I. Introduction

1. Review of Postulates and Formalism

Postulate 1.
The state of a quantum system can be described by a wavefunction in a Hilbert space.

We denote the elements of the Hilbert space as $|\psi\rangle$, and the complex conjugate by $\langle\psi|$.

Postulate 2.
Every observable/measurable quantity is described by an operator $\hat{A}$ on a Hilbert space.

Postulate 3.
The result of measuring an observable is one of its eigenvalues.

Postulate 4. Born's Rule
For an observable $\hat{A}$ and its eigenvalue a correspoinding to $|a\rangle$, the probability of measuring a is

$$\mathbb{P}(a) = |\langle a|\psi\rangle|^2 ,$$

where $|\psi\rangle$ is the quantum state before the measurement.

Postulate 5.
For an observable $\hat{A}$, if an eigenvalue a is measured, then the quantum state after the measurement is $|a\rangle$.

Postulate 6. Schrodinger Equation
The time evolution of a quantum state $|\psi\rangle$ satisfies

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle ,$$

where H is the Hamiltonian operator.

2. Time Evolution Operator

Consider the time dependent Schrodinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle .$$

Then the state $|\psi\rangle$ evolves over time according to the *time evolution operator* $U(t, t_0)$ by

$$|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle$$

with the initial condition $U(t_0, t_0) = I$, the identity operator. Note that U has to be unitary, since we want to preserve the norm. Therefore, the differential equation for U is given by

$$i\hbar\frac{\partial}{\partial t}U(t, t_0)|\psi(t_0)\rangle = HU(t, t_0)|\psi(t_0)\rangle .$$

Rewriting in terms of U,

$$\frac{\partial}{\partial t}U(t, t_0) = -\frac{i}{\hbar}HU(t, t_0) .$$

In case H is independent of the time,

$$U = Ae^{-\frac{i}{\hbar}Ht},$$

where A is a constant depending on the initial condition $U(t_0, t_0) = I$. That is,

$$I = U(t_0, t_0) = Ae^{-\frac{i}{\hbar}Ht_0} \implies A = e^{\frac{i}{\hbar}Ht_0}.$$

This solution is useful when paired with the *time independent Schrodinger equation*:

$$H|E_n\rangle = E_n|E_n\rangle.$$

In this case, for any analytic function f,

$$f(H)|E_n\rangle = f(E_n)|E_n\rangle,$$

so in particular,

$$U(t, t_0)|E_n\rangle = e^{-\frac{i}{\hbar}E_n(t-t_0)}|E_n\rangle.$$

For a general state $|\psi(t_0)\rangle$, we write it down as a linear combination of energy eigenstates,

$$|\psi(t_0)\rangle = \left(\sum_n |E_n\rangle\langle E_n|\right)|\psi(t_0)\rangle = \sum_n c_n|E_n\rangle,$$

where $c_n = \langle E_n|\psi(t_0)\rangle$. Then,

$$|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle = e^{-\frac{i}{\hbar}H(t-t_0)}\sum_n c_n|E_n\rangle = \sum_n c_n e^{-\frac{i}{\hbar}E_n(t-t_0)}|E_n\rangle.$$

Thus

$$U(t, t_0) = \sum_n e^{-\frac{i}{\hbar}E_n(t-t_0)}|E_n\rangle\langle E_n|.$$

Example 1.1. Time Evolution of Spin-$\frac{1}{2}$

Let

$$H = \hbar\Omega\sigma_x = \hbar\Omega(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|) = \hbar\Omega\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},$$

where $|\uparrow\rangle = \begin{bmatrix}1\\0\end{bmatrix}, |\downarrow\rangle = \begin{bmatrix}0\\1\end{bmatrix}$. Note that the energy eigenvectors are

$$\begin{aligned}|E_1\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \text{ corresponding to } \hbar\Omega = E_1\\ |E_2\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle) \text{ corresponding to } -\hbar\Omega = E_2\end{aligned}.$$

Consider

$$|\psi(0)\rangle = |\downarrow\rangle.$$

Then the coefficients with respect to the basis $(|E_1\rangle, |E_2\rangle)$ are

$$\begin{aligned}c_1 &= \langle E_1|\downarrow\rangle = \frac{1}{\sqrt{2}}(\langle\uparrow| + \langle\downarrow|)|\downarrow\rangle = \frac{1}{\sqrt{2}}\\ c_2 &= \langle E_2|\downarrow\rangle = \frac{1}{\sqrt{2}}(\langle\uparrow| - \langle\downarrow|)|\downarrow\rangle = -\frac{1}{\sqrt{2}}\end{aligned},$$

which means

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|E_1\rangle - \frac{1}{\sqrt{2}}|E_2\rangle.$$

It follows that

$$
\begin{aligned}
|\psi(t)\rangle = U(t,0)\,|\psi(0)\rangle = \left(\sum_n e^{-\frac{i}{\hbar}E_n t}\,|E_n\rangle\langle E_n|\right)|\psi(0)\rangle &= \left(e^{-i\Omega t}\,|E_1\rangle\langle E_1| + e^{i\Omega t}\,|E_2\rangle\langle E_2|\right)\frac{1}{\sqrt{2}}\left(|E_1\rangle - |E_2\rangle\right) \\
&= \frac{1}{\sqrt{2}}e^{-i\Omega t}\,|E_1\rangle - \frac{1}{\sqrt{2}}e^{i\Omega t}\,|E_2\rangle = \frac{1}{2}e^{-i\Omega t}\left(|\uparrow\rangle + |\downarrow\rangle\right) - \frac{1}{2}e^{i\Omega t}\left(|\uparrow\rangle - |\downarrow\rangle\right) = \frac{1}{2}\left(e^{-i\Omega t} - e^{i\Omega t}\right)|\uparrow\rangle + \frac{1}{2}\left(e^{-\Omega t} + e^{i\Omega t}\right)|\downarrow\rangle \\
&= i\sin(\Omega t)\,|\uparrow\rangle + \cos(\Omega t)\,|\downarrow\rangle .
\end{aligned}
$$

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3. Harmonic Oscillator

Consider the Hamiltonian operator

$$H = \frac{p^2}{2m} = \frac{1}{2}m\omega^2 x^2. \tag*{[1.1]}$$

Suppose $H|\psi\rangle = E|\psi\rangle$, where E is an eigenvalue of H. How do we calculate E, $|\psi\rangle$?

First approach: substitute

$$p = -i\hbar\frac{\partial}{\partial x} \tag*{[1.2]}$$

into [1.1]. We won't use this approach, as it does not give much intuition to the problem.

Second approach: rewrite Hamiltonian in terms of *ladder operators*. Let

$$a = \frac{1}{\sqrt{2\hbar m\omega}}\left(ip + m\omega x\right),$$

so that

$$a^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}\left(-ip + m\omega x\right).$$

Then, using the commutator formula $[x,p] = i\hbar$,

$$
\begin{aligned}
a^\dagger a &= \frac{1}{2\hbar m\omega}\left(-ip + m\omega x\right)\left(ip + m\omega x\right) = \frac{1}{2\hbar m\omega}\left(p^2 + im\omega\left(xp - px\right) + m^2\omega^2 x^2\right) \\
&= \frac{1}{2\hbar m\omega}\left(p^2 + im\omega\,[x,p] + m^2\omega^2 x^2\right) = \frac{1}{2\hbar m\omega}\left(p^2 + im\omega i\hbar + m^2\omega^2 x^2\right) \\
&= \frac{1}{2\hbar m\omega}\left(p^2 + m^2\omega^2 x^2 - \hbar m\omega\right) = \frac{1}{\hbar\omega}\left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2\right) - \frac{1}{2} = \frac{1}{\hbar\omega}H - \frac{1}{2}.
\end{aligned}
$$

Thus we conclude

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right). \tag*{[1.3]}$$

Observe that [1.3] tells us that the eigenvectors of H are precisely the eigenvectors of $a^\dagger a$;

$$a^\dagger a\,|E_n\rangle = n\,|E_n\rangle \implies H\,|E_n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|E_n\rangle .$$

We can find n using commutators. First

$$
\begin{aligned}
\left[a, a^\dagger\right] &= \frac{1}{2\hbar m\omega}\left[ip + m\omega x, -ip + m\omega x\right] = \frac{1}{2\hbar m\omega}\left([ip,-ip] + [ip, m\omega x] + [m\omega x, -ip] + [m\omega x, m\omega x]\right) \\
&= \frac{1}{2\hbar m\omega}\left(im\omega\,[p,x] - im\omega\,[x,p]\right) = \frac{i}{2\hbar}\left(-i\hbar - i\hbar\right) = 1.
\end{aligned}
$$

Now, observe that $a|E_n\rangle$ is an eigenstate of $a^\dagger a$ corresponding to $n-1$: since $\left[a, a^\dagger\right] = 1$,

$$a^\dagger a\left(a|E_n\rangle\right) = \left(aa^\dagger - 1\right)a|E_n\rangle = a\left(a^\dagger a\right)|E_n\rangle - a|E_n\rangle = an|E_n\rangle - a|E_n\rangle = (n-1)\,a|E_n\rangle\,.$$

Hence

$$a|E_n\rangle = c_n|E_{n-1}\rangle\,.$$

Similarly,

$$a^\dagger|E_n\rangle = b_n|E_{n+1}\rangle\,.$$

Let us find what c_n, b_n are. Observe

$$n = \langle E_n|\,n\,|E_n\rangle = \langle E_n|\,a^\dagger a\,|E_n\rangle = \langle E_{n-1}|\,c_n^* c_n\,|E_{n-1}\rangle = |c_n|^2\,.$$

Therefore n is nonnegative and

$$c_n = \sqrt{n}$$

or

$$a|E_n\rangle = \sqrt{n}\,|E_{n-1}\rangle\,.$$

In a similar fashion, we find out

$$b_n = \sqrt{n+1},$$

so that

$$a^\dagger|E_n\rangle = \sqrt{n+1}\,|E_{n+1}\rangle\,.$$

Note that we have to have n to be an integer, so that

$$a^n|E_n\rangle = 0.$$

Since we may write x, p in terms of the ladder operators as

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$$

and

$$p = i\sqrt{\frac{\hbar m\omega}{2}}\left(a^\dagger - a\right),$$

we have

$$\left\langle x^2\right\rangle = \langle E_n|\,x^2\,|E_n\rangle = \frac{\hbar}{2m\omega}\langle E_n|\,aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger\,|E_n\rangle = \frac{\hbar}{2m\omega}\langle E_n|\,aa + 2a^\dagger a + 1 + a^\dagger a^\dagger\,|E_n\rangle\,.$$

But, using orthonormality of $\{|E_n\rangle\}_{n=0}^{\infty}$, we obtain

$$\langle E_n|\,aa\,|E_n\rangle = \langle E_n|\,a^\dagger a^\dagger\,|E_n\rangle = 0,$$

so that

$$\left\langle x^2\right\rangle = \frac{\hbar}{2m\omega}\langle E_n|\,2a^\dagger a + 1\,|E_n\rangle = \frac{\hbar}{2m\omega}(2n+1)\langle E_n|E_n\rangle = \frac{\hbar}{2m\omega}(2n+1)\,.$$

Summary 1.1. Harmonic Oscillator, Ladder Operators
Given Hamiltonian

$$H = \frac{1}{2}m\omega^2 x^2,$$

we have that

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right),$$

where

$$a=\frac{1}{\sqrt{2\hbar m\omega}}\left(ip+m\omega x\right)$$

is the *ladder operator*. $a, a^{\dagger}$ acts on the energy eigenstates by

$$a\left|E_n\right\rangle=\sqrt{n}\left|E_{n-1}\right\rangle$$

and

$$a^{\dagger}\left|E_n\right\rangle=\sqrt{n+1}\left|E_{n+1}\right\rangle .$$

With respect to the ladder operators, x, p can be written as

$$x=\sqrt{\frac{\hbar}{2m\omega}}\left(a+a^{\dagger}\right)$$

and

$$p=i\sqrt{\frac{\hbar m\omega}{2}}\left(a^{\dagger}-a\right),$$

Example 1.2.

Consider the harmonic oscillator states $\left|0\right\rangle=\left|E_0\right\rangle,\ldots$ and suppose $\left|\varphi\left(0\right)\right\rangle=\frac{1}{\sqrt{2}}\left(\left|n\right\rangle+\left|n+1\right\rangle\right)$. What is $\left\langle x\left(t\right)\right\rangle$?

Answer. Observe that

$$\left|\varphi\left(t\right)\right\rangle=\frac{1}{\sqrt{2}}\left(e^{-\frac{i}{\hbar}E_nt}\left|n\right\rangle+e^{-\frac{i}{\hbar}E_{n+1}t}\left|n+1\right\rangle\right)=\frac{1}{\sqrt{2}}\left(e^{-i\omega\left(n+\frac{1}{2}\right)t}\left|n\right\rangle+e^{-i\omega\left(n+\frac{3}{2}\right)t}\left|n+1\right\rangle\right),$$

so that

$$\begin{aligned}
\left\langle x\left(t\right)\right\rangle&=\left\langle\psi\left(t\right)\right|x\left|\psi\left(t\right)\right\rangle=\sqrt{\frac{\hbar}{2m\omega}}\left\langle\psi\left(t\right)\right|a+a^{\dagger}\left|\psi\left(t\right)\right\rangle\\
&=\sqrt{\frac{\hbar}{2m\omega}}\frac{1}{2}\left(\left\langle n\right|e^{i\omega\left(n+\frac{1}{2}\right)t}+\left\langle n+1\right|e^{i\omega\left(n+\frac{3}{2}\right)t}\right)\left(a+a^{\dagger}\right)\sqrt{\frac{\hbar}{2m\omega}}\frac{1}{2}\left(e^{-i\omega\left(n+\frac{1}{2}\right)t}\left|n\right\rangle+e^{-i\omega\left(n+\frac{3}{2}\right)t}\left|n+1\right\rangle\right)\\
&=\frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(e^{i\omega+\left(n+\frac{3}{2}\right)t}e^{-i\omega\left(n+\frac{1}{2}\right)t}\left\langle n+1\right|a^{\dagger}\left|n\right\rangle+e^{i\omega\left(n+\frac{1}{2}\right)t}e^{-i\omega\left(n+\frac{3}{2}\right)t}\left\langle n\right|a\left|n+1\right\rangle\right)\\
&=\frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(e^{i\omega t}\sqrt{n+1}+e^{-i\omega t}\sqrt{n+1}\right)=\sqrt{\frac{\hbar\left(n+1\right)}{2m\omega}}\cos\left(\omega t\right).
\end{aligned}$$

Thus

$$\left\langle x\left(t\right)\right\rangle=\sqrt{\frac{\hbar\left(n+1\right)}{2m\omega}}\cos\left(\omega t\right).$$

QED

4. Angular Momentum Commutators

We start with classical *orbital angular momentum* (OAM):

$$\vec{L}=\vec{r}\times\vec{p}.$$

By definition,

$$\vec{L}=\begin{bmatrix}ypz-zp_y\\zp_x-xp_z\\xp_y-yp_x\end{bmatrix}.$$

We are going to use *canonical commutation relations*

$$\begin{aligned}\left[r_i, r_j\right] &= 0\\ \left[p_i, p_j\right] &= 0 \qquad ,\\ \left[r_i, p_j\right] &= i\hbar\delta_{i,j}\end{aligned}$$

where r_1, r_2, r_3 are position components (say x, y, z for instance) and p_1, p_2, p_3 are the momentum components, to figure out $\left[L_i, L_j\right]$. For instance,

$$\left[L_x, L_y\right] = \left[yp_z - zp_y, zp_x - xp_z\right] = \left[yp_z, zp_x\right] - \left[zp_y, zp_x\right] - \left[yp_z, xp_z\right] + \left[zp_y, xp_z\right].$$

But note that z commutes with itself and p_y, p_z commutes, so that $\left[zp_y, zp_x\right] = 0$. Similarly, x, y commutes so that $\left[yp_z, xp_z\right] = 0$. Hence,

$$\left[L_x, L_y\right] = \left[yp_z, zp_x\right] + \left[zp_y, xp_z\right] = y\left[p_z, zp_x\right] + \left[y, zp_x\right]p_z + \left[zp_y, xp_z\right].$$

Since y commutes with z, p_x, it follows $\left[y, zp_x\right] = 0$. This means

$$\left[L_x, L_y\right] = y\left[p_z, zp_x\right] + \left[zp_y, xp_z\right] = yz\underbrace{\left[p_z, p_x\right]}_{=0} + y\left[p_z, z\right]p_x + xz\underbrace{\left[p_y, p_z\right]}_{=0} + x\left[z, p_z\right]p_y = i\hbar\left(xp_y - yp_x\right) = i\hbar L_z.$$

Similarly, we have

$$\left[L_y, L_z\right] = i\hbar L_x, \left[L_z, L_x\right] = i\hbar L_y.$$

Summary 1.2. Orbital Angular Momentum Commutation Relation
Let $L = \left(L_x, L_y, L_z\right)$ be the orbital angular momentum operator. Then

$$\left[L_x, L_y\right] = i\hbar L_z, \left[L_y, L_z\right] = i\hbar L_x, \left[L_z, L_x\right] = i\hbar L_y.$$

We are going to *define* angular momentum operator J as $J = \left(J_x, J_y, J_z\right)$ satisfying the above relations. That is,

$$\left[J_i, J_j\right] = i\hbar J_k \varepsilon_{ijk}, \qquad \forall i, j, k.$$

ε_{ijk} is the *Levi-Civita symbol.*

Along with

$$J^2 = J_x^2 + J_y^2 + J_z^2,$$

we can find quantization restrictions on amount of angular momentums.

Claim 1. $\left[J^2, J_z\right] = 0$.

Observe that

$$\begin{aligned}\left[J^2, J_z\right] &= \left[J_x^2 + J_y^2 + J_z^2, J_z\right] = \left[J_x^2, J_z\right] + \left[J_y^2, J_z\right] + \underbrace{\left[J_z^2, J_z\right]}_{=0}\\ &= J_x\underbrace{\left[J_x, J_z\right]}_{=-i\hbar L_y} + \left[J_x, J_z\right]J_x + J_y\underbrace{\left[J_y, J_z\right]}_{=i\hbar J_x} + \left[J_y, J_z\right]J_y = -i\hbar J_x J_y - i\hbar J_y J_x + i\hbar J_y J_x + i\hbar J_x J_y = 0.\end{aligned}$$

(End of Claim 1)

In a similar manner, we obtain that

$$\left[J^2, J_x\right] = \left[J^2, J_y\right] = \left[J^2, J_z\right] = 0.$$

Hence we can find a common eigenbasis for J^2, J_i for all i.

We are also going to introduce *ladder operators* for angular momentum (these operators are not the same as in quantum harmonic oscillators):

$$J_{\pm} = J_x \pm iJ_y.$$

Observe that

$$\left[J^2, J_{\pm}\right] = 0,$$

since $J_{\pm}$ is a linear combination of J_x, J_y which commute with J^2. Also,

$$[J_z, J_{\pm}] = [J_z, J_x] \pm i\left[J_z, J_y\right] = i\hbar J_y \pm i\left(-i\hbar J_x\right) = i\hbar J_y \pm \hbar J_x = \pm\hbar\left(J_x \pm iJ_y\right) = \pm\hbar J_{\pm}.$$

Let us find the common eigenbasis for J^2, J_z first. Consider the system of eigenvalue equations

$$\begin{aligned} J^2 \left|a, b\right\rangle &= a \left|a, b\right\rangle \\ J_z \left|a, b\right\rangle &= b \left|a, b\right\rangle \end{aligned}.$$

Similar to how we did to harmonic oscillators, apply the raising operator J_+ to $\left|a, b\right\rangle$. Observe that

$$J_z\left(J_+\left|a, b\right\rangle\right) = J_zJ_+\left|a, b\right\rangle = \left(J_+J_z + \hbar J_+\right)\left|a, b\right\rangle = J_+ b\left|a, b\right\rangle + \hbar J_+\left|a, b\right\rangle = \left(b + \hbar\right) J_+\left|a, b\right\rangle.$$

That is, $J_+\left|a, b\right\rangle$ is an eigenvector of J_z corresponding to $b + \hbar$, so we may write

$$J_+\left|a, b\right\rangle = c\left|a', b + \hbar\right\rangle$$

for some a'. In a similar manner, we have

$$J^2 J_+ = aJ_+\left|a, b\right\rangle,$$

so that $J_+\left|a, b\right\rangle$ is an eigenvector of J^2 corresponding to a, which means

$$J_+\left|a, b\right\rangle = c\left|a, b + \hbar\right\rangle. \qquad [1.4]$$

We can also get:

$$J_-\left|a, b\right\rangle = c'\left|a, b - \hbar\right\rangle. \qquad [1.5]$$

Let us utilize [1.4], [1.5] to find which values of a, b are allowed.

Claim 2. $\left\langle J^2 - J_z^2\right\rangle_{\left|a,b\right\rangle} \geq 0$.

Since $\left|a, b\right\rangle$ is an eigenvector of J^2, J_z corresponding to a, b, respectively,

$$\left\langle J^2 - J_z^2\right\rangle_{\left|a,b\right\rangle} = a - b^2.$$

We also have

$$J^2 - J_z^2 = J_x^2 + J_y^2 + J_z^2 - J_z^2 = J_x^2 + J_y^2 = \frac{1}{2}\left(J_+J_- + J_-J_+\right),$$

so that

$$\left\langle J^2 - J_z^2\right\rangle = \left\langle \frac{1}{2}\left(J_+J_- + J_-J_+\right)\right\rangle = \frac{1}{2}\left\langle J_+J_- + J_-J_+\right\rangle.$$

Since $J_{\pm} = J_x \pm iJ_y$ where J_x, J_y are Hermitian, it follows that $J_{\pm}^{\dagger} = J_{\mp}$. Therefore, J_+J_-, J_-J_+ are Hermitian. This means

$$\left\langle J_+J_- + J_-J_+\right\rangle = \left\langle J_+J_+^{\dagger} + J_+^{\dagger}J_+\right\rangle = \left\langle J_+J_+^{\dagger}\right\rangle + \left\langle J_+^{\dagger}J_+\right\rangle = \left\|J_+^{\dagger}\left|a, b\right\rangle\right\| + \left\|J_+\left|a, b\right\rangle\right\| \geq 0.$$

(End of Claim 2)

Consequently,

$$a - b^2 \geq 0 \implies a \geq b^2.$$

Because of this define b_+ be the smallest value such that

$$(b_+ + \hbar)^2 > a.$$

Using the properties of raising and lowering operators, we can write the value of b_+ in terms of a. For this b_+, we have

$$J_+ |a, b_+\rangle = 0.$$

Similarly, let b_- be the largest value such that $(b_- - \hbar)^2 > a$, so that

$$J_- |a, b_-\rangle = 0.$$

Then

$$J_-J_+ |a, b_+\rangle = 0,$$

where

$$J_-J_+ = (J_x - iJ_y)(J_x + iJ_y) = \underbrace{J_x^2 + J_y^2}_{=J^2 - J_z^2} + i \underbrace{(J_xJ_y - J_yJ_x)}_{=[J_x, J_y] = i\hbar J_z} = J^2 - J_z^2 - \hbar J_z.$$

This means

$$0 = J_-J_+ |a, b_+\rangle = (J^2 - J_z^2 - \hbar J_z) |a, b_+\rangle = (a - b_+^2 - \hbar b_+) |a, b_+\rangle,$$

solving which gives

$$a = b_+ (b_+ + \hbar). \tag{1.6}$$

In a similar manner, we obtain

$$a = b_- (b_- - \hbar). \tag{1.7}$$

Note [1.6], [1.7] are satisfied if $b_- = -b_+$.

Assuming that *we can obtain b_+ by applying J_+ $n \in \mathbb{N}$ times on b_-*, we gain

$$b_+ = b_- + n\hbar = -b_+ + n\hbar \implies b_+ = \frac{n\hbar}{2}.$$

It follows that

$$a = \frac{n\hbar}{2}\left(\frac{n\hbar}{2} + \hbar\right) = \hbar^2 \frac{n}{2}\left(\frac{n}{2} + 1\right). \tag{1.8}$$

Since the expression $\hbar \frac{n}{2}\left(\frac{n}{2} + 1\right)$ in [1.8] is *ugly*, we substitute $j = \frac{n}{2}, m = \frac{b}{\hbar}$ so that

$$\begin{aligned} J^2 |j, m\rangle &= \hbar^2 j (j+1) |j, m\rangle \\ J_z |j, m\rangle &= \hbar m |j, m\rangle \end{aligned}, \tag{1.9}$$

where j is a half-integer and m satisfy $m \in \{-j, -j+1, \ldots, j-1, j\}$.

[1.9] is the usual convention for angular momentum operators.

Summary 1.3. Angular Momentum Operators

We define angular momentum operators $J = (J_x, J_y, J_z)$ as operators satisfying

$$[J_i, J_j] = i\hbar J_k \varepsilon_{ijk}, \qquad \forall i, j, k.$$

We define $J^2 = J_x^2 + J_y^2 + J_z^2$, which commutes with J_x, J_y, J_z:

$$[J^2, J_x] = [J^2, J_y] = [J^2, J_z] = 0.$$

The ladder operators are

$$J_+ = J_x + iJ_y, J_- = J_x - iJ_y,$$

which are also called the raising and lowering operators. Then

$$[J_z, J_\pm] = \pm\hbar J_\pm.$$

The eigenvalues of J^2, J_z are

$$\begin{aligned} J^2 \left|j, m\right\rangle &= \hbar^2 j\,(j+1) \left|j, m\right\rangle \\ J_z \left|j, m\right\rangle &= \hbar m \left|j, m\right\rangle \end{aligned},$$

where j is a half-integer and $m \in \{-j, -j+1, \ldots, j\}$.

Notationally, we write $J = L$ and $j = l$ for orbital angular momentum and $J = S$ and $j = s$ for spin angular momentum.

In case $J = L$, an orbital angular momentum, the only allowed values of $j = l$ are integers. But in case $J = S$, a spin angular momentum, any half integer $j = s$ is allowed.

Let us find the matrix representation of the operators. Since

$$J_\pm \left|j, m\right\rangle = c_\pm \left|j, m \pm 1\right\rangle$$

for some constants $c_\pm$,

$$\left\langle j, m\right| J_- J_+ \left|j, m\right\rangle = |c_+|^2 \left\langle j, m | j, m\right\rangle = |c_+|^2 .$$

But we also know that

$$J_- J_+ = \cdots = J^2 - J_z^2 - \hbar J_z,$$

so that

$$|c_+|^2 = \left\langle j, m\right| J_- J_+ \left|j, m\right\rangle = \hbar^2 j\,(j+1) - (\hbar m)^2 - \hbar^2 m = \hbar^2 \left(j\,(j+1) - m\,(m+1)\right).$$

Ignoring global phase, it follows

$$c_+ = \hbar\sqrt{j\,(j+1) - m\,(m+1)},$$

so that

$$J_+ \left|j, m\right\rangle = \hbar\sqrt{j\,(j+1) - m\,(m+1)}.$$

In a similar manner, we find

$$J_- \left|j, m\right\rangle = \hbar\sqrt{j\,(j+1) - m\,(m-1)}.$$

We can use this to find matrix representations.

Example 1.3.

Suppose $j = \frac{1}{2}$, so that we have two eigenstates

$$\begin{aligned} \left|1\right\rangle &= \left|\frac{1}{2}, \frac{1}{2}\right\rangle \\ \left|0\right\rangle &= \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \end{aligned}.$$

Then $J_z \left|1\right\rangle = \frac{\hbar}{2} \left|1\right\rangle, J_z \left|0\right\rangle = \frac{\hbar}{2} \left|0\right\rangle$, so that

$$J_z = \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

is the matrix representation of J_z.

For J_x, recall

$$J_x = \frac{1}{2}\left(J_+ - J_-\right).$$

The matrix representation of J_+ is

$$J_+ = \begin{bmatrix} 0 & \hbar \\ 0 & 0 \end{bmatrix}.$$

On the other hand,

$$J_- = \begin{bmatrix} 0 & 0 \\ \hbar & 0 \end{bmatrix},$$

so that

$$J_x = \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In a similar manner,

$$J_y = -\frac{i}{2} \left(J_+ - J_-\right) = \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Squaring each J_x, J_y, J_z, we obtain

$$J^2 = \frac{3}{4}\hbar^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

---

5. Spherical Harmonics

Consider $\vec{L} = \vec{r} \times \vec{p}$. We expect to see

$$\begin{aligned} J^2 \left|j, m\right\rangle &= \hbar^2 j \left(j+1\right) \left|j, m\right\rangle \\ J_z \left|j, m\right\rangle &= \hbar m \left|j, m\right\rangle \end{aligned}.$$

Moreover, we want an explicit form of $\left|l, m\right\rangle$ in 3D coorinates. To do so, we can use

$$L_z = -i\hbar x \partial_y + i\hbar y \partial_x$$

or use *spherical coordinates*. That is

$$\vec{L} = \vec{r} \times \vec{p} = -i\hbar \vec{r} \times \vec{\nabla},$$

where

$$\vec{\nabla} = \hat{r}\partial_r + \frac{1}{r}\hat{\theta}\partial_\theta + \frac{1}{r\sin\left(\theta\right)}\hat{\varphi}\partial_\varphi.$$

Then

$$\begin{aligned} L^2 &= -\hbar^2 \left(\frac{\partial^2}{\partial\theta^2} + \cot\left(\theta\right)\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\left(\theta\right)}\frac{\partial^2}{\partial\varphi^2} \right) \\ L_x &= -i\hbar \left(-\sin\left(\varphi\right)\partial_\theta - \cot\left(\theta\right)\cos\left(\varphi\right)\partial_\varphi \right) \\ L_y &= -i\hbar \left(\cos\left(\varphi\right)\partial_\theta - \cot\left(\theta\right)\sin\left(\varphi\right)\partial_\varphi \right) \\ L_z &= -i\hbar\partial_\varphi \end{aligned}.$$

Observe that L^2, L_x, L_y, L_z *does not depend* on the radial coordinate r. Hence an eigenfunction of L^2, L_z depends only on θ, φ, say

$$\begin{aligned} L^2 Y\left(\theta, \varphi\right) &= \hbar^2 l \left(l+1\right) Y\left(\theta, \varphi\right) \\ L_z Y\left(\theta, \varphi\right) &= \hbar m Y\left(\theta, \varphi\right) \end{aligned}.$$

As an *ansatz*, suppose

$$Y\left(\theta, \varphi\right) = \Theta\left(\theta\right)\Phi\left(\varphi\right).$$

In this case,

$$L_z Y\left(\theta, \varphi\right) = \hbar m Y\left(\theta, \varphi\right)$$

becomes a differential equation

$$-i\hbar\partial_{\varphi}\Theta\Phi = \hbar m\Theta\Phi.$$

But Θ only depends on θ, so that

$$-i\hbar\partial_{\varphi}\Phi = \hbar m\Phi \implies \Phi(\theta) = e^{im\varphi}.$$

By the 2π periodicy of $e^{i\theta}$, we have

$$e^{im\varphi} = e^{im\varphi}e^{2\pi mi}. \tag{1.10}$$

Observe that [1.10] is satisfied if and only if m is an integer. This is why for orbital angular momentums integer values of m are only permitted values.

Now that we know how Φ looks like, we have

$$L^2 Y(\theta, \varphi) = L^2 e^{im\varphi}\Theta(\theta) . = \hbar^2 l(l+1) e^{im\varphi}\Theta(\theta)$$

Recall that

$$L^2 = -\hbar^2\left(\partial_\theta^2 + \cot(\theta)\partial_\theta + \frac{1}{\sin^2(\theta)}\partial_\varphi^2\right),$$

so that we have

$$-\hbar^2\left(\partial_\theta^2 + \cot(\theta)\partial_\theta + \frac{1}{\sin^2(\theta)}\partial_\varphi^2\right)e^{im\varphi}\Theta(\theta) = \hbar^2 l(l+1)\Phi(\theta).$$

Cancelling out $\hbar^2$, $e^{im\varphi}$,

$$-\partial_\theta^2\Theta - \cot(\theta)\partial_\theta\Theta - \frac{1}{\sin^2(\theta)}\partial_\varphi^2\left(-m^2\right)\Theta = l(l+1)\Theta,$$

or

$$\left(-\partial_\theta^2 - \cot(\theta)\partial_\theta + \frac{m^2}{\sin^2(\theta)}\partial_\varphi^2 - l(l+1)\right)\Theta = 0, \tag{1.11}$$

which is of the form of the *general Legendre equation*. A general solution to [1.11] is

$$\Theta = P_l^m(\cos(\theta)),$$

where the *Legendre polynomial* P_l^m is

$$P_l^m = \left(1-x^2\right)^{\frac{|m|}{2}}\left(\frac{d}{dx}\right)^{|m|}P_l$$

and

$$P_l = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l\left(x^2-1\right)^l. \tag{1.12}$$

Since [1.12] is *usually* defined for nonnegative integer l only, it puts additional restriction on l: $l \in \mathbb{N}\cup\{0\}$.

In conclusion, the spherical harmonics $Y_l^m(\theta, \varphi)$ have the form

$$Y_l^m = AP_l^m(\cos(\theta))e^{im\varphi},$$

where A is a normalization constant. $Y_l^m(\theta, \varphi)$ are the eigenstates of L^2, L_z with eigenvalues $\hbar^2 l(l+1)$, $\hbar m$. The collection $\left\{Y_l^m\right\}$ is an orthonormal basis for the space of functions of θ, φ.

– LECTURE 15 –

Recall that symmetric 2D/3D systems feature degeneracy (i.e. there are different states with same energy). Hence we need another observable to *unambiguously* label eigenstates. That is, we are going to pick an observable A with

$$[A, H] = 0$$

and use eigenvalues of A as a second *label*.

We are going to consider *complete set of commuting operators*, CSCO: all operators commute with each other and eigenvalues of all operators are sufficient to fully distinguish basis states. Recall that

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{m\omega^2}{2}\left(x^2 + y^2\right) = \hbar\omega\left(a_x^\dagger a_x + a_y^\dagger a_y + 1\right)$$

with energy eigenvalues $E_n = \hbar\omega\,(n+1)$ where $n = n_x + n_y$. To uniquely label eigenstates, we could use $|n, n_x\rangle$ or use L_z,

$$[L_z, H] = 0,$$

as done previously. That is, we can find basis with fixed energy and angular momentum for each eigenstate.

The states $|n_x\rangle\,|n_y\rangle$ are not eigenstates of L_z:

$$L_z\,|n_x\rangle\,|n_y\rangle = \left(xp_y - yp_x\right)|n_x\rangle\,|n_y\rangle\,,$$

where

$$xp_y - yp_x = i\hbar\left(a_x a_y^\dagger - a_x^\dagger a_y\right).$$

Observe

$$a_x\,|n_x\rangle\,|n_y\rangle = \sqrt{n_x}\,|n_x - 1\rangle\,|n_y\rangle\,, a_x^\dagger = \sqrt{n_x+1}\,|n_x+1\rangle\,|n_y\rangle$$

and

$$a_y\,|n_x\rangle\,|n_y\rangle = \sqrt{n_y}\,|n_x\rangle\,|n_y - 1\rangle\,, a_y^\dagger = \sqrt{n_y+1}\,|n_x\rangle\,|n_y+1\rangle\,.$$

Therefore,

$$L_z\,|n_x\rangle\,|n_y\rangle = i\hbar\left(\sqrt{n_x}\sqrt{n_y+1}\,|n_x-1\rangle\,|n_y+1\rangle - \sqrt{n_x+1}\sqrt{n_y}\,|n_x+1\rangle\,|n_y-1\rangle\right).$$

Observe that $|n_x-1\rangle\,|n_y+1\rangle\,, |n_x+1\rangle\,|n_y-1\rangle$ are orthonormal to $|n_x\rangle\,|n_y\rangle$, so $|n_x\rangle\,|n_y\rangle$ is not an eigenstate of L_z.

To make things work out, we'd like to use different ladder operators a_1, a_2 such that

$$H = \hbar\omega\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right)$$

and

$$L_z = \alpha a_1^\dagger a_1 + \beta a_2^\dagger a_2.$$

If we can find such ladder operators, then $\hbar\omega\,(n_1 + n_2 + 1)$ and $\alpha n_1 + \beta n_2$ are the eigenvalue of H, L_z, respectively, corresponding to $|n_1\rangle\,|n_2\rangle$. Of course, a_1, a_2 should satisfy *obvious* commutation relations for ladder operators,

$$\left[a_1, a_1^\dagger\right] = 1\left[a_2, a_2^\dagger\right]$$

and

$$[a_1, a_2] = \left[a_1, a_2^\dagger\right] = \left[a_1^\dagger, a_2\right] = \left[a_1^\dagger, a_2^\dagger\right] = 0.$$

Define

$$a_1 = \frac{1}{\sqrt{2}}\left(a_x - ia_y\right), a_2 = \frac{1}{\sqrt{2}}\left(a_x + ia_y\right).$$

Then
$$\left[a_1, a_1^\dagger\right] = \left[\frac{1}{\sqrt{2}}\left(a_x - ia_y\right), \frac{1}{\sqrt{2}}\left(a_x^\dagger + ia_y^\dagger\right)\right] = \frac{1}{2}\left(\left[a_x, a_x^\dagger\right] - i\left[a_y, a_x^\dagger\right] + i\left[a_x, a_y^\dagger\right] + \left[a_y, a_y^\dagger\right]\right) = 1.$$
In a similar manner, a_1, a_2 satisfy other commutation relations.

Moreover, observe that
$$\begin{aligned}
a_1^\dagger a_1 &= \frac{1}{2}\left(a_x^\dagger + ia_y^\dagger\right)\left(a_x - ia_y\right) = \frac{1}{2}\left(a_x^\dagger a_x + a_y^\dagger a_y + ia_y^\dagger a_x - ia_x^\dagger a_y\right), \\
a_2^\dagger a_2 &= \frac{1}{2}\left(a_x^\dagger - ia_y^\dagger\right)\left(a_x + ia_y\right) = \frac{1}{2}\left(a_x^\dagger a_x + a_y^\dagger a_y - ia_y^\dagger a_x + ia_x^\dagger a_y\right).
\end{aligned}$$
This means
$$\begin{aligned}
a_1^\dagger a_1 + a_2^\dagger a_2 &= a_x^\dagger a_x + a_y^\dagger a_y, \\
a_1^\dagger a_1 - a_2^\dagger a_2 &= ia_x^\dagger a_x - ia_y^\dagger a_y.
\end{aligned}$$
It follows
$$\begin{aligned}
L_z &= \hbar\left(a_1^\dagger a_1 - a_2^\dagger a_2\right), \\
H &= \hbar\omega\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right).
\end{aligned}$$
This means the eigenvalues of L_z, H corresponding to $|n_1\rangle|n_2\rangle$ are $\hbar(n_1 - n_2), \hbar\omega(n_1 + n_2 + 1)$.

Consider spherically symmetric Hamiltonian
$$H = \frac{p^2}{2m} + V(r) = -\frac{\hbar^2}{2m}\nabla^2 + V(r).$$
Also recall
$$\nabla^2 = \frac{1}{r^2}\partial_r\left(r^2\partial_r\right) + \frac{1}{r^2\sin(\theta)}\partial_\theta\left(\sin(\theta)\partial_\theta\right) + \frac{1}{r^2\sin(\theta)^2}\partial_\varphi^2 \tag{1.13}$$
with respect to the spherical coordinates. Let's compare [1.13] with
$$L^2 = -\hbar^2\left(\partial_\theta^2 + \cot\partial_\theta + \frac{1}{\sin(\theta)^2}\partial_\varphi^2\right).$$
Observe that
$$\frac{1}{\sin(\theta)}\partial_\theta\left(\sin(\theta)\partial_\theta\right) = \cdots = \cot(\theta)\partial_\theta + \partial_\theta^2.$$
Consequently,
$$\nabla^2 = \frac{1}{r^2}\partial_r\left(r^2\partial_r\right) - \frac{1}{\hbar^2 r^2}L^2.$$
Hence
$$H = \underbrace{-\frac{\hbar^2}{2mr^2}\partial_r\left(r^2\partial_r\right)}_{\text{radial kinetic energy}} + \underbrace{\frac{1}{2mr^2}L^2}_{\text{rotational energy}} + V(r).$$
To find eigenfunctions ψ of H, we use an ansatz
$$\psi(r, \theta, \varphi) = R(r)\, Y_l^m(\theta, \varphi).$$
Then
$$L^2 Y_l^m(\theta, \varphi) = \hbar^2 l(l+1)\, Y_l^m(\theta, \varphi).$$
Observe that the equtaion
$$H\psi = E_n\psi$$

becomes

$$-\frac{\hbar^2}{2mr^2}\partial_r\left(r^2\partial_r\right)RY_l^m+\underbrace{\frac{1}{2mr^2}L^2RY_l^m}_{=\frac{\hbar^2 l(l+1)}{2mr^2}RY_l^m}+VRY_l^m=E_nRY_l^m.$$

But we are dealing with a spherically symmetrical system, so that we may cancel out Y_l^m and obtain

$$-\frac{\hbar^2}{2mr^2}\partial_r\left(r^2\partial_r\right)R+\frac{\hbar^2 l\left(l+1\right)}{2mr^2}R+VR=E_nR. \qquad [1.14]$$

Hence the radial wavefunction R depends on E_n and l, so we are going to index R with n, l.

Using change of variables $u = rR$,

$$\frac{dR}{dr}=\frac{d}{dr}\left(\frac{u}{r}\right)=\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}=\frac{1}{r^2}\left(r\frac{du}{dr}-u\right).$$

Hence

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=\frac{d}{dr}\left(r\frac{du}{dr}-\frac{u}{r^2}\right)=\frac{du}{dr}+r\frac{d^2u}{dr^2}-\frac{du}{dr}=r\frac{d^2u}{dr^2}.$$

Using this, [1.14] can be written as

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+\left(V+\frac{\hbar^2}{2mr^2}l\left(l+1\right)\right)u=E_nu. \qquad [1.15]$$

[1.15] looks like energy eigenvalue equation for 1-dimensional Hamiltonian with *effective* potential energy

$$V+\underbrace{\frac{\hbar^2}{2mr^2 l\left(l+1\right)}}_{\text{centrifugal term}}.$$

We call [1.15] the *radial equation for central potential.*

Summary 1.4. Eigenfunctions for Central Potential

The eigenfunctions for central potential has the form

$$\psi_{nlm}\left(r,\theta,\varphi\right)=R_{nl}\left(r\right)Y_l^m\left(\theta,\varphi\right),$$

where n is the energy state, l is the quantum number in L^2 eigenvalue $\hbar l\left(l+1\right)$ and m is the quantum number in L_z eigenvalue $\hbar m$. Assuming a spherically symmetrical system, we have a differential equation called *radial eqaution*

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+\left(V+\frac{\hbar^2}{2mr^2}l\left(l+1\right)\right)u=E_nu.$$

Example 1.4. Hydrogen Atom

Hydrogen atom has potential

$$V\left(r\right)=-\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r},$$

where r is the distance between proton and electron.

The radial equation becomes

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+\left(-\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r}+\frac{\hbar^2}{2mr^2}l\left(l+1\right)\right)u=E_nu.$$

Rearranging,

$$\frac{d^2u}{dr^2}+\left(\frac{2me^2}{4\pi\varepsilon_0\hbar^2}\frac{1}{r}-\frac{1}{r^2}l\left(l+1\right)\right)u=\frac{-2mE_n}{\hbar^2}u.$$

Let

$$k = \sqrt{\frac{-2mE_n}{\hbar^2}}$$

so that

$$E_n = \frac{-\hbar^2 k^2}{2m}.$$

Observe that $\hbar k$ is momentum and

$$k = \frac{2\pi}{\lambda},$$

where λ is the *de Broglie wavelength*. Also, recall the *Bohr radius*

$$a_0 = \frac{4\pi\varepsilon_0\hbar^2}{me^2}.$$

Using the introduced constants and variables, the radial eqaution can be written as

$$\frac{d^2u}{dr^2} + \left(\frac{2}{a_0 r} - \frac{l(l+1)}{r^2}\right) u = k^2 u. \tag{1.16}$$

We are going to attact [1.16] by

(a) finding dimensionless variables;

(b) looking at asymtotic behavior; and

(c) hypothesizing power series and finding relation between coefficients.

First, we introduce the *dimensionless* version of r,

$$\rho = kr,$$

to turn [1.16] into

$$\frac{d^2u}{d\rho^2} = u\left(1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right). \tag{1.17}$$

When we express u as a power series in ρ, it does not play well with limits $\rho \to 0$ or $\rho \to \infty$. We hence look at the asymtotic behaviors of the solutions.

In limit $\rho \to \infty$, [1.17] becomes

$$\frac{d^2u}{d\rho^2} = u \implies u = Ae^{-\rho} + Be^{\rho}.$$

But e^ρ blows up to ∞ as $\rho \to \infty$, so we conclude $u \to Ae^{-\rho}$ as $\rho \to \infty$.

In the limit $\rho \to 0$,

$$\frac{d^2u}{d\rho^2} = \frac{l(l+1)}{\rho^2} u \implies u = C\rho^{l+1} + D\rho^{-l}.$$

But $\rho^{-l} \to \infty$ as $\rho \to 0$, it follows $u \to C\rho^{l+1}$ as $\rho \to 0$.

Hence consider the ansatz

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho)$$

for some power series $v(\rho)$ in ρ.

Observe that

$$\frac{du}{d\rho} = \rho^l e^{-\rho}\left(v(\rho)(l+1-\rho) + \rho\frac{dv}{d\rho}\right)$$

and

$$\frac{d^2u}{d\rho^2} = \rho^l e^{-\rho}\left(v(\rho)\left(\frac{l(l+1)}{\rho} - 2l - 2 + \rho\right) + \frac{dv}{d\rho}\left(2(l+1-\rho)\right) + \frac{d^2v}{d\rho^2}\rho\right). \tag{1.18}$$

Plugging [1.18] into [1.17] gives

$$\rho\frac{d^2v}{d\rho^2}+2\left(l+1-\rho\right)\frac{dv}{d\rho}+\left(\rho_0-2l-2\right)=0. \tag{1.19}$$

Write v explicitly:

$$v\left(\rho\right)=\sum_{j=0}^{\infty}c_j\rho^j$$

to find out the recurrence relation between the coefficients. Then

$$\frac{dv}{d\rho}=\sum_{j=1}^{\infty}c_jj\rho^{j-1}=\sum_{j=0}^{\infty}\left(j+1\right)c_{j+1}\rho^j$$

and

$$\frac{d^2v}{d\rho^2}=\sum_{j=0}^{\infty}j\left(j+1\right)c_{j+1}\rho^{j-1.}$$

Using this, [1.19] becomes

$$\sum_{j=0}^{\infty}\left(j\left(j+1\right)c_{j+1}+2\left(l+1\right)\left(j+1\right)c_{j+1}-2jc_j+\left(\rho_0-2l-2\right)c_j\right)\rho^j=0. \tag{1.20}$$

This means

$$\left(j\left(j+1\right)+2\left(l+1\right)\left(j+1\right)\right)c_{j+1}+\left(\rho_0-2l-2-2j\right)c_j=0, \qquad \forall j\geq 0 \tag{1.21}$$

or

$$c_{j+1}=\frac{2j+2l+2-\rho_0}{\left(j+1\right)\left(j+2l+2\right)}c_j, \qquad \forall j\geq 0$$

is the recurrence relation.

As $j\to\infty$,

$$c_{j+1}\to\frac{2j}{\left(j+1\right)j}c_j=\frac{2}{\left(j+1\right)}c_j,$$

so that

$$c_{j+1}\to\frac{2^{j+1}}{\left(j+1\right)!}c_0.$$

This means

$$v\left(\rho\right)\approx e^{2\rho}\implies u\left(\rho\right)=\rho^{l+1}e^{-\rho}Ce^{2\rho}=C\rho^{l+1}e^{\rho}.$$

But this $v\left(\rho\right)$ is not normalizable, so there is $j_{\max}$ for which $c_{j_{\max}+1}=0$. Then

$$0=c_{j_{\max}+1}=\frac{2j_{\max}+2l+2-\rho_0}{\left(j_{\max}+1\right)\left(j_{\max}+2l+2\right)}c_{j_{\max}}\implies 2j_{\max}+2l+2-\rho_0=0.$$

It follows that

$$\rho_0=2\left(j_{\max}+l+1\right)\implies n=j_{\max}+l+1.$$

Recalling

$$\rho_0=\frac{me^2}{2\pi\varepsilon_0\hbar^2k},k=\frac{\sqrt{-2mE}}{\hbar},$$

we get

$$E_n=\frac{-\hbar^2k^2}{2m}=-\frac{\hbar^2}{2m}\left(\frac{me^2}{2\pi\varepsilon_0\hbar^2\rho_0}\right)^2=\frac{-\hbar^2}{2m}\left(\frac{me^2}{2\pi\varepsilon_0\hbar^2}\right)\frac{1}{\left(2n\right)^2}=-\frac{R_0}{n^2},$$

where

$$R_0 = \frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \approx 13.6\text{eV}$$

is the *Rydberg energy*, the energy required to *unbind* the electron from a ground state hydrogen atom.

Here is a summary of the relations for the allowed quantum numbers n, l, m_l, which index the eigenvalues $E_n, \hbar^2 l\,(l+1)\,, \hbar m_l$ of H, L^2, L_z:

$$n \text{ is a positive integer}, n = j_{\max} + l + 1, n \geq l + 1, -l \leq m_l \leq l.$$

Moreover,

$$R_{nl}\,(r) = a_0 + a_1 r + a_2 r^2 + \cdots + a_{j_{\max}} r^{j_{\max}}.$$

Example 1.5. Degeneracy

For a hydrogen atom, when $n = 3$, what are the number of degeneracies?

Answer. Since $n = 3$, the allowed values of l are $2, 1, 0$. For $l = 2$, there are 5 allowed values for m_l; for $l = 1$, there are 3 allowed values for m_l; and for $l = 0$ there is only one allowed value for m_l.

Thus there are 9 degeneracies for $n = 3$.

QED

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6. Continuous Wavefunctions

Suppose that a $|\psi\rangle$ in an infinite-dimensional Hilbert space $\mathcal{H}$ and a position operator $\hat{x}$ are given. Then for any position eigenvalue x of $\hat{x}$, we define

$$\psi\,(x) = \langle x|\psi\rangle\,.$$

In other words, the function $\psi\,(\cdot)$ is a representation of $|\psi\rangle$ in an infinite-dimensional space with a fixed ONB.

The orthonormality condition and completeness relation are modified to

$$\langle n|m\rangle = \delta_{n,m} \to \left\langle x|x'\right\rangle = \delta\left(x - x'\right) \qquad \textit{orthonormality}$$

and

$$\sum_n |n\rangle\,\{n\} = I \to \int_{-\infty}^{\infty} |x\rangle\,\langle x|\,dx = I, \qquad \textit{completeness relation}$$

where $\delta\,(\cdot)$ is the *Direc delta distribution*.

Using completeness relation, we can expand $|\psi\rangle$ in $\{|x\rangle\}_x$ basis:

$$|\psi\rangle = \int_{-\infty}^{\infty} |x\rangle\,\langle x|\psi\rangle\,dx = \int_{-\infty}^{\infty} \psi\,(x)\,|x\rangle\,dx$$

analogous to the finite-dimensional case.

Similar to how we use coefficients c_n's of $|\psi\rangle$ in an ONB to find out useful quantities about $|\psi\rangle$, we can use $\psi\,(\cdot)$ to find calculate inner products, probabilities, expectation values, and so on. For instance,

$$\langle\varphi|\psi\rangle = \int_{-\infty}^{\infty} \varphi\,(x)^*\,\psi\,(x)\,dx. \qquad \textit{inner product}$$

When we stick in an operator in between,

$$\langle\varphi|\,A\,|\psi\rangle = \langle\varphi|\,IAI\,|\psi\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle\varphi|x\rangle\,\langle x|\,A\,|y\rangle\,\langle y|\psi\rangle\,dxdy.$$

For simplicity, consider the case where $A = \hat{x}$, the position operator. Then

$$\langle \varphi | \hat{x} | \psi \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle \varphi | x \rangle \langle x | \hat{x} | y \rangle \langle y | \psi \rangle \, dxdy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(x)^* y \delta(x-y) \psi(x) \, dxdy = \int_{-\infty}^{\infty} \varphi(x)^* x \psi(x) \, dx.$$

More generally,

$$\langle \varphi | \hat{A} | \psi \rangle = \int_{-\infty}^{\infty} \varphi(x)^* A(x) \psi(x) \, dx.$$

We interprete ψ as the *probability density* (or to be more precisely, $|\varphi|^2$ is a pdf). That is,

$$\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = 1.$$

7. AdditionofAngular Momentum

Suppose $\vec{J}_1, \vec{J}_2$ are angular momenta. Consider $\vec{J} = \vec{J}_1 + \vec{J}_2$.[1] Then $\vec{J}$ obeys the same commutation relation

$$\left[J^i, J^j\right] = i\hbar J_k \varepsilon_{ijk}, \qquad \forall i, j, k.$$

What basis states should we use?

An *obvious* choice is uncoupled basis. That is, the common eigenbasis of $J_1^2, J_2^2, J_{1,z}, J_{2,z}$, with

$$\begin{aligned} J_1^2 |j_1, j_2, m_1, m_2\rangle &= \hbar^2 j_1 (j_1+1) |j_1, j_2, m_1, m_2\rangle, \\ J_2^2 |j_1, j_2, m_1, m_2\rangle &= \hbar^2 j_2 (j_2+1) |j_1, j_2, m_1, m_2\rangle, \\ J_{1,z} |j_1, j_2, m_1, m_2\rangle &= \hbar m_1 |j_1, j_2, m_1, m_2\rangle, \\ J_{2,z} |j_1, j_2, m_1, m_2\rangle &= \hbar m_2 |j_1, j_2, m_1, m_2\rangle, \end{aligned}$$

where

$$|j_1, j_2, m_1, m_2\rangle = |j_1, m_1\rangle |j_2, m_2\rangle.$$

Another choice is better if J_1, J_2 interact: common eigenbasis of J_1^2, J_2^2, J^2, J_z^2.

$$\begin{aligned} J_1^2 |j_1, j_2, j, m\rangle &= \hbar^2 j_1 (j_1+1) |j_1, j_2, j, m\rangle, \\ J_2^2 |j_1, j_2, j, m\rangle &= \hbar^2 j_2 (j_2+1) |j_1, j_2, j, m\rangle, \\ J^2 |j_1, j_2, j, m\rangle &= \hbar^2 j (j+1) |j_1, j_2, j, m\rangle, \\ J_z^2 |j_1, j_2, j, m\rangle &= \hbar m |j_1, j_2, j, m\rangle. \end{aligned}$$

Often we write

$$\begin{aligned} |m_1, m_2\rangle &= |j_1, j_2, m_1, m_2\rangle \\ |j, m\rangle &= |j_1, j_2, j, m\rangle \end{aligned}$$

provided we know j_1, j_2.

Then we know

$$|j, m\rangle = \sum_{m_1, m_2} \langle m_1, m_2 | j, m\rangle |m_1, m_2\rangle.$$

The coefficients $\langle m_1, m_2 | j, m\rangle$ are caled the *Clebsch-Gorden coefficient* of $|j, m\rangle$.

The following selection rules tell us when the coefficients are nonzero.

Fact 1.1. Selection Rule I

$m = m_1 + m_2$ or $\langle m_1, m_2 | j, m\rangle = 0$.

[1] Since $\vec{J}_1, \vec{J}_2$ are angular momenta, they belong to *different systems*, so more precisely $\vec{J} = \vec{J}_1 \otimes I_2 + I_1 \otimes \vec{J}_2$.

Example 1.6.

Consider $j_1 = 2, j_2 = 1$. Then there are $(2j_1 + 1)(2j_2 + 1) = 15$ total basis states.

The allowed values of m are $-3, \ldots, 3$ and we know $-j \leq m \leq j$. This means $j = 3, 2, 1$ must be allowed, and we have

$$\begin{aligned} j = 3 &\implies 7 \text{ total states} \\ j = 2 &\implies 5 \text{ total states} \\ j = 1 &\implies 3 \text{ total states} \end{aligned}$$

Hence here is an upgraded version of the selection rule

Fact 1.2. Selection Rule II

$m = m_1 + m_2$, $|j_1 - j_2| \leq j \leq j_1 + j_2$ or $\langle m_1, m_2 | j, m \rangle = 0$.

Consider writing

$$|j, m\rangle = \sum_{m_1, m_2} \langle m_1, m_2 | j, m\rangle \, |m_1, m_2\rangle \, .$$

To find the coefficients, start with the state that has maximum projection along z:

$$|m_1 = j_1, m_2 = j_2\rangle = |j = j_1 + j_2, m = j_1 + j_2\rangle \, ,$$

which is unique in both bases.

Start with

$$|j = 1, m = 1\rangle = \left| m_1 = \frac{1}{2}, m_2 = \frac{1}{2} \right\rangle .$$

Using

$$J_- = J_{1,-} + J_{2,-}, J_{\alpha,-} \, |j_\alpha, m_\alpha\rangle \quad = \hbar\sqrt{j_\alpha (j_\alpha + 1) - m_\alpha (m_\alpha - 1)} \, |j_\alpha, m_\alpha - 1\rangle \, ,$$

we get

$$\begin{aligned} J_- &= J_{1,-} \left| m_1 = \frac{1}{2}, m_2 = \frac{1}{2} \right\rangle + J_{2,-} \left| m_1 = \frac{1}{2}, m_2 = \frac{1}{2} \right\rangle \\ &= \hbar\sqrt{\frac{1}{2}\left(\frac{1}{2} + 1\right) - \frac{1}{2}\left(\frac{1}{2} - 1\right)} \left(\left| m_1 = -\frac{1}{2}, m_2 = \frac{1}{2} \right\rangle + \left| m_1 = \frac{1}{2}, m_2 = -\frac{1}{2} \right\rangle \right) . \end{aligned}$$

It follows

$$|j = 1, m = 0\rangle = \frac{1}{\sqrt{2}} \left(\left| m_1 = -\frac{1}{2}, m_2 = \frac{1}{2} \right\rangle + \left| m_1 = \frac{1}{2}, m_2 = -\frac{1}{2} \right\rangle \right) .$$