal like $R/I$ is a field, so $F/IF$ is a vector space. Hence every basis of $F/IF$ has the same cardinality, which means every basis of $F$ has the same cardinality.

**QED**

Def'n 3.7. **Rank** of a Free Module

The size of a basis of a free module $F$ is called the ***rank*** of $F$.

Def'n 3.8. **Homomorphism** of Modules

Let $M, N$ be $R$-modules. We say $\varphi : M \to N$ is a ***homomorphism*** if $\varphi$ is a homomorphism of abelian groups with

$$\varphi(rm) = r\varphi(m), \qquad \forall r \in R, m \in M.$$

We say $\varphi$ is an ***isomorphism*** if $\varphi$ is bijective in addition.

**Proposition 3.3.** $R$-modules Have Enough Free Modules

Let $M$ be an $R$-module. Then $M$ is a quotient of a free module. That is, there is a free module $F$ with a submodule $U \subseteq F$ such that $M \cong F/U$.

If $M$ is finitely generated, then $F$ is finitely generated.

**Proof.** Let $G$ be a generating set for $R$ and let

$$F = \{f : G \to R : f \text{ has a finite support}\}.$$

That is, $f \in F$ if and only if $f : G \to R$ with $f(g) = 0$ for all but finitely many $g \in G$.

Then $F$ is an abelian group under addition under componentwise addition,

$$(f_1 + f_2)(g) = f_1(g) + f_2(g),$$

and an $R$-module with scalars acting diagonally,

$$(rf)(g) = r(f(g)).$$

For each $g \in G$, there is a special element $e_g \in F$ given by

$$e_g(\hat{g}) = \delta_{g,\hat{g}}.$$

Claim 1. $\{e_g\}_{g\in G}$ *is a basis for F. In particular, F is free.*

*(End of Claim 1)*

Define

$$\varepsilon : F \to M$$
$$f \mapsto \sum_{g\in G} f(g)\, g.$$

Then $\varepsilon$ is a surjective homomorphism. So by the first isomorphism theorem, $M \cong F/\ker(\varepsilon)$.

**QED**

---

Suppose $N \subseteq K[x_1, \ldots, x_n]$ is a module and $M \subseteq N$. If $N$ is finitely generated, then so is $M$.

Let $M$ be a finitely generated $S$-module. Then we have a finitely generated free $S$-module $F_0$ such that there is a surjective homomorphism $\varepsilon : F_0 \to M$. Let $U_1 = \ker(\varepsilon)$. This means $U_1$ is finitely generated, so there is free $S$-module $F_1$ with a surjective homomorphism $\varepsilon_1 : F_1 \to U_1$. Let $\varphi_1 : F_1 \to F_0$ be $\varphi_1 = c_1 \circ \varepsilon_1$. Note

$$\operatorname{Im}(\varphi_1) = U_1 = \ker(\varepsilon).$$

Let $U_2 = \ker(\varphi_1)$. Since $U_2$ is finitely generated, there is free $F_2$ and $\varphi_2 : F_2 \to F_1$, with $\operatorname{Im}(\varphi_2) = U_2 = \ker(\varphi_1)$. By continuing, we get a diagram

$$\cdots \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varepsilon} M \to 0.$$

Such a sequence is called *exact*.

Def'n 3.9. **Exact** Sequence
We say

$$\cdots \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varepsilon} M \to 0.$$

is ***exact*** when $\varphi_1(F_1) = \ker(\varepsilon)$ and $\ker(\varphi_i) = \varphi_{i+1}(F_{i+1})$ for all $i \in \mathbb{N}$.

An exact sequence with all $F_i$ free is called a ***free resolution*** of $M$. The image of $\varphi_i$ is called the $i$th ***syzygy module***.

**Theorem 3.4.** Hilbert's Syzygy Theorem

Let $M$ be a finitely generated $S$-module. Then $M$ has a free resolution

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0.$$

---

Def'n 3.10. **Graded** Module
We say a $S$-module $M$ is ***graded*** if

$$M = \bigoplus_{i\in\mathbb{Z}} M_i$$

and $S_i M_j \subseteq M_{i+j}$ for all $i \geq 0, j \in \mathbb{Z}$.

If $M$ is graded, then $M(j)$ is some module with grading shifted by $j$ (i.e. $M(j)_i = M_{i+j}$)

**Example 3.6.**

Let $S = K[x, y]$, $M = S(-2)$. Then $\deg_S(2x + y) = 1 \implies 2x + y \in S_1$, so that $2x + y \in M_3$. That is, $\deg_M(2x + y) = 3$.

---

Def'n 3.11. **Graded Submodule** of a Graded Module
Let $M$ be a graded module. A ***graded submodule*** $U \subseteq M$ is a submodule with the induced grading; that is,

$$U_i = U \cap M_i, \qquad \forall i \in \mathbb{Z}.$$

Def'n 3.12. **Homomorphism** of Graded Modules

A ***homomorphism*** of graded modules $M, N$ is a map $\varphi : M \to N$ is a homomorphism of modules with

$$\varphi(M_i) \subseteq N_i, \qquad \forall i \in \mathbb{Z}.$$

**Example 3.7.** Graded Homomorphism

The map

$$\begin{aligned} \varphi : S(-2) &\to S \\ f &\mapsto x^2 f \end{aligned}$$

is a graded homomorphism.

**Proposition 3.5.**

The homomorphisms in Hilbert's syzyzy theorem can be rearranged to be graded.

**Example 3.8.** Koszul Complex

Let $S = K[x, y]$ and let $M = I = \langle x, y \rangle$.

$$0 \to S(-2) \overset{\begin{bmatrix} y \\ -x \end{bmatrix}}{\to} S^2(-1) \overset{\begin{bmatrix} x & y \end{bmatrix}}{\to} M \to 0.$$

Def'n 3.13. **Irrelevant Ideal** of a Polynomial Ring, **Minimal** Free Resolution

In $S = K[x_1, \dots, x_n]$, the ***irrelevant ideal*** is the maximal homogeneous ideal $B = \langle x_1, \dots, x_n \rangle$.

A graded free resolution

$$\cdots \overset{\varphi_3}{\to} F_2 \overset{\varphi_2}{\to} F_1 \overset{\varphi_1}{\to} F_0 \overset{\varepsilon}{\to} M \to 0.$$

is ***minimal*** if $\varphi_i(F_i) \subseteq BF_{i-1}$ for all $i$.

**Lemma 3.6.** Graded Nakayama's Lemma

Let $N$ be a finitely generated graded $S$-module and let $B \subseteq S$ be the irrelevant ideal. Suppose $n_1, \dots, n_k \in N$ are homogeneous elements such that $\{n_1 + BN, \dots, n_k + BN\}$ is a basis for $N/BN$ as a $S/B$-vector space.[1] Then $n_1, \dots, n_k$ generate $N$.

[1] Note that $S/B \cong K$, the field of coefficients for the polynomial ring $S$.

**Proof.** Let $U = \langle n_1, \dots, n_k \rangle \subseteq N$. Since $\{n_1 + BN, \dots, n_k + BN\}$ generate $N/BN$, it follows that

$$N = U + BN.$$

Let $n \in N$ be homogeneous. To show that $n \in U$, we proceed inductively on $\deg(n)$.

Since $N = U + BN$, $n = u + bn'$ for some $u \in U, b \in B, n' \in N$, with

$$\deg(n) = \deg(u) = \deg(bn').$$

Suppose every element of $N$ has degree at least $\deg(n)$. Then $\deg(n) = \deg(n')$, which means $n = u$.

Otherwise, $\deg(n') < \deg(n)$, so by induction $n' \in U$.

**QED**

**Corollary 3.6.1.**

Let $N$ be a $K[\vec{x}]$-module. Then a minimal generating set for $N$ has cardinality $\dim_K(N/BN)$.

**Lemma 3.7.**
Suppose $\varphi : N \to N'$ is an isomorphism of graded $S$-modules and let $\varepsilon : F \to N, \varepsilon' : F' \to N'$ be surjections from minimal rank free modules. Then there is an isomorphism $\psi : F \to F'$ such that $\varepsilon \circ \varphi = \psi \circ \varepsilon'$.

**Proof.** Let $\{e_1, \ldots, e_r\}$ be a homogeneous basis for $F$. For each $i \in \{1, \ldots, r\}$, choose $f_i' \in F'$ such that

$$\varepsilon' \left(f_i'\right) = \varphi\left(\varepsilon\left(e_i\right)\right).$$

Define $\psi : F \to F'$ by

$$\psi\left(e_i\right) = f_i', \qquad \forall i.$$

Then it remains to show that $\psi$ is an isomorphism.

Note that $\ker\left(\varepsilon\right) \subseteq BF$ and $\ker\left(\varepsilon'\right) \subseteq BF'$. Hence the induced quotient maps

$$\begin{aligned} \overline{\varepsilon} : F/BF &\to N/BN \\ f + BF &\mapsto \varepsilon\left(f\right) + BN \end{aligned}$$

and

$$\begin{aligned} \overline{\varepsilon'} : F'/BF' &\to N'/BN' \\ f' + BF' &\mapsto \varepsilon'\left(f'\right) + BN' \end{aligned}$$

are isomorphisms. This means the map

$$\begin{aligned} \overline{\psi} : F/BF &\to F'/BF' \\ f + BF &\mapsto \psi\left(f\right) + BF' \end{aligned}$$

induced by $\psi$ is an isomorphism, as $\overline{\psi} = \overline{\varepsilon} \circ \overline{\psi} \circ \overline{\varepsilon'}^{-1}$. Since $F, F'$ are free, it follows that $\psi$ is also an isomorphism.

QED

**Theorem 3.8.**
Let $M$ be a finitely generated graded $S$-module. Then

(a) $M$ has a minimal free resolution; and

(b) any two minimal free resolutions are isomorphic.[1]

[1] We say two free resolutions

$$\cdots \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varepsilon} M \to 0$$

$$\cdots \xrightarrow{\varphi_3'} F_2' \xrightarrow{\varphi_2'} F_1' \xrightarrow{\varphi_1'} F_0' \xrightarrow{\varepsilon'} M \to 0$$

are isomorphic if there are isomorphisms $\psi_i : F_i \to F_i'$ such that $\varphi_i \circ \psi_{i-1} = \psi_i \circ \varphi_i'$.

**Proof of (a).** Choose a minimal set of homogeneous generators $\{m_1, \ldots, m_r\}$ for $M$. Let

$$F_0 = \bigoplus_{i=1}^{r} Se_i = \bigoplus_{i=1}^{r} S\left(-\deg\left(m_i\right)\right)$$

with $\deg\left(e_i\right) = \deg\left(m_i\right)$. Then

$$\begin{aligned} \varepsilon : F_0 &\to M \\ e_i &\mapsto m_i \end{aligned}$$

is a graded surjective homomorphism.

Claim 1. $\ker\left(\varepsilon\right) \subseteq BF_0$, *where $B$ is the irrelevant ideal of $S$.*

Now $\varepsilon$ induces a map

$$\begin{aligned} \overline{\varepsilon} : F_0/BF_0 &\to M/BM \\ e_i + BF_0 &\mapsto m_i + BF_0, \end{aligned}$$

which is a surjective homomorphism. Moreover, both $F_0/BF_0$ and $M/BM$ are vector spaces, since $B$ is the irrelevant ideal. Note that $\dim_K(F_0/BF_0) = \dim_K(M/BM)$, so it follows that $\bar{\varepsilon}$ is an isomorphism. Hence $\bar{\varepsilon}$ has a trivial kernel, which implies

$$\ker(\varepsilon) \subseteq BF_0,$$

as required.

*(End of Claim 1)*

Now iterate this process, building the rest of a free resolution; at each step choose a minimal generating set for each $\ker(\varphi_i)$. By the same reasoning, $\ker(\varphi_i) \subseteq BF_i$. Thus minimal free resolution exists.

**Proof of (b).** Now suppose we have two minimal free resolutions

$$\begin{gathered} \cdots \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varepsilon} M \to 0 \\ \cdots \xrightarrow{\varphi_3'} F_2' \xrightarrow{\varphi_2'} F_1' \xrightarrow{\varphi_1'} F_0' \xrightarrow{\varepsilon'} M \to 0 \end{gathered}.$$

We want graded isomorphisms $\psi_i : F_i \to F_i'$ such that $\varphi_i \circ \psi_{i-1} = \psi_i \circ \varphi_i'$.

By induction using Lemma 3.7, we are done.

**QED**

Def'n 3.14. **Betti Number** of a Module

Consider a minimal free resolution of $M$

$$\cdots \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varepsilon} M \to 0$$

with

$$F_i = \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{i,j}}.$$

Then $\beta_{i,j}$'s are called the ***graded Betti numbers*** of $M$.

The ***ungraded Betti numbers*** are

$$\beta_i = \sum_{j\in\mathbb{Z}} \beta_{i,j} = \operatorname{rank}(F_i).$$

**Theorem 3.9.**

Suppose $M$ is a finitely generated $S$-module. Consider a graded free resolution of $M$ with

$$F_i = \bigoplus_{j\in\mathbb{Z}} S(-j)^{b_{i,j}}.$$

Then $b_{i,j} \geq \beta_{i,j}$ ($\beta_{i,j}$'s are the graded Betti numbers of $M$).

We can extract information from Betti numbers, as the following example shows.

**Example 3.9.**

Let $S = K[x, y, z, w]$ and let

$$M = \left\langle \underbrace{x^2 - yz}_{=f_1}, \underbrace{z^2 w}_{=f_2}, \underbrace{xyz}_{=f_3}, \underbrace{w^3}_{=f_4} \right\rangle.$$

Since $M$ has 4 generators, with degree 2, 3, 3, 3,

$$\cdots \to S(-2) \oplus S(-3)^3 \to I \to 0.$$

On the generators, we have the following relations that involves cancelling a pair:

$$\begin{aligned}(xyz)f_4 - \left(w^3\right)f_3 &= 0\\ (zw)f_3 - (xy)f_2 &= 0\\ &\vdots\\ \left(z^2\right)f_4 - \left(w^2\right)f_2 &= 0\end{aligned}.$$

There is one more relation that involves a triple:

$$(yzw)f_1 + \left(y^2\right)f_2 - (xw)f_3 = 0.$$

Hence

$$\cdots \to S(-5)^6 \oplus S(-6) \to S(-2) \oplus S(-3)^3 \to M \to 0.$$

We can continue and figure out that

$$0 \to S(-8) \to S(-6)^2 \oplus S(-7)^3 \to S(-5)^6 \oplus S(-6) \to S(-2) \oplus S(-3)^3 \to M \to 0.$$

Here's the ***Betti diagram*** for this example

| degree shifts$-i \setminus i$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 2 | 1 | 0 | 0 | 0 |
| 3 | 3 | 0 | 0 | 0 |
| 4 | 0 | 6 | 2 | 0 |
| 5 | 0 | 1 | 3 | 1 |

Note that we are subtracting degree shifts by $-i$ since whenever we are writing down relations, the $S$-coefficients must have nonzero degree.

---

There are two quantities that measures the size of a Betti diagram: *projective dimension* for width and *regularity* for height.

Def'n 3.15. **Projective Dimension** of a $S$-module
Let $M$ be a $S$-module. We define the ***projective dimension*** of $M$, denoted as $\operatorname{pd}(M)$, by

$$\operatorname{pd}(M) = \max\left\{i : \beta_{i,j} \neq 0 \text{ for some } j \in \mathbb{Z}\right\}.$$

Def'n 3.16. **Regularity** of a $S$-module
The ***regularity*** of a $S$-module $M$, denoted as $\operatorname{reg}(M)$, is defined as

$$\operatorname{reg}(M) = \max\left\{j : \beta_{i,i+j} \neq 0 \text{ for some } i \in \mathbb{Z}\right\}.$$

**Proposition 3.10.**
Let $I \subseteq S$ be a homogeneous ideal. Then

$$\operatorname{pd}(S/I) = \operatorname{pd}(I) + 1$$

and

$$\operatorname{reg}(S/I) = \operatorname{reg}(I) - 1$$

**Proof.** Let

$$0 \to F_k \to \cdots \to F_0 \to I \to 0$$

be the minimal free resolution for $I$. Then observe that

$$0 \to F_k \to \cdots \to F_0 \to S \to S/I \to 0$$

is a minimal free resolution for $S/I$.

QED

**Lemma 3.11.**
For all $S$-module $M$, $\operatorname{pd}(M) \leq n$ ($S = K[x_1, \ldots, x_n]$).

**Lemma 3.12.**
For all $S$-module $M$, $\operatorname{reg}(M) \geq \deg(g)$ for any generator $g \in M$.

Def'n 3.17. **Depth** of a $S$-module
The ***depth*** of a $S$-module $M$ is $n - \operatorname{pd}(n)$.

We have

$$n - \operatorname{pd}(n) = \min\left\{i : \beta_{n-i,j} \neq 0 \text{ for some } j \in \mathbb{Z}\right\}$$

Def'n 3.18. **Hilbert Function** of a Module
Let $M$ be a $S$-module. The ***Hilbert function*** $H_M$ of $M$ is

$$\begin{aligned} H_M : \mathbb{Z} &\to \mathbb{Z} \\ i &\mapsto \dim_K(M_i) \end{aligned}.$$

The ***Hilbert series*** of $M$ is

$$H(M,t) = \sum_{i \in \mathbb{Z}} H_M(i)\, t^i.$$

**Proposition 3.13.**
Let $M$ be a finitely generated $S$-module. Then

$$H(M,t) = \frac{\sum_{i \in \mathbb{Z}} (-1)^i \sum_{j \in \mathbb{Z}} \beta_{i,j} t^j}{(1-t)^n}.$$

**Example 3.10.**
Consider

$$I = \left\langle x^2 - yz, z^2 w, xyz, w^3 \right\rangle.$$

Recall that

| degree shifts$-i \setminus i$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 2 | 1 | 0 | 0 | 0 |
| 3 | 3 | 0 | 0 | 0 |
| 4 | 0 | 6 | 2 | 0 |
| 5 | 0 | 1 | 3 | 1 |

is the Betti table for $I$.

It follows that

| degree shifts$-i \setminus i$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 | 0 |
| 2 | 0 | 3 | 0 | 0 | 0 |
| 3 | 0 | 0 | 6 | 2 | 0 |
| 4 | 0 | 0 | 1 | 3 | 1 |

is the Betti table for $M = S/I$, so it follows that

$$H(M,t) = \frac{1 - t - 3t^2 + 6t^5 - t^6 - 3t^7 + t^8}{(1-t)^4}.$$

---

Def'n 3.19. **Grothendieck Polynomial** of a Finitely Generated Module

Let $M$ be a finitely generated $S$-module. We define the ***Grothendieck polynomial*** of $M$ by

$$\mathfrak{G}(M,t) = \sum_{i \in \mathbb{Z}} (-1)^i \sum_{j \in \mathbb{Z}} \beta_{i,j} (1-t)^j .$$

The minimum degree of $\mathfrak{G}(M,t)$ is called the ***Krull codimension*** of $M$.
The coefficient of the minimum degree term of $\mathfrak{G}(M,t)$ is called the ***degree*** or ***Hilbert-Samuel multiplicity*** of $M$.

**Example 3.11.**

The Grothendieck polynomial of $M$ in Example 3.10 is

$$\mathfrak{G}(M,t) = 12t^3 - 20t^4 + 7t^5 + 6t^6 - 5t^7 + t^8.$$

Then the Krull codimension of $M$ is 3, so the Krull dimension of $M$ is $4 - 3 = 1$.

---

Def'n 3.20. **Nonzerodivisor**

Let $f \in S$ be a homogeneous and let $I \subseteq S$ be an ideal. We say $f$ is ***nonzerodivisor*** on $S/I$ if

$$(f+I)(g+I) = 0 + I \implies g + I = 0 + I.$$

Geometrically, $f$ being a nonzerodivisor means $f$ does not vanish on any component of $V_\infty(I)$, even as embedded components. Hence, the hypersurface $V_\infty(f)$ slices each components nontrivially.

Def'n 3.21. **Homogeneours $S/I$-sequence**

A ***homogeneous*** $S/I$***-sequence*** is a sequence $f_1, \ldots, f_d$ such that $f_i$ is a nonzerodivisor on $S/\langle I + \langle f_1, \ldots, f_{i-1}\rangle\rangle$ and $I + \langle f_1 \ldots, f_d\rangle \neq S$.

Def'n 3.22. **Depth** of $S/I$

The ***depth*** of $S/I$, denoted as $\operatorname{depth}(S/I)$, is the maximum length of a homogeneous $S/I$-sequence.

**Theorem 3.14.**

Let $I \subseteq S$ be an ideal. Then

$$\operatorname{depth}(S/I) = n - \operatorname{pd}(S/I).$$

---

**Corollary 3.14.1.**

Let $I \subseteq S$ be an ideal. Then

$$\operatorname{depth}(S/I) \leq \text{dimension of the smallest component of } V_\infty(I).$$

---

**Example 3.12.**

Let $S = K[x, y]$, $I = \langle x^2, xy \rangle$. Then

$$V_\infty(I) = y\text{-axis} + \text{extra fuzz at the origin.}$$

Then

$$\dim(S/I) = 1$$

and

$$\operatorname{depth}(S/I) = 0.$$

We can find the Betti table of $S/I$ to be

| | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 1 | 0 | 0 |
| 1 | 0 | 2 | 1 |

so that

$$\operatorname{pd}(S/I) = 2$$

and

$$\operatorname{reg}(S/I) = 1.$$

The Hilbert series of $S/I$ is

$$H(S/I, t) = \frac{1 - 2t^2 - t^3}{(1-t)^2}$$

and the Grothendieck polynomial of $S/I$ is

$$\mathfrak{G}(S/I, t) = t + t^2 - t^3.$$

This means $\operatorname{codim}(S/I) = \dim(S/I) = \deg(S/I) = 1$.

---

Def'n 3.23. **Cohen-Macaulay** $S$-module

Let $M$ be a finitely generated graded $S$-module. We say $M$ is ***Cohen-Macaulay*** if

$$\operatorname{depth}(M) = \dim(M).$$

**Proposition 3.15.**

If $S/I$ is Cohen-Macaulay, then $V_\infty(I)$ is equidimensional (all components have the same dimension).

**Proof.** It suffices to note that

$$\text{dimension of a largest component} = \dim(M) = \operatorname{depth}(M) \leq \text{dimension of a smallest component.}$$

**QED**

**Example 3.13.**

Let $S = K[x, y]$, $I = \langle xy \rangle$. Then $V_\infty = x\text{-axis} \cup y\text{-axis}$ and $\dim(S/I) = 1$. The Betti table is

| | 0 | 1 |
|---|---|---|
| 0 | 1 | 0 |
| 1 | 0 | 1 |

so that $\operatorname{pd}(S/I) = \operatorname{depth}(S/I) = \operatorname{reg}(S/I) = 1$. Hence $S/I$ is Cohen-Macaulay.

Note

$$H(S/I, t) = \frac{1 - t^2}{(1-t)^2}, K(S/I, t) = 1 - t^2, \mathfrak{G}(S/I, t) = 2t - t^2.$$

This shows that $\operatorname{codim}(S/I) = 1$ and $\deg(S/I) = 2$. Note that we have $-t^2$ term; this shows that we are *counting something with codimension* 2, *namely a point, twice.*

**Proposition 3.16.**

A curve $V_\infty(I)$ is Cohen-Macaulay (i.e. $S/I$ is Cohen-Macaulay) if and only if the origin is not an embedded component (i.e. no extra fuzz at the origin).

**Proof.** A curve has $\dim(S/I) = 1$ and $\operatorname{depth}(S/I) \leq 1$. If there is an embedded point, then that point has dimension 0, so that $\operatorname{depth}(S/I) = 0$ so that $S/I$ is not Cohen-Macaulay. Otherwise, there is a nonzerodivior $f$ on $S/I$. This means $\operatorname{depth}(S/I) \geq 1$, as needed.

**QED**

**Proposition 3.17.**

If $V_\infty(I)$ is 0-dimensional, then $S/I$ is Cohen-Macaulay.

**Proof.** Note $\dim(S/I) = 0 = \operatorname{depth}(S/I)$.

**QED**

**Example 3.14.**

Let $S = K[x, y, z]$, $I = \langle xy, xz \rangle$. Then

$$V_\infty(I) = x\text{-axis} \cup x^\perp.$$

Hence

$$\begin{aligned} \dim(S/I) &= 2 \\ \operatorname{depth}(S/I) &\leq 1 \end{aligned},$$

so that $S/I$ is not Cohen-Macaulay.

The Betti table is

| | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 1 | 0 | 0 |
| 1 | 0 | 2 | 1 |

so that

$$\mathfrak{G}(S/I, t) = t + t^2 - t^3.$$

Again, note that the Grothendieck polynomial counts the components; there are one codimension 1 component ($x^\perp$) and one codimension 2 component ($x$-axis) indicated by $t$, $t^2$, respectively; $-t^3$ shows that a codimension 3 component (origin) is counted twice.

A nonzerodivisor on $S/I$ is $x - z$. Note

$$J = I + \langle x - z \rangle = \langle xy, xz, x - z \rangle = \left\langle xy, x^2, x - z \right\rangle,$$

since $x - z = 0$ if and only if $x = z$. But this means that $z$ is *useless* in a sense that it is constantly equal to $x$, and so we have

$$S/J \cong K[x, y] / \left\langle xy, x^2 \right\rangle.$$

**Example 3.15.**

Let $S = K[x, y, z, w]$ and let

$$I = \langle xz, xw, yz, yw \rangle = \langle xy \rangle \wedge \langle z, w \rangle,$$

so that

$$V_\infty(I) = x, y\text{-plane} \cup z, w\text{-plane}.$$

Hence

$$\begin{aligned} \dim(M/I) &= 2 \\ \operatorname{depth}(M/I) &\leq 2 \end{aligned}.$$

Let $f = y - w$, which is a nonzerodivisor on $S/I$, and let

$$J = I + \langle f \rangle = \langle xz, xw, yz, yw, y - w \rangle = \left\langle xz, xy, yz, y^2, y - w \right\rangle,$$

so that

$$S/J \cong K[x, y, z] / \left\langle xz, xy, yz, y^2 \right\rangle.$$

Note

$$V_\infty\left(\left\langle xz, xy, yz, y^2 \right\rangle\right) = z\text{-axis} \cup x\text{-axis} \cup \text{fuzz in } y\text{-direction at origin.}$$

Hence depth $(S/J) = 0$, so that depth $(S/I) = 1$.

Thus $S/I$ is not Cohen-Macaulay.

Note that the Betti table is

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 1 | | | |
| 1 | | 4 | 4 | 1 |

This means

$$\mathfrak{G}(S/I, t) = 2t^2 - t^4,$$

so the Grothendieck polynomial still counts things correctly; there are two 2-codimensional components $x, y$-axis and $z, w$-axis corresponding to $2t^2$ and a 4-codimensional component (origin) is counted twice corresponding to $-t^4$.

---

**Theorem 3.18.**

For all $i, j \in \mathbb{Z}$,

$$\beta_{i,j}(S/I) \leq \beta_{i,j}(S/\operatorname{in}(I)).$$

In particular,

$$\begin{aligned} \operatorname{reg}(S/I) &\leq \operatorname{reg}(S/\operatorname{in}(I)) \\ \operatorname{pd}(S/I) &\leq \operatorname{pd}(S/\operatorname{in}(I)) \\ \operatorname{depth}(S/I) &\geq \operatorname{depth}(S/\operatorname{in}(I)) \end{aligned}$$

If $S/\operatorname{in}(I)$ is Cohen-Macaulay, then so is $S/I$.

---

Def'n 3.24. **Extremal** Betti Numbers

A Betti number $\beta_{i,j}$ is ***extremal*** if $\beta_{r,s} = 0$ for all $r > i, s > j$.

**Theorem 3.19.** Conca-Varbaro

If in $(I)$ is squarefree, then all extremal Betti numbers of $S/I$ and $S/\operatorname{in}(I)$ conincide. In particular,

$$\begin{aligned} \operatorname{reg}(S/I) &= \operatorname{reg}(S/\operatorname{in}(I)) \\ \operatorname{pd}(S/I) &= \operatorname{pd}(S/\operatorname{in}(I)) \\ \operatorname{depth}(S/I) &= \operatorname{depth}(S/\operatorname{in}(I)) \end{aligned}$$

and $S/I$ is Cohen-Macaulay if and only if $S/\operatorname{in}(I)$ is.

---

Def'n 3.25. **Abstract Simplicial Complex**
An ***abstract simplicial complex*** is a collection $\Delta$ of subsets closed under taking subsets. That is, if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$. In other words, it is an order ideal of a boolean lattice.

Each element of $\Delta$ is called ***face*** and the ***dimension*** of $F \in \Delta$ is $\dim(F) = |F| - 1$. A maximal face is called a ***facet***. We write $\mathcal{F}(\Delta)$ to denote the collection of facets of $\Delta$.

The ***dimension*** of $\Delta$ $\dim(\Delta)$ is the maximum of the dimension of faces.

**Example 3.16.**
Suppose an abstract simplicial complex $\Delta \subseteq \mathcal{P}(\{1, \ldots, 5\})$ has facets

$$\{1, 2, 3\}, \{2, 4\}, \{3, 4\}, \{5\}$$

(picture)

---

Def'n 3.26. ASC **Generated** by Facets
Given a set $\mathcal{F}$ of facets, we write

$$\langle \mathcal{F} \rangle = \{G \subseteq F : F \in \mathcal{F}\}$$

to be the abstract simplicial complex ***generated*** by $\mathcal{F}$.

Def'n 3.27. **Pure** ASC
We say an ASC $\Delta$ is ***pure*** if every facets of $\Delta$ have the same dimension.

**Example 3.17.** Two Degenerate Abstract Simplicial Complexes
$\Delta = \emptyset$ is called the ***void complex***, which has no face with $\dim(\Delta) = -\infty$.

$\Delta = \{\emptyset\}$ is called the ***irrelevant complex***, which has one face $\emptyset$ with $\dim(\Delta) = -1$.

---

Def'n 3.28. $f$-**vector** of an ASC
Let $\Delta$ be an ASC and let $f_i$ be the number of faces of dimension $i$. The ***f-vector*** of $\Delta$ is

$$f(\Delta) = \left(f_{-1}, f_0, \ldots, f_{\dim(\Delta)}\right).$$

**Example 3.18.** $n$-simplex
The ***n-simplex*** is

$$\Delta = \langle 1, \ldots, n \rangle.$$

$\Delta$ has one facet $[n]$, which is an $(n-1)$-dimensional analogue of tetrahedron. Then,

$$f(\Delta) = \left(1, n, \binom{n}{2}, \ldots, \binom{n}{n-1}, 1\right) = \left(\binom{n}{k}\right)_{k=0}^{n}.$$

The ***boundary*** of the $n$-simplex is

$$\partial\Delta = \langle [n] \setminus \{1\}, \ldots, [n] \setminus \{n\} \rangle.$$

Note that

$$f(\partial\Delta) = \left(1, n, \binom{n}{2}, \ldots, \binom{n}{n-1}\right) = \left(\binom{n}{k}\right)_{k=0}^{n-1}.$$

---

Let $S = K[x_1, \dots, x_n]$. We are going to identify each variable with an element in an $n$-set, say $\{1, \dots, n\}$. For $F \subseteq [n]$, let

$$x_F = \prod_{i \in F} x_i$$

be a squarefree monomial.

For a simplicial complex, $I_\Delta$ is the squarefree monomial ideal

$$I_\Delta = \langle x_F : F \notin \Delta \rangle .$$

We are taking all the non-faces, since we are going to quotient the whole ring $S$ by $I_\Delta$, so that we can get rid of what's in $I_\Delta$. Note that it suffices to take

$$I_\Delta = \langle x_F : F \text{ is a } \textit{minimal} \text{ non-face} \rangle .$$

For instance, for $\Delta = \langle \{1, 2, 3, \}, \{2, 4\}, \{3, 4\}, \{5\} \rangle$ in Example 3.16,

$$I_\Delta = \langle x_2x_5, x_4x_5, x_1x_5, x_3x_5, x_2x_3x_4, x_1x_4 \rangle .$$

Def'n 3.29. **Stanley-Reisner Ideal**, **Stanley-Reisner Ring** of a ASC

Let $\Delta$ be an ASC. We call

$$I_\Delta = \langle x_F : F \notin \Delta \rangle$$

is called the ***Stanley-Reisner ideal*** of $\Delta$.

The quotient ring $S/I_\Delta$ is called the ***Stanley-Reisner ring*** of $\Delta$.

**Theorem 3.20.**

The mapping

$$\Delta \mapsto I_\Delta$$

is a bijection between abstract simplicial complexes on $[n]$ and squarefree monomial ideals in $K[x_1, \dots, x_n]$.

Moreover,

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} \langle x_i : i \notin F \rangle$$

**Proof.** Note that the squarefree monomials in $S$ with $m + I_\Delta \neq 0$ in $S/I_\Delta$ are exactly the monomials $x_F$ with $F \in \Delta$. So the correspondence is bijective.

Note $x_G \in \bigcap_{F \in \mathcal{F}(\Delta)} \langle x_i : i \notin F \rangle$ if and only if $G \nsubseteq F$ for all $F \in \mathcal{F}(\Delta)$ if and only if $G$ is not a face.

**QED**

For instance, the prime decomposition of

$$I_\Delta = \langle x_2x_5, x_4x_5, x_1x_5, x_3x_5, x_2x_3x_4, x_1x_4 \rangle$$

is

$$I_\Delta = \langle x_4, x_5 \rangle \cap \langle x_1, x_2, x_5 \rangle \cap \langle x_1, x_3, x_5 \rangle \cap \langle x_1, x_2, x_3, x_4 \rangle .$$

Note that each prime ideal above is the complement of a facet of $\Delta$.

**Lemma 3.21.**

Let $\Delta$ be an ASC grounded on an $n$-set. Then

$$V_\infty(I_\Delta) \text{ equidimensional} \iff \Delta \text{ is pure.}$$

**Proof.** It suffices to observe the following phenomenon.

If $F$ is a facet of $\Delta$ with $\dim(F) = k$, then $|F| = k+1$ by definition, so corresponds to a monomial prime ideal generated by $n-(k+1) = n-k-1$ variables in the prime decomposition of $I_\Delta$. Hence $F$ corresponds to a component of codimension $n-k-1$.

**QED**

---

**Example 3.19.**

Let $\Delta = \langle \{x\}, \{y\} \rangle$,

$$I_\Delta = \langle xy \rangle = \langle x \rangle \cap \langle y \rangle .$$

---

**Example 3.20.**

Let $\Delta = \langle \{x, y\}, \{z\} \rangle$. Then

$$I_\Delta = \langle xz, yz \rangle = \langle z \rangle \cap \langle x, y \rangle .$$

---

Def'n 3.30. **Cohen-Macaulay** ASC

We say an ASC $\Delta$ is ***Cohen-Macaulay*** over a field $K$ if $K[\vec{x}]/I_\Delta$ is Cohen-Macaulay.

**Lemma 3.22.**

If $\Delta$ is Cohen-Macaulay, then $\Delta$ is pure.

**Proof.** If $\Delta$ is not pure, then $V_\infty(I_\Delta)$ is not equidimensional, so $S/I_\Delta$ is not Cohen-Macaulay

**QED**

---

**Example 3.21.**

Let $\Delta = \langle \{x, y\}, \{z, w\} \rangle$. Then

$$I_\Delta = \langle xw, xz, yw, yz \rangle = \langle x, y \rangle \cap \langle z, w \rangle .$$

Hence $V_\infty(I_\Delta)$ is equidimensional (equivalently, $\Delta$ is pure), but $S/I_\Delta$ (and so $\Delta$) is not Cohen-Macaulay.

---

Def'n 3.31. **Shelling** of a Pure ASC

Let $\Delta$ be a pure ASC. A ***shelling*** of $\Delta$ is an ordering of facets $F_1, \ldots, F_k$ such that for $i > 1$, $\langle F_i \rangle \cap \langle F_1, \ldots, F_{i-1} \rangle$ is generated by a nonempty set of maximal proper faces of $F_i$.

**Example 3.22.**

Let $\Delta = \langle \{x, y\}, \{x, z\}, \{y, z\} \rangle$. Then $F_1 = \{x, y\}, F_2 = \{y, z\}, F_3 = \{x, z\}$ is a shelling.

---

**Example 3.23.**

Let $\Delta = \langle F_1 = \{x_1, x_2, x_3\}, F_2 = \{x_2, x_3, x_4\}, F_3 = \{x_3, x_4, x_5\} \rangle$. Then $F_1, F_2, F_3$ is a shelling but $F_1, F_3, F_2$ is not.

---

**Theorem 3.23.**

If $\Delta$ is shellable, then $\Delta$ is Cohen-Macaulay over every field.

---

**Theorem 3.24.**

Let $\Delta$ be an ASC. Then

$$\Delta \text{ is Cohen-Macaulay} \iff \tilde{H}_i(\Delta, K) = 0 = H_i(\Delta, \Delta \setminus \{p\}; K) \text{ for all points } p \in \Delta \text{ and } i < \dim(\Delta).$$

---

**Theorem 3.25.**

There is a triangulation of a tetrahedron (with 41 facets) that is not shellable.

---

Def'n 3.32. **Deletion**, **Link** of an ASC

Let $\Delta$ be an ASC and let $F \subseteq \Delta$. A ***deletion*** is

$$\operatorname{del}(F, \Delta) = \{G \in \Delta : G \cap F = \emptyset\}$$

and a ***link*** is

$$\operatorname{link}(F, \Delta) = \{G \in \operatorname{del}(F, \Delta) : G \cup F = \Delta\}.$$

Def'n 3.33. **Vertex-decomposable** Pure ASC

Let $\Delta$ be a pure ASC. We say $\Delta$ is ***vertex-decomposable*** if $\Delta = \{\emptyset\}$ or there is vertex $v \in \Delta$ such that both $\operatorname{del}(\{v\}, \Delta), \operatorname{link}(\{v\}, \Delta)$ are vertex-decomposable.

**Theorem 3.26.**

If $\Delta$ is vertex-decomposable, then $\Delta$ is shellable.

---

**Example 3.24.** Generalized Determinantal Varieties

Previously, we looked at the space of $a \times b$ of matrices of rank at most $r$. It was defiend by the vanishing of all $(r+1) \times (r+1)$ minors of a matrix of variables, say $Z = \left[Z_{i,j}\right]_{i,j=1}^{a,b}$. This is called the *classical determinantal varieties*.

More generally, consider a rank matrix

$$r = \left[r_{i,j}\right]_{i,j=1}^{n} \in (\mathbb{N} \cup \{\infty\})^{n \times n}.$$

The ***northwest rank variety*** is

$$n_r = \left\{M \in K^{n \times n} : \forall i, j \left[\operatorname{rank}\left(M_{[i],[j]}\right) \leq r_{i,j}\right]\right\},$$

where

$$M_{[i],[j]} = \begin{bmatrix} M_{1,1} & \cdots & M_{1,j} \\ \vdots & \ddots & \vdots \\ M_{i,1} & \cdots & M_{i,j} \end{bmatrix}.$$

Note that we can still recover the rectangular case by putting $\infty$ outside the rectangle. Classical determinental varieties can be obtained from setting all $r_{i,j}$ equal.

Note that many rank conditions define the same space of matrices. For instance,

$$X_{\begin{bmatrix} 4 & 1 \\ 3 & 7 \end{bmatrix}} = X_{\begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}} = X_{\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}} = X_{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}} = K^{2 \times 2}.$$

Unlike the above example, interesting ones have *alternating sign* matrices as their rank matrices.

---

Def'n 3.34. **Alternating Sign** Matrix

We say $A \in K^{n \times n}$ is an ***alternating sign*** matrix if

(a) each $A_{i,j} \in \{-1, 0, 1\}$;

(b) each row and column adds to 1; and

(c) in each row and column, nonzero entries alternate in sign.

**Example 3.25.**

Any permutation matrix is an alternating sign matrix. The smallest (and the only $3 \times 3$) alternating sign matrix is

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

---

**Theorem 3.27.**

For all $n \in \mathbb{N}$,

$$|\mathrm{ASM}(n)| = \text{number of alternating sign matrices of size } n \times n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

For instance, when $n = 3$,

$$|\mathrm{ASM}(3)| = \frac{1!}{3!}\frac{4!}{4!}\frac{7!}{5!} = \frac{7!}{3!5!} = 7.$$

**Example 3.26.**

Let $A$ be an ASM, associate a ***corner-sum matrix*** $r(A) \in K^{n\times n}$ by

$$r(A)_{a,b} = \sum_{i=1}^{a}\sum_{j=1}^{b} A_{i,j}, \qquad \forall a, b \in [n].$$

For instance, for

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we have

$$r(A) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

On the other hand, for

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

we have

$$r(B) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Note that both $r(A)$, $r(B)$ are nondecreasing to right and to below.

**Theorem 3.28.**

For each rank matrix $r$, there is a $d \in \mathbb{N}$ and an ASM $A$ such that

$$X_r \times K^d \cong X_{r(A)}.$$

Given $A \in \mathrm{ASM}(n)$, we call $X_{r(A)}$ an ***ASM variety***.

We can turn $\mathrm{ASM}(n)$ into a poset by $A \geq B$ if and only if

$$r(A)_{i,j} \leq r(B)_{i,j}, \qquad \forall i, j.$$

For instance

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leq \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = B,$$

since

$$r(A) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

is entrywise larger than

$$r(B) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

**Theorem 3.29.**

The poset $\mathrm{ASM}(n)$ is a lattice. That is, for any $A, B \in \mathrm{ASM}(n)$, there is a least upper bound $A \vee B$ and a greates lower bound $A \wedge B$.

In fact,

$$r(A \vee B)_{i,j} = \min\left(r(A)_{i,j}, r(B)_{i,j}\right)$$

and

$$r(A \wedge B)_{i,j} = \max\left(r(A)_{i,j}, r(B)_{i,j}\right).$$

---

Notation 3.35. $\mathrm{perm}(A)$

For $A \in \mathrm{ASM}(n)$, we define

$$\mathrm{perm}(A) = \{w \in S_n : w \geq A, \text{given } v \in S_n \text{ with } w \leq v \leq A, w = v\}.$$

Observe that, when $A$ is a permutation matrix of $w \in S_n$, then $\mathrm{perm}(A) = \{w\}$.

For $A \in \mathrm{ASM}(n)$, equations for $X_A$ are the ideal

$$I_A = \sum_{i,j=1}^{n} I_{r(A)_{i,j}+1}\left(Z_{[i][j]}\right),$$

where

$$I_k\left(Z_{[i][j]}\right) = \left(\text{all } k + k \text{ minors of } Z_{[i][j]}\right).$$

Def'n 3.36. **Diagonal**, **Antidiagonal** Term Order

A term order on $Z$ is ***(anti)diagonal*** if the leading term of each minor is the product of the variables on the main (anti)diagonal.

**Theorem 3.30.** Knutson-Miller

For any antidiagonal order on $Z$, the defining equations of $I_w$, where $w \in S_n$, are a Grobner basis for $I_w$.

---

**Corollary 3.30.1.**

For any $w \in S_n$, $I_w$ is radical.

**Proof.** Since $I_w$ admits a Grobner basis, there is a Grobner degeneration from $V_\infty(I_w)$ to $V_\infty(\in(I_w)) = V_\infty(I_\Delta)$ for some ASC $\Delta$. Since $I_\Delta$ is radical, so is $I_w$.

**QED**

The symmetric group $S_n$ has generators $s_i = (i, i+1)$ for $i \in \{1, \ldots, n-1\}$. Let

$$q = q_1 \cdots q_m$$

be a string in the alphabet $[n-1]$. We think of a substring of $q$ as a face of the ASC $\langle [m] \rangle$.

We say a string $p = p_1 \cdots p_k$ *represents* $w \in S_n$ if

$$w = s_{p_1} \cdots s_{p_k}$$

and $w$ canoot be written as a product of fewer generators. We say a string $p$ *contains* $w$ if some substring of $p$ represents $w$.

Def'n 3.37. **Subword Complex**

The ***subword complex*** $\Delta(q, w)$ associated to a string $q$ and a permutation $w$ is

$$\Delta(q, w) = \{q \setminus p : p \text{ contains } w\}.$$

Note that the facets of $\Delta(q, w)$ are $q \setminus p$ where *$p$ represents $w$*.

**Example 3.27.**

Let $q = 123121$, $w = s_1 s_2 s_1 = 3214 = s_2 s_1 s_2$. Then observe that

$$\begin{gathered} \underline{12}3\underline{1}21 \\ \underline{12}312\underline{1} \\ \underline{1}231\underline{21} \\ 123\underline{121} \\ 1\underline{2}3\underline{12}1 \end{gathered} \tag{3.1}$$

are the substrings 121, 212 that we can find inside $q$. The simplicial complex $\Delta(q, w)$ is

(picture)

Observe that [3.1] is a shelling of $\Delta(q, w)$.

**Theorem 3.31.**

Every subword complex $\Delta(q, w)$ is vertex-decomposable. Thus $\Delta(q, w)$ is shellable and Cohen-Macaulay.

**Proof.** Write $q = q_1 \cdots q_m$, which we want to vertex decompose at vertex $m$. Note that

$$\operatorname{link}(m, \Delta(q, w)) = \Delta(q_1 \cdots q_{m-1}, w)$$

is a subword complex.

If no minimal factorization of $w$ ends with $s_{q_m}$, then

$$\operatorname{del}(m, \Delta(q, w)) = \operatorname{link}(m, \Delta(q, w)).$$

Otherwise,

$$\operatorname{del}(m, \Delta(q, w)) = \Delta\left(q_1 \cdots q_{m-1}, w s_{q_m}\right).$$

By induction, the result holds.

**QED**

**Example 3.28.**

Consider

$$w = 2143$$

which is represented as

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then

$$r_w = \begin{bmatrix} \underline{0} & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & \underline{2} & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

Observe that, every entries except for the two underlined entries are imposing trivial rank conditions. Hence

$$I_w = \left\langle z_{1,1}, \det \begin{bmatrix} z_{1,1} & z_{1,2} & z_{1,3} \\ z_{2,1} & z_{2,2} & z_{2,3} \\ z_{3,1} & z_{3,2} & z_{3,3} \end{bmatrix} \right\rangle$$

which means, using the antidiagonal term order,

$$\mathrm{in}\,(I_w) = \langle z_{1,1}, z_{1,3}z_{2,2}z_{3,1} \rangle = \langle z_{1,1}, z_{1,3} \rangle \cap \langle z_{1,1}, z_{2,2} \rangle \cap \langle z_{1,1}, z_{3,1} \rangle .$$

Using ***pipe dream*** $P$, the generators of in $(I_w)$ are

$$\underbrace{\begin{bmatrix} + & \cdot & + & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}}_{=P_1} \underbrace{\begin{bmatrix} + & \cdot & \cdot & \cdot \\ \cdot & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}}_{=P_2}, \underbrace{\begin{bmatrix} + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}}_{=P_3}.$$

We label the cells by

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}.$$

By reading from right to left, top to bottom, we obtain words

$$Q\,(P_1) = 31, Q\,(P_2) = Q\,(P_3) = 13.$$

The ***square word*** $Q$ is

$$Q = 4321543265437654.$$

Note that every subword of $Q$ corresponds to a pipe dream $P$. Conversely, a pipe dream $P$ with $k$ $+$'s represents $w \in S_n$ if

$$w = s_{Q(P)1} s_{Q(P)2} \cdots s_{Q(P)_k}$$

is a reduced word for $w$.

Note that, $P_1$ turns into $w = s_3 s_1$ and $P_2, P_3$ turn into $w = s_1 s_3$. It turns out $s_3 s_1, s_1 s_3$ are the only reduced words for 2143 and $P_1, P_2, P_3$ are the only pipe dreams that represent 2143.

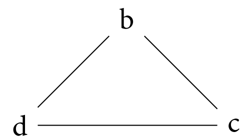
Consider the ASC

$$\Delta = \Delta\,(Q, 2143)$$

called the ***pipe dream complex***. This corresponds to 13-dimensional three facets corresponding to (the complements of) $P_1, P_2, P_3$. To make things easier, we observe that the only positions where $+$'s can occur in the pipe dreams corresponding to 2143 are

$$\begin{bmatrix} a & \cdot & b & \cdot \\ \cdot & c & \cdot & \cdot \\ d & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

so that by considering only four positions $a, b, c, d$, $\Delta$ looks like




except for the fact that we have to include 12 additional vertices for each line segment, which is a *puffed up version* of a triangle.

---

**Theorem 3.32.** Knutson-Miller

For $w \in S_n$ and any antidiagonal term order,

$$\operatorname{in}(I_w) = I_{\Delta(Q,w)} = \bigcap_{P \in \operatorname{pd}(w)} \langle z_{i,j} : P \text{ has a } + \text{ in position } (i,j) \rangle = \left\langle \prod_{\substack{P \text{ has a} \\ + \text{ in } (i,j)}} z_{i,j} : P \in \operatorname{pd}(w) \right\rangle.$$

where $\operatorname{pd}(w)$ is the set of pipe dreams corresponding to $w$.

---

**Corollary 3.32.1.**

For $w \in S_n$, $\operatorname{in}(I_w)$ is Cohen-Macaulay. Consequently, $I_w$ is Cohen-Macaulay.

**Proof.** Note $\operatorname{in}(I_w)$ is Cohen-Macaulay since it is a Stanley-Reisner ideal for a vertex-decomposable ASC $\Delta(Q, w)$.

This means $I_w$ is also Cohen-Macaulay, because it Grobner degenerates to $\operatorname{in}(I_w)$.

**QED**

**Corollary 3.32.2.**

For $w \in S_n$,

$$\operatorname{codim}(R/I_w) = \operatorname{codim}\left(R/I_{\Delta(Q,w)}\right) = \text{number of } +\text{'s in a pipe dream representing } w = l(w),$$

where $l(w)$ is the length of $w$, the length of a reduced word for $w$.

Moreover,

$$\deg(R/I_w) = \deg\left(R/I_{\Delta(Q,w)}\right) = |\operatorname{pd}(w)| = \frac{1}{l(w)!} \sum_{s \in \mathcal{R}(w)} \prod_{j=1}^{l(w)} s_j,$$

where $\mathcal{R}(w)$ is the set of reduced words for $w$.

---

Note that the key idea for Corollary 3.32.2 is that, codim, deg are things that we can read-off from Hilbert series, so is invariant of Grobner degeneration.

**Example 3.29.**

For $w = 2143$, the reduced words are

$$\mathcal{R}(w) = \{13, 31\},$$

so that

$$\deg(R/I_w) = |\operatorname{pd}(w)| = 3 = \frac{1}{2!}(1 \cdot 3 + 3 \cdot 1).$$

Similarly, for $w = 3214$, the reduced words are

$$\mathcal{R}(w) = \{121, 212\},$$

so that

$$\deg(R/I_w) = \frac{1}{3!}(1 \cdot 2 \cdot 1 + 2 \cdot 1 \cdot 2) = 1.$$

This means there is only one pipe dream corresponding to $w$. But note that we cannot find a pipe dream for 121, since there is only one 1 in the labeling of the cells. The only pipe dream that corresponds to 212 is

$$\begin{bmatrix} + & + & \cdot & \cdot \\ + & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

---

**Theorem 3.33.**
For any ASC $\Delta$, the $K$-polynomial of $S/I_\Delta$ is

$$K(S/I_\Delta, t) = \sum_{\sigma \in \Delta} t^{|\sigma|} (1-t)^{n-|\sigma|} = \sum_{j=0}^{n} f_{j-1} t^j (1-t)^{n-j},$$

where $n = \dim(\Delta)$ and $f = (f_{-1}, \ldots, f_n)$ is the $f$-vector of $\Delta$.

**Proof.** The squarefree monomials not in $I_\Delta$ correspond to faces of $\Delta$. In general, a monomial $u$ is not in $I_\Delta$ exactly if $\sqrt{u}$ is not in $I_\Delta$. So

$$H(S/I_\Delta, t) = \sum_{\substack{\text{monomial } u: \\ \sqrt{u} \in \Delta}} t^{|u|} = \sum_{\sigma \in \Delta} \sum_{\substack{\text{monomial } u: \\ \sqrt{u} = \sigma}} t^{|u|} = \sum_{\sigma \in \Delta} t^{|\sigma|} (1-t)^{-|\sigma|},$$

since, given a face

$$\sigma = \{\sigma_1, \ldots, \sigma_l\},$$

observe that

$$u = x_{\sigma_1}^{p_1} \cdots x_{\sigma_l}^{p_l}$$

for any $p_1, \ldots, p_l \geq 0$ is such that $\sqrt{u} = \sigma$ (i.e. $\sqrt{u} = x_{\sigma_1} \cdots x_{\sigma_l}$), so the corresponding generating series is

$$\left(1 + t + t^2 + \cdots\right)^l = (1-t)^{|\sigma|}.$$

It follows that

$$K(S/I_\Delta, t) = \sum_{\sigma \in \Delta} t^{|\sigma|} (1-t)^{n-|\sigma|}.$$

**QED**

---

**Corollary 3.33.1.**
For any ASC $\Delta$,

$$\mathfrak{G}(S/I_\Delta, t) = \sum_{\sigma \in \Delta} (1-t)^{|\sigma|} t^{n-|\sigma|}.$$

---

**Proof of Corollary 3.32.2**
We have

$$K\left(S/I_{\Delta(Q,w)}, t\right) = \sum_{j=0}^{n^2} f_{j-1} t^j (1-t)^{n^2-j}$$

by Theorem 3.33, so

$$\mathfrak{G}\left(S/I_{\Delta(Q,w)}, t\right) = \sum_{j=0}^{n^2} f_{j-1} (1-t)^j t^{n^2-j}.$$

Hence the loweset degree term is

$$f_{j-1}t^{n^2-j},$$

where $j = \max\left\{0 \le i \le n^2 : f_{i-1} \ne 0\right\}$. Hence

$$\operatorname{codim}\left(S/I_{\Delta(Q,w)}\right) = n^2 - j,$$

where $j - 1 = \dim(\Delta)$. Hence

$$\operatorname{codim}\left(S/I_{\Delta(Q,w)}\right) = \text{number of } +\text{'s in a pipe dream for } w = l(w).$$

Moreover,

$$\deg\left(S/I_{\Delta(Q,w)}\right) = f_{\dim(\Delta(Q,w))} = \text{number of facets of } \Delta(Q,w) = |\mathrm{pd}(w)|.$$

For the proof of last formula, read Hamaker-Pechenik-Speyer-Weigandt (2020).[1]

[1] According to Prof. Pechenik, this paper has the best proof!!

**QED**

## 4. Regularity

**Theorem 3.34.**
If $S/I$ is Cohen-Macaulay, then

$$\operatorname{reg}(S/I) = \deg(K(S/I,t)) - \operatorname{codim}(S/I).$$

**tl;dr**

**Corollary 3.34.1.**
For $w \in S_n$,

$$\operatorname{reg}(S/I_w) = \deg(K(S/I_w)) - l(w).$$

Hence we need to understand the igh degree part of $K(S/I_w)$.

Recall that, given an ASC $\Delta$,

$$I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} \left\langle x_j : j \notin F\right\rangle = \langle x_\sigma : \sigma \notin \Delta\rangle.$$

If we *get confused* and mix the definitions, we obtain the following.

Def'n 3.38. **Alexander Dual**, **Alexander Dual Ideal** of an ASC
Let $\Delta$ be an ASC. The ***Alexander dual*** of $\Delta$, denoted as $\Delta^*$, is

$$\Delta^* = \{[n] \setminus \sigma : \sigma \notin \Delta\}.$$

The ***Alexander dual ideal*** of $\Delta$, denoted as $I_\Delta^*$, is

$$I_\Delta^* = \left\langle x_{[n]\setminus\sigma} : \sigma \in \Delta\right\rangle = \left\langle x_{[n]\setminus F} : F \in \mathcal{F}(\Delta)\right\rangle = \bigcap_{\sigma \notin \Delta} \left\langle x_j : j \in \sigma\right\rangle.$$

It is immediate that.

$$I_\Delta^* = I_{\Delta^*}.$$

Def'n 3.39. **O-Hecke Algebra**

The ***O-Hecke algebra*** $H_n$ is a vector space with basis

$$\{\pi_w : w \in S_n\}$$

and a multiplication given by

$$\pi_w \pi_{s_i} = \begin{cases} \pi_{ws_i} & \text{if } l(ws_i) > l(w) \\ \pi_w & \text{otherwise} \end{cases}, \qquad \forall w \in S_n, i \in \{1, \ldots, n-1\}.$$

For a string $p = p_1 \cdots p_k$ in the alphabet $[n-1]$, if $\pi_{s_{p_1}} \pi_{s_{p_2}} \cdots \pi_{s_{p_k}} = \pi_w$, then define

$$\delta(p) = w.$$

**Theorem 3.35.** Knutson-Miller

For any substring $Q'$ of $Q$ and any $w \in S_n$, $\Delta(Q', w)$ is homeomorphic to either a ball or a sphere.

More precisely, $\Delta(Q', w)$ is a sphere if and only if $\delta(Q') = w$ and $\Delta(Q', w)$ is a ball if and only if $\delta(Q') \neq w$.

A face $Q' \setminus P$ is in the bounday of the ball $\Delta(Q', w)$ if $\delta(P) \neq w$.

**Corollary 3.35.1.**

$\Delta(Q, w)$ is (homeomorphic to) a ball.

**Lemma 3.36.**

Let $\Delta = \Delta(Q, w)$. Then

$$K(I_\Delta^*) = \sum_{P \subseteq Q : \delta(P) = w} (-1)^{|P| - l(w)} t^{|P|}.$$

**Theorem 3.37.** Alexander Inversion Theorem

**Lemma 3.38.**

Suppose $P \subset Q$ represents $w \in S_n$ and $q_i \in Q \setminus P$ satisfies

$$w = \delta(P) = \delta(P \cup \{q_i\}).$$

Then there is a unique $p_j \in p$ such that

$$\delta(P) = w = \delta\left(P \cup \{q_i\} \setminus \{p_j\}\right).$$

We say $q_i$ is ***absorbable*** if $j > i$. We define

$$\text{abs}(P) = \text{number of absorbable letters of } Q \setminus P.$$

**Theorem 3.39.**

Let $\Delta = \Delta(Q, w)$. Then

$$K(S/I_\Delta, t) = \sum_{P \subseteq, P \text{ represents } w} (1-t)^{|p|} t^{\text{abs}(P)}$$

**Corollary 3.39.1.**
For $\Delta = \Delta(Q, w)$,
$$\operatorname{reg}(S/I_\Delta) = \max\{\operatorname{abs}(P) : P \subseteq Q \text{ represents } w\}.$$

**Corollary 3.39.2.**
For $w \in S_n$,
$$\operatorname{reg}(S/I_w) = \operatorname{reg}\left(S/I_{\Delta(Q,w)}\right) = \max_{p \in \operatorname{pd}(w)} \operatorname{abs}(w).$$

Def'n 3.40. **Rajchgot Index** of a Permutation
Let $w \in S_n$. For $i \in [n]$, find a maximal increasing subsequence of $w(i)\, w(i+1) \cdots w(n)$ containing $w(i)$. Let $r_i$ be the number of entries omitted. Then the ***Rajchgot index*** of $w$ is
$$\operatorname{raj}(w) = \sum_{i=1}^{n} r_i.$$

**Example 3.30.**
$\operatorname{raj}(2143) = 2 + 1 + 1 + 0 = 4.$

**Theorem 3.40.** Pechenik-Speyer-Weigandt
For any $w \in S_n$,
$$\operatorname{reg}(S/I_w) = \operatorname{raj}(w) - l(w).$$

### Summary of Known Things about $I_w$
For $I_w$, we know:
- $S/I_w$ is Cohen-Macaulay;
- $\operatorname{codim}(S/I_w) = \operatorname{pd}(S/I_w) = l(w)$;
- $\dim(S/I_w) = \operatorname{depth}(S/I_w) = n^2 - l(w)$;
- $\deg(S/I_w) = |\operatorname{pd}(w)|$; and
- $\operatorname{reg}(S/I_w) = \operatorname{raj}(w) - l(w)$.

On the other hand, here are some open questions.

### Open Questions
- What are other Betti numbers of $S/I_w$?
- *Regulartiy* of other complexes?
- For an ASM $A$, which $I_A$ are Cohen-Macaulay? For instance, when
$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$
we have
$$I_A = I_{4123} \cap I_{3412},$$
$V_\infty(I_{4123})$ has codimension 3 since there are three pairs out of order in 4123 but $V_\infty(I_{3412})$ has codimension 4.

- What are Grobner basis for $I_w$ for other term orders?

---

We look closely into the last question.

For

$$a = a_1 \cdots a_k$$

of distinct numbers, let

$$\text{flat}\,(a) = f_1 \cdots f_k \in S_k$$

be such that $f_i < f_j \iff a_i < a_j$ for all $i, j$.

For instance, if $a = 2, 7, -4, \pi, 0, 18$, then

$$\text{flat}\,(a) = 351426.$$

Def'n 3.41. Permutation **Containing**, **Avoiding** Another Permutation

We say a permutation $w = w_1 \cdots w_n$ ***contains*** permutation $p = p_1 \cdots p_k$ if there is a subsequence $w' = w_{i_1} \cdots w_{i_k}$ that flats to $p$.

If $w$ doesn't contain $p$, then we say we say $w$ ***avoids*** $p$.

For instance, if $w = 31425$, then the substring 145 flats to $p = 123$ and 342 flats to $q = 231$, so that $w$ contains $p, q$.

**Theorem 3.41.** Knutson-Miller-Yong

For any diagonal term order, the defining equations of $I_w$ form a Grobner basis if and only if $w$ avoid 2143. [1]

[1] Knutson-Miller-Yong also obtained an explicit combinatorial description of $\text{in}_{\text{diag}}\,(I_w)$ in this setting.

---

Def'n 3.42. **CDG Generators** of $I_w$

The ***CDG Generators*** of $I_w$ are the $1 \times 1$ minors of the defining equations, together with the minors of size $r\,(w)_{i,j} + 1$ in $\left[\hat{z}_{i,j}\right]_{1 \le i,j \le n}$, where

$$\hat{z}_{i,j} = \begin{cases} 0 & \text{if } r\,(w)_{i,j} = 0 \\ z_{i,j} & \text{otherwise} \end{cases}.$$

**Example 3.31.**

Let $w = 2143$. Then

$$r\,(w) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

and

$$I_w = \left\langle z_{1,1}, \det \begin{bmatrix} z_{1,1} & z_{1,2} & z_{1,3} \\ z_{2,1} & z_{2,2} & z_{2,3} \\ z_{3,1} & z_{3,2} & z_{3,3} \end{bmatrix} \right\rangle.$$

But we saw that

$$\left\{ z_{1,1}, \det \begin{bmatrix} z_{1,1} & z_{1,2} & z_{1,3} \\ z_{2,1} & z_{2,2} & z_{2,3} \\ z_{3,1} & z_{3,2} & z_{3,3} \end{bmatrix} \right\}$$

is not a Grobner basis, but the CDG generators

$$\left\{ z_{1,1}, \det \begin{bmatrix} 0 & z_{1,2} & z_{1,3} \\ z_{2,1} & z_{2,2} & z_{2,3} \\ z_{3,1} & z_{3,2} & z_{3,3} \end{bmatrix} \right\}$$

is a Grobner basis.

---

**Theorem 3.42.**

Fix a diagonal term order. Then the CDG generators of $I_w$ form a diagonal Grobner basis if and only if $w$ avoids

$$13254, 21543, 214635, 215364, 315264, 215634, 4261735.$$

Other diagonal Grobner bases are unkown so far.

**Theorem 3.43.**

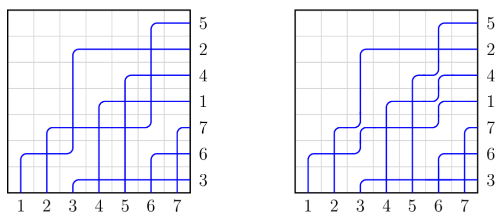
Suppose all degree $> 1$ defining equations of $I_w$ come from a single position $(i, j)$ in $r(w)$. Suppose further that

$$r(w)_{i,j} = \min(i, j) - 1.$$

Then the CDG generators of $I_w$ are a Grobner basis for every term order.

Def'n 3.43. **Bumpless Pipe Dream**

A ***bumpless pipe dream*** is a tiling of the $n \times n$ grid that look like this:




**Example 3.32.**

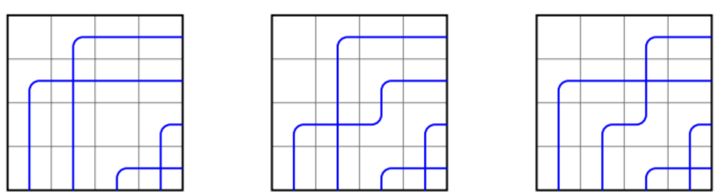
For $w = 2143$, recall that

$$z_{1,1}, \det \begin{bmatrix} 0 & z_{1,2} & z_{1,3} \\ z_{2,1} & z_{2,2} & z_{2,3} \\ z_{3,1} & z_{3,2} & z_{3,3} \end{bmatrix}$$

is a diagonal Grobner basis, and the initial ideal is

$$\langle z_{1,1}, z_{1,2}z_{2,1}z_{3,3} \rangle = \langle z_{1,1}, z_{1,2} \rangle \cap \langle z_{1,1}, z_{2,1} \rangle \cap \langle z_{1,1}, z_{3,3} \rangle .$$

The associated bumpless pipe dreams are



Observe that the prime ideals $\langle z_{1,1}, z_{1,2} \rangle$ , $\langle z_{1,1}, z_{2,1} \rangle$ , $\langle z_{1,1}, z_{3,3} \rangle$ biject with the pipe dreams by the coordinates of blank tiles.

**Theorem 3.44.** Klein-Weigandt

For $w \in S_n$ and many diagonal orders, the radical of the initial ideal

$$\sqrt{\mathrm{in}_{\mathrm{diag}}(I_w)} = \bigcap_{P \in \mathrm{bpd}(w)} \langle z_{i,j} : P \text{ has a blank tile in position } (i, j) \rangle .$$

where $\mathrm{bpd}(w)$ is the set of bumpless pipe dreams associated with $w$.

Moreover, the degree of a component equals the number of bumpless pipe dreams with blank tiles in these positions.

**Example 3.33.**
Consider
$$w = 321654 \in S_6.$$
The corresponding bpd's are

(picture)

Observe that the six northwest corners are blank in both diagrams, which means the corresponding variables have *quadratic fuzz* in the scheme.

---

We still do not know when $\mathrm{in}_{\mathrm{diag}}(I_w)$ is radical. Clearly permutations such that there are multiple bpd's with the same blank tile position are not radical. Even if different bpd's have blank tiles in different positisions, there could be some embedded components. The best guess so far is *pattern avoidance*. From Example 3.33, it is clear that 321654 is *very bad*. The blank tiles appear in the same position in every bpd's.

## 5. Skew-symmetric Matries with Northwest Rank Conditions

Consider
$$\mathcal{F} = \left\{X \in \mathbb{F}^{n\times n} : X^T = -X, \mathrm{rank}\left(X_{[i],[j]}\right) \leq r_{i,j}\right\}$$
for some rank matrix $r \in \mathbb{N}^{n\times n}$. In case $\mathbb{F} = \mathbb{Z}_2$, we impose an extra condition that the diagonal entries of $X \in \mathcal{F}$ are 0.

The irreducibles of $\mathcal{F}$ are indexed by fixed-point free involutions $w \in S_n$. That is, $w^2(i) = i \neq w(i)$ for all $i \in [n]$.

**Theorem 3.45.** Marberg-Pawlowski
The defining determinants of $\mathcal{F}$ do not generate a radical ideal.
Instead, the radical ideal is generated by Pfaffians.[1] That is,
$$\left\langle \sqrt{\det(A)} : A \in \mathcal{F}\right\rangle$$
is a radical ideal.
A Grobner basis for the radical ideal is a set of Pfaffians to a *particular antidiagonal order*. For this term order, the initial ideal is is squarefree and Cohen-Macaulay (the corresponding ASC is vertex-decomposable). The prime decomposition of the initial ideal is generated by *pfp-involution pipe dreams*.

[1] The square roots of $\det(A)$, $A \in \mathcal{F}$, which make sense since determinants of skew-symmetric matrices are perfect squares

---

Here are some related questions.
- *Skew symmetric matrices with southwest rank conditions?*
- *Symmetric matrices? With southwest rank conditions?*

Turns out symmetric matrices with southwest rank conditions are easier (but still hard) to deal with.

**Theorem 3.46.** Fink-Rajchgot-Sullivant
For symmetric matrices with southwest rank conditions, defining equations form a diagoanal Grobner basis. The generated diagonal initial ideal is squarefree.

---

**Theorem 3.47.**
Consider the setting of Theorem 3.46. The diagonal ideal is Cohen-Macaulay.

**Proof Sketch.** The diagonal initial is Stanley-Reisner ideal of a type C subword complex.
**QED**

The facets are given by type C pipe dreams.

**Example 3.34.** Double Determinantal Variety

A ***double determinantal variety*** is

$$D_{m,n,r,s,t} = \left\{ (X_1, \ldots, X_r) \in \left(\mathbb{F}^{m\times n}\right)^r : \operatorname{rank}\left(\begin{bmatrix} X_1 & \cdots & X_r \end{bmatrix}\right) \leq s, \operatorname{rank}\left(\begin{bmatrix} X_1 \\ \vdots \\ X_r \end{bmatrix}\right) \leq t \right\}.$$

**Theorem 3.48.** Fieldstell-Klein

The defining equations of $D_{m,n,r,s,t}$ form a Grobner basis under any diagonal or antidiagonal term order, which generates a prime ideal. The initial ideal is Stanlye-Reisner ideal of a vertex-decomposable ASC so both ideals are Cohen-Macaulay.

## 6. Edge Ideal of a Graph

Let $G$ be a simple graph on $[n]$ and let

$$I_G = \left\langle x_i x_j : ij \in E(G) \right\rangle,$$

called the ***edge ideal***.

**Proposition 3.49.**

$\dim(S/I_G) = \alpha(G)$, where $\alpha(G)$ is the size of the largest independent set in $G$ (i.e. set of vertices that are not adjacent).

**Theorem 3.50.** Hibi-Matsuda

For any $r, r' \geq 1$, there is a monomial ideal $I$ with

$$\operatorname{reg}(S/I) = r$$

and

$$\deg(K(S/I, t)) - \operatorname{codim}(S/I) = r'.$$

**Theorem 3.51.** Hibi-Matsuda-van Tvyl

For any $r, r' \geq 1$, there is a graph $G$ with

$$\operatorname{reg}(S/I_G) = r$$

and

$$\deg(K(S/I_G, t)) - \operatorname{codim}(S/I_G) = r'.$$

However, if $|V(G)| = n$, then

$$\dim(S/I_G) + \deg(K(S/I, t)) - \operatorname{codim}(S/I_G) \leq n.$$

# IV. Application of Grobner Basis

We are going to discuss about how Grobner basis can be used for real-world computations.

## 1. Sudoku

When we are writing down polynomial equations

$$x = 2, 2 - x = 0, (2 - x)^{100} = 0, \dots,$$

they are incoding some information.

Let's see how we can represent a sudoku board by a polynomial system, and use Grobner basis to solve the system. Recall that sudoku involves a $9 \times 9$ grid where each square has a number in $\{1, \dots, 9\}$ and each row, column, $3 \times 3$ block has distinct numbers.

---

First Try

Make variables $\left\{x_{i,j}\right\}_{i,j=1}^{9}$, each representing a square. We are going to build a polynomial ideal $I$ as follows.

Add polynomials

$$\left(x_{i,j} - 1\right)\left(x_{i,j} - 2\right) \cdots \left(x_{i,j} - 9\right), \qquad \forall i, j,$$

to $I$. To encode the row / column / block constraints, observe that if $\{y_1, \dots, y_9\} = \{1, \dots, 9\}$, then

$$\sum_{i=1}^{9} y_i = 45$$
$$\prod_{i=1}^{9} y_i = 9! = 362880$$

are satisfied.

Question:

*does the above constraints make $y_i$'s unique?'*

The answer is *unfortunately false*; observe that 1, 3, 4, 4, 4, 5, 7, 9, 9 also satisfy the constraints but is different from $1, \dots, 9$.

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Now that the first try has failed, we have two choices: either *add more constraints* or *come up with more clever constraints*.

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Second Try

Add polynomials

$$\left(x_{i,j} + 2\right)\left(x_{i,j} + 1\right)\left(x_{i,j} - 1\right) \cdots \left(x_{i,j} - 7\right), \qquad \forall i, j,$$

to $I$ instead. Then observe that the constraints

$$\sum_{j=1}^{9} y_i = 25$$
$$\prod_{j=1}^{9} y_i = 10080$$

are constraints uniquely satisfied by $-2, -1, 1, \dots, 7$.

---

This gives us a polynomial system in 81 variables $x_{i,j}$ with 135 polynomials of degree at most 9. Question:

*how do Grobner bases help solve a system like this?*

We are going to utilize the *elimination theorem*.

**Theorem 4.1.** Elimination Theorem

Let $I \subseteq \mathbb{Q}[x_1, \ldots, x_d]$ be an ideal with a Grobner basis $G = \{g_1, \ldots, g_r\}$ with respect to the monomial lexicographic order $>$, where $x_1 > \cdots > x_d$. Then for all $l \in \{0, \ldots, d-1\}$, the set

$$I_l = I \cap \mathbb{Q}[x_{l+1}, \ldots, x_d]$$

is an ideal in $Q[x_{l+1}, \ldots, x_d]$, and with respect to $>$, it has a Grobner basis

$$G_l = G \cap \mathbb{Q}[x_{l+1}, \ldots, x_d].$$

**Proof.** For each $l$, note $I_l$ is an ideal since $I$ is an ideal.

Let $f \in I_l$. Then the leading term in $(f) \in \langle \text{in}(g) : g \in G \rangle$ since $G$ is a Grobner basis. Hence

$$\text{in}(g) \mid \text{in}(f)$$

for some $g \in G$, so that

$$\text{in}(g) \in \mathbb{Q}[x_{l+1}, \ldots, x_n]$$

as well. But recall that we have a lexicographic order $>$, which means every other term of $g$ must be in $\mathbb{Q}[x_{l+1}, \ldots, x_n]$ as well. Hence

$$f \in I_l \implies \text{in}(f) \in \langle \text{in}(g) : g \in G_l \rangle.$$

We know $\langle G_l \rangle \subseteq I_l$. Suppose there is $f \in I_l$ but $f \notin \langle G_l \rangle$ for contradiction. We may assume $f$ has the minimal leading term in $(f)$ among elements in $I_l \setminus \langle G_l \rangle$. Since $f \in I_l$,

$$\text{in}(f) = \text{in}(g)$$

for some $g \in G_l$. But then

$$\text{in}(f - g) \in I_l$$

has a smaller leading term than $f$, so that $f - g \in \langle G_l \rangle$. But $g \in \langle G_l \rangle$, so this means $f \in \langle G_l \rangle$, which is a contradiction.

Thus $I_l = \langle G_l \rangle$ and $f \in I_l$ implies in $(f) \in \langle \text{in}(g) : g \in G_l \rangle$; that is, $G_l$ is a Grobner basis for $I_l$.

**QED**

Lexicographic basis are nice in the above sense, but they are expensive; they can be sometimes computationally infeasible to calculate. It turns out computing Grevlex basis is usually the most efficient choice, and there is an algorithm which allows one to change a Grobner basis with respect to a monomial order to a Grobner basis with respect to another monomial order.

Boolean System

Our sum-product system won't terminate in reasonable time, so we are going to adapt an alternative approach called *boolean system*. Add polynomials

$$w_i^{(a,b)}\left(w_i^{(a,b)} - 1\right), \qquad \forall a, b, i \in [9],$$

and

$$\sum_{i=1}^{9} w_i^{(a,b)} - 1, \qquad \forall a, b \in [9]$$

to the ideal. That is, the first polynomial allow $w_i^{(a,b)}$ to take only values $0, 1$ and the second polynomial allow one and only one of $w_1^{(a,b)}, \ldots, w_9^{(a,b)}$ to take nonzero value, indicating which sqaure to fill in. To ensure that distinct values are taken for a row / column / block, if $y, z$ are in the same row / column / block, then add

$$y_1 z_1 + \cdots + y_9 z_9 = 0.$$

This makes 1620 polynomials of degree 2 in 729 variables. That is, we reduced the degree of polynomials by increasing number of variables.