I. Polynomial Rings and Ideals

1. Preliminaries

Def'n 1.1. **Algebraically Closed** Field
We say a field K is ***algebraically closed*** if every nonconstant $f \in K[\vec{x}]$ has a root.

Def'n 1.2. **Degree**, **Support** of a Polynomial
Let $f = \sum_i c_{vec}\vec{x}^{\vec{a}}$ be a polynomial. Then the **degree** of f is

$$\deg(f) = \max\left\{\sum_i a_i : c_{\vec{a}} \neq 0\right\}.$$

The ***support*** of f is

$$\text{supp}(f) = \{\vec{a} : c_{\vec{a}} \neq 0\}.$$

Def'n 1.3. **Homogeneous** Polynomial
We say f is **homogeneous** of degree i if it is only supported in degree i.

Def'n 1.4. **Hilbert Series** of a Graded Ring
A ***Hilbert series*** of a graded ring S is

$$H(S;t) = \sum_{i\in\mathbb{N}} \dim(S_i)\, t^i.$$

Def'n 1.5. **Ideal** of a Polynomial Ring
An ***ideal*** is a nonempty subset $I \subseteq K[\vec{x}]$ such that

$$f, h \in I, g \in K[\vec{x}] \implies fg + h \in I.$$

Def'n 1.6. Ideal **Generated** by a Subset
Let $F \subseteq K[\vec{x}]$. We define the ideal ***generated*** by F, denoted as $\langle F\rangle$ (or (F)), to be

$$\langle F\rangle = \bigcap\{I \supseteq F : I \text{ is an ideal}\},$$

or the smallest ideal containing F.

Def'n 1.7. **Homogeneous** Ideal
An ideal $I \subseteq K[\vec{x}]$ is ***homogeneous*** if there exist homogeneous polynomials that generate I.[1]

[1] Polynomials need not have the same degree.

Def'n 1.8. **Quotient Ring**
Let $I \subseteq K[\vec{x}]$ be an ideal. Given any $f \in K[\vec{x}]$, the ***residue class*** of f modulo I is the set

$$f + I = \{f + i : i \in I\}.$$

Then there is an equivalence relation $\sim$ on $K[\vec{x}]$ by

$$f \sim g \iff f + I = g + I.$$

The ***quotient ring*** S/I is the ring structure on the set of all residue classes $S/\sim$, with addition

$$(f+I)+(g+I)=(f+g)+I$$

and multiplication

$$(f+I)(g+I)=fg+I.$$

Def'n 1.9. **Product Ideal**, **Colon Ideal** of Two Ideals

Let I, J be ideals. We define the ***product ideal*** of I, J, denoted as IJ, to be

$$IJ=\langle fg : f\in I, g\in J\rangle .$$

We also define the ***colon ideal*** of I, J, denoted as $I:J$, as

$$I:J=\{f\in S : \forall j\in J\,[fj\in I]\}=\{f\in S : fJ\subseteq I\}.$$

Def'n 1.10. **Radical** of an Ideal

Let I be an ideal. The ***radical*** of I is

$$\sqrt{I}=\left\{f\in S : \exists k\in\mathbb{N}\left[f^k\in I\right]\right\}.$$

II. Algebraic Geometry

1. Introduction

Def'n 2.1. **Vanishing Locus** of a Subset of a Polynomial Ring
For any $F \subseteq S$, we define a ***variety*** $V(F)$ (or $V_K(F)$ when we want to specify the field K) by

$$V(F) = \{\vec{p} \in K^n : \forall f \in F\,[f(\vec{p}) = 0]\},$$

called the ***vanishing locus*** of F.

2. Schemes

Def'n 2.2. **Radical Ideal** of a Variety
Let $X = V(F)$ be a variety. Then the ***radical ideal*** of X is the set

$$I(X) = \{f \in K[\vec{x}] : \forall x \in X\,[f(x) = 0]\}.$$

Def'n 2.3. **Scheme** of a Set of Polynomials
For every $F \subseteq K[\vec{x}]$, we define the ***scheme*** of F, denoted as $V_\infty(F)$, by

$$V_\infty(F) = \{V_R(F) : R \supseteq K \text{ is a ring extension of } K\}.$$

3. Monomial Ideals

Def'n 2.4. **Monomial Ideal**
A ***monomial ideal*** in $K[\vec{x}]$ is an ideal generated by monomials.

Def'n 2.5. **Hyperplane** of a Vector Space
A ***hyperplane*** is a codimension 1 vecor subspace.
We say a hyperplane is ***coordinate*** if it is spanned by axes.

Def'n 2.6. **Minimal** Set of Monomials
Let M be a set of monomials. We say M is ***minimal*** if for every proper subset $N \subset M$, $\langle N\rangle \subset \langle M\rangle$.

Def'n 2.7. **Canonical Generating Set** of a Monomial Ideal
Let I be a monomial ideal. The ***canonical generating set*** of I, denoted as $G(I)$, is the unique minimal set of monomial generators in Proposition 2.9.

4. Operations on Monomial Ideals

5. Grobner Bases

Def'n 2.8. **Monomial Order**
A ***monomial order*** is a total order on $\mathbb{N}^n$ such that

(a) if $a < b$ in $\mathbb{N}^n$, then $a + c < b + c$ for all $c \in \mathbb{N}^n$; and *shifting*

(b) for all $a \in \mathbb{N}^n$, $a \geq (0, \ldots, 0)$.

Def'n 2.9. **Initial Monomial** of a Polynomial

Let $S = K[\vec{x}]$ and fix a monomial order $<$. For $f \in S$, if f is nonzero, we define the ***initial monomial*** (or ***leading monomial***) of f, denoted as $\text{in}_<(f)$, as

$$\text{in}_<(f) = <\text{-greatest monomial of supp}(f).$$

The coefficient k of $\text{in}_<$ is called the ***leading coefficient*** (or ***initial coefficient***), and $k\,\text{in}_<(f)$ is called the ***leading term*** (or ***initial term***).

In case $f = 0$, we set $\text{in}_<(f) = 0$.

Def'n 2.10. **Initial Ideal** of an Ideal

Let I be an ideal. We define the ***initial ideal*** of I, denoted as $\text{in}_<(I)$, by

$$\text{in}_<(I) = \langle \text{in}_<(f) : f \in I \rangle .$$

Def'n 2.11. **Grobner Basis** for an Ideal

We say $G = \{g_i\}_{i=1}^{k}$ is a ***Grobner basis*** for an ideal I if

$$\text{in}(I) = \langle \text{in}(g_i) \rangle_{i=1}^{k} .$$

6. Division Algorithm

7. Buchberger's Algorithm for Finding a Grobner Basis

Def'n 2.12. S**-polynomial**

Fix a monomial order. For all $f, g \in K[x]$, the S***-polynomial*** of f by g, denoted as $S(f,g)$, is defined as

$$S(f,g) = \frac{\text{lcm}(\text{in}(f), \text{in}(g))}{c\,\text{in}(f)} f - \frac{\text{lcm}(\text{in}(f), \text{in}(g))}{d\,\text{in}(g)} g,$$

where $c, d \in K$ are the leading coefficients of f, g, respectively.

Def'n 2.13. **Reduces to** 0 Modulo $g_1, \ldots, g_m$

We say $f \in K[\vec{x}]$ ***reduces to*** 0 modulo $g_1, \ldots, g_m$ if

$$f = \sum_{i=1}^{m} q_i g_i$$

for some $q_1, \ldots, q_m$ with $\text{in}(f) \geq \text{in}(q_i g_i)$.

8. Reduced Grobner Bases

Def'n 2.14. **Reduced** Grobner Basis

Let $G = \{g_k\}_{j=1}^{m}$ be a Grobner basis for an ideal I. We say G is ***reduced*** if

(a) all leading coefficients of g_j's are monic; and

(b) for $i \neq j$, no $u \in \text{supp}(g_i)$ is divisible by $\text{in}(g_j)$.