I. Polynomial Rings and Ideals

1. Preliminaries

Theorem 1.1.
Every field K has an *algebraic closure* $\overline{K}$, the unique smallest algebraically closed field containinig K.

Proposition 1.2.
Let $S = K[\vec{x}]$ and let

$$S_i = \{p \in S : \deg(p) = i, p \text{ is homogeneous}\}, \qquad \forall i \in \mathbb{N} \cup \{0\}.$$

Then

$$S = \bigoplus_{i=0}^{\infty} S_i$$

as a vector space.

Proposition 1.3.
For any $F \subseteq K[\vec{x}]$, let I_F be the set of all finite $K[\vec{x}]$-linear combinations of elements of F. Then

$$I_F = \langle F \rangle.$$

Proposition 1.4. Characterization of Homogeneous Ideals
Let $S = K[\vec{x}]$ and let $I \subseteq S$ be an ideal. The following are equivalent.

(a) I is homogeneous.

(b) For every $f \in I$, every homogeneous components are in I.[1]

(c) $I = \bigoplus_{j=0}^{\infty} I_j$, where $I_j = I \cap S_j$, the homogeneous subspace of I of degree j.

[1] e.g. If $f(x, y) = xy - y$, then it must be the case that $xy \in I, y \in I$.

Proposition 1.5.
Let I be an ideal of $K[\vec{x}]$ and let $f, g \in K[\vec{x}]$. Then

$$f + I = g + I \iff f - g \in I.$$

Proposition 1.6. Multiplication Is Well-defined on Quotient Rings
Let $I \subseteq K[\vec{x}]$ be an ideal. If $f + I = \hat{f} + I, g + I = \hat{g} + I$, then $fg + I = \widehat{fg} + I$.

Proposition 1.7. Operations on Ideals
Let $I, J \subseteq K[\vec{x}]$ be ideals. Then

(a) $I + J = \{f + g : f \in I, g \in J\}$; and

(b) $I \cap J$

are ideals.

Proposition 1.8.
Let I, J be ideals of $K[\vec{x}]$. Then

$$IJ \subseteq I \cap J.$$

Example 1.1.

Let $S = \mathbb{Q}[x]$, $I = \langle x^3 + 6x^2 + 12x + 8 \rangle$, $J = \langle x^2 + x - 2 \rangle$. By factoring,

$$I = \left\langle (x+2)^3 \right\rangle, J = \langle (x+2)(x-1) \rangle.$$

This means

(a) $I \cap J = \left\langle (x+2)^3 (x-1) \right\rangle$, the ideal generated by the *lcm* of the generators of I, J;

(b) $I + J = \langle (x+2) \rangle$, the ideal generated by the *gcd* of the generators;

(c) $IJ = \left\langle (x+2)^4 (x-1) \right\rangle$; and

(d) $I : J = \left\langle (x+2)^2 \right\rangle$.

Example 1.2. Examples of Radicals of Ideals

Consider I, J from Example 1.2. Then

$$\sqrt{\langle I \rangle} = \langle x+2 \rangle, \sqrt{J} = \langle (x+2)(x-1) \rangle.$$

Proposition 1.9. Every Radical of an Ideal Is an Ideal

Let I be an ideal. Then $\sqrt{I}$ is also an ideal.

Proposition 1.10.

For all ideal I, $\sqrt{\sqrt{I}} = \sqrt{I}$.

II. Algebraic Geometry

1. Introduction

Example 2.1.
Consider $K = \mathbb{R}$. Then
$$V\left(x^2 + y^2 - 1\right) = \left\{(p_1, p_2) \in \mathbb{R}^2 : p_1^2 + p_2^2 = 1\right\},$$
which is the set of points on the unit circle.

Example 2.2.
Consider $K = \mathbb{R}, S = \mathbb{R}[x, y, z]$. Then by inspection,
$$(0,0,0), (1,1,1) \in V(xy - z, yz - x, xz - y).$$
Note that, when at least one of x, y, z is 0, then the others must be also 0.

If two variables have absolute value larger than 1, say $|x|, |y| > 1$, then since $xy - z = 0$, we see that $z > |x|\,|y|$. But this makes it impossible to satisfy $yz - x = 0$.

Similar problem arises when $|x|, |y| < 1$ or $|x| > 1, |y| < 1$.

So the conclusion is that, other than $(0,0,0), (1,1,1)$, every point (p_1, p_2, p_3) in the variety must have that: $|p_1| = |p_2| = |p_3| = 1$. By checking all 8 possibilities (or 7 if we discard $(1,1,1)$), we see that
$$(-1,-1,1), (-1,1,-1), (1,-1,-1) \in V(xy - z, yz - x, xz - y).$$

Example 2.3.
Note that, when $K = \mathbb{R}, S = K[x, y]$,
$$V(1) = \emptyset.$$

Example 2.4. Sadnesses

(a) Consider $K = \mathbb{R}, S = K[x, y]$. Then
$$V\left(x^2 + y^2 + 1\right) = \emptyset.$$
Hence, in some sense, *some information is lost*. *sadness 1*

(b) Consider $K = \mathbb{R}, S = K[x]$. Then $V(x) = \{0\} = V\left(x^2\right)$. So two different polynomials give the same variety. *sadness 2*

We are going to fix both *sadness* simultaneously.

Lemma 2.1.
Let $I = \langle F \rangle$. Then $V(F) = V(I)$.

2. Schemes

Lemma 2.2.
Let I be an ideal. Then
$$V(I) = V\left(\sqrt{I}\right).$$

Lemma 2.3.
Every radical ideal of a variety is a radical ideal.

Lemma 2.4.
Given $F \subseteq K[\vec{x}]$,

$$\sqrt{\langle F \rangle} \subseteq I(V(F)).$$

Consequently, $\langle F \rangle \subseteq I(V(F))$.

Theorem 2.5. Nullstellensatz
Suppose K is algebraically closed. Then for any $F \subseteq K[\vec{x}]$, $I(V(F)) = \sqrt{\langle F \rangle}$.

Example 2.5. $V_{\mathbb{R}}\left(x^2+y^2+1\right)$
We have

$$V_{\mathbb{R}}\left(x^2+y^2+1\right) = \emptyset = V_{\mathbb{R}}(1)$$

which is bad, but

$$V_{\mathbb{C}}\left(x^2+y^2+1\right) = \text{a conic section}$$

and

$$V_{\mathbb{C}}(1) = \emptyset.$$

Example 2.6.
We have

$$V_{\mathbb{R}}(x) = \{0\} = V_{\mathbb{R}}\left(x^2\right).$$

For this time, we cannot solve this problem by entering the world of complex numbers, as

$$V_{\mathbb{C}}(x) = \{0\} = V_{\mathbb{C}}\left(x^2\right).$$

But we can use commutative rings instead. Let $R = \mathbb{R}[\varepsilon] / \left\langle \varepsilon^2 \right\rangle = \left\{a + b\varepsilon + \left\langle \varepsilon^2 \right\rangle : a, b \in \mathbb{R}\right\}$.[1] This makes:

$$V_R(x) = \left\{0 + \left\langle \varepsilon^2 \right\rangle\right\} \neq \left\{b\varepsilon + \left\langle \varepsilon^2 \right\rangle\right\} = V_R\left(x^2\right).$$

[1] The intution behind quotienting with $\left\langle \varepsilon^2 \right\rangle$ is that, ε is too small such that when it is squared, it vanishes. Of course such an element does not exist in $\mathbb{R}$, so we just throw in one.

Theorem 2.6.
There is a bijection

$$\text{ideals of } K[\vec{x}] \leftrightarrow \text{schemes}.$$

3. Monomial Ideals

Example 2.7. Monomials Are Easy
If $u = \vec{x}^{\vec{a}}, v = \vec{x}^{\vec{b}}$, then

$$u | v \iff \forall i \left[a_i \leq b_i\right].$$

Moreover,

$$\gcd(u, v) = \vec{x}^{\min\left(\vec{a}, \vec{b}\right)},$$

and

$$\operatorname{lcm}(u, v) = \vec{x}^{\max\left(\vec{a}, \vec{b}\right)}.$$

Proposition 2.7. Characterization of Monomial Ideals

Let $I \subseteq K[\vec{x}]$ be an ideal. The following are equivalent.

(a) I is a monomial ideal.

(b) For every $f \in I$, every monomial components are in I. That is, $\operatorname{supp}(f) \subseteq I$.

Corollary 2.7.1.

Let I be a monomial ideal and let $M \subseteq I$ be a set of monomials. Then

$$\langle M \rangle = I \iff \text{for every monomial } v \in I \text{, there exists } m \in M \text{ such that } m | v.$$

Example 2.8.

Let $S = K[x, y, z, w]$. Then

$$\begin{aligned} V_\infty(x) &= y, z, w\text{-coordinate hyperplane} = x^\perp\text{-coordinate hyperplane}, \\ V_\infty(xy) &= V_\infty(x) \cup V_\infty(y) = x^\perp \cup y^\perp, \\ V_\infty(x) &= x^\perp \text{ (but fuzzy – we have a \textit{very small} thickness)}, \\ V_\infty\left(x^2 y\right) &= \left(\text{fuzzy } x^\perp\right) \cup y^\perp, \\ V_\infty\left(xy^3\right) &= x^\perp \cup \left(\text{very fuzzy } y^\perp\right) \end{aligned}$$

In general,

$$V_\infty(\text{monomial}) = \text{union of fuzzy coordinate hyperplanes}.$$

More precisely, if the monomial is $\vec{x}^{\vec{a}}$, then

$$\begin{aligned} a_i > 0 &\iff x_i^\perp \text{ appears}, \\ a_i &\longleftrightarrow \text{fuzziness of } x_i^\perp. \end{aligned}$$

Example 2.9.

In $K[x, y, z, w]$,

$$V(xy, xz) = V(\langle xy \rangle + \langle xz \rangle) = V(xy) \cap V(xz) = \left(x^\perp \cup y^\perp\right) \cap \left(x^\perp \cup z^\perp\right) = x^\perp \cup \left(y^\perp \cap z^\perp\right),$$

which is the union of y, z, w-hyperplane and x, w-subspace. In general,

$$V(\text{monomial ideal}) = \text{fuzzy union of coordinate subspaces}.$$

Theorem 2.8. Dickson's Lemma

Let $S \subseteq \mathbb{N}^n$. Then S has finitely many minimal elements.

Corollary 2.8.1.

Let I be a monomial ideal and let M be a generating set ofmonomials. Then there is a finite $M' \subseteq M$ such that $I = \langle M' \rangle$.

Proposition 2.9.

Every monomial ideal I has a unique minimal set of monomial generators.

Proposition 2.10.

Every ascending chain $(I_n)_{n=1}^\infty$ of monomial ideals stabilizes. That is, there is $N \in \mathbb{N}$ such that $I_{n+1} = I_n$ for all $n > N$.

Example 2.10.

Let $I = \left(xy, yz^2\right), J = \left(x^2y, yz\right) \subseteq K[x, y, z]$. Then

$$I = y^{\perp} \cup (y\text{-axis with some fuzz in } z\text{-direction})$$

and

$$J = y^{\perp} \cup (y\text{-axis with some fuzz in } x\text{-direction}) .$$

It turns out that,

$$I + J = \left\langle xy, yz^2, x^2y, yz \right\rangle .$$

This means

$$I + J = y^{\perp} \cup (y\text{-axis}) ,$$

since the *fuzziness in I, J are in orthogonal direction*. Thus

$$I + J = \langle xy, yz \rangle .$$

Example 2.11.

Consider I, J from Example 2.11. Then

$$IJ = \left\langle x^3y^2, xy^2z, x^2y^2z^2, y^2z^3 \right\rangle .$$

Since $x^2y^2z^2$ is a multiple of xy^2z, we can get rid of it:

$$IJ = \left\langle x^3y^2, xy^2z, y^2z^3 \right\rangle .$$

This means we can make all three generators vanish by

(a) making y^2 vanish (i.e. $y^{\perp}$ with some fuzz); or

(b) making x, z, x^3, z^3 vanish (i.e. y-axis with lots of fuzz).

Hence

$$IJ = \left(y^{\perp} \text{ with some fuzz}\right) \cup (y\text{-axis with lots of fuzz}) .$$

Proposition 2.11.

Let I, J be monomial ideals. Then $I \cap J$ is monomial and

$$I \cap J = \langle \operatorname{lcm}(u, v) : u \in G(I), v \in G(J) \rangle .$$

Example 2.12.

Consider I, J from Example 2.11. Then

$$I \cap J = \left\langle x^2y, xyz, x^2yz^2, yz^2 \right\rangle$$

and we can get rid of x^2yz^2 since it is a multiple of every other generator. Hence

$$I \cap J = \left\langle x^2y, xyz, yz^2 \right\rangle .$$

To make every generator vanish, we can make

(a) y vanish ($y^{\perp}$); or

(b) x, z, xz, x^2, z^2 vanish (y-axis with fuzz in any other direction!).

Proposition 2.12.
Let I, J be monomial ideals. Then $I : J$ is monomial with

$$I : J = \bigcap_{v \in G(J)} I : \langle v \rangle$$

and

$$I : \langle v \rangle = \left\langle \frac{u}{\gcd(u, v)} : u \in G(I) \right\rangle .$$

Example 2.13.
Consider I, J from Example 2.11. Then

$$I : J = \left(I : \langle x^2 y \rangle\right) \cap (I : \langle yz \rangle) = \left\langle 1, z^2 \right\rangle \cap \langle x, z \rangle = \langle x, z \rangle = (y\text{-axis}) .$$

Proposition 2.13.
Let $I \subseteq S$ be an ideal. Then

$$I \text{ is prime} \iff S/I \text{ is a domain.}$$

Proposition 2.14.
Let I be a squarefree monomial ideal. Then I is a finite intersection of monomial prime ideals.

Corollary 2.14.1.
Let I be a monomial ideal. Then

$$I \text{ is radical} \iff I \text{ is squarefree.}$$

Theorem 2.15.
Let I be monomial. Then

$$\sqrt{I} = \left\langle \sqrt{u} : u \in G(I) \right\rangle ,$$

where $\sqrt{u}$ is obtained by re-writing every nonzero exponent of u to 1.

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5. Grobner Bases

Example 2.14. Lexicographic Order (Lex)
The ***lexicographic order*** (***lex***) given by

$$a < b \iff \text{first nonzero entry of } b - a > 0$$

is a monomial order.

Example 2.15. Graded Lexicographic Order (Grlex)
The ***graded lexicographic order*** (***grlex***) given by

$$a < b \iff |a| < |b| \text{ or } (|a| = |b| \text{ and } a < b \text{ using the lexicographic order})$$

is a monomial order.

Example 2.16. Reverse Lexicographic Order (Grevlex)

The ***reverse lexicographic order*** (***grevlex***) given by

$$a < b \iff |a| < |b| \text{ or } (|a| = |b| \text{ and last nonzero entry of } b - a \text{ is negative})$$

is a monomial order.

Example 2.17.

Consider the order $x > y > z$ on three variables x, y, z and consider monomials

$$\left\{x^2, xz^2, y^3\right\}.$$

According to lex:

$$x^2 > xz^2 > y^3.$$

According to grlex:

$$xz^2 > y^3 > x^2.$$

According to grevlex:

$$y^3 > xz^2 > x^2.$$

Example 2.18.

Again consider the order $x > y > z$ and consider monomials of degree 2. Then according to grlex:

$$x^2 > xy > xz > y^2 > yz > z^2.$$

According to grevlex:

$$x^2 > xy > y^2 > xz > yz > z^2.$$

Hence we see that it is *impossible* to permute the variables to obtain the same order on grlex and grevlex.

Proposition 2.16.

Let $<$ be a monomial order on $\mathbb{N}^n$. Then $<$ can be extended to a partial order $\leq$ such that

(a) if $u, v \in \mathbb{N}^n$ are monomi $u_j = u_{i_k} = u_{i_r}$.als with $u|v$, then $u \leq v$; and

(b) if there is a decreasing sequence $(u_i)_{i=1}^{\infty} \in \mathbb{N}^n$, then there is $N \in \mathbb{N}$ such that $u_i = u_N$ for all $i \geq N$. *Well-ordering*

Example 2.19.

Let $S = K[x, y, z, w]$ ordered by grevlex with $x > y > z > w$. Let

$$I = \left\langle xy - zw, xz - y^2 \right\rangle.$$

Then the leading terms of the generators are xy, y^2, so

$$\left\langle xy, y^2 \right\rangle \subseteq \operatorname{in}(I).$$

Moreover,

$$y(xy - zw) + x\left(xz - y^2\right) = x^2 z$$

which is not a multiple of xy, y^2, so

$$\left\langle xy, y^2, x^2 z\right\rangle \subseteq \operatorname{in}(I).$$

But how are we supposed to know when we are done?

Theorem 2.17. Macaulay (1927)

Let $I \subseteq S = K[\vec{x}]$ be an ideal. Then the residue classes of monomials in $S \setminus \operatorname{in}(I)$ form a K-basis for S/I.

Corollary 2.17.1.

If I is a homogeneous ideal, then

$$S/I = \bigoplus_{k=0}^{\infty} S_k/I_k$$

is graded and

$$\dim (S/I)_k = \dim (S/\operatorname{in}(I))_k .$$

Corollary 2.17.2.

If I is a homogeneous ideal, then

$$H(S/I, t) = H(S/\operatorname{in}(I), t) .$$

Theorem 2.18.

An ideal I has only finitely many initial ideals.

Theorem 2.19. Hilbert Basis

Every ideal $I \subseteq K[\vec{x}]$ is finitely generated. Precisely, if $g_1, \ldots, g_m \in K[\vec{x}]$ form a Grobner basis of I, then $\langle g_1, \ldots, g_m \rangle = I$.

Corollary 2.19.1.

Let

$$I_1 \subseteq I_2 \subseteq \cdots$$

be an ascending chain of ideals in S. Then there is $N \in \mathbb{N}$ such that $I_k = I_N$ for all $k \geq N$.

Example 2.20. Commuting Matrices Problem -- Still Open!

Let

$$V = \left\{ (A, B) \in \left(\mathbb{R}^{n \times n}\right)^2 : AB = BA \right\} \subseteq K^{2n^2}.$$

What is $I = I(V)$?

Obviously, we can write down homogeneous quadratics concerning dot products of rows and columns. But no one knows if these quadratics are enough.

6. Division Algorithm

Example 2.21. Division Algorithm for Univariate Polynomials

Consider the case

$$f = x^3 + 4x^2 + 3x - 7, g = x - 1 \in K[x] .$$

Then by doing long division, we obtain

$$f = g\left(x^2 + 5x + 8\right) + 1.$$

Example 2.22. Division Algorithm for Multivariate Polynomials

Let

$$f = xy^2 + 1, g_1 = xy + 1, g_2 = y + 1 \in K[x, y] .$$

We want to write f in terms of

$$f = q_1 g_1 + q_2 g_2 + r,$$

where r is a *remainder*, which should be *small* (but what do we even mean by saying small?).

We are going to divide f by g_1 first and then divide the remainder of f/g_1 by g_2.

In this case, the leading terms are clear, xy^2 for f, xy for g_1, and y for g_2. Note that the leading term xy for g_1 goes into the leading term xy^2 for f y times, so we have

$$f - g_1 y = \left(xy^2 + 1\right) - (xy + 1)\, y = -y + 1.$$

Note that the leading term xy does not divide $-y + 1$, so $-y + 1$ is the remainder of the division f/g_1. But note that the leading term y for g_2 goes into $-y$ -1 times, so that

$$(f - g_1 y) - g_2 (-1) = (-y + 1) - (y + 1)(-1) = 2.$$

Thus

$$f = y (xy + 1) - (y + 1) + 2.$$

Example 2.23.

Consider

$$f = x^2 y + xy^2 + y^2, g_1 = xy - 1, g_2 = y^2 - 1 \in K[x, y].$$

Then

$$f - xg_1 - yg_1 = \left(x^2 y + xy^2 + y^2\right) - x (xy - 1) - y (xy - 1) = y^2 + x + y.$$

Note that the leading term is x now, which is not divisible by the leading terms xy, y^2 of g_1, g_2. So we record it as a part of the remainder, and proceed the division with $y^2 + y$. Now the leading term y^2 is divisible by the leading term y^2 of g_2, so that

$$\left(y^2 + y\right) - \left(y^2 - 1\right) = y + 1.$$

Now the terms $y, 1$ are not divisible by the leading terms xy, y^2, so we conclude that $x + y + 1$ is the remainder.

Theorem 2.20. Division Algorithm for Multivariate Polynomials

Let $f \in S = K[\vec{x}]$ and let $g_1, \ldots, g_m \in S$ be nonzero. The division algorithm produces polynomials $q_1, \ldots, q_m, r \in S$ such that

(a) $f = \left(\sum_{j=1}^{m} q_j g_j\right) + r$;

(b) $\operatorname{supp}(r) \cap \langle \operatorname{in}(g_1), \ldots, \operatorname{in}(g_m) \rangle$;[1] and

(c) $\operatorname{in}\left(q_j g_j\right) \leq \operatorname{in}(f)$.

[1] This makes sure that r is small.

Theorem 2.21.

Fix a monomial order. Suppose $\{g_1, \ldots, g_m\}$ is a Grobner basis for a monomial ideal I. Then every $f \in S$ has a unique remainder aon division by $g_1, \ldots, g_m$. That is, no matter which g_j we start dividing f by, the quotients q_j's in Theorem 2.20 may change, but r stays the same.

Corollary 2.21.1. Algorithm for Ideal Membership

Let $\{g_1, \ldots, g_m\}$ be a Grobner basis for a monomial ideal I and let $f \in S$. Then

$$f \in I \iff f \text{ has remainder } 0 \text{ on division by } g_1, \ldots, g_m.$$

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7. Buchberger's Algorithm for Finding a Grobner Basis

Algorithm 2.1. Buchberger's Algorithm

```
INPUT:   monomial ideal I and a generating set F = {f_1, ..., f_k}
01.  for pair (f,f') ∈ F^2:
02.       compute S(f,f')
03.       for j ∈ {1, ..., k}:
04.            use division algorithm to divide S(f,f') by f_j to get a remainder r_j
05.            if r_j ≠ 0:
06.                 update F ← F ∪ {r_j}
07.                 restart from line 1
08.  return F, which is a Grobner basis
```

Proposition 2.22.

Buchberger's algorithm terminates in finite time.

Theorem 2.23.

Fix a monomial order and let $I = \langle g_1, \ldots, g_m \rangle$ with each $g_i \neq 0$. The following are equivalent.

(a) $\{g_1, \ldots, g_m\}$ is a Grobner basis for I.

(b) Every $S\left(g_i, g_j\right)$ reduce to 0 modulo $g_1, \ldots, g_m$.

Lemma 2.24.

Suppose $\gcd\left(\operatorname{in}(f), \operatorname{in}(g)\right) = 1$. Then $S(f,g)$ reduces to 0 modulo f, g.

Example 2.24.

Let $f = xy - zw, g = xz - y^2 \in K[x,y,z,w]$ and let

$$I = \langle f, g \rangle .$$

We compute a Grobner basis with respect to grevlex $x > y > z > w$.

We start with the generating set

$$F = \{f, g\} .$$

Observe that

$$\operatorname{in}(f) = xy, \operatorname{in}(g) = y^2,$$

which are not coprime. Hence Lemma 2.24 does not apply.

We have

$$S(f,g) = yf - (-x)g = x^2z - yzw.$$

We then run a division algorithm on $S(f,g)$ by f, g. Note that the leading terms xy, y^2 do not divide x^2z, so we throw the leading term x^2z of $S(f,g)$ into remainder. But xy, y^2 do not divide the new lead term yzw either, so it follows that the remainder is $x^2z - yzw$.

Hence define

$$h = x^2z - yzw$$

and update

$$F \leftarrow F \cup \{h\} = \{f, g, h\} .$$

For this time, we can skip $S(f,g)$, which was already computed.

For f, h, note that the leading terms xy, x^2z are not coprime, and we have

$$S(f,h) = xzf - yh = \left(x^2yz - xz^2w\right) - \left(x^2yz - y^2zw\right) = y^2zw - xz^2w.$$

Now note that

$$-zwg = -zw\left(xz - y^2\right) = y^2zw - xz^2w = S(f,h),$$

so that $S(f,h)$ reduces to 0 modulo f,g,h.

But for g,h, note that

$$\text{in}(g)=y^2, \text{in}(h)=x^2z,$$

so that the leading terms are coprime. Then Lemma 2.24 applies and $S(g,h)$ reduces to 0 modulo f,g,h, so that we can skip computing $S(g,h)$.

Thus

$$F=\{f,g,h\}$$

is a Grobner basis for I.

Example 2.25. Classical Determinantal Variety

Let

$$D_r(m,n)=\left\{M\in K^{m\times n} : \text{rank}(M)\le r\right\}.$$

Then observe that

$$D_r(m,n)=V_K\left(\left\{\underbrace{\text{all the } (r+1)\times(r+1) \text{ minors of } m\times n \text{ matrix of variables } x_{i,j}, 1\le i\le m, 1\le j\le n.}_{=J}\right\}\right)$$

We ask:

is J the set of all relations (i.e. $J=I(D_r(m,n))$)?

Equivalently,

is $\langle J\rangle$ radical?

Consider the special case $D_1(2,n)$ and let

$$J=\left\{2\times 2 \text{ minors of } \begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix}\right\}.$$

For convenience, define

$$P_{i,j}=x_iy_j-x_jy_i=\text{the minor of columns } i,j, \qquad \forall i<j.$$

We are going to understand this case by finding a Grobner basis. Fix a lexicographic order

$$x_1>\cdots>x_n>y_1>\cdots>y_n.$$

Then

$$\text{in}\left(P_{i,j}\right)=x_iy_j=\text{the diagonal term}, \qquad \forall i<j.$$

To find a Grobner basis, we observe that

$$i\ne k, j\ne l \implies \gcd\left(\text{in}\left(P_{i,j}\right), \text{in}\left(P_{k,l}\right)\right)=1,$$

so that we can skip those S-polynomials.

Suppose $i=k$. Without loss of generality, say $j<l$. Then

$$S\left(P_{i,j},P_{i,l}\right)=\left(x_iy_jy_l-x_jy_iy_l\right)-\left(x_iy_jy_l-x_ly_iy_j\right)=x_ly_iy_j-x_jy_iy_l=y_i\left(x_ly_j-x_jy_l\right)=-y_iP_{j,l}.$$

So these S-polynomials reduce to 0.

In case $j=l$ with $i<k$, an analogous calculation results in

$$x_jP_{i,k}$$

so that $S\left(P_{i,j},P_{k,j}\right)$ reduces to 0.

Thus J is a Grobner basis, so that

$$\text{in}(J) = \left\langle \text{in}\left(P_{i,j}\right) \right\rangle_{i<j} = \left\langle x_i y_j \right\rangle_{i<j}.$$

But then note that $\left\langle x_i y_j \right\rangle_{i<j}$ is radical. Hence it follows that

$$D_1(2,n) = \langle J \rangle \text{ is also radical.}$$

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$D_r(m,n)$ is radical.

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8. Reduced Grobner Bases

Theorem 2.26.

Let I be an ideal and fix a monomial order. Then there is a unique reduced Grobner basis for I.
