I. Introduction

1. Review of Postulates and Formalism

Postulate 1.
The state of a quantum system can be described by a wavefunction in a Hilbert space.

We denote the elements of the Hilbert space as $|\psi\rangle$, and the complex conjugate by $\langle\psi|$.

Postulate 2.
Every observable/measurable quantity is described by an operator $\hat{A}$ on a Hilbert space.

Postulate 3.
The result of measuring an observable is one of its eigenvalues.

Postulate 4. Born's Rule
For an observable $\hat{A}$ and its eigenvalue a correspoinding to $|a\rangle$, the probability of measuring a is

$$\mathbb{P}\left(a\right) = \left|\langle a|\psi\rangle\right|^2 ,$$

where $|\psi\rangle$ is the quantum state before the measurement.

Postulate 5.
For an observable $\hat{A}$, if an eigenvalue a is measured, then the quantum state after the measurement is $|a\rangle$.

Postulate 6. Schrodinger Equation
The time evolution of a quantum state $|\psi\rangle$ satisfies

$$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = H\left|\psi\right\rangle ,$$

where H is the Hamiltonian operator.

2. Time Evolution Operator

Consider the time dependent Schrodinger equation

$$i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = H\left|\psi\right\rangle .$$

Then the state $|\psi\rangle$ evolves over time according to the *time evolution operator* $U\left(t, t_0\right)$ by

$$\left|\psi\left(t\right)\right\rangle = U\left(t, t_0\right)\left|\psi\left(t_0\right)\right\rangle$$

with the initial condition $U\left(t_0, t_0\right) = I$, the identity operator. Note that U has to be unitary, since we want to preserve the norm. Therefore, the differential equation for U is given by

$$i\hbar\frac{\partial}{\partial t}U\left(t, t_0\right)\left|\psi\left(t_0\right)\right\rangle = HU\left(t, t_0\right)\left|\psi\left(t_0\right)\right\rangle .$$

Rewriting in terms of U,

$$\frac{\partial}{\partial t}U\left(t, t_0\right) = -\frac{i}{\hbar}HU\left(t, t_0\right) .$$

In case H is independent of the time,

$$U = Ae^{-\frac{i}{\hbar}Ht},$$

where A is a constant depending on the initial condition $U(t_0, t_0) = I$. That is,

$$I = U(t_0, t_0) = Ae^{-\frac{i}{\hbar}Ht_0} \implies A = e^{\frac{i}{\hbar}Ht_0}.$$

This solution is useful when paired with the *time independent Schrodinger equation*:

$$H|E_n\rangle = E_n|E_n\rangle.$$

In this case, for any analytic function f,

$$f(H)|E_n\rangle = f(E_n)|E_n\rangle,$$

so in particular,

$$U(t, t_0)|E_n\rangle = e^{-\frac{i}{\hbar}E_n(t-t_0)}|E_n\rangle.$$

For a general state $|\psi(t_0)\rangle$, we write it down as a linear combination of energy eigenstates,

$$|\psi(t_0)\rangle = \left(\sum_n |E_n\rangle\langle E_n|\right)|\psi(t_0)\rangle = \sum_n c_n|E_n\rangle,$$

where $c_n = \langle E_n|\psi(t_0)\rangle$. Then,

$$|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle = e^{-\frac{i}{\hbar}H(t-t_0)}\sum_n c_n|E_n\rangle = \sum_n c_n e^{-\frac{i}{\hbar}E_n(t-t_0)}|E_n\rangle.$$

Thus

$$U(t, t_0) = \sum_n e^{-\frac{i}{\hbar}E_n(t-t_0)}|E_n\rangle\langle E_n|.$$

Example 1.1. Time Evolution of Spin-$\frac{1}{2}$

Let

$$H = \hbar\Omega\sigma_x = \hbar\Omega(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|) = \hbar\Omega\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},$$

where $|\uparrow\rangle = \begin{bmatrix}1\\0\end{bmatrix}, |\downarrow\rangle = \begin{bmatrix}0\\1\end{bmatrix}$. Note that the energy eigenvectors are

$$\begin{aligned}|E_1\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \text{ corresponding to } \hbar\Omega = E_1\\ |E_2\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle) \text{ corresponding to } -\hbar\Omega = E_2\end{aligned}.$$

Consider

$$|\psi(0)\rangle = |\downarrow\rangle.$$

Then the coefficients with respect to the basis $(|E_1\rangle, |E_2\rangle)$ are

$$\begin{aligned}c_1 &= \langle E_1|\downarrow\rangle = \frac{1}{\sqrt{2}}(\langle\uparrow| + \langle\downarrow|)|\downarrow\rangle = \frac{1}{\sqrt{2}}\\ c_2 &= \langle E_2|\downarrow\rangle = \frac{1}{\sqrt{2}}(\langle\uparrow| - \langle\downarrow|)|\downarrow\rangle = -\frac{1}{\sqrt{2}}\end{aligned},$$

which means

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|E_1\rangle - \frac{1}{\sqrt{2}}|E_2\rangle.$$

It follows that

$$
\begin{aligned}
|\psi(t)\rangle &= U(t,0)|\psi(0)\rangle = \left(\sum_n e^{-\frac{i}{\hbar}E_n t}|E_n\rangle\langle E_n|\right)|\psi(0)\rangle = \left(e^{-i\Omega t}|E_1\rangle\langle E_1| + e^{i\Omega t}|E_2\rangle\langle E_2|\right)\frac{1}{\sqrt{2}}\left(|E_1\rangle - |E_2\rangle\right) \\
&= \frac{1}{\sqrt{2}}e^{-i\Omega t}|E_1\rangle - \frac{1}{\sqrt{2}}e^{i\Omega t}|E_2\rangle = \frac{1}{2}e^{-i\Omega t}\left(|\uparrow\rangle + |\downarrow\rangle\right) - \frac{1}{2}e^{i\Omega t}\left(|\uparrow\rangle - |\downarrow\rangle\right) = \frac{1}{2}\left(e^{-i\Omega t} - e^{i\Omega t}\right)|\uparrow\rangle + \frac{1}{2}\left(e^{-\Omega t} + e^{i\Omega t}\right)|\downarrow\rangle \\
&= i\sin(\Omega t)|\uparrow\rangle + \cos(\Omega t)|\downarrow\rangle.
\end{aligned}
$$

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3. Harmonic Oscillator

Consider the Hamiltonian operator

$$H = \frac{p^2}{2m} = \frac{1}{2}m\omega^2 x^2. \qquad [1.1]$$

Suppose $H|\psi\rangle = E|\psi\rangle$, where E is an eigenvalue of H. How do we calculate $E, |\psi\rangle$?

First approach: substitute

$$p = -i\hbar\frac{\partial}{\partial x} \qquad [1.2]$$

into [1.1]. We won't use this approach, as it does not give much intuition to the problem.

Second approach: rewrite Hamiltonian in terms of *ladder operators*. Let

$$a = \frac{1}{\sqrt{2\hbar m\omega}}\left(ip + m\omega x\right),$$

so that

$$a^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}\left(-ip + m\omega x\right).$$

Then, using the commutator formula $[x,p] = i\hbar$,

$$
\begin{aligned}
a^\dagger a &= \frac{1}{2\hbar m\omega}\left(-ip + m\omega x\right)\left(ip + m\omega x\right) = \frac{1}{2\hbar m\omega}\left(p^2 + im\omega\left(xp - px\right) + m^2\omega^2x^2\right) \\
&= \frac{1}{2\hbar m\omega}\left(p^2 + im\omega[x,p] + m^2\omega^2x^2\right) = \frac{1}{2\hbar m\omega}\left(p^2 + im\omega i\hbar + m^2\omega^2x^2\right) \\
&= \frac{1}{2\hbar m\omega}\left(p^2 + m^2\omega^2x^2 - \hbar m\omega\right) = \frac{1}{\hbar\omega}\left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2\right) - \frac{1}{2} = \frac{1}{\hbar\omega}H - \frac{1}{2}.
\end{aligned}
$$

Thus we conclude

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right). \qquad [1.3]$$

Observe that [1.3] tells us that the eigenvectors of H are precisely the eigenvectors of $a^\dagger a$;

$$a^\dagger a|E_n\rangle = n|E_n\rangle \implies H|E_n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|E_n\rangle.$$

We can find n using commutators. First

$$
\begin{aligned}
\left[a, a^\dagger\right] &= \frac{1}{2\hbar m\omega}\left[ip + m\omega x, -ip + m\omega x\right] = \frac{1}{2\hbar m\omega}\left([ip,-ip] + [ip, m\omega x] + [m\omega x, -ip] + [m\omega x, m\omega x]\right) \\
&= \frac{1}{2\hbar m\omega}\left(im\omega[p,x] - im\omega[x,p]\right) = \frac{i}{2\hbar}\left(-i\hbar - i\hbar\right) = 1.
\end{aligned}
$$

Now, observe that $a|E_n\rangle$ is an eigenstate of $a^\dagger a$ corresponding to $n-1$: since $\left[a, a^\dagger\right] = 1$,

$$a^\dagger a\left(a|E_n\rangle\right) = \left(aa^\dagger - 1\right)a|E_n\rangle = a\left(a^\dagger a\right)|E_n\rangle - a|E_n\rangle = an|E_n\rangle - a|E_n\rangle = (n-1)\,a|E_n\rangle\,.$$

Hence

$$a|E_n\rangle = c_n|E_{n-1}\rangle\,.$$

Similarly,

$$a^\dagger|E_n\rangle = b_n|E_{n+1}\rangle\,.$$

Let us find what c_n, b_n are. Observe

$$n = \langle E_n|\,n\,|E_n\rangle = \langle E_n|\,a^\dagger a\,|E_n\rangle = \langle E_{n-1}|\,c_n^* c_n\,|E_{n-1}\rangle = |c_n|^2\,.$$

Therefore n is nonnegative and

$$c_n = \sqrt{n}$$

or

$$a|E_n\rangle = \sqrt{n}\,|E_{n-1}\rangle\,.$$

In a similar fashion, we find out

$$b_n = \sqrt{n+1},$$

so that

$$a^\dagger|E_n\rangle = \sqrt{n+1}\,|E_{n+1}\rangle\,.$$

Note that we have to have n to be an integer, so that

$$a^n|E_n\rangle = 0.$$

Since we may write x, p in terms of the ladder operators as

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$$

and

$$p = i\sqrt{\frac{\hbar m\omega}{2}}\left(a^\dagger - a\right),$$

we have

$$\left\langle x^2\right\rangle = \langle E_n|\,x^2\,|E_n\rangle = \frac{\hbar}{2m\omega}\langle E_n|\,aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger\,|E_n\rangle = \frac{\hbar}{2m\omega}\langle E_n|\,aa + 2a^\dagger a + 1 + a^\dagger a^\dagger\,|E_n\rangle\,.$$

But, using orthonormality of $\{|E_n\rangle\}_{n=0}^{\infty}$, we obtain

$$\langle E_n|\,aa\,|E_n\rangle = \langle E_n|\,a^\dagger a^\dagger\,|E_n\rangle = 0,$$

so that

$$\left\langle x^2\right\rangle = \frac{\hbar}{2m\omega}\langle E_n|\,2a^\dagger a + 1\,|E_n\rangle = \frac{\hbar}{2m\omega}(2n+1)\langle E_n|E_n\rangle = \frac{\hbar}{2m\omega}(2n+1)\,.$$

Summary 1.1. Harmonic Oscillator, Ladder Operators
Given Hamiltonian

$$H = \frac{1}{2}m\omega^2 x^2,$$

we have that

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right),$$

where

$$a=\frac{1}{\sqrt{2\hbar m\omega}}\left(ip+m\omega x\right)$$

is the *ladder operator*. $a, a^\dagger$ acts on the energy eigenstates by

$$a\left|E_n\right\rangle=\sqrt{n}\left|E_{n-1}\right\rangle$$

and

$$a^\dagger\left|E_n\right\rangle=\sqrt{n+1}\left|E_{n+1}\right\rangle.$$

With respect to the ladder operators, x, p can be written as

$$x=\sqrt{\frac{\hbar}{2m\omega}}\left(a+a^\dagger\right)$$

and

$$p=i\sqrt{\frac{\hbar m\omega}{2}}\left(a^\dagger-a\right),$$

Example 1.2.

Consider the harmonic oscillator states $\left|0\right\rangle=\left|E_0\right\rangle,\ldots$ and suppose $\left|\varphi\left(0\right)\right\rangle=\frac{1}{\sqrt{2}}\left(\left|n\right\rangle+\left|n+1\right\rangle\right)$. What is $\left\langle x\left(t\right)\right\rangle$?

Answer. Observe that

$$\left|\varphi\left(t\right)\right\rangle=\frac{1}{\sqrt{2}}\left(e^{-\frac{i}{\hbar}E_nt}\left|n\right\rangle+e^{-\frac{i}{\hbar}E_{n+1}t}\left|n+1\right\rangle\right)=\frac{1}{\sqrt{2}}\left(e^{-i\omega\left(n+\frac{1}{2}\right)t}\left|n\right\rangle+e^{-i\omega\left(n+\frac{3}{2}\right)t}\left|n+1\right\rangle\right),$$

so that

$$\begin{aligned}\left\langle x\left(t\right)\right\rangle&=\left\langle\psi\left(t\right)\right|x\left|\psi\left(t\right)\right\rangle=\sqrt{\frac{\hbar}{2m\omega}}\left\langle\psi\left(t\right)\right|a+a^\dagger\left|\psi\left(t\right)\right\rangle\\&=\sqrt{\frac{\hbar}{2m\omega}}\frac{1}{2}\left(\left\langle n\right|e^{i\omega\left(n+\frac{1}{2}\right)t}+\left\langle n+1\right|e^{i\omega\left(n+\frac{3}{2}\right)t}\right)\left(a+a^\dagger\right)\sqrt{\frac{\hbar}{2m\omega}}\frac{1}{2}\left(e^{-i\omega\left(n+\frac{1}{2}\right)t}\left|n\right\rangle+e^{-i\omega\left(n+\frac{3}{2}\right)t}\left|n+1\right\rangle\right)\\&=\frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(e^{i\omega+\left(n+\frac{3}{2}\right)t}e^{-i\omega\left(n+\frac{1}{2}\right)t}\left\langle n+1\right|a^\dagger\left|n\right\rangle+e^{i\omega\left(n+\frac{1}{2}\right)t}e^{-i\omega\left(n+\frac{3}{2}\right)t}\left\langle n\right|a\left|n+1\right\rangle\right)\\&=\frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(e^{i\omega t}\sqrt{n+1}+e^{-i\omega t}\sqrt{n+1}\right)=\sqrt{\frac{\hbar\left(n+1\right)}{2m\omega}}\cos\left(\omega t\right).\end{aligned}$$

Thus

$$\left\langle x\left(t\right)\right\rangle=\sqrt{\frac{\hbar\left(n+1\right)}{2m\omega}}\cos\left(\omega t\right).$$

QED

4. Angular Momentum Commutators

We start with classical *orbital angular momentum* (OAM):

$$\vec{L}=\vec{r}\times\vec{p}.$$

By definition,

$$\vec{L}=\begin{bmatrix}ypz-zp_y\\zp_x-xp_z\\xp_y-yp_x\end{bmatrix}.$$

We are going to use *canonical commutation relations*

$$\begin{aligned} [r_i, r_j] &= 0 \\ [p_i, p_j] &= 0 \quad , \\ [r_i, p_j] &= i\hbar\delta_{i,j} \end{aligned}$$

where r_1, r_2, r_3 are position components (say x, y, z for instance) and p_1, p_2, p_3 are the momentum components, to figure out $[L_i, L_j]$. For instance,

$$[L_x, L_y] = [yp_z - zp_y, zp_x - xp_z] = [yp_z, zp_x] - [zp_y, zp_x] - [yp_z, xp_z] + [zp_y, xp_z] .$$

But note that z commutes with itself and p_y, p_z commutes, so that $[zp_y, zp_x] = 0$. Similarly, x, y commutes so that $[yp_z, xp_z] = 0$. Hence,

$$[L_x, L_y] = [yp_z, zp_x] + [zp_y, xp_z] = y[p_z, zp_x] + [y, zp_x]p_z + [zp_y, xp_z] .$$

Since y commutes with z, p_x, it follows $[y, zp_x] = 0$. This means

$$[L_x, L_y] = y[p_z, zp_x] + [zp_y, xp_z] = yz\underbrace{[p_z, p_x]}_{=0} + y[p_z, z]p_x + xz\underbrace{[p_y, p_z]}_{=0} + x[z, p_z]p_y = i\hbar(xp_y - yp_x) = i\hbar L_z.$$

Similarly, we have

$$[L_y, L_z] = i\hbar L_x, [L_z, L_x] = i\hbar L_y.$$

Summary 1.2. Orbital Angular Momentum Commutation Relation
Let $L = (L_x, L_y, L_z)$ be the orbital angular momentum operator. Then

$$[L_x, L_y] = i\hbar L_z, [L_y, L_z] = i\hbar L_x, [L_z, L_x] = i\hbar L_y.$$

We are going to *define* angular momentum operator J as $J = (J_x, J_y, J_z)$ satisfying the above relations. That is,

$$[J_i, J_j] = i\hbar J_k \varepsilon_{ijk}, \qquad \forall i, j, k.$$

ε_{ijk} is the *Levi-Civita symbol.*

Along with

$$J^2 = J_x^2 + J_y^2 + J_z^2,$$

we can find quantization restrictions on amount of angular momentums.

Claim 1. $[J^2, J_z] = 0.$

Observe that

$$\begin{aligned} [J^2, J_z] &= [J_x^2 + J_y^2 + J_z^2, J_z] = [J_x^2, J_z] + [J_y^2, J_z] + \underbrace{[J_z^2, J_z]}_{=0} \\ &= J_x\underbrace{[J_x, J_z]}_{=-i\hbar L_y} + [J_x, J_z]J_x + J_y\underbrace{[J_y, J_z]}_{=i\hbar J_x} + [J_y, J_z]J_y = -i\hbar J_xJ_y - i\hbar J_yJ_x + i\hbar J_yJ_x + i\hbar J_xJ_y = 0. \end{aligned}$$

(End of Claim 1)

In a similar manner, we obtain that

$$[J^2, J_x] = [J^2, J_y] = [J^2, J_z] = 0.$$

Hence we can find a common eigenbasis for J^2, J_i for all i.

We are also going to introduce *ladder operators* for angular momentum (these operators are not the same as in quantum harmonic oscillators):

$$J_\pm = J_x \pm iJ_y.$$

Observe that

$$\left[J^2, J_\pm\right] = 0,$$

since $J_\pm$ is a linear combination of J_x, J_y which commute with J^2. Also,

$$\left[J_z, J_\pm\right] = \left[J_z, J_x\right] \pm i\left[J_z, J_y\right] = i\hbar J_y \pm i\left(-i\hbar J_x\right) = i\hbar J_y \pm \hbar J_x = \pm\hbar\left(J_x \pm iJ_y\right) = \pm\hbar J_\pm.$$

Let us find the common eigenbasis for J^2, J_z first. Consider the system of eigenvalue equations

$$\begin{aligned} J^2 \left|a, b\right\rangle &= a\left|a, b\right\rangle \\ J_z \left|a, b\right\rangle &= b\left|a, b\right\rangle \end{aligned}.$$

Similar to how we did to harmonic oscillators, apply the raising operator J_+ to $\left|a, b\right\rangle$. Observe that

$$J_z\left(J_+\left|a, b\right\rangle\right) = J_zJ_+\left|a, b\right\rangle = \left(J_+J_z + \hbar J_+\right)\left|a, b\right\rangle = J_+b\left|a, b\right\rangle + \hbar J_+\left|a, b\right\rangle = \left(b + \hbar\right)J_+\left|a, b\right\rangle.$$

That is, $J_+\left|a, b\right\rangle$ is an eigenvector of J_z corresponding to $b + \hbar$, so we may write

$$J_+\left|a, b\right\rangle = c\left|a', b + \hbar\right\rangle$$

for some a'. In a similar manner, we have

$$J^2J_+ = aJ_+\left|a, b\right\rangle,$$

so that $J_+\left|a, b\right\rangle$ is an eigenvector of J^2 corresponding to a, which means

$$J_+\left|a, b\right\rangle = c\left|a, b + \hbar\right\rangle. \qquad [1.4]$$

We can also get:

$$J_-\left|a, b\right\rangle = c'\left|a, b - \hbar\right\rangle. \qquad [1.5]$$

Let us utilize [1.4], [1.5] to find which values of a, b are allowed.

Claim 2. $\left\langle J^2 - J_z^2\right\rangle_{\left|a,b\right\rangle} \geq 0.$

Since $\left|a, b\right\rangle$ is an eigenvector of J^2, J_z corresponding to a, b, respectively,

$$\left\langle J^2 - J_z^2\right\rangle_{\left|a,b\right\rangle} = a - b^2.$$

We also have

$$J^2 - J_z^2 = J_x^2 + J_y^2 + J_z^2 - J_z^2 = J_x^2 + J_y^2 = \frac{1}{2}\left(J_+J_- + J_-J_+\right),$$

so that

$$\left\langle J^2 - J_z^2\right\rangle = \left\langle \frac{1}{2}\left(J_+J_- + J_-J_+\right)\right\rangle = \frac{1}{2}\left\langle J_+J_- + J_-J_+\right\rangle.$$

Since $J_\pm = J_x \pm iJ_y$ where J_x, J_y are Hermitian, it follows that $J_\pm^\dagger = J_\mp$. Therefore, J_+J_-, J_-J_+ are Hermitian. This means

$$\left\langle J_+J_- + J_-J_+\right\rangle = \left\langle J_+J_+^\dagger + J_+^\dagger J_+\right\rangle = \left\langle J_+J_+^\dagger\right\rangle + \left\langle J_+^\dagger J_+\right\rangle = \left\|J_+^\dagger\left|a, b\right\rangle\right\| + \left\|J_+\left|a, b\right\rangle\right\| \geq 0.$$

(End of Claim 2)

Consequently,

$$a - b^2 \geq 0 \implies a \geq b^2.$$

Because of this define b_+ be the smallest value such that

$$(b_+ + \hbar)^2 > a.$$

Using the properties of raising and lowering operators, we can write the value of b_+ in terms of a. For this b_+, we have

$$J_+ \left| a, b_+ \right\rangle = 0.$$

Similarly, let b_- be the largest value such that $(b_- - \hbar)^2 > a$, so that

$$J_- \left| a, b_- \right\rangle = 0.$$

Then

$$J_- J_+ \left| a, b_+ \right\rangle = 0,$$

where

$$J_- J_+ = \left(J_x - iJ_y\right)\left(J_x + iJ_y\right) = \underbrace{J_x^2 + J_y^2}_{=J^2 - J_z^2} + i \underbrace{\left(J_x J_y - J_y J_x\right)}_{=\left[J_x, J_y\right] = i\hbar J_z} = J^2 - J_z^2 - \hbar J_z.$$

This means

$$0 = J_- J_+ \left| a, b_+ \right\rangle = \left(J^2 - J_z^2 - \hbar J_z\right) \left| a, b_+ \right\rangle = \left(a - b_+^2 - \hbar b_+\right) \left| a, b_+ \right\rangle,$$

solving which gives

$$a = b_+ \left(b_+ + \hbar\right). \tag{1.6}$$

In a similar manner, we obtain

$$a = b_- \left(b_- - \hbar\right). \tag{1.7}$$

Note [1.6], [1.7] are satisfied if $b_- = -b_+$.

Assuming that *we can obtain b_+ by applying J_+ $n \in \mathbb{N}$ times on b_-*, we gain

$$b_+ = b_- + n\hbar = -b_+ + n\hbar \implies b_+ = \frac{n\hbar}{2}.$$

It follows that

$$a = \frac{n\hbar}{2}\left(\frac{n\hbar}{2} + \hbar\right) = \hbar^2 \frac{n}{2}\left(\frac{n}{2} + 1\right). \tag{1.8}$$

Since the expression $\hbar \frac{n}{2}\left(\frac{n}{2} + 1\right)$ in [1.8] is *ugly*, we substitute $j = \frac{n}{2}, m = \frac{b}{\hbar}$ so that

$$\begin{aligned} J^2 \left| j, m \right\rangle &= \hbar^2 j (j+1) \left| j, m \right\rangle \\ J_z \left| j, m \right\rangle &= \hbar m \left| j, m \right\rangle \end{aligned}, \tag{1.9}$$

where j is a half-integer and m satisfy $m \in \{-j, -j+1, \ldots, j-1, j\}$.

[1.9] is the usual convention for angular momentum operators.

Summary 1.3. Angular Momentum Operators

We define angular momentum operators $J = \left(J_x, J_y, J_z\right)$ as operators satisfying

$$\left[J_i, j_j\right] = i\hbar J_k \varepsilon_{ijk}, \qquad \forall i, j, k.$$

We define $J^2 = J_x^2 + J_y^2 + J_z^2$, which commutes with J_x, J_y, J_z:

$$\left[J^2, J_x\right] = \left[J^2, J_y\right] = \left[J^2, J_z\right] = 0.$$

The ladder operators are

$$J_+ = J_x + iJ_y, J_- = J_x - iJ_y,$$

which are also called the raising and lowering operators. Then

$$[J_z, J_\pm] = \pm\hbar J_\pm.$$

The eigenvalues of J^2, J_z are

$$\begin{aligned} J^2 \left|j, m\right\rangle &= \hbar^2 j\left(j+1\right)\left|j, m\right\rangle \\ J_z \left|j, m\right\rangle &= \hbar m \left|j, m\right\rangle \end{aligned},$$

where j is a half-integer and $m \in \{-j, -j+1, \ldots, j\}$.

Notationally, we write $J = L$ and $j = l$ for orbital angular momentum and $J = S$ and $j = s$ for spin angular momentum.

In case $J = L$, an orbital angular momentum, the only allowed values of $j = l$ are integers. But in case $J = S$, a spin angular momentum, any half integer $j = s$ is allowed.

Let us find the matrix representation of the operators. Since

$$J_\pm \left|j, m\right\rangle = c_\pm \left|j, m \pm 1\right\rangle$$

for some constants $c_\pm$,

$$\left\langle j, m\right| J_- J_+ \left|j, m\right\rangle = \left|c_+\right|^2 \left\langle j, m | j, m\right\rangle = \left|c_+\right|^2.$$

But we also know that

$$J_- J_+ = \cdots = J^2 - J_z^2 - \hbar J_z,$$

so that

$$\left|c_+\right|^2 = \left\langle j, m\right| J_- J_+ \left|j, m\right\rangle = \hbar^2 j\left(j+1\right) - \left(\hbar m\right)^2 - \hbar^2 m = \hbar^2\left(j\left(j+1\right) - m\left(m+1\right)\right).$$

Ignoring global phase, it follows

$$c_+ = \hbar\sqrt{j\left(j+1\right) - m\left(m+1\right)},$$

so that

$$J_+ \left|j, m\right\rangle = \hbar\sqrt{j\left(j+1\right) - m\left(m+1\right)}.$$

In a similar manner, we find

$$J_- \left|j, m\right\rangle = \hbar\sqrt{j\left(j+1\right) - m\left(m-1\right)}.$$

We can use this to find matrix representations.

Example 1.3.

Suppose $j = \frac{1}{2}$, so that we have two eigenstates

$$\begin{aligned} \left|1\right\rangle &= \left|\frac{1}{2}, \frac{1}{2}\right\rangle \\ \left|0\right\rangle &= \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \end{aligned}.$$

Then $J_z\left|1\right\rangle = \frac{\hbar}{2}\left|1\right\rangle, J_z\left|0\right\rangle = \frac{\hbar}{2}\left|0\right\rangle$, so that

$$J_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

is the matrix representation of J_z.

For J_x, recall

$$J_x = \frac{1}{2}\left(J_+ - J_-\right).$$

The matrix representation of J_+ is

$$J_+ = \begin{bmatrix} 0 & \hbar \\ 0 & 0 \end{bmatrix}.$$

On the other hand,

$$J_- = \begin{bmatrix} 0 & 0 \\ \hbar & 0 \end{bmatrix},$$

so that

$$J_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In a similar manner,

$$J_y = -\frac{i}{2}\left(J_+ - J_-\right) = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Squaring each J_x, J_y, J_z, we obtain

$$J^2 = \frac{3}{4}\hbar^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

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5. Spherical Harmonics

Consider $\vec{L} = \vec{r} \times \vec{p}$. We expect to see

$$\begin{aligned} J^2 \left|j, m\right\rangle &= \hbar^2 j\left(j+1\right)\left|j, m\right\rangle \\ J_z \left|j, m\right\rangle &= \hbar m \left|j, m\right\rangle \end{aligned}.$$

Moreover, we want an explicit form of $\left|l, m\right\rangle$ in 3D coorinates. To do so, we can use

$$L_z = -i\hbar x \partial_y + i\hbar y \partial_x$$

or use *spherical coordinates*. That is

$$\vec{L} = \vec{r} \times \vec{p} = -i\hbar \vec{r} \times \vec{\nabla},$$

where

$$\vec{\nabla} = \hat{r}\partial_r + \frac{1}{r}\hat{\theta}\partial_\theta + \frac{1}{r\sin\left(\theta\right)}\hat{\varphi}\partial_\varphi.$$

Then

$$\begin{aligned} L^2 &= -\hbar^2 \left(\frac{\partial^2}{\partial\theta^2} + \cot\left(\theta\right)\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\left(\theta\right)}\frac{\partial^2}{\partial\varphi^2}\right) \\ L_x &= -i\hbar\left(-\sin\left(\varphi\right)\partial_\theta - \cot\left(\theta\right)\cos\left(\varphi\right)\partial_\varphi\right) \\ L_y &= -i\hbar\left(\cos\left(\varphi\right)\partial_\theta - \cot\left(\theta\right)\sin\left(\varphi\right)\partial_\varphi\right) \\ L_z &= -i\hbar\partial_\varphi \end{aligned}.$$

Observe that L^2, L_x, L_y, L_z *does not depend* on the radial coordinate r. Hence an eigenfunction of L^2, L_z depends only on θ, φ, say

$$\begin{aligned} L^2 Y\left(\theta, \varphi\right) &= \hbar^2 l\left(l+1\right) Y\left(\theta, \varphi\right) \\ L_z Y\left(\theta, \varphi\right) &= \hbar m Y\left(\theta, \varphi\right) \end{aligned}.$$

As an *ansatz*, suppose

$$Y\left(\theta, \varphi\right) = \Theta\left(\theta\right)\Phi\left(\varphi\right).$$

In this case,

$$L_z Y\left(\theta, \varphi\right) = \hbar m Y\left(\theta, \varphi\right)$$

becomes a differential equation

$$-i\hbar\partial_{\varphi}\Theta\Phi = \hbar m\Theta\Phi.$$

But Θ only depends on θ, so that

$$-i\hbar\partial_{\varphi}\Phi = \hbar m\Phi \implies \Phi\left(\theta\right) = e^{im\varphi}.$$

By the 2π periodicy of $e^{i\theta}$, we have

$$e^{im\varphi} = e^{im\varphi}e^{2\pi mi}. \tag{1.10}$$

Observe that [1.10] is satisfied if and only if m is an integer. This is why for orbital angular momentums integer values of m are only permitted values.

Now that we know how Φ looks like, we have

$$L^2 Y\left(\theta,\varphi\right) = L^2 e^{im\varphi}\Theta\left(\theta\right). = \hbar^2 l\left(l+1\right)e^{im\varphi}\Theta\left(\theta\right)$$

Recall that

$$L^2 = -\hbar^2\left(\partial_\theta^2 + \cot\left(\theta\right)\partial_\theta + \frac{1}{\sin^2\left(\theta\right)}\partial_\varphi^2\right),$$

so that we have

$$-\hbar^2\left(\partial_\theta^2 + \cot\left(\theta\right)\partial_\theta + \frac{1}{\sin^2\left(\theta\right)}\partial_\varphi^2\right)e^{im\varphi}\Theta\left(\theta\right) = \hbar^2 l\left(l+1\right)\Phi\left(\theta\right).$$

Cancelling out $\hbar^2$, $e^{im\varphi}$,

$$-\partial_\theta^2\Theta - \cot\left(\theta\right)\partial_\theta\Theta - \frac{1}{\sin^2\left(\theta\right)}\partial_\varphi^2\left(-m^2\right)\Theta = l\left(l+1\right)\Theta,$$

or

$$\left(-\partial_\theta^2 - \cot\left(\theta\right)\partial_\theta + \frac{m^2}{\sin^2\left(\theta\right)}\partial_\varphi^2 - l\left(l+1\right)\right)\Theta = 0, \tag{1.11}$$

which is of the form of the *general Legendre equation*. A general solution to [1.11] is

$$\Theta = P_l^m\left(\cos\left(\theta\right)\right),$$

where the *Legendre polynomial* P_l^m is

$$P_l^m = \left(1-x^2\right)^{\frac{|m|}{2}}\left(\frac{d}{dx}\right)^{|m|}P_l$$

and

$$P_l = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l\left(x^2-1\right)^l. \tag{1.12}$$

Since [1.12] is *usually* defined for nonnegative integer l only, it puts additional restriction on l: $l \in \mathbb{N}\cup\{0\}$.

In conclusion, the spherical harmonics $Y_l^m\left(\theta,\varphi\right)$ have the form

$$Y_l^m = AP_l^m\left(\cos\left(\theta\right)\right)e^{im\varphi},$$

where A is a normalization constant. $Y_l^m\left(\theta,\varphi\right)$ are the eigenstates of L^2, L_z with eigenvalues $\hbar^2 l\left(l+1\right)$, $\hbar m$. The collection $\left\{Y_l^m\right\}$ is an orthonormal basis for the space of functions of θ,φ.