I. Polynomial Rings and Ideals

1. Preliminaries

Theorem 1.1. Algebraic Closure of a Field

Proposition 1.2.

Let $S = K[\vec{x}]$ and let

$$S_i = \{p \in S : \deg(p) = i, p \text{ is homogeneous}\}, \qquad \forall i \in \mathbb{N} \cup \{0\}.$$

Then

as a vector space.

Proposition 1.3. Alternative Definition of $\langle F \rangle$

For any $F \subseteq K[\vec{x}]$

$$\langle F \rangle =$$

Proposition 1.4. Characterization of Homogeneous Ideals

Let $S = K[\vec{x}]$ and let $I \subseteq S$ be an ideal. The following are equivalent.

(a) I is homogeneous.

(b)

(c)

Proposition 1.5. Operations on Ideals

Let $I, J \subseteq K[\vec{x}]$ be ideals. Which are ideals?

(a) $\{f + g : f \in I, g \in J\}$.

(b) $I \cap J$.

(c) $\{fg : f \in I, g \in J\}$.

(d) $\{f \in S : fJ \subseteq I\}$.

Proposition 1.6.

Let I, J be ideals of $K[\vec{x}]$. Then

$$IJ \quad I \cap J.$$

Proposition 1.7.

Let I be an ideal. Then $\sqrt{I}$ is

Proposition 1.8.

For all ideal I, $\sqrt{\sqrt{I}} =$

II. Algebraic Geometry

1. Introduction

Lemma 2.1.

Let $I = \langle F \rangle$. Then $V(F) =$

2. Schemes

Lemma 2.2.

Let I be an ideal. Then

$$V(I) =$$

Lemma 2.3.

Every radical ideal of a variety is

Lemma 2.4.

Given $F \subseteq K[\vec{x}]$,

$$\sqrt{\langle F \rangle} \quad I(V(F)).$$

Consequently, $\langle F \rangle$

Theorem 2.5. Nullstellensatz

Suppose K is algebraically closed. Then for any $F \subseteq K[\vec{x}]$,

$$I(V(F)) =$$

Theorem 2.6.

There is a bijection

$$\text{ideals of } K[\vec{x}] \leftrightarrow$$

3. Monomial Ideals

Proposition 2.7. Characterization of Monomial Ideals

Let $I \subseteq K[\vec{x}]$ be an ideal. The following are equivalent.

(a) I is a monomial ideal.

(b)

Corollary 2.7.1.

Let I be a monomial ideal and let $M \subseteq I$ be a set of monomials. Then

$$\langle M \rangle = I \iff$$

Theorem 2.8. Dickson's Lemma

Let $S \subseteq \mathbb{N}^n$. Then

Corollary 2.8.1.
Let I be a monomial ideal and let M be a generating set ofmonomials. Then

Proposition 2.9.
Every monomial ideal I

Proposition 2.10.
Every ascending chain $(I_n)_{n=1}^{\infty}$ of monomial ideals

4. Operations on Monomial Ideals

Proposition 2.11.
Let I, J be monomial ideals. Then $I \cap J$ is monomial and

$$I \cap J =$$

Proposition 2.12.
Let I, J be monomial ideals. Then $I : J$ is monomial with

$$I : J =$$

and

$$I : \langle v \rangle =$$

Proposition 2.13.
Let $I \subseteq S$ be an ideal. Then

$$I \text{ is prime } \iff$$

Proposition 2.14.
Let I be a squarefree monomial ideal. Then I is

Corollary 2.14.1.
Let I be a monomial ideal. Then

$$I \text{ is radical } \iff$$

Theorem 2.15.
Let I be monomial. Then

$$\sqrt{I} =$$

where $\sqrt{u}$ is obtained by re-writing every nonzero exponent of u to 1.

5. Grobner Bases

Proposition 2.16.
Let $<$ be a monomial order on $\mathbb{N}^n$. Then $<$ can be extended to a partial order $\leq$ such that

(a)

(b)

Theorem 2.17. Macaulay (1927)
Let $I \subseteq S = K[\vec{x}]$ be an ideal. Then

Corollary 2.17.1.
If I is a homogeneous ideal, then

$$S/I =$$

is and

$$\dim (S/I)_k =$$

Corollary 2.17.2.
If I is a homogeneous ideal, then

$$H(S/I, t) =$$

Theorem 2.18.
An ideal I has only initial ideals.

Theorem 2.19. Hilbert Basis
Every ideal $I \subseteq K[\vec{x}]$ is . Precisely, if $g_1, \ldots, g_m \in K[\vec{x}]$ form a Grobner basis of I, then

Corollary 2.19.1.
Let

$$I_1 \subseteq I_2 \subseteq \cdots$$

be an ascending chain of ideals in S. Then

6. Division Algorithm

Theorem 2.20. Division Algorithm for Multivariate Polynomials
Let $f \in S = K[\vec{x}]$ and let $g_1, \ldots, g_m \in S$ be nonzero. The division algorithm produces polynomials $q_1, \ldots, q_m, r \in S$ such that

(a)

(b)

(c)

Theorem 2.21.
Fix a monomial order. Suppose $\{g_1, \ldots, g_m\}$ is a Grobner basis for a monomial ideal I. Then every $f \in S$ has

Corollary 2.21.1. Algorithm for Ideal Membership
Let $\{g_1, \ldots, g_m\}$ be a Grobner basis for a monomial ideal I and let $f \in S$. Then

$$f \in I \iff$$

7. Buchberger's Algorithm for Finding a Grobner Basis

Algorithm 2.1. Buchberger's Algorithm

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INPUT:   monomial ideal I and a generating set F = {f_1, ..., f_k}
for :
      for :
            if :
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Proposition 2.22.

Buchberger's algorithm terminates in time.

Theorem 2.23.

Fix a monomial order and let $I = \langle g_1, \ldots, g_m \rangle$ with each $g_i \neq 0$. The following are equivalent.

(a) $\{g_1, \ldots, g_m\}$ is a Grobner basis for I.

(b)

Lemma 2.24.

Suppose $\gcd(\mathrm{in}(f), \mathrm{in}(g)) = 1$. Then $S(f, g)$

Theorem 2.25. Hochster-Eagen

$D_r(m, n)$ is radical.

8. Reduced Grobner Bases

Theorem 2.26.

Let I be an ideal and fix a monomial order. Then