I. Measures

1. Motivation

Let X be a set and let $A \subseteq X$. We aim to develop a *meaningful* theory of integration that is

$$\int_A f,$$

where $f : X \to \mathbb{R}$.

There are a bunch of natural question that come out here.

(a) *Which A are appropirate?*

(b) *Which f are appropirate?*

(c) *What does $\int_A f$ even mean?*

Moreover, we want the following:

$$\mu(A) = \int_A 1$$

to be some meaningful idea of size/volume/measure. Some μ's do this better than others. Here are some properties we want μ to satisfy:

(a) $\mu(\emptyset) = 0$.

(b) $\mu(A \mathbin{\dot\cup} B) = \mu(A) + \mu(B)$.

(c) $\mu(A \cup B) \leq \mu(A) + \mu(B)$.

(d) $A \subseteq B \implies \mu(A) \leq \mu(B)$.

(e) $\mu(X) \in [0, \infty]$.

(f) $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$.

(g) $\mu\left(\dot{\bigcup}_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$.

Let's take a step back. If we are going to achieve those things, we want some basics. Let D be the domain of μ – the *nonprecise measure function* handed on us. We need:

(a) $\emptyset \in D$; and

(b) if $A_1, A_2, \ldots \in D$, then $\bigcup_{n=1}^{\infty} A_n \in D$.

2. σ-algebras

Def'n 1.1. σ-**algebra** of Subsets of X

Let X be a set and let $\mathcal{A} \subseteq \mathcal{P}(X)$. We say $\mathcal{A}$ is an ***algebra***[1] of subsets of X if

(a) $\emptyset \in \mathcal{A}$;

(b) $A \in \mathcal{A}$ implies $X \setminus A \in \mathcal{A}$; and *closure under complements*

(c) $A, B \in \mathcal{A}$ implies $A \cup B \in \mathcal{A}$. *closure under finite union*

Moreover, we say $\mathcal{A}$ is a ***σ-algebra*** if it satisfies in addition

$$\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A} \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}.$$

That is, $\mathcal{A}$ is *closed under countable unions*.

[1] The word *algebra* comes from boolean algebra, one of the most universal objects in abstract math.

Question 1.1.
Are all algebra a σ-algebra?

Answer. To answer this question, we should think about:

what is preserved for finite sets but not infinite sets?

The easiest answer is *finiteness*. Let X be an infinite set and let

$$\mathcal{A} = \{A \subseteq X : A \text{ is finite or } X \setminus A \text{ is finite}\}.$$

Then $\mathcal{A}$ is an algebra but not a σ-algebra.

QED

Let $\mathcal{A} \subseteq \mathcal{P}$ be an algebra. Then, as a corollary to Def'n 1.1,

(a) $A, B \in \mathcal{A}$ implies $X \setminus A, X \setminus B \in \mathcal{A}$, so that $A \cap B = X \setminus ((X \setminus A) \cup (X \setminus B)) \in \mathcal{A}$; *closure under closure*

(b) $X = X \setminus \emptyset \in \mathcal{A}$;

(c) $A, B \in \mathcal{A}$ implies $A \setminus B = A \cap (X \setminus B) \in \mathcal{A}$; and *closure under set difference*

(d) $A, B \in \mathcal{A}$ implies $A \triangle B \in \mathcal{A}$. *closure under symmetric set difference*

Moreover, if $\mathcal{A}$ is a σ-algebra, then (a) holds with countable number of sets.

Proposition 1.1. Generating σ-algebra from a Collection of Subsets
Let X be a set and let $\mathcal{E} \subseteq \mathcal{P}(X)$. Then

$$\langle \mathcal{E} \rangle = \bigcap \{\mathcal{A} \supseteq \mathcal{E} : \mathcal{A} \text{ is a } \sigma\text{-algebra}\}$$

is a σ-algebra.

Exercise

Def'n 1.2. σ-algebra **Generated** by $\mathcal{E}$
Consider Proposition 1.1. We call $\langle \mathcal{E} \rangle$ the σ-algebra ***generated*** by $\mathcal{E}$.

Def'n 1.3. **Borel σ-algebra** of a Topological Space
Let (X, τ) be a topological space. Then

$$\mathrm{Bor}(X) = \langle \tau \rangle$$

is called the ***Borel σ-algebra*** of (X, τ).

We call elements of $\mathrm{Bor}(X)$ the ***Borel sets***.

Def'n 1.4. **Measurable Space**
Let X be a set and let $\mathcal{A}$ be a σ-algebra of X. Then we call $(X, \mathcal{A})$ a ***measurable space***.

The elements of $\mathcal{A}$ are called the ***measurable sets***.

3. Measures

In this course, we often work in the extend real numbers $\mathbb{R} \cup \{-\infty, \infty\} = [-\infty, \infty]$. Here are things that we assume.

Assumption 1. Assumptions about Extended Real Numbers
For all $a \in \mathbb{R}$,

(a) $a + \infty = \infty$;

(b) $a - \infty = -\infty$;

(c) $\infty + \infty = \infty$; and

(d) $-\infty - \infty = -\infty$.

However, we leave the following expressions to be *undefined*:

(a) $\infty - \infty$;

(b) $\frac{\infty}{\infty}$; and

(c) 0∞.

Def'n 1.5. **Measure** on a Measurable Space

Let $(X, \mathcal{A})$ be a measurable space. A ***measure*** on $(X, \mathcal{A})$[1] is a function $\mu : \mathcal{A} \to [0, \infty]$ such that

(a) $\mu(\emptyset) = 0$; and

(b) we have

$$\mu\left(\dot{\bigcup_{n\in\mathbb{N}}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

for every $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ with $A_n \cap A_m$ for $n \neq m$. *countable additivity*

In case μ is a measure on $(X, \mathcal{A})$, we call $(X, \mathcal{A}, \mu)$ a ***measure space***.

[1] Or, ***measure*** on X if we are lazy.

Example 1.2. Examples of Measures

Let X be a set.

(a) $\mu(A) = 0$ for all $A \in \mathcal{P}(X)$ is a measure on $(X, \mathcal{P}(X))$. *zero measure*

(b) $\mu(\emptyset) = 0, \mu(A) = \infty$ for all $A \in \mathcal{P}(X) \setminus \{\emptyset\}$ is a measure on $(X, \mathcal{P}(X))$.

(c) $\mu(A) = |A|$ (where $|A| = \infty$ if A is infinite) is a measure on $(X, \mathcal{P}(X))$. *counting measure*

(d) Fix $x \in X$ and define

$$\mu(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

for all $A \in \mathcal{P}(X)$. Then μ is a measure on $(X, \mathcal{P}(X))$. *point-mass measure*

Proposition 1.2.

Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) For all $A, B \in \mathcal{A}$ and $A \subseteq B$, $\mu(A) \leq \mu(B)$. *monotonicity*

(b) For all $A, B \in \mathcal{A}$ with $A \subseteq B$ and $\mu(A) < \infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$. *excision*

(c) If $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$, then $\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu(A_n)$. *countable subadditivity*

Proof.

(a) Consider $B \setminus A$, which is measurable since $\mathcal{A}$ is closed under set difference. Hence we have

$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A).$$

(b) We have

$$\mu(A) + \mu(B \setminus A) = \mu(B)$$

as seen in (a). Since $\mu(A) < \infty$, we can freely subtract $\mu(A)$ from both sides to obtain that $\mu(B \setminus A) = \mu(B) - \mu(A)$.

(c) Let $B_1 = A_1$ and let $B_n = A_n \setminus \bigcup_{m=1}^{n-1} A_m$ for all $n \geq 2$. Then each B_n is measurable with $B_n \subseteq A_n$ and we have

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \mu\left(\dot{\bigcup_{n\in\mathbb{N}}} B_n\right) = \sum_{n\in\mathbb{N}} \mu(B_n) \leq \sum_{n\in\mathbb{N}} \mu(A_n).$$

QED

Proposition 1.3. Continuity of Measure

Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) Let $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be an ascending chain. That is,

$$A_1 \subseteq A_2 \subseteq \cdots .$$

Then

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n). \qquad \textit{continuity from below}$$

(b) Let $\{B_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be a decending chain with $\mu(B_1) < \infty$. That is,

$$B_1 \supseteq B_2 \supseteq \cdots .$$

Then

$$\mu\left(\bigcap_{n\in\mathbb{N}} B_n\right) = \lim_{n\to\infty} \mu(B_n). \qquad \textit{continuity from above}$$

Proof.

(a) Let $C_1 = A_1$ and let $C_n = A_n \setminus A_{n-1} = A_n \setminus \bigcup_{m=1}^{n-1} A_m$ for all $n \geq 2$, where the last equality follows from the ascending chain condition.

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \mu\left(\dot{\bigcup_{n\in\mathbb{N}}} C_n\right) = \sum_{n\in\mathbb{N}} \mu(C_n) = \lim_{N\to\infty} \sum_{n=1}^{N} \mu(C_n) = \lim_{N\to\infty} \mu\left(\bigcup_{n=1}^{N} C_n\right) = \lim_{N\to\infty} \mu(A_N).$$

(b) Let $D_n = B_1 \setminus B_n$ for all $n \in \mathbb{N}$, so that $\{D_n\}_{n\in\mathbb{N}}$ is an ascending chain. Then

$$B_1 \setminus \bigcap_{n\in\mathbb{N}} B_n = \bigcup_{n\in\mathbb{N}} D_n,$$

so that

$$\mu\left(B_1 \setminus \bigcap_{n\in\mathbb{N}} B_n\right) = \mu\left(\bigcup_{n\in\mathbb{N}} D_n\right) = \lim_{n\to\infty} \mu(D_n) = \lim_{n\to\infty} \mu(B_1) - \mu(B_n) = \mu(B_1) - \lim_{n\to\infty} \mu(B_n).$$

The result then follows from excision property of μ.

QED

Def'n 1.6. **Finite**, **Probability**, **σ-finite**, **Semifinite**, **Complete** Measure

Let $(X, \mathcal{A}, \mu)$ be a measure space. We say μ is

(a) ***finite*** if $\mu(X) < \infty$;

(b) a ***probability*** measure if $\mu(X) = 1$;

(c) ***σ-finite*** if

$$X = \bigcup_{n=1}^{\infty} A_n$$

for some $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ with $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$;

(d) ***semifinite*** if

$$\forall A \in \mathcal{A}\,[\mu(A) \neq 0 \implies B \in \mathcal{A}\,[B \subseteq A, 0 < \mu(B) < \infty]];$$

and

(e) ***complete*** if

$$\forall A \in \mathcal{A}\,[\mu(A) = 0 \implies \forall B \subseteq A\,[B \in \mathcal{A}]].$$

Example 1.3. An Example of Non-complete Measure

Let $X = \{a, b\}$, $\mathcal{A} = \{\emptyset, \{a, b\}\}$, $\mu = 0$. Then μ is not complete, as $\{a\} \in \mathcal{A}$.

The goal of this section is:

given a measure space $(X, \mu, \mathcal{A})$, if μ is not complete, we extend $\mathcal{A}$ and μ so that the result is complete.

A natural way of doing this is throw every subsets of measure-zero sets into $\mathcal{A}$.

Proposition 1.4. Completion of a Measure Space

Let $(X, \mu, \mathcal{A})$ be a measure space. Let

$$\overline{\mathcal{A}} = \{A \cup F : A \in \mathcal{A}, \exists N \in \mathcal{A}\,[F \subseteq N, \mu(N) = 0]\}$$

and define

$$\begin{aligned} \overline{\mu} : \overline{\mathcal{A}} &\to [0, \infty] \\ A \cup F &\mapsto \mu(A) \end{aligned}.$$

Then

(a) $\overline{\mathcal{A}}$ is a σ-algebra;

(b) $\overline{\mu}$ is a measure;

(c) $\overline{\mu}|_{\mathcal{A}} = \mu$; and

(d) $\overline{\mu}$ is complete.

Proof.

(a) Note that $\emptyset = \emptyset \cup \emptyset$ with $\emptyset \subseteq \emptyset$ where $\mu(\emptyset) = 0$. Hence $\emptyset \in \overline{\mathcal{A}}$.

Let $E = A \cup F$ with $A \in \mathcal{A}, F \subseteq N \in \mathcal{A}$ where $\mu(N) = 0$. Then

$$X \setminus E = \underbrace{X \setminus (A \cup N)}_{\in \mathcal{A}} \cup \underbrace{(N \setminus (A \cup F))}_{\subseteq N} \in \overline{\mathcal{A}}.$$

Let $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ with $E_n = A_n \cup F_n$ where $F_n \subseteq N_n$ for some $n \in \mathbb{N}$. Then

$$\bigcup_{n=1}^{\infty} E_n = \left(\bigcup_{n=1}^{\infty} A_n\right) \cup \left(\bigcup_{n=1}^{\infty} F_n\right).$$

But $\bigcup_{n=1}^{\infty} F_n \subseteq \bigcup_{n=1}^{\infty} N_n$ with $\mu\left(\bigcup_{n=1}^{\infty} N_n\right) \leq \sum_{n=1}^{\infty} \mu(N_n) = 0$. Thus $\bigcup_{n=1}^{\infty} E_n \in \overline{\mathcal{A}}$.

(b) We first check that $\overline{\mu}$ is well-defined. Let

$$E = A_1 \cup F_1 = A_2 \cup F_2$$

for some $A_1, A_2 \in \mathcal{A}$ and $F_1 \subseteq N_1, F_2 \subseteq N_2$ with $\mu(N_1) = \mu(N_2) = 0$.

Then note that

$$A_1 \cap A_2 \subseteq A_i \subseteq E \subseteq (A_1 \cup F_1) \cap (A_2 \cup F_2) \subseteq (A_1 \cap A_2) \cup N_1 \cup N_2.$$

Hence

$$\mu(A_1 \cap A_2) \leq \mu(A_i) \leq \mu(E_1 \cap E_2).$$

This means $\mu(A_i) = \mu(A_1 \cap A_2)$, sot hat $\mu(E_1) = \mu(E_2)$.

Thus $\overline{\mu}$ is well-defined.

To show $\overline{\mu}$ is a measure, note that

$$\overline{\mu}(\emptyset) = \overline{\mu}(\emptyset \cup \emptyset) = \mu(\emptyset) = 0.$$

Say we have a collection of disjoint sets in $\overline{\mathcal{A}}$, $\{E_n\}_{n=1}^{\infty} \subseteq \overline{\mathcal{A}}$, with

$$E_n = A_n \cup F_n$$

for some $E_n \subseteq N_n$ with $\mu(N_n) = 0$. Then

$$\dot{\bigcup_{n=1}^{\infty}} E_n = \left(\dot{\bigcup_{n=1}^{\infty}} A_n\right) \cup \underbrace{\left(\dot{\bigcup_{n=1}^{\infty}} F_n\right)}_{\subseteq \dot{\bigcup}_{n=1}^{\infty} N_n}.$$

Thus

$$\overline{\mu}\left(\dot{\bigcup_{n=1}^{\infty}} E_n\right) = \mu\left(\dot{\bigcup_{n=1}^{\infty}} A_n\right) = \sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} \overline{\mu}(A_n),$$

so $\overline{\mu}$ is a measure.

(c) Given $A \in \mathcal{A}$, $A = A \,\dot\cup\, \emptyset$, so that $\overline{\mu}(A) = \mu(A)$.

(d) Let $A \subseteq B \in \overline{A}$ with $\overline{\mu}(B) = 0$. We are going to show $A \in \overline{\mathcal{A}}$.

We can write

$$B = E \cup F$$

for some $F \subseteq N \in \mathcal{A}$ with $\mu(N) = 0$. Then

$$\overline{\mu}(B) = \mu(E) = 0.$$

Since $A \subseteq B \subseteq E \cup N$ with $\mu(E \cup N) = 0$ (complete this).

QED

Def'n 1.7. **Completion** of a Measure Space

Let $(X, \mu, \mathcal{A})$ be a measure space. We call $\left(X, \overline{\mu}, \overline{\mathcal{A}}\right)$ the ***completion*** of $(X, \mu, \mathcal{A})$.

5. Construction of Measures

Def'n 1.8. **Outer Measure** on a Set

Let X be a nonempty set. An ***outer measure*** on X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that

(a) $\mu^*(\emptyset) = 0$;

(b) $A \subseteq B$ implies $\mu^*(A) \leq \mu^*(B)$; and *monotonicity*

(c) $\{A_n\}_{n=1}^{\infty} \mathcal{P}(X)$ implies $\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$. *countable subadditivity*

The idea is that

outer measures are naive approaches to measure **every** *subset of X.*

We start with $\mathcal{E} \subseteq \mathcal{P}(X)$ which are *easy* to measure. We use the outer measure μ^* and $\mathcal{E}$ to construct a measure.

Proposition 1.5. Construction of an Outer Measure

Suppose $\{\emptyset, X\} \subseteq \mathcal{E} \subseteq \mathcal{P}(X)$ and $\mu : \mathcal{E} \to [0, \infty]$ satisfies $\mu(\emptyset) = 0$. For $A \subseteq X$, define

$$\mu^*(A) = \inf\left\{\sum_{n=1}^{\infty} \mu(E_n) : \{E_n\}_{n=1}^{\infty} \subseteq \mathcal{E}, A \subseteq \bigcup_{n=1}^{\infty} E_n\right\}.$$

Then μ^* is an outer measure on X.

Example 1.4. Lebesgue Outer Measure

Let $X = \mathbb{R}, \mathcal{E} = \{(a,b) : -\infty < a < b < \infty\} \cup \{\emptyset, X\}$. Define

$$\mu((a,b)) = b - a, \mu(X) = \infty.$$

Then μ^* as said in Proposition 1.5 is called the ***Lebesgue outer measure***.

Proposition 1.6.

Suppose $\{\emptyset, X\} \subseteq \mathcal{E} \subseteq X$ and let $\mu : \mathcal{E} \to [0,\infty]$. If $\mu(\emptyset) = 0$, then $\mu^* : \mathcal{P}(X) \to [0,\infty]$ by

$$\mu^*(A) = \inf\left\{\sum_{n=1}^{\infty} \mu(A_n) : \{A_n\}_{n=1}^{\infty} \subseteq \mathcal{E}, A \subseteq \bigcup_{n=1}^{\infty} A_n\right\}, \qquad \forall A \in \mathcal{E}.$$

is an outer measure.

Proof. We verify few things.

(a) Note that $\emptyset \subseteq \bigcup_{n=1}^{\infty} \emptyset$ and so $0 \le \mu^*(\emptyset) \le \sum_{n=1}^{\infty} \mu(\emptyset) = 0$.

(b) Say $A \subseteq B \subseteq X$. Then

$$\left\{\sum_{n=1}^{\infty} \mu(A_n) : \forall n \in \mathbb{N}[A_n \in \mathcal{E}], A \subseteq \bigcup_{n=1}^{\infty} A_n\right\} \supseteq \left\{\sum_{n=1}^{\infty} \mu(A_n) : \forall n \in \mathbb{N}[A_n \in \mathcal{E}], B \subseteq \bigcup_{n=1}^{\infty} A_n\right\}$$

by definition. By taking infimum, we see that

$$\mu^*(A) \le \mu^*(B).$$

(c) Say $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(X)$ and consider $\bigcup_{n=1}^{\infty} A_n$. We claim that

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} \mu^*(A_n).$$

We may assume $\sum_{n=1}^{\infty} \mu^*(A_n) < \infty$.

Let $\varepsilon > 0$ be given. For every A_i, we may find $\{E_{i,j}\}_{j=1}^{\infty} \subseteq \mathcal{E}$ such that

$$A_i \subseteq \bigcup_{n=1}^{\infty} E_{i,j}$$

and

$$\sum_{j=1}^{\infty} \mu(E_{i,j}) < \mu^*(A_i) + \frac{\varepsilon}{2^i}$$

We then have

$$\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i,j=1}^{\infty} E_{i,j}.$$

Hence

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \overset{\inf}{\le} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mu(E_{i,j}) \le \sum_{i=1}^{\infty} \mu^*(A_i) + \frac{\varepsilon}{2^i} = \left(\sum_{i=1}^{\infty} \mu^*(A_i)\right) + \varepsilon.$$

Since ε is an arbitary positive number, we see that μ^* is countably subadditive.

QED

Def'n 1.9. μ^***-measurable** Set

Let μ^* be an outer measure on X. We say $A \subseteq X$ is μ^****-measurable*** if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap (X \setminus A))$$

for all $E \subseteq X$.

Let $A, E \subseteq X$.

(a) Note

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap (X \setminus A)).$$

Hence it suffices to prove the reverse inequality to show that A is μ^*-measurable.

(b) As a corollary to (a), we may assume $\mu^*(E) < \infty$ when proving A is μ^*-measurable.

(c) When $A = \emptyset$,

$$\mu^*(E \cap \emptyset) + \mu^*(E \cap (X \setminus \emptyset)) = 0 + \mu^*(E) = \mu^*(E).$$

Thus $\emptyset$ is μ^*-measurable.

(d) If A is μ^*-measurable, then $X \setminus A$ is also μ^*-measurable. This is direct from the definition of μ^*-measurability.

Theorem 1.7. Caratheodory

Let μ^* be an outer measure on X. Then the collection of μ^*-measurable subsets of X,

$$\mathcal{A} = \{A \subseteq X : A \text{ is } \mu^*\text{-measurable}\},$$

is a σ-algebra.

Moreover, $\mu = \mu^*|_{\mathcal{A}}$ is a complete measure on $(X, \mathcal{A})$.

Proof. Let $A, B \in \mathcal{A}$ and let $E \subseteq X$. Then

$$\begin{aligned}
\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap (X \setminus A) \cap B) + \mu^*(E \cap (X \setminus A) \cap (X \setminus B)) && \text{since } A, B \text{ are } \mu^*\text{-measurable} \\
&\geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (X \setminus (A \cup B))). && \text{by subadditivity of } \mu^* \text{ and de Morgan's Law}
\end{aligned}$$

Since we know the other direction of the above inequality, we see that $A \cup B \in \mathcal{A}$. Inductively, $\mathcal{A}$ is closed under finite union, which means $\mathcal{A}$ is an algebra on X (we know $\emptyset \in \mathcal{A}$).

Now assume $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$. For any $E \subseteq X$,

$$\mu^*(E \cap (A \cupdot B)) = \mu^*(E \cap (A \cupdot B) \cap A) + \mu^*(E \cap (A \cupdot B) \cap (X \setminus A)) = \mu^*(E \cap A) + \mu^*(E \cap B).$$

By taking $E = X$, we see that

$$\mu^*(A \cupdot B) = \mu^*(A) + \mu^*(B),$$

so that μ^* is finitely additive.

Assume $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$, let $B_n = \bigcup_{k=1}^{n} A_k$, and let $A_n' = A_1 \setminus \bigcup_{k=1}^{n-1} A_k$ for all $n \in \mathbb{N}$. Since $\mathcal{A}$ is an algebra, each $A_n', B_n \in \mathcal{A}$. Then $B_n = \bigcupdot_{n=1}^{\infty} A_k'$ and $B = \bigcup_{n=1}^{\infty} A_n = \bigcupdot_{n=1}^{\infty} A_n'$. For any $E \subseteq X$,

$$\begin{aligned}
\mu^*(E) &= \mu^*(E \cap B_n) + \mu^*(E \cap (X \setminus B_n)) \\
&\geq \mu^*(E \cap B_n) + \mu^*(E \cap (X \setminus B)) && \text{by monotonicity of } \mu^* \\
&= \sum_{k=1}^{n} \mu^*\left(E \cap A_k'\right) + \mu^*(E \cap (X \setminus B)) \\
&\geq \lim_{n\to\infty} \sum_{k=1}^{n} \mu^*\left(E \cap A_k'\right) + \mu^*(E \cap (X \setminus B)) \\
&\geq \mu^*(E \cap B) + \mu^*(E \cap (X \setminus B)) \\
&\geq \mu^*(E). && \text{by subadditivity of } \mu^*
\end{aligned}$$

This means $\mu^*(E) = \mu^*(E \cap B) + \mu^*(E \cap (X \setminus B))$, so $\bigcup_{n=1}^{\infty} A_n = B \in \mathcal{A}$. Hence $\mathcal{A}$ is a σ-algebra.

Assume $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ is a collection of disjoint sets in $\mathcal{A}$. By taking $A'_n = A_n$ for all $n \in \mathbb{N}$ and $E = B$, we see that

$$\mu^*(B) \geq \sum_{n=1}^{\infty} \mu^*(B \cap A_n) + \underbrace{\mu^*(B \cap (X \setminus B))}_{=0} \geq \mu^*(B) \implies \mu^*(B) = \sum_{n=1}^{\infty} \mu^*(B \cap A_n)$$

from the series of inequalities we used for proving closure of $\mathcal{A}$ under countable union.

We now show that μ is complete. Let $A \subseteq X$ with $\mu^*(A) = 0$. For any $E \subseteq X$,

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap (X \setminus A)) \leq \underbrace{\mu^*(A)}_{=0} + \mu^*(E).$$

This means every set A with $\mu^*(A) = 0$ is measurable. But given any $B \in \mathcal{A}$ with $\mu(B) = 0$, we have

$$0 \leq \mu^*(A) \leq \mu^*(B) = \mu(B) = 0, \qquad \forall A \subseteq B,$$

so that $\mu^*(A) = 0$ and that A is measurable.

QED

We can construct a measure as follows. Given $\mathcal{E} \subseteq \mathcal{P}(X)$ with $\{\emptyset, X\} \subseteq \mathcal{E}$ and $\mu : \mathcal{E} \to [0, \infty]$, we let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be an outer measure as defind in Proposition 1.6.

In general, $\mathcal{A} = \{A \subseteq X : A \text{ is } \mu^*\text{-measurable}\}$ and $\mu^*|_{\mathcal{A}}$ are very different from $\mathcal{E}, \mu$. To resolve this, we introduce the following notion.

Def'n 1.10. **Premeasure** on an Algebra of Subsets

Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra of subsets of X. We say $\mu : \mathcal{A} \to [0, \infty]$ is a ***premeasure*** on $\mathcal{A}$ if

(a) $\mu(\emptyset) = 0$; and

(b) for any $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ with $\dot{\bigcup}_{n=1}^{\infty} A_n \in \mathcal{A}$, we have

$$\mu\left(\dot{\bigcup_{n=1}^{\infty}} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Theorem 1.8. Constructing Measure from Premeasure I

Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra and let $\mu : \mathcal{A} \to [0, \infty]$ be a premeasure on $\mathcal{A}$. Let μ^* be the outer measure constructed with $\mathcal{A}$:

$$\mu^*(A) = \inf\left\{\sum_{n=1}^{\infty} \mu(A_n) : \{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}, A \subseteq \bigcup_{n=1}^{\infty} A_n\right\}, \qquad \forall A \in \mathcal{P}(X).$$

Then

(a) $\mu^*|_{\mathcal{A}} = \mu$; and

(b) every $A \in \mathcal{A}$ is μ^*-measurable.

Proof.

(a) We show $\mu^*|_{\mathcal{A}} = \mu$. Let $E \in \mathcal{A}$. Say

$$E \subseteq \bigcup_{n=1}^{\infty} A_n$$

where each $A_n \in \mathcal{A}$. Then by taking $A'_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k$,

$$E = \bigcup_{n=1}^{\infty} (A_n \cap E) = \dot{\bigcup_{n=1}^{\infty}} (A'_n \cap E).$$

But each $A'_n \cap E \in \mathcal{A}$, so that

$$\mu(E) = \sum_{n=1}^{\infty} \mu\left(A'_n \cap E\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

by the monotonicity of μ.[1] Therefore, $\mu(E) \leq \mu^*(E)$ by taking infimum.

On the other hand, by letting $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ with $A_1 = E, A_2 = A_3 = \cdots = \emptyset$, we see that $\mu^*(E) \geq \mu(E)$. Hence $\mu^*|_{\mathcal{A}} = \mu$.

(b) Let $A \in \mathcal{A}$. We show A is μ^*-measurable. Let $E \subseteq X$ and let $\varepsilon > 0$ be given. We may find $\{B_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ such that $E \subseteq \bigcup_{n=1}^{\infty} B_n$ and

$$\sum_{n=1}^{\infty} \mu(B_n) < \mu^*(E) + \varepsilon.$$

Then,

$$\begin{aligned}
\mu^*(E) + \varepsilon &\geq \sum_{n=1}^{\infty} \mu(B_n) \\
&= \sum_{n=1}^{\infty} \mu(B_n \cap A) + \mu(B_n \cap (X \setminus A)) \\
&= \sum_{n=1}^{\infty} \mu^*(B_n \cap A) + \sum_{n=1}^{\infty} \mu^*(B_n \cap (X \setminus A)) && \text{by (a)} \\
&\geq \mu^*\left(\left(\bigcup_{n=1}^{\infty} B_n\right) \cap A\right) + \mu^*\left(\left(\bigcup_{n=1}^{\infty} B_n\right) \cap (X \setminus A)\right) && \text{by subadditivity of } \mu^* \\
&\geq \mu^*(E \cap A) + \mu^*(E \cap (X \setminus A)). && \text{by monotonicity of } \mu^* \text{ since } E \subseteq \bigcup_{n=1}^{\infty} B_n
\end{aligned}$$

[1] It suffices to note that premeasures are finitely additive, which implies monotonicity.

QED

Theorem 1.9. Constructing Measure from Premeasure II

Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra and let μ^* be as in Theorem 1.8. Let $\mathcal{B} = \sigma(\mathcal{A})$. Then

(a) $\overline{\mu} = \mu^*|_{\mathcal{B}}$ is a complete measure with $\overline{\mu}|_{\mathcal{A}} = \mu$.

(b) Let ν be another measure on $\mathcal{B}$ with $\nu|_{\mathcal{A}} = \mu$. Then $\nu \leq \overline{\mu}$. That is,

$$\nu(A) \leq \overline{\mu}(A), \qquad \forall A \in \mathcal{B}.$$

(c) For any $E \in \mathcal{B}$, if $\overline{\mu}(E) < \infty$, then $\nu(E) = \overline{\mu}(E)$.

(d) If μ is σ-finite,[1] then $\overline{\mu} = \nu$.

[1] We say a premeasure is ***σ-finite*** if $X = \bigcup_{n=1}^{\infty} A_n$ for some $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ with $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$.

Proof.

(a) Let

$$\mathcal{C} = \{A \subseteq \mathcal{P}(X) : A \text{ is } \sigma\text{-measurable}\},$$

which is a σ-algebra. Then by Theorem 1.8, $\mathcal{A} \subseteq \mathcal{C}$, and so $\mathcal{B} \subseteq \mathcal{C}$ by minimality of $\mathcal{B}$. Therefore,

$$\overline{\mu} = \mu^*|_{\mathcal{B}}$$

is the restriction of $\mu^*|_{\mathcal{C}}$ to $\mathcal{B}$. Since $\mu^*|_{\mathcal{C}}$ is a complete measure on $(X, \mathcal{C})$, it follows $\overline{\mu} = \mu^*|_{\mathcal{B}}$ is a complete measure on $(X, \mathcal{B})$. Since $\mu^*|_{\mathcal{A}} = \mu$, $\overline{\mu}|_{\mathcal{A}} = \mu$ as well.

(b) Let $A \in \mathcal{B}$ and let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ be such that $A \subseteq \bigcup_{n=1}^{\infty} A_n$. Since ν is a measure extending μ,

$$\nu(A) \leq \nu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \nu(A_n) \overset{\nu|_{\mathcal{A}}=\mu}{=} \sum_{n=1}^{\infty} \mu(A_n).$$

By recalling that μ^* is defined as the *greatest* lower bound, it follows

$$\nu(A) \leq \mu^*(A) = \overline{\mu}(A).$$

(c) Let $A \in \mathcal{B}$ with $\overline{\mu}(A) < \infty$. Let $\varepsilon > 0$ be given. We may find $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ such that $A \subseteq \bigcup_{n=1}^{\infty} A_n$ and

$$\sum_{n=1}^{\infty} \mu(A_n) < \overline{\mu}(A) + \varepsilon.$$

Let $B = \bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$. Note that

$$\nu(B) = \nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{k\to\infty} \nu\left(\bigcup_{n=1}^{k} A_n\right) = \lim_{k\to\infty} \overline{\mu}\left(\bigcup_{n=1}^{k} A_n\right) = \overline{\mu}\left(\bigcup_{n=1}^{\infty} A_n\right) = \overline{\mu}(B).$$

Moreover

$$\overline{\mu}(B) \leq \sum_{n=1}^{\infty} \mu(A_n) < \overline{\mu}(A) + \varepsilon < \infty.$$

It follows

$$\overline{\mu}(B \setminus A) < \varepsilon,$$

so that

$$\overline{\mu}(A) \leq \overline{\mu}(B) = \nu(B) = \nu(A) + \nu(B \setminus A) \leq \nu(A) + \overline{\mu}(B \setminus A) < \nu(A) < \varepsilon.$$

Since ε was given arbitrarily, we have $\overline{\nu}(A) \leq \nu(A)$. Since the reverse inequality is given in (b), we thus conclude $\overline{\mu}(A) = \nu(A)$.

(d) Say $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ is such that $X = \bigcup_{n=1}^{\infty} A_n$ with $\mu(A_n) < \infty$. Write $A_n' = A_n \setminus \bigcup_{m=1}^{n-1} A_m$ so that

$$X = \dot{\bigcup_{n=1}^{\infty}} A_n'.$$

Therefore,

$$\overline{\mu}(A) = \overline{\mu}\left(\dot{\bigcup_{n=1}^{\infty}} \left(A \cap A_n'\right)\right) = \sum_{n=1}^{\infty} \overline{\mu}\left(A \cap A_n'\right) = \sum_{n=1}^{\infty} \nu\left(A \cap A_n'\right) = \nu(A).$$

QED

6. Lebesgue-Stieltjes Measures on $\mathbb{R}$

Suppose we have a measure space $(\mathbb{R}, \mathrm{Bor}(\mathbb{R}), \mu)$, where we are working with the usual topology on $\mathbb{R}$. We further assume that

for all compact $K \subseteq \mathbb{R}$, $\mu(K) < \infty$.

We consider

$$F : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \begin{cases} \mu([0,x]) & x \geq 0 \\ -\mu((x,0)) & x < 0 \end{cases}.$$

Then by definition, F is increasing.

Let $(x_n)_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ be a decreasing sequence with $x_n \to x \in \mathbb{R}$. In case $x \geq 0$,

$$F(x) = \mu([0, x]) = \mu\left(\bigcap_{n=1}^{\infty} [0, x_n]\right) = \lim_{n\to\infty} \mu([0, x_n]) = \lim_{n\to\infty} F(x_n),$$

where we are using the compactness assumption to use the continuity from above. Hence F is ***right-continuous*** on $[0, \infty)$.

Exercise 1.5.

Show that F is right-continuous on $(-\infty, 0)$. That is, when $x < 0$,

$$F(x) = \lim_{n\to\infty} F(x_n).$$

Example 1.6.

Consider the point-mass measure

$$\begin{aligned} \mu_0 : \mathrm{Bor}(\mathbb{R}) &\to [0, \infty] \\ A &\mapsto \begin{cases} 0 & \text{if } 0 \notin A \\ 1 & \text{if } 0 \in A \end{cases} \end{aligned}$$

and the measure space $(\mathbb{R}, \mathrm{Bor}(\mathbb{R}), \mu_0)$.

Then note that,

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases},$$

which is right-continuous but not left-continuous.

The goal of this section is, then:

given an increasing right-continuous $F : \mathbb{R} \to \mathbb{R}$, we make a measure μ_F on $(\mathbb{R}, \mathrm{Bor}(\mathbb{R}))$.

That is, we are doing the converse of the motivation for this section.

The idea is to start with

$$\mu_F((a, b]) = F(b) - F(a), \qquad \forall a, b \in \mathbb{R}, a < b.$$

Let $\mathcal{A}$ be the set of finite unions of half-open intervals of the form $(a, b]$, where $a \in [-\infty, \infty), b \in (-\infty, \infty]$ (we note that when $b = \infty$, we are taking (a, ∞) instead of $(a, \infty]$, since we are working with subsets of $\mathbb{R}$).

We note that

$$\mathbb{R} \setminus (a, b] = (-\infty, a] \cup (b, \infty) \in \mathcal{A}$$

so that $\mathcal{A}$ is an algebra.

In addition, we insist

(a) $F(\infty) = \lim_{x\to\infty} F(x)$ and $F(-\infty) = \lim_{x\to-\infty} F(x)$; and

(b) $\mu_F\left(\bigcup_{k=1}^{n} (a_k, b_k]\right) = \sum_{k=1}^{n} F(b_k) - F(a_k)$.

In this way we get a fuction $\mu_F : \mathcal{A} \to [0, \infty]$.

Fact 1.10.

μ_F is a premeasure on $(\mathbb{R}, \mathcal{A})$.

Theorem 1.11.

Consider the above setting. There is a complete measure space $(\mathbb{R}, \mathcal{B}, \overline{\mu_F})$ such that

(a) $\overline{\mu_F}|_{\mathcal{A}} = \mu_F$; and

(b) $\mathrm{Bor}(\mathbb{R}) \subseteq \mathcal{B}$.

Proof. Consider μ_F^* be the outer measure constructed as in Theorem 1.8 and let $\mathcal{B}$ be the σ-algebra of μ_F^*-measurable sets. We set $\overline{\mu_F} = \mu_F^*|_{\mathcal{B}}$. By Theorem 1.8, we know that $(\mathbb{R}, \mathcal{B}, \overline{\mu_F})$ is complete and $\overline{\mu_F}|_{\mathcal{A}} = \mu_F$.

By Theorem 1.8 again, $\mathcal{A} \subseteq \mathcal{B}$ (which was implicit in restricting $\overline{\mu_F}$ to $\mathcal{A}$). In particular, half-open intervals are $\mathcal{B}$, so that

$$(a, b) = \bigcup_{n=1}^{\infty} \left(a, b - \frac{1}{n}\right] \in \mathcal{B}$$

for all $a < b$ in $\mathbb{R}$. Since $\mathcal{B}$ has every open intervals, which generate the Borel σ-algebra on $\mathbb{R}$, it follows $\text{Bor}\,(\mathbb{R}) \subseteq \mathcal{B}$.

QED

Theorem 1.12.

When $F(x) = x$ for all $x \in \mathbb{R}$, then

(a) $\overline{\mu_F}$ is the Lebesgue measure; and

(b) $\mathcal{B}$ is the set of Lebesgue measurable sets.

Def'n 1.11. **Lebesgue-Steltjes Measure**

Any measure of the form $\overline{\mu_F}$ is called a ***Lebesgue-Steltjes measure***.

Theorem 1.13. Regularity of Lebesgue-Steltjes Measures

Let $(\mathbb{R}, \mathcal{B}, \overline{\mu_F})$ as above and let $A \subseteq \mathbb{R}$. The following are equivalent.

(a) $A \in \mathcal{B}$ (i.e. A is μ_F^*-measurable).

(b) For all $\varepsilon > 0$, there is open $U \subseteq \mathbb{R}$ such that $A \subseteq U$ and $\mu_F^*(U \setminus A) < \varepsilon$.

(c) For all $\varepsilon > 0$, there is closed $C \subseteq \mathbb{R}$ such that $C \subseteq A$ and $\mu_F^*(A \setminus C) < \varepsilon$.

(d) There exists a G_δ-set[1] such that $A \subseteq G$ and $\mu_F^*(G \setminus A) = 0$.

(e) There exists a F_σ-set[2] such that $F \subseteq A$ and $\mu_F^*(A \setminus F) = 0$.

[1] A set is G_δ if it is a countable intersection of open sets.
[2] A set is F_σ if it is a countable union of closed sets.

Proof. (1) $\Longrightarrow$ (2) Assume $A \in \mathcal{B}$ and let $\varepsilon > 0$ be given.

Case 1. *Suppose A is bounded.*

Then $A \subseteq (a, b]$ and $\overline{\mu_F}(A) \leq F(b) - F(a) < \infty$. We may find $\{(a_n, b_n]\}_{n=1}^{\infty}$ such that

$$B = \bigcup_{n=1}^{\infty} (a_n, b_n]$$

contains A and

$$\overline{\mu_F}(B) < \overline{\mu_F}(A) + \frac{\varepsilon}{2}.$$

Now, choose $c_n > b_n$ such that

$$F(c_n) < F(b_n) + \frac{\varepsilon}{2^{n+1}}$$

by the right-continuity of F. Let $U = \bigcup_{n=1}^{\infty} (a_n, c_n)$. Since $A \in \mathcal{B}$, we have

$$\overline{\mu_F}(B) = \overline{\mu_F}(A) + \overline{\mu_F}(B \setminus A)$$

by Caratheodory measurability condition (Def'n 1.9). So by excision,

$$\overline{\mu_F}(B \setminus A) = \overline{\mu_F}(B) - \overline{\mu_F}(A) < \frac{\varepsilon}{2}.$$

Hence

$$\overline{\mu_F}\left(U \setminus A\right) \leq \overline{\mu_F}\left(U \setminus B\right) + \overline{\mu_F}\left(B \setminus A\right) < \overline{\mu_F}\left(\bigcup_{n=1}^{\infty}\left(b_n, c_n\right)\right) + \frac{\varepsilon}{2} \leq \sum_{n=1}^{\infty}\frac{\varepsilon}{2^{n+1}} + \frac{\varepsilon}{2} = \varepsilon.$$

(End of Case 1)

Case 2. Let $A \in \mathcal{B}$ and consider $A_n = A \cap [-n, n]$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ be given and choose open U_n such that $A_n \subseteq U_n$ and

$$\mu_F^*\left(U_n \setminus A_n\right) < \frac{\varepsilon}{2^n}$$

for all $n \in \mathbb{N}$. Consider $U = \bigcup_{n=1}^{\infty} U_n$. Then $A = \bigcup_{n=1}^{\infty} A_n \subseteq U$ and

$$\mu_F^*\left(U \setminus A\right) \leq \mu_F^*\left(\bigcup_{n=1}^{\infty}\left(U_n \setminus A_n\right)\right) \leq \sum_{n=1}^{\infty}\mu_F^*\left(U_n \setminus A_n\right) < \sum_{n=1}^{\infty}\frac{\varepsilon}{2^n} = \varepsilon.$$

(End of Case 2)

(2) $\Longrightarrow$ (4) For every $n \in \mathbb{N}$, find open $U_n \subseteq \mathbb{R}$ containing A such that

$$\mu_F^*\left(U_n \setminus A\right) < \frac{1}{n}.$$

Take

$$G = \bigcap_{n=1}^{\infty} U_n.$$

Then $A \subseteq G$ and

$$\mu_F^*\left(G \setminus A\right) \leq \mu_F^*\left(U_n \setminus A\right) < \frac{1}{n}$$

for all $n \in \mathbb{N}$. Thus $\mu_F^*\left(G \setminus A\right) = 0$.

(4) $\Longrightarrow$ (1) Take a G_δ-set $G \subseteq \mathbb{R}$ containing A with $\mu^*\left(G \setminus A\right) = 0$. In particular, we have that $G \setminus A \in \mathcal{B}$.[1] Since every open set is in $\mathcal{B}$ and $\mathcal{B}$ is closed under countable intersection, $G \in \mathcal{B}$ as a countable intersection of open sets, and

$$A = G \setminus \left(G \setminus A\right) \in \mathcal{B}.$$

(1) $\Longrightarrow$ (3) Let $A \in \mathcal{B}$ and let $\varepsilon > 0$. Since $X \setminus A \in \mathcal{B}$, we may find open $U \supseteq X \setminus A$ such that

$$\mu_F^*\left(U \setminus \left(X \setminus A\right)\right) < \varepsilon.$$

Letting $C = X \setminus U$, $C \subseteq A$ and

$$\mu_F^*\left(A \setminus C\right) = \mu_F^*\left(U \setminus \left(X \setminus A\right)\right) < \varepsilon.$$

(3) $\Longrightarrow$ (5) Choose $C_n \subseteq A$ such that

$$\mu_F^*\left(A \setminus C_n\right) < \frac{1}{n}$$

for all $n \in \mathbb{N}$ and let

$$K = \bigcup_{n=1}^{\infty} C_n.$$

(5) $\Longrightarrow$ (1) Let K be a F_σ-set contained in A with $\mu_F^*\left(A \setminus F\right) = 0$. Then we observe that $A = \left(A \setminus F\right) \cup F \in \mathcal{B}$.

[1] See the proof of Theorem 1.7, Caratheodory theorem.

QED

II. Measurable Functions

1. Measurable Functions

Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be measurable spaces. We care about functions $f : X \to Y$ which relay information about the measurable spaces.

Def'n 2.1. **Measurable** Function

Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be measurable spaces. We say $f : X \to Y$ is ***measurable*** if

$$f^{-1}(B) = \{x \in X : f(x) \in B\} \in \mathcal{A}, \qquad \forall B \in \mathcal{B}.$$

Before we proceed, here is a convention that we are going to use. Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$ and let $(X, \mathcal{A})$. We say

$$f : X \to Y \text{ is measurable} \iff f \text{ is measurable with respect to } (X, \mathcal{A}), (\mathbb{F}, \mathrm{Bor}(\mathbb{F})).$$

By Assignment 1, we see that

$$f : X \to Y \text{ is measurable} \iff \text{for all open } B, f^{-1}(B) \in \mathcal{A},$$

since $\mathrm{Bor}(\mathbb{F})$ is generated by open subsets of $\mathbb{F}$. In case $\mathbb{F} = \mathbb{R}$, we can replace B with open interval, since every open subset of $\mathbb{R}$ is a countable union of open intervals.

Recall the following trick for analysis. Let $a < b$ in $\mathbb{R}$. Then

$$\begin{aligned}
(a, b] &= \bigcap_{n=1}^{\infty} \left(a, b + \frac{1}{n}\right) \\
(a, b) &= \bigcup_{n=1}^{\infty} \left(a, b - \frac{1}{n}\right] \\
[a, b] &= \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right). \\
(a, \infty) &= \bigcup_{n=1}^{\infty} (a, a + n) \\
(a, b] &= (-\infty, b] \cap (a, \infty) \\
&\ \ \vdots
\end{aligned}$$

That is, all interval types independently generate $\mathrm{Bor}(\mathbb{R})$.

Proposition 2.1.

Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$. The following are equivalent.

(a) f is measurable.

(b) For all $\alpha \in \mathbb{R}, f^{-1}((\alpha, \infty)) \in \mathcal{A}$.

(c) For all $\alpha \in \mathbb{R}, f^{-1}([\alpha, \infty)) \in \mathcal{A}$.

(d) For all $\alpha \in \mathbb{R}, f^{-1}((-\infty, \alpha)) \in \mathcal{A}$.

(e) For all $\alpha \in \mathbb{R}, f^{-1}((-\infty, \alpha]) \in \mathcal{A}$.

Proposition 2.2.

Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{C}$. The following are equivalent. Then

$$f \text{ is measurable} \iff \operatorname{Re} \circ f \text{ and } \operatorname{Im} \circ f \text{ are measurable.}$$

Proof Sketch. $(\Longleftarrow)$ Every open $U \subseteq \mathbb{C}$ can be written as a countable union of open rectangles $(a, b) \times (c, d)$. Then

$$f^{-1}\left((a, b) \times (c, d)\right) = (\operatorname{Re} \circ f)^{-1}\left((a, b)\right) \cap (\operatorname{Im} \circ f)^{-1}\left((c, d)\right).$$

$(\Longrightarrow)$ Note that

$$(\operatorname{Re} \circ f)^{-1}\left((a, b)\right) = f^{-1}(V)$$

where

$$V = \{x + iy : a < x < b\}.$$

Similarly,

$$(\operatorname{Im} \circ f)^{-1}\left((c, d)\right) = f^{-1}(H)$$

where

$$H = \{x + iy : c < y < d\}.$$

QED

Proposition 2.3.

Let (X, τ) be a topological space. If $f : X \to \mathbb{F}$ is continuous, then f is measurable.

Proof. It suffices to check that $f^{-1}(U) \in \operatorname{Bor}(X)$ for all open $U \subseteq \mathbb{F}$, which is guaranteed by the continuity of f.

QED

Proposition 2.4.

Let $(X, \mathcal{A})$ be a measurable space and let $f, g : X \to \mathbb{F}$ be measurable.

(a) For any $\lambda \in \mathbb{F}$, $\lambda f + g$ is measurable.

(b) fg is measurable.

(c) If $g(x) \neq 0$ for all $x \in X$, then $\frac{1}{g}$ is measurable.

Proof. By considering Proposition 2.2, we assume $\mathbb{F} = \mathbb{R}$.

(a) Suppose $\lambda > 0$. Then given $\alpha \in \mathbb{R}$,

$$(\lambda f)^{-1}\left((\alpha, \infty)\right) = \{x \in X : \lambda f(x) > \alpha\} = \left\{x \in X : f(x) > \frac{\alpha}{\lambda}\right\} = f^{-1}\left(\left(\frac{\alpha}{\lambda}, \infty\right)\right),$$

which is measurable.

In case $\lambda < 0$,

$$(\lambda f)^{-1}\left((\alpha, \infty)\right) = f^{-1}\left(\left(-\infty, \frac{\alpha}{\lambda}\right)\right)$$

is measurable.

When $\lambda = 0$, λf is the constant 0 function, which is trivially measurable.

Let $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}(f + g)^{-1}\left((\alpha, \infty)\right) &= \{x \in X : f(x) + g(x) > \alpha\} = \{x \in X : f(x) > \alpha - g(x)\} \\ &= \bigcup_{q \in \mathbb{Q}} \left(\{x \in X : f(x) > q\} \cap \{x \in X : g(x) > \alpha - q\}\right) = \bigcup_{q \in \mathbb{Q}} \left(f^{-1}\left((q, \infty)\right) \cap g^{-1}\left(\alpha - q, \infty\right)\right),\end{aligned}$$

which is measurable as a countable union of measurable sets.

(b) Note
$$(f+g)^2 = f^2 + 2fg + g^2.$$
Hence it suffices to show that f^2 is measurable. Let $\alpha \in \mathbb{R}$.

Suppose $\alpha \geq 0$. Then
$$\begin{aligned} f^{-1}((\alpha,\infty)) = \left\{x \in X : f(x)^2 > \alpha\right\} &= \{x \in X : f(x) > \sqrt{\alpha}\} \cup \{x \in X : f(x) < -\sqrt{\alpha}\} \\ &= f^{-1}\left((\sqrt{\alpha},\infty)\right) \cup f^{-1}\left((-\infty,-\sqrt{\alpha})\right) \end{aligned}$$
is a union of measurable of measurable sets.

If $\alpha < \infty$, then
$$\left(f^2\right)^{-1}((\alpha,\infty)) = \left\{x \in X : f(x)^2 > \alpha\right\} = X$$
is measurable.

(c) Let $\alpha \in \mathbb{R}$. Suppose $\alpha > 0$. Then
$$\begin{aligned} \left(\frac{1}{g}\right)^{-1}((-\infty,\alpha)) = \left\{x \in X : \frac{1}{g(x)} < \alpha\right\} &= \left\{x \in X : g(x) > \frac{1}{\alpha}\right\} \cup \{x \in X : g(x) < 0\} \\ &= g^{-1}\left(\left(\frac{1}{\alpha},\infty\right)\right) \cup g^{-1}((-\infty,0)). \end{aligned}$$
The cases where $\alpha < 0, \alpha = 0$ are similar.

QED

Notation 2.2. $\overline{\mathbb{R}}$

We write $\overline{\mathbb{R}}$ to denote
$$\overline{\mathbb{R}} = [-\infty,\infty].$$

Def'n 2.3. **Borel σ-algebra** of Subsets of $\overline{\mathbb{R}}$

We define the ***Borel σ-algebra*** of subsets of $\overline{\mathbb{R}}$, denoted as $\operatorname{Bor}\left(\overline{\mathbb{R}}\right)$, by
$$\operatorname{Bor}\left(\overline{\mathbb{R}}\right) = \left\{A \subseteq \overline{\mathbb{R}} : A \cap \mathbb{R} \in \operatorname{Bor}(\mathbb{R})\right\}.$$

To show that $\operatorname{Bor}\left(\overline{\mathbb{R}}\right)$ is *really Borel*, we consider the following metric on $\overline{\mathbb{R}}$. Define
$$\begin{aligned} d : \overline{\mathbb{R}}^2 &\to [0,\infty) \\ (x,y) &\mapsto |\arctan(x) - \arctan(y)|, \end{aligned}$$
where $\arctan(-\infty) = -\frac{\pi}{2}$, $\arctan(\infty) = \frac{\pi}{2}$.

Exercise 2.1.

Show that $\operatorname{Bor}\left(\overline{\mathbb{R}}\right)$ is generated by the open subsets of $\left(\overline{\mathbb{R}}, d\right)$.

$\operatorname{Bor}\left(\overline{\mathbb{R}}\right)$ is (independently) generated by intervals of the form $(\alpha,\infty]$, $[-\infty,\alpha)$.

Proposition 2.5.

Let $(f_n)_{\mathbb{R}}^{\infty}$ be a sequence of measurable functions from X to $\mathbb{R}$.

(a) $\sup_{n\in\mathbb{N}} f_n$ is measurable.

(b) $\inf_{n\in\mathbb{N}} f_n$ is measurable.

(c) $\limsup_{n\in\mathbb{N}} f_n$ is measurable.

(d) $\liminf_{n\in\mathbb{N}} f_n$ is measurable.

Proof.

(a) Note that, given $\alpha \in \mathbb{R}$,

$$\left(\sup_{n\in\mathbb{N}} f_n\right)^{-1}((\alpha,\infty]) = \left\{x \in X : \sup_{n\in\mathbb{N}} f_n(x) > \alpha\right\} = \bigcup_{n\in\mathbb{N}} \{x \in X : f_n(x) > \alpha\} = \bigcup_{n\in\mathbb{N}} f_n^{-1}((\alpha,\infty)).$$

(b) It suffices to note that $\inf_{n\in\mathbb{N}} f_n = -\sup_{n\in\mathbb{N}}(-f_n)$.

(c) Recall that

$$\limsup_{n\in\mathbb{N}} f_n = \lim_{n\to\infty} \sup_{k\geq n} f_k = \inf_{n\in\mathbb{N}} \sup_{k\geq n} f_k.$$

Hence by (a), (b), $\limsup_{n\in\mathbb{N}} f_n$ is measurable.

(d) Similar to (c),

$$\liminf_{n\in\mathbb{N}} f_n = \sup_{n\in\mathbb{N}} \inf_{k\geq n} f_k.$$

Hence $\liminf_{n\in\mathbb{N}} f_n$ is measurable.

QED

Corollary 2.5.1.
Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions from X to $\mathbb{R}$. If $f_n \to x$ pointwise, then f is measurable.

Proof. Note that

$$f_n \to x \iff \liminf_{n\in\mathbb{N}} f_n = \limsup_{n\in\mathbb{N}} f_n = \lim_{n\to\infty} f_n.$$

QED

Let $(X, \mathcal{A})$ be a measurable space. Then given measurable $f : X \to \mathbb{F}$ and continuous $g : \mathbb{F} \to \mathbb{F}$, $g \circ f$ is measurable, as for any open $U \subseteq \mathbb{F}$,

$$(g \circ f)^{-1}(U) = f^{-1}\left(g^{-1}(U)\right),$$

where $g^{-1}(U)$ is open.

In particular, this gives alternative proofs that $f^2, \frac{1}{f}$, $\mathrm{Re} \circ f$, $\mathrm{Im} \circ f$ are measurable. Moreover, $|f|$ is measurable.

Def'n 2.4. μ-**almost Everywhere** Predicate
Let $(X, \mathcal{A}, \mu)$ be a measure space and let P be a predicate on X. We say P is true ***μ-almost everywhere*** (or ***μ-ae***) if there exists $N \in \mathcal{A}$ with $\mu(N) = 0$ such that $P(x)$ is true for all $x \in X \setminus N$.

Note that the definition of μ-almost everywhere does not say that

$$N = \{x \in X : P(x) \text{ is false}\}$$

is measurable. But in case μ is complete, N is measurable with $\mu(N) = 0$.

Proposition 2.6.
Let $(X, \mathcal{A}, \mu)$ be a complete measure space and let $f : X \to \mathbb{F}$ be measurable. Suppose that $g : X \to \mathbb{F}$ is such that $f = g$ μ-ae. Then g is measurable.

Proof. Let $N \in \mathcal{A}$ be such that $\mu(N) = 0$ with $f = g$ on $X \setminus N$. Then given any measurable $U \subseteq \mathbb{R}$,

$$g^{-1}(U) = \left(g^{-1}(U) \cap N\right) \cup \left(g^{-1}(U) \setminus N\right).$$

Note that $g^{-1}(U) \cap N \subseteq N$ so has measure 0, which means $g^{-1}(U) \cap N \in \mathcal{A}$ by the completeness of μ. Moreover, $f = g$ on $X \setminus N$ so that $g^{-1}(U) \setminus N = f^{-1}(U) \setminus N$, which is measurable. Thus $g^{-1}(U)$ is measurable, as required.

QED

Def'n 2.5. **Characteristic Function** of a Subset
Let X be a set and let $A \subseteq X$. The ***characteristic function*** of A, denoted as χ_A, is defined as

$$\chi_A : X \to \mathbb{R}$$
$$x \mapsto \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

Note that, given $A \subseteq X$,

$$\chi_A \text{ is measurable} \iff A \text{ is measurable.}$$

Def'n 2.6. **Simple** Function
Let $(X, \mathcal{A})$ be a measurable space. We say $\varphi : X \to \mathbb{F}$ is ***simple*** if

$$\varphi = \sum_{k=1}^{n} a_k \chi_{A_k}$$

where $a_1, \ldots, a_n \in \mathbb{F}$ and $A_1, \ldots, A_n \in \mathcal{A}$ are pairwise disjoint.

Let $(X, \mathcal{A})$ be a measurable space and let $\varphi : X \to \mathbb{F}$. Then

$$\varphi \text{ is simple} \iff \varphi \text{ is measurable and } \varphi(X) \text{ is finite.}$$

To see the reverse direction, suppose φ is measurable and $\varphi(X)$ is finite, say

$$\varphi(X) = \{a_k\}_{k=1}^{n}.$$

Then each $A_k = \varphi^{-1}(\{a_k\})$ is measurable and $\varphi = \sum_{k=1}^{n} a_k \chi_{a_k}$.

The goal of this subsection is to show

$$f : X \to \mathbb{R} \text{ is measurable} \iff f \text{ is a pointwise limit of simple functions.}$$

Proposition 2.7.
Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$ be measurable and bounded. Then for all $\varepsilon > 0$, there are simple $\varphi_\varepsilon, \psi_\varepsilon : X \to \mathbb{R}$ such that

(a) $\varphi_\varepsilon \leq f \leq \psi_\varepsilon$; and
(b) $0 \leq \psi_\varepsilon - \varphi_\varepsilon < \varepsilon$.

Proof. Let $\varepsilon > 0$. Say $f(X) \subseteq [a, b)$. Let $y_0, \ldots, y_n$ be given such that

$$a = y_0 < y_1 < \cdots < y_n = b,$$

where each $y_k - y_{k-1} < \varepsilon$. Let $I_k = [y_{k-1}, y_k)$. Then each $A_k = f^{-1}(I_k)$ is measurable. Define

$$\varphi = \sum_{k=1}^{n} y_{k-1} \chi_{A_k}, \psi = \sum_{k=1}^{n} y_k \chi_{A_k}.$$

Then for any $x \in X$, we have $x \in I_k$ for some k, so that $\varphi(x) = y_{k-1} \leq f(x) \leq y_k = \psi(x)$.
Moreover,

$$0 < \psi(x) - \varphi(x) = y_k - y_{k-1} < \varepsilon.$$

QED

Theorem 2.8. Simple Approximation

Let $(X, \mathcal{A})$ be a measure space and let $f : X \to \mathbb{R}$. Then

$$f \text{ is measurable} \iff \text{there are simple } \varphi_1, \varphi_2, \ldots : X \to \mathbb{R} \text{ with } \varphi_n \to f \text{ pointwise and } |\varphi_n| \leq f \text{ for all } n \in \mathbb{N}.$$

Proof. ($\Longleftarrow$) Recall that pointwise limit of measurable functions is measurable, where each φ_n is measurable.

($\Longrightarrow$) We split into few cases.

Case 1. *Suppose $f \geq 0$.*

Let

$$A_n = \{x \in X : f(x) \leq n\}.$$

Note that

$$\mathcal{A}' = \{B \cap A_n : B \in \mathcal{A}\}$$

is a σ-algebra of subsets of A_n. Then $(A_n, \mathcal{A}')$ is a measurable space and $f|_{A_n}$ is measurable, since

$$(f|_{A_n})^{-1}(U) = f^{-1}(U) \cap A_n \in \mathcal{A}'$$

for all measurable $U \subseteq \mathbb{R}$. Moreover, by definition $f|_{A_n}$ is bounded.

Hence by Proposition 2.7, we can find simple $\varphi_m, \psi_m : A_n \to \mathbb{R}$, $m \in \mathbb{N}$, such that

$$0 \leq \varphi_m \leq f \leq \psi_m$$

and

$$0 \leq \psi_m - \varphi_m < \frac{1}{m}$$

for all $m \in \mathbb{N}$ on A_n.

Extend $\varphi_m(x) = n$ for all $x \in X \setminus A_n$, so that $\varphi_m \leq f$ on X.

Now fix $x \in X$. Then $x \in A_N$ for some N, and so

$$0 \leq f(x) - \varphi_N(x) \leq \psi_N(x) - \varphi_N(x) < \frac{1}{N}.$$

This means given any $\varepsilon > 0$ we can take $N' > N$ so that $\frac{1}{N'} < \varepsilon$, which means for all $m \geq N'$,

$$0 \leq f(x) - \varphi_m(x) < \frac{1}{N'} < \varepsilon.$$

Thus $\varphi_m \to f$ pointwise.

(End of Case 1)

Case 2. *Consider the general case on f. That is, we only assume that f is measurable.*

Let

$$A = \{x \in X : f(x) \geq 0\} \in \mathcal{A}$$
$$B = \{x \in X : f(x) < 0\} \in \mathcal{A}$$

and let $g = f\chi_A$, $h = -f\chi_B$, so that both $g, h \geq 0$. By Case 1, there exist $(\varphi_n)_{n=1}^{\infty}$, $(\psi_n)_{n=1}^{\infty}$ such that $\varphi_n \nearrow g$ and $\psi_n \nearrow h$ pointwise as $n \to \infty$. Then $f = g - h$ so that $\varphi_n - \psi_n \to g - h = f$ pointwise. Moreover,

$$|\varphi_n - \psi_n| \leq |\varphi_n| + |\psi_n| = \varphi_n + \psi_n \leq g + h = |f|.$$

(End of Case 2)

QED

Note that in the proof, we know that, given a fixed $n \in \mathbb{N}$, we have

$$0 \leq f - \varphi_m \leq \frac{1}{m}$$

on A_n. That is,

$$0 \leq f(x) - \varphi_m(x) \leq \frac{1}{m}, \qquad \forall x \in A_n,$$

so that $\varphi_m \to f$ *uniformly* as $m \to \infty$ on A_n.

Suppose that $f \geq 0$ is measurable and that

$$0 \leq \varphi_n \leq f, \qquad \forall n \in \mathbb{N}$$

with $\varphi_n \to f$ pointwise. Then by taking $\psi_n = \max\{\varphi_1, \ldots, \varphi_n\}$, φ_n is still simple. Then

$$0 \leq \psi_n \leq f, \qquad \forall n \in \mathbb{N}$$

as well, so that $\psi_n \nearrow f$ pointwise as $n \to \infty$.

3. Two Theorems

We are going to prove two useful theorems in measure theory in this subsection.

Lemma 2.9.

Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $(f_n)_{n=1}^{\infty} \in \left(\mathbb{R}^X\right)^{\mathbb{N}}$ be a sequence of measurable functions such that $f_n \to f$ pointwise for some measurable $f : X \to \mathbb{R}$. Then for every $\alpha, \beta > 0$, there exist $B \in \mathcal{A}$, $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \alpha, \qquad \forall x \in B, n \geq N$$

and

$$\mu(X \setminus B) < \beta.$$

Proof Sketch. Let

$$A_n = \{x \in X : \forall k \geq n\, [f_k(x) - f(x) < \alpha]\}, \qquad \forall n \in \mathbb{N}.$$

Then

$$A_n = \bigcap_{k \geq n} |f_k - f|^{-1}((-\infty, \alpha)),$$

which is measurable. Since $f_n \to f$ pointwise, we have

$$X = \bigcup_{n=1}^{\infty} A_n.$$

We also have an increasing chain

$$A_1 \subseteq A_2 \subseteq \cdots,$$

so that

$$\lim_{n \to \infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu(X) < \infty$$

by the continuity from below. Hence we may find $N \in \mathbb{N}$ such that

$$\mu(X) - \mu(A_n) < \beta, \qquad \forall n \geq N.$$

Since $\mu(X) < \infty$, each $\mu(A_n) < \infty$ as well, so that

$$\mu(X \setminus A_n) < \beta, \qquad \forall n \geq N.$$

By taking $B = A_N$, we are done.

QED

Theorem 2.10. Egoroff

Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $(f_n)_{n=1}^{\infty} \in \left(\mathbb{R}^X\right)^{\mathbb{N}}$ be a sequence of measurable functions such that $f_n \to f$ pointwise for some measurable $f : X \to \mathbb{R}$. Then for all $\varepsilon > 0$ there exists $A \in \mathcal{A}$ such that

(a) $f_n \to f$ uniformly on A; and

(b) $\mu(X \setminus A) < \varepsilon$.

Proof. Let $\varepsilon > 0$ be given. For all $n \in \mathbb{N}$, we may find $A_n \in \mathcal{A}$ and $N_n \in \mathbb{N}$ such that

$$\forall x \in A_n, k \geq N_n \left[|f_k(x) - f(x)| < \frac{1}{n}\right]$$

and

$$\mu(X \setminus A_n) < \frac{\varepsilon}{2^n}.$$

Let

$$A = \bigcap_{n=1}^{\infty} A_n.$$

Given any $\varepsilon' > 0$, by taking $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon'$, we have, for all $k \geq N_n$ and $x \in A$,

$$|f_k(x) - f(x)| < \frac{1}{n} < \varepsilon'.$$

Hence $f_k \to f$ uniformly on A. Finally,

$$\mu(X \setminus A) = \mu\left(\bigcup_{n=1}^{\infty}(X \setminus A_n)\right) \leq \sum_{n=1}^{\infty} \mu(X \setminus A_n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

QED

Let m be the Lebesgue measure on $\mathbb{R}$ and let $A \subseteq \mathbb{R}$ with $m(A) < \infty$. Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions from A to $\mathbb{R}$ that converges to $f : A \to \mathbb{R}$. Then by Egoroff's theorem, for every $\varepsilon > 0$, there is $B \subseteq A$ such that

$$f_n \to f \text{ uniformly on } B$$

and

$$m(A \setminus B) < \frac{\varepsilon}{2}.$$

Then we can find a closed subset $C \subseteq B$ with

$$m(B \setminus C) < \frac{\varepsilon}{2}$$

by the regularity of Lebesgue measure. Then

$$f_n \to f \text{ uniformly on } C$$

and

$$m(A \setminus C) = m(A \setminus B) + m(B \setminus C) < \varepsilon.$$

Hence for the Lebesgue measure (in fact, any Lebesgue-Steltjes measure), we can assume that $f_n \to f$ uniformly on a closed set with arbitrarily small difference.

Lemma 2.11.

Let $A \subseteq \mathbb{R}$ be Lebesgue measurable and let $\varphi : A \to \mathbb{R}$ be Lebesgue-simple. Then for all $\varepsilon > 0$, there exists closed $C \subseteq \mathbb{R}$ and a continuous $g : \mathbb{R} \to \mathbb{R}$ such that

(a) $C \subseteq A$;

(b) $\varphi = g$ on C; and

(c) $m(A \setminus C) < \varepsilon$.

Proof. Write

$$\varphi = \sum_{i=1}^{n} a_i \chi_{A_i},$$

where each $a_i \neq 0$ and $A_i = \varphi^{-1}(\{a_i\})$. Let $A_0 = \varphi^{-1}(\{0\})$. We also insist that $a_i \neq a_j$ for $i \neq j$. Then

$$A = \dot{\bigcup_{i=0}^{n}} A_i.$$

Let $\varepsilon > 0$ be given. For each i, let C_i be a closed such that $C_i \subseteq A_i$ and

$$m(A_i \setminus C_i) < \frac{\varepsilon}{n+1}$$

by regularity of Lebesgue measure. Let

$$C = \dot{\bigcup_{i=0}^{n}} C_i,$$

which is closed. Since φ is continuous on each C_i and $C_i \cap C_j = \emptyset$, φ is continuous on C. Then there is continuous $g : \mathbb{R} \to \mathbb{R}$ that extends $\varphi : C \to \mathbb{R}$. Finally,

$$m(A \setminus C) = m\left(\dot{\bigcup_{i=0}^{n}} A_i \setminus C_i\right) = \sum_{i=0}^{n} m(A_i \setminus C_i) < \varepsilon.$$

QED

Theorem 2.12. Lusin

Let $f : A \to \mathbb{R}$ be Lebesgue measurable. Then for all $\varepsilon > 0$, there exists continuous $g : \mathbb{R} \to \mathbb{R}$ and closed $C \subseteq \mathbb{R}$ such that

(a) $C \subseteq A$;

(b) $f = g$ on C; and

(c) $m(A \setminus C) < \varepsilon$.

Proof. We split the proof into two cases. Let $\varepsilon > 0$ be given.

Case 1. *Suppose $m(A) < \infty$.*

Let $(\varphi_n)_{n=1}^{\infty}$ be a sequence of simple functions such that $\varphi_n \to f$ pointwise by simple approximation. For each $n \in \mathbb{N}$, let $C_n \subseteq \mathbb{R}$ be closed and $g_n : \mathbb{R} \to \mathbb{R}$ be continuous such that $\varphi_n = g_n$ on C_n and

$$m(A \setminus C_n) < \frac{\varepsilon}{2^{n+1}}.$$

By Egoroff, let C_0 be the closed set such that

$$\varphi_n \to f \text{ uniformly on } C_0$$

and

$$m(A \setminus C_0) < \frac{\varepsilon}{2}.$$

Let

$$C = \bigcap_{n=0}^{\infty} C_n.$$

Then,

$$g_n = \varphi_n \to f \text{ uniformly on } C.$$

In particular, f is continous on C. This means we can extend $f|_C$ to continuous $g : \mathbb{R} \to \mathbb{R}$. Finally,

$$m(A \setminus C) = m\left(A \setminus \bigcap_{n=0}^{\infty} C_n\right) = m\left(\bigcup_{n=0}^{\infty} (A \setminus C_n)\right) \leq m(A \setminus C_0) + \sum_{n=1}^{\infty} m(A \setminus C_n) < \varepsilon.$$

(End of Case 1)

Case 2. *Suppose $m(A) < \infty$.*

This is left as an exercise.

(End of Case 2)

QED

III. Integration

1. Nonnegative Measurable Functions

Def'n 3.1. **Integral** of a Nonnegative Simple Function
Let $(X, \mathcal{A}, \mu)$ be a measure space and let

$$\varphi = \sum_{i=1}^{n} a_i \chi_{A_i} : X \to [0, \infty]$$

be simple. We define the ***integral*** of φ, denoted as $\int \varphi d\mu$, by

$$\int \varphi d\mu = \sum_{i=1}^{n} a_i \mu\left(A_i\right).$$[1]

[1] For this, we use the convention $0\infty = \infty 0 = 0$.

Proposition 3.1.
Let $\varphi : X \to [0, \infty]$ be simple. Then $\int \varphi d\mu$ is well-defined.

Proof Sketch. Say

$$\varphi = \sum_{i=1}^{n} a_i \chi_{E_i} = \sum_{j=1}^{m} b_j \chi_{F_j}.$$

Suppose that $\varphi(X) = \left\{c_1, \ldots, c_p\right\}$ and let

$$A_k = \varphi^{-1}\left(\{c_k\}\right), \qquad \forall k \in \{1, \ldots, p\}.$$

Then

$$\sum_{i=1}^{n} a_i \mu\left(E_i\right) = \sum_{k=1}^{p} c_k \sum_{i:a_i=c_k} \mu\left(E_i\right) = \sum_{k=1}^{p} c_k \mu\left(\dot{\bigcup_{i:a_i=c_k}} E_i\right) = \sum_{k=1}^{p} c_k \mu\left(A_k\right).$$

By symmetry, $\sum_{j=1}^{m} b_j \chi_{F_j} = \sum_{k=1}^{p} c_k \mu\left(A_k\right)$. Thus $\int \varphi d\mu$ is well-defined.

QED

Proposition 3.2.
Let $\varphi, \psi : X \to [0, \infty]$ be simple.

(a) If $\alpha \geq 0$, then

$$\int \alpha\varphi d\mu = \alpha \int \varphi d\mu.$$

(b)

$$\int \varphi + \psi d\mu = \int \varphi d\mu + \int \psi d\mu.$$

(c) $\varphi \leq \psi \implies \int \varphi d\mu \leq \int \psi d\mu$.

Proof. Write

$$\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}, \psi = \sum_{j=1}^{m} b_j \chi_{F_j}$$

and let $a_0 = b_0 = 0$, with $E_0 = X \setminus \bigcup_{i=1}^n E_i, F_0 = X \setminus \bigcup_{j=1}^m F_j$. This means

$$\varphi = \sum_{i=0}^{n} a_i \chi_{E_i}, \psi = \sum_{j=0}^{m} b_j \chi_{F_j}$$

as well.

(a) Note that

$$\int \alpha\varphi d\mu = \sum_{i=1}^{n} \alpha a_i \mu\left(A_i\right) = \alpha \sum_{i=1}^{n} a_i \mu\left(A_i\right) = \alpha \int \varphi d\mu.$$

(b) For all $i \in \{0, \ldots, n\}, j \in \{0, \ldots, n\}$, let

$$A_{i,j} = E_i \cap F_j.$$

Then it follows that

$$\varphi = \sum_{i=0}^{n} \sum_{j=0}^{m} a_i \chi_{A_{i,j}}$$

and

$$\psi = \sum_{j=0}^{m} \sum_{i=0}^{n} b_j \chi_{A_{i,j}}.$$

Thus

$$\int \varphi + \psi d\mu = \sum_{i=0}^{n} \sum_{j=0}^{m} \left(a_i + b_j\right) \mu\left(A_{i,j}\right) = \sum_{i=0}^{n} \sum_{j=0}^{m} a_i \mu\left(A_{i,j}\right) + \sum_{j=0}^{m} \sum_{i=0}^{n} b_j \mu\left(A_{i,j}\right) = \int \varphi d\mu + \int \psi d\mu.$$

(c) Given $i \in \{0, \ldots, n\}, j \in \{0, \ldots, m\}$, if $A_{i,j} \neq \emptyset$, then $a_i \leq b_j$. Otherwise, $\mu\left(A_{i,j}\right) = 0$. This means

$$a_i \mu\left(A_{i,j}\right) \leq b_j \mu\left(A_{i,j}\right), \qquad \forall i \in \{0, \ldots, n\}, j \in \{0, \ldots, m\},$$

so that

$$\int \varphi d\mu = \sum_{i=0}^{n} \sum_{j=0}^{m} a_i \mu\left(A_{i,j}\right) \leq \sum_{j=0}^{m} \sum_{i=0}^{n} b_j \mu\left(A_{i,j}\right) = \int \psi d\mu.$$

QED

Def'n 3.2. **Integral** of a Nonnegative Simple Function over a Measurable Subset

Let $\varphi : X \to [0, \infty]$ be simple and let $A \in \mathcal{A}$. We define the ***integral*** of φ over A, denoted as $\int_A \varphi d\mu$, by

$$\int_A \varphi d\mu = \int \varphi \chi_A d\mu.$$

Proposition 3.3.

Let $\varphi : X \to [0, \infty]$ be simple. Define $\nu : \mathcal{A} \to [0, \infty]$ by

$$\nu(A) = \int_A \varphi d\mu.$$

Then ν is a measure on $(X, \mathcal{A})$.

Proof. Write

$$\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}.$$

We have

$$\nu(\emptyset) = \int \chi_{\emptyset}\varphi d\mu = 0.$$

Let $\{A_m\}_{m=1}^{\infty} \subseteq \mathcal{A}$ be a collection of disjoint sets and $A = \dot{\bigcup}_{m=1}^{\infty} A_m$. Then

$$\begin{aligned}\nu(A) &= \int_A \varphi d\mu = \int \varphi\chi_A d\mu = \int \sum_{i=1}^{n} a_i\chi_{E_i}\chi_A d\mu = \int \sum_{i=1}^{n} a_i\chi_{E_i\cap A} d\mu = \sum_{i=1}^{n} a_i\mu(E_i\cap A) = \sum_{i=1}^{n} a_i\mu\left(\dot{\bigcup_{m=1}^{\infty}}(E_i\cap A_m)\right)\\ &= \sum_{i=1}^{n} a_i\sum_{m=1}^{\infty}\mu(E_i\cap A_m) = \sum_{m=1}^{\infty}\sum_{i=1}^{n} a_i\mu(E_i\cap A_m) = \sum_{m=1}^{\infty}\int_{A_m}\varphi d\mu = \sum_{m=1}^{\infty}\nu(A_m).\end{aligned}$$

QED

Notation 3.3. $\mathrm{L}^+(X, \mathcal{A}, \mu)$

We write $\mathrm{L}^+(X, \mathcal{A}, \mu)$, or simply L^+ when $(X, \mathcal{A}, \mu)$ is understood, to mean

$$\mathrm{L}^+(X, \mathcal{A}, \mu) = \{f : X \to [0, \infty] : f \text{ is measurable}\}.$$

Def'n 3.4. **Integral** of a L^+-function

Let $f \in \mathrm{L}^+$. We define the ***integral*** of f, denoted as $\int f d\mu$, by

$$\int f d\mu = \sup\left\{\int \varphi d\mu : \varphi : [0, \infty] \to X, \varphi \le f, \varphi \text{ is simple}\right\}.$$

If $A \in \mathcal{A}$, we define the ***integral*** of f over A, denoted as $\int_A f d\mu$, by

$$\int_A f d\mu = \int f\chi_A d\mu.$$

Proposition 3.4.

Let $f, g \in \mathrm{L}^+$.

(a) If $\alpha \ge 0$, then

$$\int \alpha f d\mu = \alpha\int f d\mu.$$

(b) If $f \le g$, then

$$\int f d\mu \le \int g d\mu.$$

Proof.

(a) This is trivial when $\alpha = 0$. For $\alpha > 0$,

$$\begin{aligned}\{\varphi : X \to [0, \infty] : \varphi \le \alpha f, \varphi \text{ is simple}\} &= \left\{\varphi : X \to [0, \infty] : \frac{1}{\alpha}\varphi \le f, \varphi \text{ is simple}\right\}\\ &= \{\alpha\psi : \psi : X \to [0, \infty], \psi \le f, \psi \text{ is simple}\}.\end{aligned}$$

By taking sup, we have the desired equality.

(b) It suffices to note

$$\{\varphi : X \to [0, \infty] : \varphi \le f, \varphi \text{ is simple}\} \subseteq \{\psi : X \to [0, \infty] : \psi \le g, \psi \text{ is simple}\}.$$

QED

We are leaving (a one-liner!) proof of $\int f + g d\mu = \int f d\mu + \int g d\mu$ for later.

Lemma 3.5.
Let $\varphi : X \to [0, \infty]$ be simple and let $(A_n)_{n=1}^{\infty} \in \mathcal{A}^{\mathbb{N}}$ be an ascending chain with $X = \bigcup_{n=1}^{\infty} A_n$. Then

$$\lim_{n\to\infty} \int_{A_n} \varphi d\mu = \int \varphi d\mu.$$

Proof. Recall that $\nu : \mathcal{A} \to [0, \infty]$ by

$$\nu(A) = \int_A \varphi d\mu, \qquad \forall A \in \mathcal{A}$$

is a measure. Hence by the continuity from below,

$$\lim_{n\to\infty} \int_{A_n} \varphi d\mu = \lim_{n\to\infty} \nu(A_n) = \nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \nu(X) = \int \varphi d\mu.$$

QED

Theorem 3.6. Monotone Convergence Theorem (MCT)
Let $(f_n)_{n=1}^{\infty} \in \mathrm{L}^{+\mathbb{N}}$ be an increasing sequence and define $f \in \mathrm{L}^+$ by

$$f(x) = \lim_{n\to\infty} f_n(x), \qquad \forall x \in X.$$

Then

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

Proof. For every $x \in X$, $(f_n(x))_{n=1}^{\infty}$ is an increasing sequence. Hence by the MCT for sequences, $\lim_{n\to\infty} f_n(x)$ converges in $[0, \infty]$. Define

$$f(x) = \lim_{n\to\infty} f_n(x), \qquad \forall x \in X.$$

In fact, MCT for sequences tells us that

$$f(x) = \sup_{n\in\mathbb{N}} f_n(x), \qquad \forall x \in X,$$

so that

$$f_1 \leq f_2 \leq \cdots \leq f.$$

This means

$$\int f_1 d\mu \leq \int f_2 d\mu \leq \cdots \leq \int f d\mu$$

using monotonicity of integral, so that

$$\lim_{n\to\infty} \int f_n d\mu = \sup_{n\in\mathbb{N}} \int f_n d\mu \leq \int f d\mu.$$

Let $\varphi : X \to [0, \infty]$ be a simple function with $\varphi \leq f$. Let $\varepsilon \in (0, 1)$ and let

$$A_n = \{x \in X : (1-\varepsilon)\varphi(x) \leq f_n(x)\}, \qquad \forall n \in \mathbb{N}.$$

Then

$$A_1 \subseteq A_2 \subseteq \cdots$$

and

$$X = \bigcup_{n=1}^{\infty} A_n,$$

since $f_n(x) \to f(x)$ means there must be $N \in \mathbb{N}$ such that $(1-\varepsilon)\varphi(x) \le f_n(x)$, as $(1-\varepsilon)\varphi(x) < \varphi(x) \le f(x)$. This means

$$(1-\varepsilon)\int \varphi d\mu = \int (1-\varepsilon)\varphi d\mu = \lim_{n\to\infty}\int_{A_n}(1-\varepsilon)\varphi d\mu \le \lim_{n\to\infty}\int_{A_n} f_n d\mu \le \lim_{n\to\infty}\int f_n d\mu.$$

Since the choice of ε was arbitrary, we conclude

$$\int \varphi d\mu \le \lim_{n\to\infty}\int f_n d\mu.$$

But $\int f d\mu$ is the supremum of such φ, so it follows that

$$\int f d\mu \le \lim_{n\to\infty}\int f_n d\mu,$$

as required.

QED

Proposition 3.7.
Let $f, g \in \mathrm{L}^+$. Then

$$\int f + g d\mu = \int f d\mu + \int g d\mu.$$

Proof. By simple approximation, we can find increasing sequence of simple functions $(\varphi_n)_{n=1}^{\infty}, (\psi_n)_{n=1}^{\infty}$ such that $\varphi_n \nearrow f$, $\psi_n \nearrow g$ pointwise. Thus by the MCT,

$$\int f + g d\mu = \lim_{n\to\infty}\int \varphi_n + \psi_n d\mu = \lim_{n\to\infty}\int \varphi_n d\mu + \int \psi_n d\mu = \int f d\mu + \int g d\mu.$$

QED

Proposition 3.8.
Let $(f_n)_{n=1}^{\infty} \in \mathrm{L}^{+\mathbb{N}}$. Then

$$\int \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty}\int f_n d\mu.$$

Proof. Note that $\left(\sum_{n=1}^{k} f_n\right)_{k=1}^{\infty} \in \mathrm{L}^{+\mathbb{N}}$ is increasing, so that

$$\int \sum_{n=1}^{\infty} f_n d\mu = \int \lim_{k\to\infty}\sum_{n=1}^{k} f_n d\mu = \lim_{k\to\infty}\int \sum_{n=1}^{k} f_n d\mu = \lim_{k\to\infty}\int \sum_{n=1}^{k} f_n d\mu = \sum_{n=1}^{\infty}\int f_n d\mu.$$

QED

Proposition 3.9.
Let $f \in \mathrm{L}^+$. Then

$$\begin{aligned} \nu : \mathcal{A} &\to [0, \infty] \\ A &\mapsto \int_A f d\mu \end{aligned}$$

is a measure.

Proof. Clearly $\nu(\emptyset) = \int_\emptyset f d\mu = 0$.

Write $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ be a collection of disjoint sets and let $A = \dot\bigcup_{n=1}^{\infty} A_n$. Then

$$\nu(A) = \int f\chi_A d\mu = \int \sum_{n=1}^{\infty} f\chi_{A_n} d\mu = \sum_{n=1}^{\infty}\int_{A_n} f d\mu = \sum_{n=1}^{\infty}\nu(A).$$

QED

Lemma 3.10.
Let $f \in \mathrm{L}^+$. Then
$$\int f d\mu = 0 \iff f = 0 \ \mu\text{-ae.}$$

Proof. ($\Longleftarrow$) Suppose $f = 0$ μ-ae. Let $\varphi : X \to [0, \infty]$ be simple with $\varphi \leq f$, say
$$\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}.$$
Then $\varphi = 0$ ae. This means each $a_i > 0$ implies $\mu(A_i) = 0$. Thus
$$\int \varphi d\mu = 0$$
so that
$$\int f d\mu = 0.$$

($\Longrightarrow$) Suppose $\int f d\mu = 0$. Let
$$A = \{x \in X : f(x) > 0\}$$
and let
$$A_n = \left\{x \in X : f(x) \geq \frac{1}{n}\right\}, \qquad \forall n \in \mathbb{N}.$$
Then
$$A_1 \subseteq A_2 \subseteq \cdots$$
with
$$\bigcup_{n=1}^{\infty} A_n = A.$$
Therefore
$$\mu(A) = \lim_{n \to \infty} \mu(A_n)$$
and
$$0 = \int f d\mu \geq \int \frac{1}{n} \chi_{A_n} d\mu = \frac{1}{n} \mu(A_n),$$
so that each $\mu(A_n) = 0$. Thus $\mu(A) = 0$, as required.

QED

Proposition 3.11.
Let $f \in \mathrm{L}^+$ and let $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$. Then
$$\int_{A \cup B} f d\mu = \int_A f d\mu + \int_B f d\mu.$$

Proof. Note that
$$\int_{A \cup B} f d\mu = \int f(\chi_A + \chi_B) d\mu = \int f\chi_A d\mu + \int f\chi_B d\mu = \int_A f d\mu + \int_B f d\mu.$$

QED

Proposition 3.12.
Let $f \in \mathrm{L}^+$ and let $A \in \mathcal{A}$ with $\mu(A) = 0$. Then
$$\int_A f d\mu = 0.$$

Proof. Note that $f\chi_A = 0$ μ-ae.

QED

Proposition 3.13.
Let $(f_n)_{n=1}^{\infty} \in \mathrm{L}^{+\mathbb{N}}$ be such that

$$f_n \leq f_{n+1}\ \mu\text{-ae}, \qquad \forall n \in \mathbb{N}$$

and let $f \in \mathrm{L}^{+\mathbb{N}}$ be such that

$$\lim_{n\to\infty} f_n = f \text{ pointwise } \mu\text{-ae}.$$

Then

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

Proof. Let

$$A_n = \{x \in X : f_n(x) > f_{n+1}(x)\}$$

and let

$$A_0 = \left\{x \in X : \lim_{n\to\infty} f_n(x) \neq f(x)\right\}.$$

Then $\mu(A_n) = 0$ for all $n \in \mathbb{N} \cup \{0\}$. Let $A = \bigcup_{n=0}^{\infty} A_n$, so that $\mu(A) = 0$ as well. We have

$$f_n \chi_{X \setminus A} \leq f_{n+1} \chi_{X \setminus A}, \qquad \forall n \in \mathbb{N}$$

and

$$f_n \chi_{X \setminus A} \to f \chi_{X \setminus A} \text{ pointwise.}$$

By the MCT,

$$\int_{X \setminus A} f_n d\mu \to \int_{X \setminus A} f d\mu.$$

The result then follows from Proposition 3.11 and 3.12.

QED

Theorem 3.14. Fatou's Lemma
Let $(f_n)_{n=1}^{\infty} \in \mathrm{L}^{+\mathbb{N}}$. Then

$$\int \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int f_n d\mu.$$

Proof. Let

$$g_n = \inf_{k \geq n} f_k.$$

Then $(g_n)_{n=1}^{\infty}$ is an increasing sequence in L^+ such that

$$\lim_{n\to\infty} g_n = \liminf_{n\to\infty} f_n$$

pointwise. By the monotone convergence theorem,

$$\int \liminf_{n\to\infty} f_n d\mu = \int \lim_{n\to\infty} g_n d\mu = \lim_{n\to\infty} \int g_n d\mu = \liminf_{n\to\infty} \int g_n d\mu \leq \liminf_{n\to\infty} \int f_n d\mu.$$

QED

Corollary 3.14.1.
Let $(f_n)_{n=1}^{\infty} \in \mathrm{L}^{+\mathbb{N}}$ such that $f_n \to f$ pointwise for some $f \in \mathrm{L}^+$. Then

$$\int f d\mu \leq \liminf_{n\to\infty} \int f_n d\mu.$$

Def'n 3.5. **Integrable** Complex-valued Function
Let $f : X \to \mathbb{C}$ be measurable. We say f is ***integrable*** if

$$\int |f|\, d\mu < \infty.$$

In case $f : X \to \mathbb{R}$ is integrable, we consider the ***positive part*** f^+ and ***negative part*** f^- of f defined as

$$\begin{aligned} f^+ &= \max\{f, 0\},\\ f^- &= -\min\{f, 0\}. \end{aligned}$$

Note that both f^+, f^- are nonnegative and we define the ***integral*** of f, denoted as $\int f d\mu$, by

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu.$$[1]

Let $f : X \to \mathbb{C}$ be integrable. Then we define the ***integral*** of f, denoted as $\int f d\mu$, by

$$\int f d\mu = \int \mathrm{Re} \circ f d\mu + i \int \mathrm{Im} \circ f d\mu.$$[2]

In case $f : X \to \mathbb{C}$ is measurable, we define

$$\|f\|_1 = \int |f|\, d\mu.$$

[1]Note that $f = f^+ - f^-$ and $|f| = f^+ + f^-$. Hence $f^+, f^- \leq |f|$ so that both $\int f^+ d\mu$, $\int f^- d\mu$ are finite.
[2]Observe that $|\mathrm{Re} \circ f|, |\mathrm{Im} \circ f| \leq |f|$, so that $\mathrm{Re} \circ f$, $\mathrm{Im} \circ f$ are integrable.

Notation 3.6. $L^1(X, \mathcal{A}, \mu)$
We define

$$L^1(X, \mathcal{A}, \mu) = \{f : X \to \mathbb{C} : f \text{ is measurable and } \|f\|_1 < \infty\}.$$

We shall write L^1 when $(X, \mathcal{A}, \mu)$ are understood.

We state few results without proof.

Proposition 3.15. Linearity
Let $f, g \in L^1$ and $\alpha \in \mathbb{C}$. Then $\alpha f + g \in L^1$ with

$$\int \alpha f + g d\mu = \alpha \int f d\mu + \int g d\mu.$$

Proposition 3.16. Monotonicity
Let $f, g \in L^1$ be real-valued functions. If $f \leq g$, then

$$\int f d\mu \leq \int g d\mu.$$

Def'n 3.7. **Integral** over a Measurable Set
Let $f \in L^1$. For $A \in \mathcal{A}$, we define the ***integral*** of f over A, denoted as $\int_A f d\mu$, by

$$\int_A f d\mu = \int f \chi_A d\mu.$$

Proposition 3.17.
Let $f \in L^1$ and let $A, B \in \mathcal{A}$ be disjoint. Then

$$\int_{A \cup B} f d\mu = \int_A f d\mu + \int_B f d\mu.$$

The following proposition is surprisingly non-trivial.

Proposition 3.18.
Let $f \in L^1$. Then

$$\left| \int f d\mu \right| \leq \int |f| \, d\mu.$$

Proof. The case when f is real-valued is trivial:

$$\left| \int f d\mu \right| = \left| \int f^+ d\mu - \int f^- d\mu \right| \leq \int f^+ d\mu + \int f^- d\mu = \int f^+ + f^- d\mu = \int |f| \, d\mu.$$

Let

$$z = \int f d\mu.$$

Write

$$z = re^{i\theta}$$

in polar form, so that $r = |z|$. Therefore,

$$\left| \int f d\mu \right| = r = e^{-i\theta} z = \int e^{-i\theta} f d\mu = \operatorname{Re} \int e^{-i\theta} f d\mu = \int \underbrace{\operatorname{Re} \circ e^{-i\theta} f}_{=g} d\mu \leq \int |g| \, d\mu \leq \int |f| \, d\mu.$$

QED

Theorem 3.19. Lebesgue Dominated Convergence Theorem (LDCT)
Let $(f_n)_{n=1}^{\infty} \in \left(L^1\right)^{\mathbb{N}}$ and let $g \in L^1$. If $f_n \to f$ pointwise for some $f : X \to \mathbb{C}$ and $|f_n| \leq g$ for all $n \in \mathbb{N}$, then $f \in L^1$ with

$$\int \lim_{n \to \infty} f_n d\mu = \lim_{n \to \infty} \int f_n d\mu.$$

Proof. Case 1. *Suppose f, g, f_n are real-valued.*

Since $|f| \leq g$ by taking limits as $n \to \infty$,

$$\int |f| \, d\mu \leq \int g d\mu < \infty.$$

Hence $f \in L^1$. Then

$$\int g + f d\mu \overset{\text{Fatou}}{\leq} \liminf_{n \to \infty} \int g + f_n d\mu = \int g d\mu + \liminf_{n \to \infty} \int f_n d\mu.$$

Similarly,

$$\int g - f d\mu \overset{\text{Fatou}}{\leq} \liminf_{n \to \infty} \int g - f_n d\mu = \int g d\mu - \limsup_{n \to \infty} \int f_n d\mu.$$

(End of Case 1)

QED

Proposition 3.20.
$L^1(X, \mathcal{A}, \mu)$ is a Banach space.

Here are some ideas for the proof.
Suppose that V is a normed linear space and let $(a_n)_{n=1}^{\infty} \in V^n$ be Cauchy. Then we know

$$\text{there is a subsequence } (a_{n_k})_{k=1}^{\infty} \text{ such that } a_{n_k} \to a \in V \implies a_n \to a.$$

Let $(f_n)_{n=1}^{\infty} \in L^1(X, \mathcal{A}, \mu)^{\mathbb{N}}$ be Cauchy. Then

$$|\mathrm{Re} \circ f_n - \mathrm{Re} \circ f_m|^2 \leq |\mathrm{Re} \circ f_n - \mathrm{Re} \circ f_m|^2 + |\mathrm{Im} \circ f_n - \mathrm{Im} \circ f_m| = |f_n - f_m|^2$$

so that

$$|\mathrm{Re} \circ f_n - \mathrm{Re} \circ f_m| \leq |f_n - f_m|.$$

Hence by monotonicity,

$$\|\mathrm{Re} \circ f_n - \mathrm{Re} \circ f_m\|_1 \leq \|f_n - f_m\|_1,$$

which means $(\mathrm{Re} \circ f_n)_{n=1}^{\infty}$ is Cauchy. Similarly, $(\mathrm{Im} \circ f_n)_{n=1}^{\infty}$ is also Cauchy.

Proof of Proposition 3.20
Let $(f_n)_{n=1}^{\infty} \in L^1(X, \mathcal{A}, \mu)$ be Cauchy. Assume each f_n is real-valued without loss of generality. For all $k \in \mathbb{N}$, there is $n_k \in \mathbb{N}$ such that

$$\|f_n - f_m\|_1 < \frac{1}{2^k}, \qquad \forall n, m \geq n_k.$$

Without loss of generality assume $(n_k)_{k=1}^{\infty}$ is increasing. Let

$$\hat{g} = |f_{n_1}| + \sum_{k=1}^{\infty} \left|f_{n_{k+1}} - f_{n_k}\right|.$$

By the MCT,

$$\int \hat{g} d\mu = \int |f_{n_1}| \, d\mu + \sum_{k=1}^{\infty} \int \left|f_{n_{k+1}} - f_{n_k}\right| d\mu = \|f_{n_1}\|_1 + \sum_{k=1}^{\infty} \left\|f_{n_{k+1}} - f_{n_k}\right\|_1 = \|f_{n_1}\|_1 + 1 < \infty.$$

This means $\hat{g}$ is finite almost everywhere – that is, there is $N \in \mathcal{A}$ such that $\hat{g}|_{X \setminus N}$ is finite and $\mu(N) = 0$. Hence define $g : X \to \mathbb{R}$ by

$$g(x) = \begin{cases} \hat{g}(x) & \text{if } x \in X \setminus N \\ 0 & \text{if } x \in N \end{cases}, \qquad \forall x \in X.$$

Let $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} f_{n_1}(x) + \sum_{k=1}^{\infty} f_{n_{k+1}}(x) - f_{n_k}(x) & \text{if } x \in X \setminus N \\ 0 & \text{if } x \in N \end{cases}, \qquad \forall x \in X.$$

Then $f_{n_k} \to f$ pointwise almost everywhere and we have that $|f| \leq g$. Then by the LDCT,

$$f \in L^1(X, \mathcal{A}, \mu).$$

Moreover,

$$|f_{n_k}| \leq |f_{n_1}| + \sum_{j=1}^{k-1} \left|f_{n_{j+1}} - f_{n_j}\right| \overset{\text{ae}}{\leq} g, \qquad \forall k \in \mathbb{N}.$$

Finally,

$$|f - f_{n_k}| \leq 2g, \qquad \forall k \in \mathbb{N},$$

so by the LDCT,

$$\|f - f_{n_k}\|_1 = \int |f - f_{n_k}| \, d\mu \to 0.$$

QED

IV. Product Measures

1. Product Measures

Def'n 4.1. **Measurable Rectangle**
Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be measure spaces. For every $A \in \mathcal{A}, B \in \mathcal{B}$, we call $A \times B$ a ***measurable rectangle***.

Lemma 4.1.
Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be measure spaces and let $\{A_k \times B_k\}_{k=1}^{\infty}$ be a collection of measurable rectangles that are pairwise disjoint. Also assume that

$$\dot{\bigcup_{k=1}^{\infty}} A_k \times B_k = A \times B$$

for some $A \in \mathcal{A}, B \in \mathcal{B}$. Then

$$\mu(A)\,\nu(B) = \sum_{k=1}^{\infty} \mu(A_k)\,\nu(B_k)\,.$$[1]

[1] We are using the convention $0\infty = 0$.

Proof. Fix $x \in A$. For all $y \in B$, there exists a unique $k \in \mathbb{N}$ such that $(x, y) \in A_k \times B_k$. Hence

$$B = \dot{\bigcup_{k \in \mathbb{N}: x \in A_k}} B_k$$

This means

$$\mu(B) = \sum_{k \in \mathbb{N}: x \in A_k} \mu(B_k)\,,$$

so that

$$\nu(B)\,\chi_A(x) = \sum_{k=1}^{\infty} \nu(B_k)\,\chi_{A_k}, \qquad \forall x \in X.$$

By MCT,

$$\nu(B)\,\mu(A) - \sum_{k=1}^{\infty}\sum_{k=1}^{\infty} \nu(B_k)\,\mu(A_k)\,.$$

QED

Let

$$\mathcal{R} = \left\{ \dot{\bigcup_{k=1}^{n}} A_k \times B_k : n \geq 0, \forall k \in \{1, \ldots, n\}\,[A_k \in \mathcal{A}, B_k \in \mathcal{B}] \right\}.$$

Proposition 4.2.
Let

$$\lambda : \mathcal{R} \to [0, \infty]$$
$$\dot{\bigcup_{k=1}^{n}} A_k \times B_k \mapsto \sum_{k=1}^{n} \mu(A_k)\,\nu(B_k)^{\cdot}$$

Then λ is a premeasure.

By Caratheodory, there is a complete measure

$$\left(X \times Y, \overline{\mathcal{A} \times \mathcal{B}}, \mu \times \nu\right)$$

on $X \times Y$ such that

$$\mathcal{A} \times \mathcal{B} \subseteq \overline{\mathcal{A} \times \mathcal{B}} = \{A \times B \in \mathcal{A} \times \mathcal{B} : A \times B \text{ is } \lambda^*\text{-measurable}\}.$$

and

$$(\mu \times \nu)(A \times B) = \mu(A)\,\nu(B), \qquad \forall A \in \mathcal{A}, B \in \mathcal{B}.$$

Def'n 4.2. **Product Measure**
Consider the above setting. We call $\mu \times \nu$ the ***product measure*** on $\mathcal{A} \times \mathcal{B}$.

2. Product Integration

Theorem 4.3. Fubini

Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be complete measure spaces. If $f \in \mathrm{L}^1\left(X \times Y, \overline{\mathcal{A} \times \mathcal{B}}, \mu \times \nu\right)$, then

(a) For all $x \in X$, let

$$\begin{aligned} f_x : Y &\to \mathbb{F} \\ y &\mapsto f(x,y) \end{aligned}.$$

Then $f_x \in \mathrm{L}^1(Y, \mathcal{B}, \nu)$ for almost all x.

(b) For all $y \in Y$, let

$$\begin{aligned} f^y : X &\to \mathbb{F} \\ x &\mapsto f(x,y) \end{aligned}.$$

Then $f^y \in \mathrm{L}^1(X, \mathcal{A}, \mu)$ for almost all y.

(c) Let

$$\begin{aligned} F : X &\to \mathbb{F} \\ x &\mapsto \int f_x d\nu \end{aligned}.$$

Then $F \in \mathrm{L}^1(X, \mathcal{A}, \nu)$.

(d) Let

$$\begin{aligned} G : X &\to \mathbb{F} \\ y &\mapsto \int f^y d\mu \end{aligned}.$$

Then $G \in \mathrm{L}^1(X, \mathcal{A}, \nu)$.

(e) We have

$$\int_{X \times Y} fd(\mu \times \nu) = \int_X \int_Y f(x,y)\, d\nu d\mu = \int_Y \int_X f(x,y)\, d\mu d\nu.$$

Given $E \subseteq X \times Y$, let us write write

$$E_x = \{y \in Y : (x,y) \in E\}, \qquad \forall x \in X$$

and

$$E^y = \{x \in X : (x,y) \in E\}, \qquad \forall y \in Y.$$

Lemma 4.4.
Let $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ be measure spaces and let

$$\mathcal{R} = \left\{ \dot{\bigcup_{k=1}^{n}} A_k \times B_k : n \geq 0, \forall k \in \{1, \ldots, n\} \left[A_k \in \mathcal{A}, B_k \in \mathcal{B} \right] \right\}.$$

Let $E \in \mathcal{R}_{\sigma\delta}$ with $(\mu \times \nu)(E) < \infty$. Then

(a) $g : X \to \mathbb{R}$ by $g(x) = \nu(E_x)$ for all $x \in X$ is μ-measurable;

(b) $g \in \mathrm{L}^+ \cap \mathrm{L}^1$; and

(c) $\int g d\mu = (\mu \times \nu)(E)$.

Proof.

Case 1. *Suppose $E = A \times B$ for some $A \in \mathcal{A}, B \in \mathcal{B}$.*
Then

$$E_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases} \in \mathcal{B}, \qquad \forall x \in X$$

Now

$$g(x) = \nu(E_x) = \nu(B)\, \chi_A(x), \qquad \forall x \in X$$

so that g is a nonnegative measurable function, with

$$\int g d\mu = \int \nu(B)\, \chi_A d\mu = \nu(B)\, \mu(A) = (\mu \times \nu)(E) < \infty,$$

as needed.

(End of Case 1)

Case 2. *Consider $E = \bigcup_{i=1}^{\infty} A_i \times B_i$ for some $A_1, \ldots \in \mathcal{A}, B_1, \ldots \in \mathcal{B}$.*
Without loss of generality, we may assume that the union is disjoint, since intersection of rectangles is still a rectangle.
Define $g_i = \nu(B_i)\, \chi_{A_i}$ for all $i \in \mathbb{N}$. Then

$$g = \sum_{i=1}^{\infty} g_i$$

so that g is μ-measurable. Moreover, every $E_x = \dot{\bigcup}_{i=1}^{\infty} (A_i \times B_i)_x$ is measurable.
Then by the MCT,

$$\int g d\mu = \sum_{i=1}^{\infty} \int g_i d\mu = \sum_{i=1}^{\infty} \mu(A_i)\, \nu(B_i) = \sum_{i=1}^{\infty} (\mu \times \nu)(A_i \times B_i) = (\mu \times \nu)(E) < \infty.$$

(End of Case 2)

Case 3. *Consider $E = \bigcap_{n=1}^{\infty} E_n$, where each $E_n \in \mathcal{R}_\sigma$.*
Without loss of generality, we may assume

$$E_1 \supseteq E_2 \supseteq \cdots.$$

Moreover, we may also assume that

$$(\mu \times \nu)(E_1) < \infty,$$

since $(\mu \times \nu)(E) < \infty$.

Then we have that

$$E_x = \bigcap_{n=1}^{\infty} (E_n)_x$$

and

$$(E_1)_x \supseteq (E_2)_x \supseteq \cdots ,$$

so

$$\lim_{n\to\infty} \nu\left((E_n)_x\right) = \nu(E)$$

and

$$\lim_{n\to\infty} (\mu \times \nu)(E_n) = (\mu \times \nu)(E) .$$

Let

$$\begin{aligned} g_n : X &\to \mathbb{R} \\ x &\mapsto \nu\left((E_n)_x\right) \end{aligned}, \qquad \forall n \in \mathbb{N} .$$

Then $0 \geq g$ and $g_n \searrow g$ pointwise with

$$\int g_1 d\nu = (\mu \times \nu)(E_1) < \infty,$$

so by the LDCT,

$$\int g d\mu = \lim_{n\to\infty} \int g_n d\mu = \lim_{n\to\infty} (\mu \times \nu)(E_n) = (\mu \times \nu)(E) .$$

(End of Case 3)

QED

V. Differentiation

VI. L^p Spaces

VII. Application on Probability Theory