I. Polynomial Rings and Ideals

1. Preliminaries

Def'n 1.1. **Algebraically Closed** Field
We say a field K is ***algebraically closed*** if

Def'n 1.2. **Degree**, **Support** of a Polynomial
Let $f = \sum_i c_{vec} \vec{x}^{\vec{a}}$ be a polynomial. Then the **degree** of f is

$$\deg(f) =$$

The ***support*** of f is

$$\operatorname{supp}(f) =$$

Def'n 1.3. **Homogeneous** Polynomial
We say f is **homogeneous** of degree i if

Def'n 1.4. **Hilbert Series** of a Graded Ring
A ***Hilbert series*** of a graded ring S is

$$H(S;t) =$$

Def'n 1.5. **Ideal** of a Polynomial Ring
An ***ideal*** is

Def'n 1.6. Ideal **Generated** by a Subset
Let $F \subseteq K[\vec{x}]$. We define the ideal ***generated*** by F, denoted as $\langle F \rangle$ (or (F)), to be

$$\langle F \rangle =$$

or

Def'n 1.7. **Homogeneous** Ideal
An ideal $I \subseteq K[\vec{x}]$ is ***homogeneous*** if

Def'n 1.8. **Quotient Ring**
Let $I \subseteq K[\vec{x}]$ be an ideal. Given any $f \in K[\vec{x}]$, the ***residue class*** of f modulo I is

Then there is an equivalence relation $\sim$ on $K[\vec{x}]$ by

The ***quotient ring*** S/I is the ring structure on the set of all residue classes $S/\sim$, with addition and multiplication

Def'n 1.9. **Product Ideal**, **Colon Ideal** of Two Ideals

Let I, J be ideals. We define the ***product ideal*** of I, J, denoted as IJ, to be

$$IJ =$$

We also define the ***colon ideal*** of I, J, denoted as $I : J$, as

$$I : J =$$

Def'n 1.10. **Radical** of an Ideal

Let I be an ideal. The ***radical*** of I is

$$\sqrt{I} =$$

II. Algebraic Geometry

1. Introduction

Def'n 2.1. **Vanishing Locus** of a Subset of a Polynomial Ring
For any $F \subseteq S$, we define a ***variety*** $V(F)$ (or $V_K(F)$ when we want to specify the field K) by

$$V(F) =$$

called the ***vanishing locus*** of F.

2. Schemes

Def'n 2.2. **Radical Ideal** of a Variety
Let $X = V(F)$ be a variety. Then the ***radical ideal*** of X is the set

$$I(X) =$$

Def'n 2.3. **Scheme** of a Set of Polynomials
For every $F \subseteq K[\vec{x}]$, we define the ***scheme*** of F, denoted as $V_\infty(F)$, by

$$V_\infty(F) =$$

3. Monomial Ideals

Def'n 2.4. **Monomial Ideal**
A ***monomial ideal*** in $K[\vec{x}]$ is

Def'n 2.5. **Hyperplane** of a Vector Space
A ***hyperplane*** is
We say a hyperplane is ***coordinate*** if

Def'n 2.6. **Minimal** Set of Monomials
Let M be a set of monomials. We say M is ***minimal*** if

Def'n 2.7. **Canonical Generating Set** of a Monomial Ideal
Let I be a monomial ideal. The ***canonical generating set*** of I, denoted as $G(I)$, is

4. Operations on Monomial Ideals

5. Grobner Bases

Def'n 2.8. **Monomial Order**
A ***monomial order*** is a total order on $\mathbb{N}^n$ such that
(a) *shifting*
(b)

Def'n 2.9. **Initial Monomial** of a Polynomial
Let $S = K[\vec{x}]$ and fix a monomial order $<$. For $f \in S$, if f is nonzero, we define the ***initial monomial*** (or ***leading monomial***) of f, denoted as $\mathrm{in}_<(f)$, as

$$\mathrm{in}_<(f) =$$

In case $f = 0$, we set

$$\mathrm{in}_<(f) =$$

The ***leading coefficient*** (or ***initial coefficient***) is
The ***leading term*** (or ***initial term***)is

Def'n 2.10. **Initial Ideal** of an Ideal
Let I be an ideal. We define the ***initial ideal*** of I, denoted as $\mathrm{in}_<(I)$, by

$$\mathrm{in}_<(I) =$$

Def'n 2.11. **Grobner Basis** for an Ideal
We say $G = \{g_i\}_{i=1}^{k}$ is a ***Grobner basis*** for an ideal I if

6. Division Algorithm

7. Buchberger's Algorithm for Finding a Grobner Basis

Def'n 2.12. **S-polynomial**
Fix a monomial order. For all $f, g \in K[x]$, the S-***polynomial*** of f by g, denoted as $S(f, g)$, is defined as

$$S(f, g) =$$

Def'n 2.13. **Reduces to** 0 Modulo $g_1, \ldots, g_m$
We say $f \in K[\vec{x}]$ ***reduces to*** 0 modulo $g_1, \ldots, g_m$ if

8. Reduced Grobner Bases

Def'n 2.14. **Reduced** Grobner Basis
Let $G = \{g_k\}_{j=1}^{m}$ be a Grobner basis for an ideal I. We say G is ***reduced*** if

(a)

(b)