

I. Measures

1. Motivation

Let X be a set and let $A \subseteq X$. We aim to develop a *meaningful* theory of integration that is

$$\int_A f,$$

where $f: X \rightarrow \mathbb{R}$.

There are a bunch of natural question that come out here.

- (a) Which A are appropriate?
- (b) Which f are appropriate?
- (c) What does $\int_A f$ even mean?

Moreover, we want the following:

$$\mu(A) = \int_A 1$$

to be some meaningful idea of size/volume/measure. Some μ 's do this better than others. Here are some properties we want μ to satisfy:

- (a) $\mu(\emptyset) = 0$.
- (b) $\mu(A \cup B) = \mu(A) + \mu(B)$.
- (c) $\mu(A \cup B) \leq \mu(A) + \mu(B)$.
- (d) $A \subseteq B \implies \mu(A) \leq \mu(B)$.
- (e) $\mu(X) \in [0, \infty]$.
- (f) $\mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$.
- (g) $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

Let's take a step back. If we are going to achieve those things, we want some basics. Let D be the domain of μ – the *nonprecise measure function* handed on us. We need:

- (a) $\emptyset \in D$; and
- (b) if $A_1, A_2, \dots \in D$, then $\bigcup_{n=1}^{\infty} A_n \in D$.

2. σ -algebras

Def'n 1.1. **σ -algebra** of Subsets of X

Let X be a set and let $\mathcal{A} \subseteq \mathcal{P}(X)$. We say \mathcal{A} is an **algebra**¹ of subsets of X if

- (a) $\emptyset \in \mathcal{A}$;
- (b) $A \in \mathcal{A}$ implies $X \setminus A \in \mathcal{A}$; and
- (c) $A, B \in \mathcal{A}$ implies $A \cup B \in \mathcal{A}$.

closure under complements
closure under finite union

Moreover, we say \mathcal{A} is a **σ -algebra** if it satisfies in addition

$$\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A} \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}.$$

That is, \mathcal{A} is *closed under countable unions*.

¹The word *algebra* comes from boolean algebra, one of the most universal objects in abstract math.

Question 1.1.

Are all algebra a σ -algebra?

Answer. To answer this question, we should think about:

what is preserved for finite sets but not infinite sets?

The easiest answer is *finiteness*. Let X be an infinite set and let

$$\mathcal{A} = \{A \subseteq X : A \text{ is finite or } X \setminus A \text{ is finite}\}.$$

Then \mathcal{A} is an algebra but not a σ -algebra.

QED

Let $\mathcal{A} \subseteq \mathcal{P}$ be an algebra. Then, as a corollary to Def'n 1.1,

(a) $A, B \in \mathcal{A}$ implies $X \setminus A, X \setminus B \in \mathcal{A}$, so that $A \cap B = X \setminus ((X \setminus A) \cup (X \setminus B)) \in \mathcal{A}$;

closure under closure

(b) $X = X \setminus \emptyset \in \mathcal{A}$;

(c) $A, B \in \mathcal{A}$ implies $A \setminus B = A \cap (X \setminus B) \in \mathcal{A}$; and

closure under set difference

(d) $A, B \in \mathcal{A}$ implies $A \triangle B \in \mathcal{A}$.

closure under symmetric set difference

Moreover, if \mathcal{A} is a σ -algebra, then (a) holds with countable number of sets.

Proposition 1.1. Generating σ -algebra from a Collection of Subsets

Let X be a set and let $\mathcal{E} \subseteq \mathcal{P}(X)$. Then

$$\langle \mathcal{E} \rangle = \bigcap \{ \mathcal{A} \supseteq \mathcal{E} : \mathcal{A} \text{ is a } \sigma\text{-algebra} \}$$

is a σ -algebra.

Exercise

Def'n 1.2. σ -algebra **Generated** by \mathcal{E}

Consider Proposition 1.1. We call $\langle \mathcal{E} \rangle$ the σ -algebra *generated* by \mathcal{E} .

Def'n 1.3. **Borel σ -algebra** of a Topological Space

Let (X, τ) be a topological space. Then

$$\text{Bor}(X) = \langle \tau \rangle$$

is called the *Borel σ -algebra* of (X, τ) .

We call elements of $\text{Bor}(X)$ the *Borel sets*.

Def'n 1.4. **Measurable Space**

Let X be a set and let \mathcal{A} be a σ -algebra of X . Then we call (X, \mathcal{A}) a *measurable space*.

The elements of \mathcal{A} are called the *measurable sets*.

3. Measures

In this course, we often work in the extend real numbers $\mathbb{R} \cup \{-\infty, \infty\} = [-\infty, \infty]$. Here are things that we assume.

Assumption 1. Assumptions about Extended Real Numbers

For all $a \in \mathbb{R}$,

(a) $a + \infty = \infty$;

(b) $a - \infty = -\infty$;

(c) $\infty + \infty = \infty$; and

(d) $-\infty - \infty = -\infty$.

However, we leave the following expressions to be *undefined*:

- (a) $\infty - \infty$;
- (b) $\frac{\infty}{\infty}$; and
- (c) 0∞ .

Def'n 1.5. **Measure** on a Measurable Space

Let (X, \mathcal{A}) be a measurable space. A **measure** on (X, \mathcal{A}) ¹ is a function $\mu : \mathcal{A} \rightarrow [0, \infty]$ such that

- (a) $\mu(\emptyset) = 0$; and
- (b) we have

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$$

for every $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{A}$ with $A_n \cap A_m = \emptyset$ for $n \neq m$.

countable additivity

In case μ is a measure on (X, \mathcal{A}) , we call (X, \mathcal{A}, μ) a **measure space**.

¹Or, **measure** on X if we are lazy.

Example 1.2. Examples of Measures

Let X be a set.

- (a) $\mu(A) = 0$ for all $A \in \mathcal{P}(X)$ is a measure on $(X, \mathcal{P}(X))$.
- (b) $\mu(\emptyset) = 0, \mu(A) = \infty$ for all $A \in \mathcal{P}(X) \setminus \{\emptyset\}$ is a measure on $(X, \mathcal{P}(X))$.
- (c) $\mu(A) = |A|$ (where $|A| = \infty$ if A is infinite) is a measure on $(X, \mathcal{P}(X))$.
- (d) Fix $x \in X$ and define

$$\mu(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

for all $A \in \mathcal{P}(X)$. Then μ is a measure on $(X, \mathcal{P}(X))$.

zero measure

counting measure

point-mass measure

Proposition 1.2.

Let (X, \mathcal{A}, μ) be a measure space.

- (a) For all $A, B \in \mathcal{A}$ and $A \subseteq B$, $\mu(A) \leq \mu(B)$.
- (b) For all $A, B \in \mathcal{A}$ with $A \subseteq B$ and $\mu(A) < \infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$.
- (c) If $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{A}$, then $\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \mu(A_n)$.

monotonicity

excision

countable subadditivity

Proof.

- (a) Consider $B \setminus A$, which is measurable since \mathcal{A} is closed under set difference. Hence we have

$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A).$$

- (b) We have

$$\mu(A) + \mu(B \setminus A) = \mu(B)$$

as seen in (a). Since $\mu(A) < \infty$, we can freely subtract $\mu(A)$ from both sides to obtain that $\mu(B \setminus A) = \mu(B) - \mu(A)$.

- (c) Let $B_1 = A_1$ and let $B_n = A_n \setminus \bigcup_{m=1}^{n-1} A_m$ for all $n \geq 2$. Then each B_n is measurable with $B_n \subseteq A_n$ and we have

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \mu\left(\bigcup_{n \in \mathbb{N}} B_n\right) = \sum_{n \in \mathbb{N}} \mu(B_n) \leq \sum_{n \in \mathbb{N}} \mu(A_n).$$

Proposition 1.3. Continuity of Measure

Let (X, \mathcal{A}, μ) be a measure space.

(a) Let $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{A}$ be an ascending chain. That is,

$$A_1 \subseteq A_2 \subseteq \cdots.$$

Then

$$\mu \left(\bigcup_{n \in \mathbb{N}} A_n \right) = \lim_{n \rightarrow \infty} \mu(A_n). \quad \text{continuity from below}$$

(b) Let $\{B_n\}_{n \in \mathbb{N}} \subseteq \mathcal{A}$ be a decending chain with $\mu(B_1) < \infty$. That is,

$$B_1 \supseteq B_2 \supseteq \cdots.$$

Then

$$\mu \left(\bigcap_{n \in \mathbb{N}} B_n \right) = \lim_{n \rightarrow \infty} \mu(B_n). \quad \text{continuity from above}$$

Proof.

(a) Let $C_1 = A_1$ and let $C_n = A_n \setminus A_{n-1} = A_n \setminus \bigcup_{m=1}^{n-1} A_m$ for all $n \geq 2$, where the last equality follows from the ascending chain condition.

$$\mu \left(\bigcup_{n \in \mathbb{N}} A_n \right) = \mu \left(\bigcup_{n \in \mathbb{N}} C_n \right) = \sum_{n \in \mathbb{N}} \mu(C_n) = \lim_{N \rightarrow \infty} \sum_{n=1}^N \mu(C_n) = \lim_{N \rightarrow \infty} \mu \left(\bigcup_{n=1}^N C_n \right) = \lim_{N \rightarrow \infty} \mu(A_N).$$

(b) Let $D_n = B_1 \setminus B_n$ for all $n \in \mathbb{N}$, so that $\{D_n\}_{n \in \mathbb{N}}$ is an ascending chain. Then

$$B_1 \setminus \bigcap_{n \in \mathbb{N}} B_n = \bigcup_{n \in \mathbb{N}} D_n,$$

so that

$$\mu \left(B_1 \setminus \bigcap_{n \in \mathbb{N}} B_n \right) = \mu \left(\bigcup_{n \in \mathbb{N}} D_n \right) = \lim_{n \rightarrow \infty} \mu(D_n) = \lim_{n \rightarrow \infty} \mu(B_1) - \mu(B_n) = \mu(B_1) - \lim_{n \rightarrow \infty} \mu(B_n).$$

The result then follows from excision property of μ .

QED

Def'n 1.6. **Finite, Probability, σ -finite, Semifinite, Complete** Measure

Let (X, \mathcal{A}, μ) be a measure space. We say μ is

(a) **finite** if $\mu(X) < \infty$;

(b) a **probability** measure if $\mu(X) = 1$;

(c) **σ -finite** if

$$X = \bigcup_{n=1}^{\infty} A_n$$

for some $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ with $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$;

(d) **semifinite** if

$$\forall A \in \mathcal{A} [\mu(A) \neq 0 \implies \exists B \in \mathcal{A} [B \subseteq A, 0 < \mu(B) < \infty]];$$

and

(e) **complete** if

$$\forall A \in \mathcal{A} [\mu(A) = 0 \implies \forall B \subseteq A [B \in \mathcal{A}]].$$

4. Completion of Measure Spaces

Example 1.3. An Example of Non-complete Measure

Let $X = \{a, b\}$, $\mathcal{A} = \{\emptyset, \{a, b\}\}$, $\mu = 0$. Then μ is not complete, as $\{a\} \in \mathcal{A}$.

The goal of this section is:

given a measure space (X, μ, \mathcal{A}) , if μ is not complete, we extend \mathcal{A} and μ so that the result is complete.

A natural way of doing this is throw every subsets of measure-zero sets into \mathcal{A} .

Proposition 1.4. Completion of a Measure Space

Let (X, μ, \mathcal{A}) be a measure space. Let

$$\overline{\mathcal{A}} = \{A \cup F : A \in \mathcal{A}, \exists N \in \mathcal{A} [F \subseteq N, \mu(N) = 0]\}$$

and define

$$\begin{aligned} \overline{\mu} : \overline{\mathcal{A}} &\rightarrow [0, \infty] \\ A \cup F &\mapsto \mu(A) \end{aligned}$$

Then

- (a) $\overline{\mathcal{A}}$ is a σ -algebra;
- (b) $\overline{\mu}$ is a measure;
- (c) $\overline{\mu}|_{\mathcal{A}} = \mu$; and
- (d) $\overline{\mu}$ is complete.

Proof.

- (a) Note that $\emptyset = \emptyset \cup \emptyset$ with $\emptyset \subseteq \emptyset$ where $\mu(\emptyset) = 0$. Hence $\emptyset \in \overline{\mathcal{A}}$.

Let $E = A \cup F$ with $A \in \mathcal{A}$, $F \subseteq N \in \mathcal{A}$ where $\mu(N) = 0$. Then

$$X \setminus E = \underbrace{X \setminus (A \cup N)}_{\in \mathcal{A}} \cup \underbrace{(N \setminus (A \cup F))}_{\subseteq N} \in \overline{\mathcal{A}}.$$

Let $\{E_n\}_{n=1}^{\infty} \subseteq \overline{\mathcal{A}}$ with $E_n = A_n \cup F_n$ where $F_n \subseteq N_n$ for some $n \in \mathbb{N}$. Then

$$\bigcup_{n=1}^{\infty} E_n = \left(\bigcup_{n=1}^{\infty} A_n \right) \cup \left(\bigcup_{n=1}^{\infty} F_n \right).$$

But $\bigcup_{n=1}^{\infty} F_n \subseteq \bigcup_{n=1}^{\infty} N_n$ with $\mu(\bigcup_{n=1}^{\infty} N_n) \leq \sum_{n=1}^{\infty} \mu(N_n) = 0$. Thus $\bigcup_{n=1}^{\infty} E_n \in \overline{\mathcal{A}}$.

- (b) We first check that $\overline{\mu}$ is well-defined. Let

$$E = A_1 \cup F_1 = A_2 \cup F_2$$

for some $A_1, A_2 \in \mathcal{A}$ and $F_1 \subseteq N_1, F_2 \subseteq N_2$ with $\mu(N_1) = \mu(N_2) = 0$.

Then note that

$$A_1 \cap A_2 \subseteq A_i \subseteq E \subseteq (A_1 \cup F_1) \cap (A_2 \cup F_2) \subseteq (A_1 \cap A_2) \cup N_1 \cup N_2.$$

Hence

$$\mu(A_1 \cap A_2) \leq \mu(A_i) \leq \mu(E_1 \cap E_2).$$

This means $\mu(A_i) = \mu(A_1 \cap A_2)$, so that $\mu(E_1) = \mu(E_2)$.

Thus $\overline{\mu}$ is well-defined.

To show $\bar{\mu}$ is a measure, note that

$$\bar{\mu}(\emptyset) = \bar{\mu}(\emptyset \cup \emptyset) = \mu(\emptyset) = 0.$$

Say we have a collection of disjoint sets in $\bar{\mathcal{A}}$, $\{E_n\}_{n=1}^{\infty} \subseteq \bar{\mathcal{A}}$, with

$$E_n = A_n \cup F_n$$

for some $E_n \subseteq N_n$ with $\mu(N_n) = 0$. Then

$$\bigcup_{n=1}^{\infty} E_n = \left(\bigcup_{n=1}^{\infty} A_n \right) \cup \underbrace{\left(\bigcup_{n=1}^{\infty} F_n \right)}_{\subseteq \bigcup_{n=1}^{\infty} N_n}.$$

Thus

$$\bar{\mu}\left(\bigcup_{n=1}^{\infty} E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} \bar{\mu}(A_n),$$

so $\bar{\mu}$ is a measure.

(c) Given $A \in \mathcal{A}$, $A = A \cup \emptyset$, so that $\bar{\mu}(A) = \mu(A)$.

(d) Let $A \subseteq B \in \bar{\mathcal{A}}$ with $\bar{\mu}(B) = 0$. We are going to show $A \in \bar{\mathcal{A}}$.

We can write

$$B = E \cup F$$

for some $F \subseteq N \in \mathcal{A}$ with $\mu(N) = 0$. Then

$$\bar{\mu}(B) = \mu(E) = 0.$$

Since $A \subseteq B \subseteq E \cup N$ with $\mu(E \cup N) = 0$ (complete this).

QED

Def'n 1.7. **Completion** of a Measure Space

Let (X, μ, \mathcal{A}) be a measure space. We call $(X, \bar{\mu}, \bar{\mathcal{A}})$ the **completion** of (X, μ, \mathcal{A}) .

5. Construction of Measures

Def'n 1.8. **Outer Measure** on a Set

Let X be a nonempty set. An **outer measure** on X is a function $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ such that

- (a) $\mu^*(\emptyset) = 0$;
- (b) $A \subseteq B$ implies $\mu^*(A) \leq \mu^*(B)$; and
- (c) $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(X)$ implies $\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$.

monotonicity

countable subadditivity

The idea is that

outer measures are naive approaches to measure every subset of X .

We start with $\mathcal{E} \subseteq \mathcal{P}(X)$ which are *easy* to measure. We use the outer measure μ^* and \mathcal{E} to construct a measure.

Proposition 1.5. Construction of an Outer Measure

Suppose $\{\emptyset, X\} \subseteq \mathcal{E} \subseteq \mathcal{P}(X)$ and $\mu : \mathcal{E} \rightarrow [0, \infty]$ satisfies $\mu(\emptyset) = 0$. For $A \subseteq X$, define

$$\mu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \mu(E_n) : \{E_n\}_{n=1}^{\infty} \subseteq \mathcal{E}, A \subseteq \bigcup_{n=1}^{\infty} E_n \right\}.$$

Then μ^* is an outer measure on X .

Example 1.4. Lebesgue Outer Measure

Let $X = \mathbb{R}$, $\mathcal{E} = \{(a, b) : -\infty < a < b < \infty\} \cup \{\emptyset, X\}$. Define

$$\mu((a, b)) = b - a, \mu(X) = \infty.$$

Then μ^* as said in Proposition 1.5 is called the *Lebesgue outer measure*.

Proposition 1.6.

Suppose $\{\emptyset, X\} \subseteq \mathcal{E} \subseteq X$ and let $\mu : \mathcal{E} \rightarrow [0, \infty]$. If $\mu(\emptyset) = 0$, then $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ by

$$\mu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n) : \{A_n\}_{n=1}^{\infty} \subseteq \mathcal{E}, A \subseteq \bigcup_{n=1}^{\infty} A_n \right\}, \quad \forall A \in \mathcal{E}.$$

is an outer measure.

Proof. We verify few things.

- (a) Note that $\emptyset \subseteq \bigcup_{n=1}^{\infty} \emptyset$ and so $0 \leq \mu^*(\emptyset) \leq \sum_{n=1}^{\infty} \mu(\emptyset) = 0$.
- (b) Say $A \subseteq B \subseteq X$. Then

$$\left\{ \sum_{n=1}^{\infty} \mu(A_n) : \forall n \in \mathbb{N} [A_n \in \mathcal{E}], A \subseteq \bigcup_{n=1}^{\infty} A_n \right\} \supseteq \left\{ \sum_{n=1}^{\infty} \mu(A_n) : \forall n \in \mathbb{N} [A_n \in \mathcal{E}], B \subseteq \bigcup_{n=1}^{\infty} A_n \right\}$$

by definition. By taking infimum, we see that

$$\mu^*(A) \leq \mu^*(B).$$

- (c) Say $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(X)$ and consider $\bigcup_{n=1}^{\infty} A_n$. We claim that

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

We may assume $\sum_{n=1}^{\infty} \mu^*(A_n) < \infty$.

Let $\varepsilon > 0$ be given. For every A_i , we may find $\{E_{i,j}\}_{j=1}^{\infty} \subseteq \mathcal{E}$ such that

$$A_i \subseteq \bigcup_{j=1}^{\infty} E_{i,j}$$

and

$$\sum_{j=1}^{\infty} \mu(E_{i,j}) < \mu^*(A_i) + \frac{\varepsilon}{2^i}$$

We then have

$$\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i,j=1}^{\infty} E_{i,j}.$$

Hence

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \inf \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu(E_{i,j}) \leq \sum_{i=1}^{\infty} \mu^*(A_i) + \frac{\varepsilon}{2^i} = \left(\sum_{i=1}^{\infty} \mu^*(A_i)\right) + \varepsilon.$$

Since ε is an arbitrary positive number, we see that μ^* is countably subadditive.

QED

Def'n 1.9. μ^* -**measurable** Set

Let μ^* be an outer measure on X . We say $A \subseteq X$ is μ^* -**measurable** if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap (X \setminus A))$$

for all $E \subseteq X$.

Let $A, E \subseteq X$.

(a) Note

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap (X \setminus A)).$$

Hence it suffices to prove the reverse inequality to show that A is μ^* -measurable.

(b) As a corollary to (a), we may assume $\mu^*(E) < \infty$ when proving A is μ^* -measurable.

(c) When $A = \emptyset$,

$$\mu^*(E \cap \emptyset) + \mu^*(E \cap (X \setminus \emptyset)) = 0 + \mu^*(E) = \mu^*(E).$$

Thus \emptyset is μ^* -measurable.

(d) If A is μ^* -measurable, then $X \setminus A$ is also μ^* -measurable. This is direct from the definition of μ^* -measurability.

Theorem 1.7. Caratheodory

Let μ^* be an outer measure on X . Then the collection of μ^* -measurable subsets of X ,

$$\mathcal{A} = \{A \subseteq X : A \text{ is } \mu^*\text{-measurable}\},$$

is a σ -algebra.

Moreover, $\mu = \mu^*|_{\mathcal{A}}$ is a complete measure on (X, \mathcal{A}) .

Proof. Let $A, B \in \mathcal{A}$ and let $E \subseteq X$. Then

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap (X \setminus A) \cap B) + \mu^*(E \cap (X \setminus A) \cap (X \setminus B)) && \text{since } A, B \text{ are } \mu^*\text{-measurable} \\ &\geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (X \setminus (A \cup B))) && \text{by subadditivity of } \mu^* \text{ and de Morgan's Law} \end{aligned}$$

Since we know the other direction of the above inequality, we see that $A \cup B \in \mathcal{A}$. Inductively, \mathcal{A} is closed under finite union, which means \mathcal{A} is an algebra on X (we know $\emptyset \in \mathcal{A}$).

Now assume $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$. For any $E \subseteq X$,

$$\mu^*(E \cap (A \cup B)) = \mu^*(E \cap (A \cup B) \cap A) + \mu^*(E \cap (A \cup B) \cap (X \setminus A)) = \mu^*(E \cap A) + \mu^*(E \cap B).$$

By taking $E = X$, we see that

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B),$$

so that μ^* is finitely additive.

Assume $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$, let $B_n = \bigcup_{k=1}^n A_k$, and let $A'_n = A_1 \setminus \bigcup_{k=1}^{n-1} A_k$ for all $n \in \mathbb{N}$. Since \mathcal{A} is an algebra, each $A'_n, B_n \in \mathcal{A}$. Then $B_n = \bigcup_{k=1}^n A'_k$ and $B = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} A'_n$. For any $E \subseteq X$,

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap B_n) + \mu^*(E \cap (X \setminus B_n)) \\ &\geq \mu^*(E \cap B_n) + \mu^*(E \cap (X \setminus B)) && \text{by monotonicity of } \mu^* \\ &= \sum_{k=1}^n \mu^*(E \cap A'_k) + \mu^*(E \cap (X \setminus B)) \\ &\geq \lim_{n \rightarrow \infty} \sum_{k=1}^n \mu^*(E \cap A'_k) + \mu^*(E \cap (X \setminus B)) \\ &\geq \mu^*(E \cap B) + \mu^*(E \cap (X \setminus B)) \\ &\geq \mu^*(E). && \text{by subadditivity of } \mu^* \end{aligned}$$

This means $\mu^*(E) = \mu^*(E \cap B) + \mu^*(E \cap (X \setminus B))$, so $\bigcup_{n=1}^{\infty} A_n = B \in \mathcal{A}$. Hence \mathcal{A} is a σ -algebra.

Assume $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ is a collection of disjoint sets in \mathcal{A} . By taking $A'_n = A_n$ for all $n \in \mathbb{N}$ and $E = B$, we see that

$$\mu^*(B) \geq \sum_{n=1}^{\infty} \mu^*(B \cap A_n) + \underbrace{\mu^*(B \cap (X \setminus B))}_{=0} \geq \mu^*(B) \implies \mu^*(B) = \sum_{n=1}^{\infty} \mu^*(B \cap A_n)$$

from the series of inequalities we used for proving closure of \mathcal{A} under countable union.

We now show that μ is complete. Let $A \subseteq X$ with $\mu^*(A) = 0$. For any $E \subseteq X$,

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap (X \setminus A)) \leq \underbrace{\mu^*(A)}_{=0} + \mu^*(E).$$

This means every set A with $\mu^*(A) = 0$ is measurable. But given any $B \in \mathcal{A}$ with $\mu(B) = 0$, we have

$$0 \leq \mu^*(A) \leq \mu^*(B) = \mu(B) = 0, \quad \forall A \subseteq B,$$

so that $\mu^*(A) = 0$ and that A is measurable.

QED

We can construct a measure as follows. Given $\mathcal{E} \subseteq \mathcal{P}(X)$ with $\{\emptyset, X\} \subseteq \mathcal{E}$ and $\mu : \mathcal{E} \rightarrow [0, \infty]$, we let $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$ be an outer measure as defined in Proposition 1.6.

In general, $\mathcal{A} = \{A \subseteq X : A \text{ is } \mu^*\text{-measurable}\}$ and $\mu^*|_{\mathcal{A}}$ are very different from \mathcal{E}, μ . To resolve this, we introduce the following notion.

Def'n 1.10. **Premeasure** on an Algebra of Subsets

Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra of subsets of X . We say $\mu : \mathcal{A} \rightarrow [0, \infty]$ is a **premeasure** on \mathcal{A} if

- (a) $\mu(\emptyset) = 0$; and
- (b) for any $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ with $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$, we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Theorem 1.8. Constructing Measure from Premeasure I

Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra and let $\mu : \mathcal{A} \rightarrow [0, \infty]$ be a premeasure on \mathcal{A} . Let μ^* be the outer measure constructed with \mathcal{A} :

$$\mu^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n) : \{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}, A \subseteq \bigcup_{n=1}^{\infty} A_n \right\}, \quad \forall A \in \mathcal{P}(X).$$

Then

- (a) $\mu^*|_{\mathcal{A}} = \mu$; and
- (b) every $A \in \mathcal{A}$ is μ^* -measurable.

Proof.

- (a) We show $\mu^*|_{\mathcal{A}} = \mu$. Let $E \in \mathcal{A}$. Say

$$E \subseteq \bigcup_{n=1}^{\infty} A_n$$

where each $A_n \in \mathcal{A}$. Then by taking $A'_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k$,

$$E = \bigcup_{n=1}^{\infty} (A_n \cap E) = \bigcup_{n=1}^{\infty} (A'_n \cap E).$$

But each $A'_n \cap E \in \mathcal{A}$, so that

$$\mu(E) = \sum_{n=1}^{\infty} \mu(A'_n \cap E) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

by the monotonicity of μ .¹ Therefore, $\mu(E) \leq \mu^*(E)$ by taking infimum.

On the other hand, by letting $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ with $A_1 = E, A_2 = A_3 = \dots = \emptyset$, we see that $\mu^*(E) \geq \mu(E)$. Hence $\mu^*|_{\mathcal{A}} = \mu$.

(b) Let $A \in \mathcal{A}$. We show A is μ^* -measurable. Let $E \subseteq X$ and let $\varepsilon > 0$ be given. We may find $\{B_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ such that $E \subseteq \bigcup_{n=1}^{\infty} B_n$ and

$$\sum_{n=1}^{\infty} \mu(B_n) < \mu^*(E) + \varepsilon.$$

Then,

$$\begin{aligned} \mu^*(E) + \varepsilon &\geq \sum_{n=1}^{\infty} \mu(B_n) \\ &= \sum_{n=1}^{\infty} \mu(B_n \cap A) + \mu(B_n \cap (X \setminus A)) \\ &= \sum_{n=1}^{\infty} \mu^*(B_n \cap A) + \sum_{n=1}^{\infty} \mu^*(B_n \cap (X \setminus A)) && \text{by (a)} \\ &\geq \mu^*\left(\left(\bigcup_{n=1}^{\infty} B_n\right) \cap A\right) + \mu^*\left(\left(\bigcup_{n=1}^{\infty} B_n\right) \cap (X \setminus A)\right) && \text{by subadditivity of } \mu^* \\ &\geq \mu^*(E \cap A) + \mu^*(E \cap (X \setminus A)). && \text{by monotonicity of } \mu^* \text{ since } E \subseteq \bigcup_{n=1}^{\infty} B_n \end{aligned}$$

¹It suffices to note that premeasures are finitely additive, which implies monotonicity.

QED

Theorem 1.9. Constructing Measure from Premeasure II

Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra and let μ^* be as in Theorem 1.8. Let $\mathcal{B} = \sigma(\mathcal{A})$. Then

- (a) $\bar{\mu} = \mu^*|_{\mathcal{B}}$ is a complete measure with $\bar{\mu}|_{\mathcal{A}} = \mu$.
- (b) Let ν be another measure on \mathcal{B} with $\nu|_{\mathcal{A}} = \mu$. Then $\nu \leq \bar{\mu}$. That is,

$$\nu(A) \leq \bar{\mu}(A), \quad \forall A \in \mathcal{B}.$$

- (c) For any $E \in \mathcal{B}$, if $\bar{\mu}(E) < \infty$, then $\nu(E) = \bar{\mu}(E)$.
- (d) If μ is σ -finite,¹ then $\bar{\mu} = \nu$.

¹We say a premeasure is σ -finite if $X = \bigcup_{n=1}^{\infty} A_n$ for some $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ with $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$.

Proof.

- (a) Let

$$\mathcal{C} = \{A \subseteq \mathcal{P}(X) : A \text{ is } \sigma\text{-measurable}\},$$

which is a σ -algebra. Then by Theorem 1.8, $\mathcal{A} \subseteq \mathcal{C}$, and so $\mathcal{B} \subseteq \mathcal{C}$ by minimality of \mathcal{B} . Therefore,

$$\bar{\mu} = \mu^*|_{\mathcal{B}}$$

is the restriction of $\mu^*|_{\mathcal{C}}$ to \mathcal{B} . Since $\mu^*|_{\mathcal{C}}$ is a complete measure on (X, \mathcal{C}) , it follows $\bar{\mu} = \mu^*|_{\mathcal{B}}$ is a complete measure on (X, \mathcal{B}) . Since $\mu^*|_{\mathcal{A}} = \mu$, $\bar{\mu}|_{\mathcal{A}} = \mu$ as well.

(b) Let $A \in \mathcal{B}$ and let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ be such that $A \subseteq \bigcup_{n=1}^{\infty} A_n$. Since ν is a measure extending μ ,

$$\nu(A) \leq \nu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \nu(A_n) \stackrel{\nu|_{\mathcal{A}} = \mu}{=} \sum_{n=1}^{\infty} \mu(A_n).$$

By recalling that μ^* is defined as the *greatest* lower bound, it follows

$$\nu(A) \leq \mu^*(A) = \bar{\mu}(A).$$

(c) Let $A \in \mathcal{B}$ with $\bar{\mu}(A) < \infty$. Let $\varepsilon > 0$ be given. We may find $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ such that $A \subseteq \bigcup_{n=1}^{\infty} A_n$ and

$$\sum_{n=1}^{\infty} \mu(A_n) < \bar{\mu}(A) + \varepsilon.$$

Let $B = \bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$. Note that

$$\nu(B) = \nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{k \rightarrow \infty} \nu\left(\bigcup_{n=1}^k A_n\right) = \lim_{k \rightarrow \infty} \bar{\mu}\left(\bigcup_{n=1}^k A_n\right) = \bar{\mu}\left(\bigcup_{n=1}^{\infty} A_n\right) = \bar{\mu}(B).$$

Moreover

$$\bar{\mu}(B) \leq \sum_{n=1}^{\infty} \mu(A_n) < \bar{\mu}(A) + \varepsilon < \infty.$$

It follows

$$\bar{\mu}(B \setminus A) < \varepsilon,$$

so that

$$\bar{\mu}(A) \leq \bar{\mu}(B) = \nu(B) = \nu(A) + \nu(B \setminus A) \leq \nu(A) + \bar{\mu}(B \setminus A) < \nu(A) + \varepsilon.$$

Since ε was given arbitrarily, we have $\bar{\nu}(A) \leq \nu(A)$. Since the reverse inequality is given in (b), we thus conclude $\bar{\mu}(A) = \nu(A)$.

(d) Say $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ is such that $X = \bigcup_{n=1}^{\infty} A_n$ with $\mu(A_n) < \infty$. Write $A'_n = A_n \setminus \bigcup_{m=1}^{n-1} A_m$ so that

$$X = \bigcup_{n=1}^{\infty} A'_n.$$

Therefore,

$$\bar{\mu}(A) = \bar{\mu}\left(\bigcup_{n=1}^{\infty} (A \cap A'_n)\right) = \sum_{n=1}^{\infty} \bar{\mu}(A \cap A'_n) = \sum_{n=1}^{\infty} \nu(A \cap A'_n) = \nu(A).$$

QED

6. Lebesgue-Stieltjes Measures on \mathbb{R}

Suppose we have a measure space $(\mathbb{R}, \text{Bor}(\mathbb{R}), \mu)$, where we are working with the usual topology on \mathbb{R} . We further assume that

for all compact $K \subseteq \mathbb{R}$, $\mu(K) < \infty$.

We consider

$$F : \mathbb{R} \rightarrow \mathbb{R}$$

$$x \mapsto \begin{cases} \mu([0, x]) & x \geq 0. \\ -\mu((x, 0)) & x < 0 \end{cases}$$

Then by definition, F is increasing.

Let $(x_n)_{n=1}^\infty \in \mathbb{R}^\mathbb{N}$ be a decreasing sequence with $x_n \rightarrow x \in \mathbb{R}$. In case $x \geq 0$,

$$F(x) = \mu([0, x]) = \mu\left(\bigcap_{n=1}^\infty [0, x_n]\right) = \lim_{n \rightarrow \infty} \mu([0, x_n]) = \lim_{n \rightarrow \infty} F(x_n),$$

where we are using the compactness assumption to use the continuity from above. Hence F is *right-continuous* on $[0, \infty)$.

Exercise 1.5.

Show that F is right-continuous on $(-\infty, 0)$. That is, when $x < 0$,

$$F(x) = \lim_{n \rightarrow \infty} F(x_n).$$

Example 1.6.

Consider the point-mass measure

$$\begin{aligned} \mu_0 : \text{Bor}(\mathbb{R}) &\rightarrow [0, \infty] \\ A &\mapsto \begin{cases} 0 & \text{if } 0 \notin A \\ 1 & \text{if } 0 \in A \end{cases} \end{aligned}$$

and the measure space $(\mathbb{R}, \text{Bor}(\mathbb{R}), \mu_0)$.

Then note that,

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases},$$

which is right-continuous but not left-continuous.

The goal of this section is, then:

given an increasing right-continuous $F : \mathbb{R} \rightarrow \mathbb{R}$, we make a measure μ_F on $(\mathbb{R}, \text{Bor}(\mathbb{R}))$.

That is, we are doing the converse of the motivation for this section.

The idea is to start with

$$\mu_F((a, b]) = F(b) - F(a), \quad \forall a, b \in \mathbb{R}, a < b.$$

Let \mathcal{A} be the set of finite unions of half-open intervals of the form $(a, b]$, where $a \in [-\infty, \infty)$, $b \in (-\infty, \infty]$ (we note that when $b = \infty$, we are taking (a, ∞) instead of $(a, \infty]$, since we are working with subsets of \mathbb{R}).

We note that

$$\mathbb{R} \setminus (a, b] = (-\infty, a] \cup (b, \infty) \in \mathcal{A}$$

so that \mathcal{A} is an algebra.

In addition, we insist

(a) $F(\infty) = \lim_{x \rightarrow \infty} F(x)$ and $F(-\infty) = \lim_{x \rightarrow -\infty} F(x)$; and

(b) $\mu_F(\bigcup_{k=1}^n (a_k, b_k]) = \sum_{k=1}^n F(b_k) - F(a_k)$.

In this way we get a function $\mu_F : \mathcal{A} \rightarrow [0, \infty]$.

Fact 1.10.

μ_F is a premeasure on $(\mathbb{R}, \mathcal{A})$.

Theorem 1.11.

Consider the above setting. There is a complete measure space $(\mathbb{R}, \mathcal{B}, \overline{\mu_F})$ such that

(a) $\overline{\mu_F}|_{\mathcal{A}} = \mu_F$; and

(b) $\text{Bor}(\mathbb{R}) \subseteq \mathcal{B}$.

Proof. Consider μ_F^* be the outer measure constructed as in Theorem 1.8 and let \mathcal{B} be the σ -algebra of μ_F^* -measurable sets. We set $\overline{\mu}_F = \mu_F^*|_{\mathcal{B}}$. By Theorem 1.8, we know that $(\mathbb{R}, \mathcal{B}, \overline{\mu}_F)$ is complete and $\overline{\mu}_F|_{\mathcal{A}} = \mu_F$.

By Theorem 1.8 again, $\mathcal{A} \subseteq \mathcal{B}$ (which was implicit in restricting $\overline{\mu}_F$ to \mathcal{A}). In particular, half-open intervals are in \mathcal{B} , so that

$$(a, b) = \bigcup_{n=1}^{\infty} \left(a, b - \frac{1}{n} \right] \in \mathcal{B}$$

for all $a < b$ in \mathbb{R} . Since \mathcal{B} has every open intervals, which generate the Borel σ -algebra on \mathbb{R} , it follows $\text{Bor}(\mathbb{R}) \subseteq \mathcal{B}$.

QED

Theorem 1.12.

When $F(x) = x$ for all $x \in \mathbb{R}$, then

- (a) $\overline{\mu}_F$ is the Lebesgue measure; and
 - (b) \mathcal{B} is the set of Lebesgue measurable sets.
-

Def'n 1.11. Lebesgue-Stieltjes Measure

Any measure of the form $\overline{\mu}_F$ is called a *Lebesgue-Stieltjes measure*.

Theorem 1.13. Regularity of Lebesgue-Stieltjes Measures

Let $(\mathbb{R}, \mathcal{B}, \overline{\mu}_F)$ as above and let $A \subseteq \mathbb{R}$. The following are equivalent.

- (a) $A \in \mathcal{B}$ (i.e. A is μ_F^* -measurable).
- (b) For all $\varepsilon > 0$, there is open $U \subseteq \mathbb{R}$ such that $A \subseteq U$ and $\mu_F^*(U \setminus A) < \varepsilon$.
- (c) For all $\varepsilon > 0$, there is closed $C \subseteq \mathbb{R}$ such that $C \subseteq A$ and $\mu_F^*(A \setminus C) < \varepsilon$.
- (d) There exists a G_δ -set¹ such that $A \subseteq G$ and $\mu_F^*(G \setminus A) = 0$.
- (e) There exists a F_σ -set² such that $F \subseteq A$ and $\mu_F^*(A \setminus F) = 0$.

¹A set is G_δ if it is a countable intersection of open sets.

²A set is F_σ if it is a countable union of closed sets.

Proof. (1) \implies (2) Assume $A \in \mathcal{B}$ and let $\varepsilon > 0$ be given.

Case 1. Suppose A is bounded.

Then $A \subseteq (a, b]$ and $\overline{\mu}_F(A) \leq F(b) - F(a) < \infty$. We may find $\{(a_n, b_n]\}_{n=1}^{\infty}$ such that

$$B = \bigcup_{n=1}^{\infty} (a_n, b_n]$$

contains A and

$$\overline{\mu}_F(B) < \overline{\mu}_F(A) + \frac{\varepsilon}{2}.$$

Now, choose $c_n > b_n$ such that

$$F(c_n) < F(b_n) + \frac{\varepsilon}{2^{n+1}}$$

by the right-continuity of F . Let $U = \bigcup_{n=1}^{\infty} (a_n, c_n)$. Since $A \in \mathcal{B}$, we have

$$\overline{\mu}_F(B) = \overline{\mu}_F(A) + \overline{\mu}_F(B \setminus A)$$

by Caratheodory measurability condition (Def'n 1.9). So by excision,

$$\overline{\mu}_F(B \setminus A) = \overline{\mu}_F(B) - \overline{\mu}_F(A) < \frac{\varepsilon}{2}.$$

Hence

$$\overline{\mu_F}(U \setminus A) \leq \overline{\mu_F}(U \setminus B) + \overline{\mu_F}(B \setminus A) < \overline{\mu_F}\left(\bigcup_{n=1}^{\infty} (b_n, c_n)\right) + \frac{\varepsilon}{2} \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} + \frac{\varepsilon}{2} = \varepsilon.$$

(End of Case 1)

Case 2. Let $A \in \mathcal{B}$ and consider $A_n = A \cap [-n, n]$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ be given and choose open U_n such that $A_n \subseteq U_n$ and

$$\mu_F^*(U_n \setminus A_n) < \frac{\varepsilon}{2^n}$$

for all $n \in \mathbb{N}$. Consider $U = \bigcup_{n=1}^{\infty} U_n$. Then $A = \bigcup_{n=1}^{\infty} A_n \subseteq U$ and

$$\mu_F^*(U \setminus A) \leq \mu_F^*\left(\bigcup_{n=1}^{\infty} (U_n \setminus A_n)\right) \leq \sum_{n=1}^{\infty} \mu_F^*(U_n \setminus A_n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

(End of Case 2)

(2) \implies (4) For every $n \in \mathbb{N}$, find open $U_n \subseteq \mathbb{R}$ containing A such that

$$\mu_F^*(U_n \setminus A) < \frac{1}{n}.$$

Take

$$G = \bigcap_{n=1}^{\infty} U_n.$$

Then $A \subseteq G$ and

$$\mu_F^*(G \setminus A) \leq \mu_F^*(U_n \setminus A) < \frac{1}{n}$$

for all $n \in \mathbb{N}$. Thus $\mu_F^*(G \setminus A) = 0$.

(4) \implies (1) Take a G_δ -set $G \subseteq \mathbb{R}$ containing A with $\mu^*(G \setminus A) = 0$. In particular, we have that $G \setminus A \in \mathcal{B}$.¹ Since every open set is in \mathcal{B} and \mathcal{B} is closed under countable intersection, $G \in \mathcal{B}$ as a countable intersection of open sets, and

$$A = G \setminus (G \setminus A) \in \mathcal{B}.$$

(1) \implies (3) Let $A \in \mathcal{B}$ and let $\varepsilon > 0$. Since $X \setminus A \in \mathcal{B}$, we may find open $U \supseteq X \setminus A$ such that

$$\mu_F^*(U \setminus (X \setminus A)) < \varepsilon.$$

Letting $C = X \setminus U$, $C \subseteq A$ and

$$\mu_F^*(A \setminus C) = \mu_F^*(U \setminus (X \setminus A)) < \varepsilon.$$

(3) \implies (5) Choose $C_n \subseteq A$ such that

$$\mu_F^*(A \setminus C_n) < \frac{1}{n}$$

for all $n \in \mathbb{N}$ and let

$$K = \bigcup_{n=1}^{\infty} C_n.$$

(5) \implies (1) Let K be a F_σ -set contained in A with $\mu_F^*(A \setminus K) = 0$. Then we observe that $A = (A \setminus K) \cup K \in \mathcal{B}$.

¹See the proof of Theorem 1.7, Caratheodory theorem.

II. Measurable Functions

1. Measurable Functions

Let (X, \mathcal{A}) , (Y, \mathcal{B}) be measurable spaces. We care about functions $f : X \rightarrow Y$ which relay information about the measurable spaces.

Def'n 2.1. **Measurable** Function

Let (X, \mathcal{A}) , (Y, \mathcal{B}) be measurable spaces. We say $f : X \rightarrow Y$ is *measurable* if

$$f^{-1}(B) = \{x \in X : f(x) \in B\} \in \mathcal{A}, \quad \forall B \in \mathcal{B}.$$

Before we proceed, here is a convention that we are going to use. Let \mathbb{F} be \mathbb{R} or \mathbb{C} and let (X, \mathcal{A}) . We say

$$f : X \rightarrow Y \text{ is measurable} \iff f \text{ is measurable with respect to } (X, \mathcal{A}), (\mathbb{F}, \text{Bor}(\mathbb{F})).$$

By Assignment 1, we see that

$$f : X \rightarrow Y \text{ is measurable} \iff \text{for all open } B, f^{-1}(B) \in \mathcal{A},$$

since $\text{Bor}(\mathbb{F})$ is generated by open subsets of \mathbb{F} . In case $\mathbb{F} = \mathbb{R}$, we can replace B with open interval, since every open subset of \mathbb{R} is a countable union of open intervals.

Recall the following trick for analysis. Let $a < b$ in \mathbb{R} . Then

$$\begin{aligned} (a, b] &= \bigcap_{n=1}^{\infty} \left(a, b + \frac{1}{n} \right) \\ (a, b) &= \bigcup_{n=1}^{\infty} \left(a, b - \frac{1}{n} \right] \\ [a, b] &= \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n} \right). \\ (a, \infty) &= \bigcup_{n=1}^{\infty} (a, a + n) \\ (a, b] &= (-\infty, b] \cap (a, \infty) \\ &\vdots \end{aligned}$$

That is, all interval types independently generate $\text{Bor}(\mathbb{R})$.

Proposition 2.1.

Let (X, \mathcal{A}) be a measurable space and let $f : X \rightarrow \mathbb{R}$. The following are equivalent.

- (a) f is measurable.
 - (b) For all $\alpha \in \mathbb{R}$, $f^{-1}((\alpha, \infty)) \in \mathcal{A}$.
 - (c) For all $\alpha \in \mathbb{R}$, $f^{-1}([\alpha, \infty)) \in \mathcal{A}$.
 - (d) For all $\alpha \in \mathbb{R}$, $f^{-1}((-\infty, \alpha)) \in \mathcal{A}$.
 - (e) For all $\alpha \in \mathbb{R}$, $f^{-1}((-\infty, \alpha]) \in \mathcal{A}$.
-

Proposition 2.2.

Let (X, \mathcal{A}) be a measurable space and let $f: X \rightarrow \mathbb{C}$. The following are equivalent. Then

$$f \text{ is measurable} \iff \operatorname{Re} \circ f \text{ and } \operatorname{Im} \circ f \text{ are measurable.}$$

Proof Sketch. (\Leftarrow) Every open $U \subseteq \mathbb{C}$ can be written as a countable union of open rectangles $(a, b) \times (c, d)$. Then

$$f^{-1}((a, b) \times (c, d)) = (\operatorname{Re} \circ f)^{-1}((a, b)) \cap (\operatorname{Im} \circ f)^{-1}((c, d)).$$

(\Rightarrow) Note that

$$(\operatorname{Re} \circ f)^{-1}((a, b)) = f^{-1}(V)$$

where

$$V = \{x + iy : a < x < b\}.$$

Similarly,

$$(\operatorname{Im} \circ f)^{-1}((c, d)) = f^{-1}(H)$$

where

$$H = \{x + iy : c < y < d\}.$$

QED

Proposition 2.3.

Let (X, τ) be a topological space. If $f: X \rightarrow \mathbb{F}$ is continuous, then f is measurable.

Proof. It suffices to check that $f^{-1}(U) \in \operatorname{Bor}(X)$ for all open $U \subseteq \mathbb{F}$, which is guaranteed by the continuity of f .

QED

Proposition 2.4.

Let (X, \mathcal{A}) be a measurable space and let $f, g: X \rightarrow \mathbb{F}$ be measurable.

- (a) For any $\lambda \in \mathbb{F}$, $\lambda f + g$ is measurable.
- (b) fg is measurable.
- (c) If $g(x) \neq 0$ for all $x \in X$, then $\frac{1}{g}$ is measurable.

Proof. By considering Proposition 2.2, we assume $\mathbb{F} = \mathbb{R}$.

- (a) Suppose $\lambda > 0$. Then given $\alpha \in \mathbb{R}$,

$$(\lambda f)^{-1}((\alpha, \infty)) = \{x \in X : \lambda f(x) > \alpha\} = \left\{x \in X : f(x) > \frac{\alpha}{\lambda}\right\} = f^{-1}\left(\left(\frac{\alpha}{\lambda}, \infty\right)\right),$$

which is measurable.

In case $\lambda < 0$,

$$(\lambda f)^{-1}((\alpha, \infty)) = f^{-1}\left(\left(-\infty, \frac{\alpha}{\lambda}\right)\right)$$

is measurable.

When $\lambda = 0$, λf is the constant 0 function, which is trivially measurable.

Let $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} (f + g)^{-1}((\alpha, \infty)) &= \{x \in X : f(x) + g(x) > \alpha\} = \{x \in X : f(x) > \alpha - g(x)\} \\ &= \bigcup_{q \in \mathbb{Q}} (\{x \in X : f(x) > q\} \cap \{x \in X : g(x) > \alpha - q\}) = \bigcup_{q \in \mathbb{Q}} (f^{-1}((q, \infty)) \cap g^{-1}(\alpha - q, \infty)), \end{aligned}$$

which is measurable as a countable union of measurable sets.

(b) Note

$$(f + g)^2 = f^2 + 2fg + g^2.$$

Hence it suffices to show that f^2 is measurable. Let $\alpha \in \mathbb{R}$.

Suppose $\alpha \geq 0$. Then

$$\begin{aligned} f^{-1}((\alpha, \infty)) &= \{x \in X : f(x)^2 > \alpha\} = \{x \in X : f(x) > \sqrt{\alpha}\} \cup \{x \in X : f(x) < -\sqrt{\alpha}\} \\ &= f^{-1}((\sqrt{\alpha}, \infty)) \cup f^{-1}((-\infty, -\sqrt{\alpha})) \end{aligned}$$

is a union of measurable of measurable sets.

If $\alpha < 0$, then

$$(f^2)^{-1}((\alpha, \infty)) = \{x \in X : f(x)^2 > \alpha\} = X$$

is measurable.

(c) Let $\alpha \in \mathbb{R}$. Suppose $\alpha > 0$. Then

$$\begin{aligned} \left(\frac{1}{g}\right)^{-1}((-\infty, \alpha)) &= \left\{x \in X : \frac{1}{g(x)} < \alpha\right\} = \left\{x \in X : g(x) > \frac{1}{\alpha}\right\} \cup \{x \in X : g(x) < 0\} \\ &= g^{-1}\left(\left(\frac{1}{\alpha}, \infty\right)\right) \cup g^{-1}((-\infty, 0)). \end{aligned}$$

The cases where $\alpha < 0$, $\alpha = 0$ are similar.

QED

Notation 2.2. $\overline{\mathbb{R}}$

We write $\overline{\mathbb{R}}$ to denote

$$\overline{\mathbb{R}} = [-\infty, \infty].$$

Def'n 2.3. **Borel σ -algebra** of Subsets of $\overline{\mathbb{R}}$

We define the **Borel σ -algebra** of subsets of $\overline{\mathbb{R}}$, denoted as $\text{Bor}(\overline{\mathbb{R}})$, by

$$\text{Bor}(\overline{\mathbb{R}}) = \{A \subseteq \overline{\mathbb{R}} : A \cap \mathbb{R} \in \text{Bor}(\mathbb{R})\}.$$

To show that $\text{Bor}(\overline{\mathbb{R}})$ is *really Borel*, we consider the following metric on $\overline{\mathbb{R}}$. Define

$$\begin{aligned} d : \overline{\mathbb{R}}^2 &\rightarrow [0, \infty) \\ (x, y) &\mapsto |\arctan(x) - \arctan(y)|, \end{aligned}$$

where $\arctan(-\infty) = -\frac{\pi}{2}$, $\arctan(\infty) = \frac{\pi}{2}$.

Exercise 2.1.

Show that $\text{Bor}(\overline{\mathbb{R}})$ is generated by the open subsets of $(\overline{\mathbb{R}}, d)$.

$\text{Bor}(\overline{\mathbb{R}})$ is (independently) generated by intervals of the form $(\alpha, \infty]$, $[-\infty, \alpha)$.

Proposition 2.5.

Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of measurable functions from X to \mathbb{R} .

(a) $\sup_{n \in \mathbb{N}} f_n$ is measurable.

- (b) $\inf_{n \in \mathbb{N}} f_n$ is measurable.
- (c) $\limsup_{n \in \mathbb{N}} f_n$ is measurable.
- (d) $\liminf_{n \in \mathbb{N}} f_n$ is measurable.

Proof.

- (a) Note that, given $\alpha \in \mathbb{R}$,

$$\left(\sup_{n \in \mathbb{N}} f_n \right)^{-1}((\alpha, \infty]) = \left\{ x \in X : \sup_{n \in \mathbb{N}} f_n(x) > \alpha \right\} = \bigcup_{n \in \mathbb{N}} \{x \in X : f_n(x) > \alpha\} = \bigcup_{n \in \mathbb{N}} f_n^{-1}((\alpha, \infty)).$$

- (b) It suffices to note that $\inf_{n \in \mathbb{N}} f_n = -\sup_{n \in \mathbb{N}} (-f_n)$.

- (c) Recall that

$$\limsup_{n \in \mathbb{N}} f_n = \lim_{n \rightarrow \infty} \sup_{k \geq n} f_k = \inf_{n \in \mathbb{N}} \sup_{k \geq n} f_k.$$

Hence by (a), (b), $\limsup_{n \in \mathbb{N}} f_n$ is measurable.

- (d) Similar to (c),

$$\liminf_{n \in \mathbb{N}} f_n = \sup_{n \in \mathbb{N}} \inf_{k \geq n} f_k.$$

Hence $\liminf_{n \in \mathbb{N}} f_n$ is measurable.

QED

Corollary 2.5.1.

Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions from X to \mathbb{R} . If $f_n \rightarrow x$ pointwise, then f is measurable.

Proof. Note that

$$f_n \rightarrow x \iff \liminf_{n \in \mathbb{N}} f_n = \limsup_{n \in \mathbb{N}} f_n = \lim_{n \rightarrow \infty} f_n.$$

QED

Let (X, \mathcal{A}) be a measurable space. Then given measurable $f: X \rightarrow \mathbb{F}$ and continuous $g: \mathbb{F} \rightarrow \mathbb{F}$, $g \circ f$ is measurable, as for any open $U \subseteq \mathbb{F}$,

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)),$$

where $g^{-1}(U)$ is open.

In particular, this gives alternative proofs that $f^2, \frac{1}{f}, \operatorname{Re} f, \operatorname{Im} f$ are measurable. Moreover, $|f|$ is measurable.

Def'n 2.4. μ -almost Everywhere Predicate

Let (X, \mathcal{A}, μ) be a measure space and let P be a predicate on X . We say P is true μ -almost everywhere (or μ -ae) if there exists $N \in \mathcal{A}$ with $\mu(N) = 0$ such that $P(x)$ is true for all $x \in X \setminus N$.

Note that the definition of μ -almost everywhere does not say that

$$N = \{x \in X : P(x) \text{ is false}\}$$

is measurable. But in case μ is complete, N is measurable with $\mu(N) = 0$.

Proposition 2.6.

Let (X, \mathcal{A}, μ) be a complete measure space and let $f: X \rightarrow \mathbb{F}$ be measurable. Suppose that $g: X \rightarrow \mathbb{F}$ is such that $f = g$ μ -ae. Then g is measurable.

Proof. Let $N \in \mathcal{A}$ be such that $\mu(N) = 0$ with $f = g$ on $X \setminus N$. Then given any measurable $U \subseteq \mathbb{F}$,

$$g^{-1}(U) = (g^{-1}(U) \cap N) \cup (g^{-1}(U) \setminus N).$$

Note that $g^{-1}(U) \cap N \subseteq N$ so has measure 0, which means $g^{-1}(U) \cap N \in \mathcal{A}$ by the completeness of μ . Moreover, $f = g$ on $X \setminus N$ so that $g^{-1}(U) \setminus N = f^{-1}(U) \setminus N$, which is measurable. Thus $g^{-1}(U)$ is measurable, as required.

QED

2. Simple Approximation

Def'n 2.5. **Characteristic Function** of a Subset

Let X be a set and let $A \subseteq X$. The *characteristic function* of A , denoted as χ_A , is defined as

$$\chi_A : X \rightarrow \mathbb{R}$$

$$x \mapsto \begin{cases} 1 & \text{if } x \in A. \\ 0 & \text{if } x \notin A \end{cases}$$

Note that, given $A \subseteq X$,

$$\chi_A \text{ is measurable} \iff A \text{ is measurable.}$$

Def'n 2.6. **Simple Function**

Let (X, \mathcal{A}) be a measurable space. We say $\varphi : X \rightarrow \mathbb{F}$ is *simple* if

$$\varphi = \sum_{k=1}^n a_k \chi_{A_k}$$

where $a_1, \dots, a_n \in \mathbb{F}$ and $A_1, \dots, A_n \in \mathcal{A}$ are pairwise disjoint.

Let (X, \mathcal{A}) be a measurable space and let $\varphi : X \rightarrow \mathbb{F}$. Then

$$\varphi \text{ is simple} \iff \varphi \text{ is measurable and } \varphi(X) \text{ is finite.}$$

To see the reverse direction, suppose φ is measurable and $\varphi(X)$ is finite, say

$$\varphi(X) = \{a_k\}_{k=1}^n.$$

Then each $A_k = \varphi^{-1}(\{a_k\})$ is measurable and $\varphi = \sum_{k=1}^n a_k \chi_{A_k}$.

The goal of this subsection is to show

$$f : X \rightarrow \mathbb{R} \text{ is measurable} \iff f \text{ is a pointwise limit of simple functions.}$$

Proposition 2.7.

Let (X, \mathcal{A}) be a measurable space and let $f : X \rightarrow \mathbb{R}$ be measurable and bounded. Then for all $\varepsilon > 0$, there are simple $\varphi_\varepsilon, \psi_\varepsilon : X \rightarrow \mathbb{R}$ such that

$$(a) \quad \varphi_\varepsilon \leq f \leq \psi_\varepsilon; \text{ and}$$

$$(b) \quad 0 \leq \psi_\varepsilon - \varphi_\varepsilon < \varepsilon.$$

Proof. Let $\varepsilon > 0$. Say $f(X) \subseteq [a, b]$. Let y_0, \dots, y_n be given such that

$$a = y_0 < y_1 < \dots < y_n = b,$$

where each $y_k - y_{k-1} < \varepsilon$. Let $I_k = [y_{k-1}, y_k)$. Then each $A_k = f^{-1}(I_k)$ is measurable. Define

$$\varphi = \sum_{k=1}^n y_{k-1} \chi_{A_k}, \quad \psi = \sum_{k=1}^n y_k \chi_{A_k}.$$

Then for any $x \in X$, we have $x \in I_k$ for some k , so that $\varphi(x) = y_{k-1} \leq f(x) \leq y_k = \psi(x)$.

Moreover,

$$0 < \psi(x) - \varphi(x) = y_k - y_{k-1} < \varepsilon.$$

QED

Theorem 2.8. Simple Approximation

Let (X, \mathcal{A}) be a measure space and let $f: X \rightarrow \mathbb{R}$. Then

$$f \text{ is measurable} \iff \text{there are simple } \varphi_1, \varphi_2, \dots : X \rightarrow \mathbb{R} \text{ with } \varphi_n \rightarrow f \text{ pointwise and } |\varphi_n| \leq f \text{ for all } n \in \mathbb{N}.$$

Proof. (\Leftarrow) Recall that pointwise limit of measurable functions is measurable, where each φ_n is measurable.

(\Rightarrow) We split into few cases.

Case 1. Suppose $f \geq 0$.

Let

$$A_n = \{x \in X : f(x) \leq n\}.$$

Note that

$$\mathcal{A}' = \{B \cap A_n : B \in \mathcal{A}\}$$

is a σ -algebra of subsets of A_n . Then (A_n, \mathcal{A}') is a measurable space and $f|_{A_n}$ is measurable, since

$$(f|_{A_n})^{-1}(U) = f^{-1}(U) \cap A_n \in \mathcal{A}'$$

for all measurable $U \subseteq \mathbb{R}$. Moreover, by definition $f|_{A_n}$ is bounded.

Hence by Proposition 2.7, we can find simple $\varphi_m, \psi_m : A_n \rightarrow \mathbb{R}$, $m \in \mathbb{N}$, such that

$$0 \leq \varphi_m \leq f \leq \psi_m$$

and

$$0 \leq \psi_m - \varphi_m < \frac{1}{m}$$

for all $m \in \mathbb{N}$ on A_n .

Extend $\varphi_m(x) = n$ for all $x \in X \setminus A_n$, so that $\varphi_m \leq f$ on X .

Now fix $x \in X$. Then $x \in A_N$ for some N , and so

$$0 \leq f(x) - \varphi_N(x) \leq \psi_N(x) - \varphi_N(x) < \frac{1}{N}.$$

This means given any $\varepsilon > 0$ we can take $N' > N$ so that $\frac{1}{N'} < \varepsilon$, which means for all $m \geq N'$,

$$0 \leq f(x) - \varphi_m(x) < \frac{1}{N'} < \varepsilon.$$

Thus $\varphi_m \rightarrow f$ pointwise.

(End of Case 1)

Case 2. Consider the general case on f . That is, we only assume that f is measurable.

Let

$$A = \{x \in X : f(x) \geq 0\} \in \mathcal{A}$$

$$B = \{x \in X : f(x) < 0\} \in \mathcal{A}$$

and let $g = f\chi_A$, $h = -f\chi_B$, so that both $g, h \geq 0$. By Case 1, there exist $(\varphi_n)_{n=1}^\infty, (\psi_n)_{n=1}^\infty$ such that $\varphi_n \nearrow g$ and $\psi_n \nearrow h$ pointwise as $n \rightarrow \infty$. Then $f = g - h$ so that $\varphi_n - \psi_n \rightarrow g - h = f$ pointwise. Moreover,

$$|\varphi_n - \psi_n| \leq |\varphi_n| + |\psi_n| = \varphi_n + \psi_n \leq g + h = |f|.$$

(End of Case 2)

Note that in the proof, we know that, given a fixed $n \in \mathbb{N}$, we have

$$0 \leq f - \varphi_m \leq \frac{1}{m}$$

on A_n . That is,

$$0 \leq f(x) - \varphi_m(x) \leq \frac{1}{m}, \quad \forall x \in A_n,$$

so that $\varphi_m \rightarrow f$ uniformly as $m \rightarrow \infty$ on A_n .

Suppose that $f \geq 0$ is measurable and that

$$0 \leq \varphi_n \leq f, \quad \forall n \in \mathbb{N}$$

with $\varphi_n \rightarrow f$ pointwise. Then by taking $\psi_n = \max \{\varphi_1, \dots, \varphi_n\}$, φ_n is still simple. Then

$$0 \leq \psi_n \leq f, \quad \forall n \in \mathbb{N}$$

as well, so that $\psi_n \nearrow f$ pointwise as $n \rightarrow \infty$.

3. Two Theorems

We are going to prove two useful theorems in measure theory in this subsection.

Lemma 2.9.

Let (X, \mathcal{A}, μ) be a finite measure space and let $(f_n)_{n=1}^\infty \in (\mathbb{R}^X)^\mathbb{N}$ be a sequence of measurable functions such that $f_n \rightarrow f$ pointwise for some measurable $f: X \rightarrow \mathbb{R}$. Then for every $\alpha, \beta > 0$, there exist $B \in \mathcal{A}, N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \alpha, \quad \forall x \in B, n \geq N$$

and

$$\mu(X \setminus B) < \beta.$$

Proof Sketch. Let

$$A_n = \{x \in X : \forall k \geq n [f_k(x) - f(x) < \alpha]\}, \quad \forall n \in \mathbb{N}.$$

Then

$$A_n = \bigcap_{k \geq n} |f_k - f|^{-1}((-\infty, \alpha)),$$

which is measurable. Since $f_n \rightarrow f$ pointwise, we have

$$X = \bigcup_{n=1}^\infty A_n.$$

We also have an increasing chain

$$A_1 \subseteq A_2 \subseteq \dots,$$

so that

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^\infty A_n\right) = \mu(X) < \infty$$

by the continuity from below. Hence we may find $N \in \mathbb{N}$ such that

$$\mu(X) - \mu(A_n) < \beta, \quad \forall n \geq N.$$

Since $\mu(X) < \infty$, each $\mu(A_n) < \infty$ as well, so that

$$\mu(X \setminus A_n) < \beta, \quad \forall n \geq N.$$

By taking $B = A_N$, we are done.

QED

Theorem 2.10. Egoroff

Let (X, \mathcal{A}, μ) be a finite measure space and let $(f_n)_{n=1}^\infty \in (\mathbb{R}^X)^\mathbb{N}$ be a sequence of measurable functions such that $f_n \rightarrow f$ pointwise for some measurable $f: X \rightarrow \mathbb{R}$. Then for all $\varepsilon > 0$ there exists $A \in \mathcal{A}$ such that

- (a) $f_n \rightarrow f$ uniformly on A ; and
- (b) $\mu(X \setminus A) < \varepsilon$.

Proof. Let $\varepsilon > 0$ be given. For all $n \in \mathbb{N}$, we may find $A_n \in \mathcal{A}$ and $N_n \in \mathbb{N}$ such that

$$\forall x \in A_n, k \geq N_n \left[|f_k(x) - f(x)| < \frac{1}{n} \right]$$

and

$$\mu(X \setminus A_n) < \frac{\varepsilon}{2^n}.$$

Let

$$A = \bigcap_{n=1}^\infty A_n.$$

Given any $\varepsilon' > 0$, by taking $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon'$, we have, for all $k \geq N_n$ and $x \in A$,

$$|f_k(x) - f(x)| < \frac{1}{n} < \varepsilon'.$$

Hence $f_k \rightarrow f$ uniformly on A . Finally,

$$\mu(X \setminus A) = \mu\left(\bigcup_{n=1}^\infty (X \setminus A_n)\right) \leq \sum_{n=1}^\infty \mu(X \setminus A_n) < \sum_{n=1}^\infty \frac{\varepsilon}{2^n} = \varepsilon.$$

QED

Let m be the Lebesgue measure on \mathbb{R} and let $A \subseteq \mathbb{R}$ with $m(A) < \infty$. Let $(f_n)_{n=1}^\infty$ be a sequence of measurable functions from A to \mathbb{R} that converges to $f: A \rightarrow \mathbb{R}$. Then by Egoroff's theorem, for every $\varepsilon > 0$, there is $B \subseteq A$ such that

$$f_n \rightarrow f \text{ uniformly on } B$$

and

$$m(A \setminus B) < \frac{\varepsilon}{2}.$$

Then we can find a closed subset $C \subseteq B$ with

$$m(B \setminus C) < \frac{\varepsilon}{2}$$

by the regularity of Lebesgue measure. Then

$$f_n \rightarrow f \text{ uniformly on } C$$

and

$$m(A \setminus C) = m(A \setminus B) + m(B \setminus C) < \varepsilon.$$

Hence for the Lebesgue measure (in fact, any Lebesgue-Stieltjes measure), we can assume that $f_n \rightarrow f$ uniformly on a closed set with arbitrarily small difference.

Lemma 2.11.

Let $A \subseteq \mathbb{R}$ be Lebesgue measurable and let $\varphi: A \rightarrow \mathbb{R}$ be Lebesgue-simple. Then for all $\varepsilon > 0$, there exists closed $C \subseteq \mathbb{R}$ and a continuous $g: \mathbb{R} \rightarrow \mathbb{R}$ such that

- (a) $C \subseteq A$;
- (b) $\varphi = g$ on C ; and
- (c) $m(A \setminus C) < \varepsilon$.

Proof. Write

$$\varphi = \sum_{i=1}^n a_i \chi_{A_i},$$

where each $a_i \neq 0$ and $A_i = \varphi^{-1}(\{a_i\})$. Let $A_0 = \varphi^{-1}(\{0\})$. We also insist that $a_i \neq a_j$ for $i \neq j$. Then

$$A = \bigcup_{i=0}^n A_i.$$

Let $\varepsilon > 0$ be given. For each i , let C_i be a closed such that $C_i \subseteq A_i$ and

$$m(A_i \setminus C_i) < \frac{\varepsilon}{n+1}$$

by regularity of Lebesgue measure. Let

$$C = \bigcup_{i=0}^n C_i,$$

which is closed. Since φ is continuous on each C_i and $C_i \cap C_j = \emptyset$, φ is continuous on C . Then there is continuous $g : \mathbb{R} \rightarrow \mathbb{R}$ that extends $\varphi : C \rightarrow \mathbb{R}$. Finally,

$$m(A \setminus C) = m\left(\bigcup_{i=0}^n A_i \setminus C_i\right) = \sum_{i=0}^n m(A_i \setminus C_i) < \varepsilon.$$

QED

Theorem 2.12. Lusin

Let $f : A \rightarrow \mathbb{R}$ be Lebesgue measurable. Then for all $\varepsilon > 0$, there exists continuous $g : \mathbb{R} \rightarrow \mathbb{R}$ and closed $C \subseteq \mathbb{R}$ such that

- (a) $C \subseteq A$;
- (b) $f = g$ on C ; and
- (c) $m(A \setminus C) < \varepsilon$.

Proof. We split the proof into two cases. Let $\varepsilon > 0$ be given.

Case 1. Suppose $m(A) < \infty$.

Let $(\varphi_n)_{n=1}^\infty$ be a sequence of simple functions such that $\varphi_n \rightarrow f$ pointwise by simple approximation. For each $n \in \mathbb{N}$, let $C_n \subseteq \mathbb{R}$ be closed and $g_n : \mathbb{R} \rightarrow \mathbb{R}$ be continuous such that $\varphi_n = g_n$ on C_n and

$$m(A \setminus C_n) < \frac{\varepsilon}{2^{n+1}}.$$

By Egoroff, let C_0 be the closed set such that

$$\varphi_n \rightarrow f \text{ uniformly on } C_0$$

and

$$m(A \setminus C_0) < \frac{\varepsilon}{2}.$$

Let

$$C = \bigcap_{n=0}^\infty C_n.$$

Then,

$$g_n = \varphi_n \rightarrow f \text{ uniformly on } C.$$

In particular, f is continuous on C . This means we can extend $f|_C$ to continuous $g : \mathbb{R} \rightarrow \mathbb{R}$. Finally,

$$m(A \setminus C) = m\left(A \setminus \bigcap_{n=0}^{\infty} C_n\right) = m\left(\bigcup_{n=0}^{\infty} (A \setminus C_n)\right) \leq m(A \setminus C_0) + \sum_{n=1}^{\infty} m(A \setminus C_n) < \varepsilon.$$

(End of Case 1)

Case 2. Suppose $m(A) < \infty$.

This is left as an exercise.

(End of Case 2)

QED

III. Integration

1. Nonnegative Measurable Functions

Def'n 3.1. **Integral** of a Nonnegative Simple Function

Let (X, \mathcal{A}, μ) be a measure space and let

$$\varphi = \sum_{i=1}^n a_i \chi_{A_i} : X \rightarrow [0, \infty]$$

be simple. We define the *integral* of φ , denoted as $\int \varphi d\mu$, by

$$\int \varphi d\mu = \sum_{i=1}^n a_i \mu(A_i).^1$$

¹For this, we use the convention $0\infty = \infty 0 = 0$.

Proposition 3.1.

Let $\varphi : X \rightarrow [0, \infty]$ be simple. Then $\int \varphi d\mu$ is well-defined.

Proof Sketch. Say

$$\varphi = \sum_{i=1}^n a_i \chi_{E_i} = \sum_{j=1}^m b_j \chi_{F_j}.$$

Suppose that $\varphi(X) = \{c_1, \dots, c_p\}$ and let

$$A_k = \varphi^{-1}(\{c_k\}), \quad \forall k \in \{1, \dots, p\}.$$

Then

$$\sum_{i=1}^n a_i \mu(E_i) = \sum_{k=1}^p c_k \sum_{i: a_i=c_k} \mu(E_i) = \sum_{k=1}^p c_k \mu\left(\bigcup_{i: a_i=c_k} E_i\right) = \sum_{k=1}^p c_k \mu(A_k).$$

By symmetry, $\sum_{j=1}^m b_j \chi_{F_j} = \sum_{k=1}^p c_k \mu(A_k)$. Thus $\int \varphi d\mu$ is well-defined.

QED

Proposition 3.2.

Let $\varphi, \psi : X \rightarrow [0, \infty]$ be simple.

(a) If $\alpha \geq 0$, then

$$\int \alpha \varphi d\mu = \alpha \int \varphi d\mu.$$

(b)

$$\int \varphi + \psi d\mu = \int \varphi d\mu + \int \psi d\mu.$$

(c) $\varphi \leq \psi \implies \int \varphi d\mu \leq \int \psi d\mu$.

Proof. Write

$$\varphi = \sum_{i=1}^n a_i \chi_{E_i}, \quad \psi = \sum_{j=1}^m b_j \chi_{F_j}$$

and let $a_0 = b_0 = 0$, with $E_0 = X \setminus \bigcup_{i=1}^n E_i$, $F_0 = X \setminus \bigcup_{j=1}^m F_j$. This means

$$\varphi = \sum_{i=0}^n a_i \chi_{E_i}, \psi = \sum_{j=0}^m b_j \chi_{F_j}$$

as well.

(a) Note that

$$\int \alpha \varphi d\mu = \sum_{i=1}^n \alpha a_i \mu(A_i) = \alpha \sum_{i=1}^n a_i \mu(A_i) = \alpha \int \varphi d\mu.$$

(b) For all $i \in \{0, \dots, n\}$, $j \in \{0, \dots, m\}$, let

$$A_{i,j} = E_i \cap F_j.$$

Then it follows that

$$\varphi = \sum_{i=0}^n \sum_{j=0}^m a_i \chi_{A_{i,j}}$$

and

$$\psi = \sum_{j=0}^m \sum_{i=0}^n b_j \chi_{A_{i,j}}.$$

Thus

$$\int \varphi + \psi d\mu = \sum_{i=0}^n \sum_{j=0}^m (a_i + b_j) \mu(A_{i,j}) = \sum_{i=0}^n \sum_{j=0}^m a_i \mu(A_{i,j}) + \sum_{j=0}^m \sum_{i=0}^n b_j \mu(A_{i,j}) = \int \varphi d\mu + \int \psi d\mu.$$

(c) Given $i \in \{0, \dots, n\}$, $j \in \{0, \dots, m\}$, if $A_{i,j} \neq \emptyset$, then $a_i \leq b_j$. Otherwise, $\mu(A_{i,j}) = 0$. This means

$$a_i \mu(A_{i,j}) \leq b_j \mu(A_{i,j}), \quad \forall i \in \{0, \dots, n\}, j \in \{0, \dots, m\},$$

so that

$$\int \varphi d\mu = \sum_{i=0}^n \sum_{j=0}^m a_i \mu(A_{i,j}) \leq \sum_{j=0}^m \sum_{i=0}^n b_j \mu(A_{i,j}) = \int \psi d\mu.$$

QED

Def'n 3.2. Integral of a Nonnegative Simple Function over a Measurable Subset

Let $\varphi : X \rightarrow [0, \infty]$ be simple and let $A \in \mathcal{A}$. We define the *integral* of φ over A , denoted as $\int_A \varphi d\mu$, by

$$\int_A \varphi d\mu = \int \varphi \chi_A d\mu.$$

Proposition 3.3.

Let $\varphi : X \rightarrow [0, \infty]$ be simple. Define $\nu : \mathcal{A} \rightarrow [0, \infty]$ by

$$\nu(A) = \int_A \varphi d\mu.$$

Then ν is a measure on (X, \mathcal{A}) .

Proof. Write

$$\varphi = \sum_{i=1}^n a_i \chi_{E_i}.$$

We have

$$v(\emptyset) = \int \chi_{\emptyset} \varphi d\mu = 0.$$

Let $\{A_m\}_{m=1}^{\infty} \subseteq \mathcal{A}$ be a collection of disjoint sets and $A = \bigcup_{m=1}^{\infty} A_m$. Then

$$\begin{aligned} v(A) &= \int_A \varphi d\mu = \int \varphi \chi_A d\mu = \int \sum_{i=1}^n a_i \chi_{E_i} \chi_A d\mu = \int \sum_{i=1}^n a_i \chi_{E_i \cap A} d\mu = \sum_{i=1}^n a_i \mu(E_i \cap A) = \sum_{i=1}^n a_i \mu\left(\bigcup_{m=1}^{\infty} (E_i \cap A_m)\right) \\ &= \sum_{i=1}^n a_i \sum_{m=1}^{\infty} \mu(E_i \cap A_m) = \sum_{m=1}^{\infty} \sum_{i=1}^n a_i \mu(E_i \cap A_m) = \sum_{m=1}^{\infty} \int_{A_m} \varphi d\mu = \sum_{m=1}^{\infty} v(A_m). \end{aligned}$$

QED

Notation 3.3. $L^+(X, \mathcal{A}, \mu)$

We write $L^+(X, \mathcal{A}, \mu)$, or simply L^+ when (X, \mathcal{A}, μ) is understood, to mean

$$L^+(X, \mathcal{A}, \mu) = \{f : X \rightarrow [0, \infty] : f \text{ is measurable}\}.$$

Def'n 3.4. **Integral** of a L^+ -function

Let $f \in L^+$. We define the *integral* of f , denoted as $\int f d\mu$, by

$$\int f d\mu = \sup \left\{ \int \varphi d\mu : \varphi : [0, \infty] \rightarrow X, \varphi \leq f, \varphi \text{ is simple} \right\}.$$

If $A \in \mathcal{A}$, we define the *integral* of f over A , denoted as $\int_A f d\mu$, by

$$\int_A f d\mu = \int f \chi_A d\mu.$$

Proposition 3.4.

Let $f, g \in L^+$.

(a) If $\alpha \geq 0$, then

$$\int \alpha f d\mu = \alpha \int f d\mu.$$

(b) If $f \leq g$, then

$$\int f d\mu \leq \int g d\mu.$$

Proof.

(a) This is trivial when $\alpha = 0$. For $\alpha > 0$,

$$\begin{aligned} \{\varphi : X \rightarrow [0, \infty] : \varphi \leq \alpha f, \varphi \text{ is simple}\} &= \left\{ \varphi : X \rightarrow [0, \infty] : \frac{1}{\alpha} \varphi \leq f, \varphi \text{ is simple} \right\} \\ &= \{\alpha \psi : \psi : X \rightarrow [0, \infty], \psi \leq f, \psi \text{ is simple}\}. \end{aligned}$$

By taking sup, we have the desired equality.

(b) It suffices to note

$$\{\varphi : X \rightarrow [0, \infty] : \varphi \leq f, \varphi \text{ is simple}\} \subseteq \{\psi : X \rightarrow [0, \infty] : \psi \leq g, \psi \text{ is simple}\}.$$

QED

We are leaving (a one-liner!) proof of $\int f + g d\mu = \int f d\mu + \int g d\mu$ for later.

2. Nonnegative Limit Theorems

Lemma 3.5.

Let $\varphi : X \rightarrow [0, \infty]$ be simple and let $(A_n)_{n=1}^\infty \in \mathcal{A}^\mathbb{N}$ be an ascending chain with $X = \bigcup_{n=1}^\infty A_n$. Then

$$\lim_{n \rightarrow \infty} \int_{A_n} \varphi d\mu = \int \varphi d\mu.$$

Proof. Recall that $\nu : \mathcal{A} \rightarrow [0, \infty]$ by

$$\nu(A) = \int_A \varphi d\mu, \quad \forall A \in \mathcal{A}$$

is a measure. Hence by the continuity from below,

$$\lim_{n \rightarrow \infty} \int_{A_n} \varphi d\mu = \lim_{n \rightarrow \infty} \nu(A_n) = \nu\left(\bigcup_{n=1}^\infty A_n\right) = \nu(X) = \int \varphi d\mu.$$

QED

Theorem 3.6. Monotone Convergence Theorem (MCT)

Let $(f_n)_{n=1}^\infty \in L^+\mathbb{N}$ be an increasing sequence and define $f \in L^+$ by

$$f(x) = \lim_{n \rightarrow \infty} f_n(x), \quad \forall x \in X.$$

Then

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu.$$

Proof. For every $x \in X$, $(f_n(x))_{n=1}^\infty$ is an increasing sequence. Hence by the MCT for sequences, $\lim_{n \rightarrow \infty} f_n(x)$ converges in $[0, \infty]$. Define

$$f(x) = \lim_{n \rightarrow \infty} f_n(x), \quad \forall x \in X.$$

In fact, MCT for sequences tells us that

$$f(x) = \sup_{n \in \mathbb{N}} f_n(x), \quad \forall x \in X,$$

so that

$$f_1 \leq f_2 \leq \cdots \leq f.$$

This means

$$\int f_1 d\mu \leq \int f_2 d\mu \leq \cdots \leq \int f d\mu$$

using monotonicity of integral, so that

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \sup_{n \in \mathbb{N}} \int f_n d\mu \leq \int f d\mu.$$

Let $\varphi : X \rightarrow [0, \infty]$ be a simple function with $\varphi \leq f$. Let $\varepsilon \in (0, 1)$ and let

$$A_n = \{x \in X : (1 - \varepsilon) \varphi(x) \leq f_n(x)\}, \quad \forall n \in \mathbb{N}.$$

Then

$$A_1 \subseteq A_2 \subseteq \cdots$$

and

$$X = \bigcup_{n=1}^{\infty} A_n,$$

since $f_n(x) \rightarrow f(x)$ means there must be $N \in \mathbb{N}$ such that $(1 - \varepsilon) \varphi(x) \leq f_n(x)$, as $(1 - \varepsilon) \varphi(x) < \varphi(x) \leq f(x)$. This means

$$(1 - \varepsilon) \int \varphi d\mu = \int (1 - \varepsilon) \varphi d\mu = \lim_{n \rightarrow \infty} \int_{A_n} (1 - \varepsilon) \varphi d\mu \leq \lim_{n \rightarrow \infty} \int_{A_n} f_n d\mu \leq \lim_{n \rightarrow \infty} \int f_n d\mu.$$

Since the choice of ε was arbitrary, we conclude

$$\int \varphi d\mu \leq \lim_{n \rightarrow \infty} \int f_n d\mu.$$

But $\int f d\mu$ is the supremum of such φ , so it follows that

$$\int f d\mu \leq \lim_{n \rightarrow \infty} \int f_n d\mu,$$

as required.

QED

Proposition 3.7.

Let $f, g \in L^+$. Then

$$\int f + g d\mu = \int f d\mu + \int g d\mu.$$

Proof. By simple approximation, we can find increasing sequence of simple functions $(\varphi_n)_{n=1}^{\infty}, (\psi_n)_{n=1}^{\infty}$ such that $\varphi_n \nearrow f, \psi_n \nearrow g$ pointwise. Thus by the MCT,

$$\int f + g d\mu = \lim_{n \rightarrow \infty} \int \varphi_n + \psi_n d\mu = \lim_{n \rightarrow \infty} \int \varphi_n d\mu + \int \psi_n d\mu = \int f d\mu + \int g d\mu.$$

QED

Proposition 3.8.

Let $(f_n)_{n=1}^{\infty} \in L^{+\mathbb{N}}$. Then

$$\int \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int f_n d\mu.$$

Proof. Note that $\left(\sum_{n=1}^k f_n\right)_{k=1}^{\infty} \in L^{+\mathbb{N}}$ is increasing, so that

$$\int \sum_{n=1}^{\infty} f_n d\mu = \int \lim_{k \rightarrow \infty} \sum_{n=1}^k f_n d\mu = \lim_{k \rightarrow \infty} \int \sum_{n=1}^k f_n d\mu = \lim_{k \rightarrow \infty} \int \sum_{n=1}^k f_n d\mu = \sum_{n=1}^{\infty} \int f_n d\mu.$$

QED

Proposition 3.9.

Let $f \in L^+$. Then

$$\nu : \mathcal{A} \rightarrow [0, \infty]$$

$$A \mapsto \int_A f d\mu$$

is a measure.

Proof. Clearly $\nu(\emptyset) = \int_{\emptyset} f d\mu = 0$.

Write $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ be a collection of disjoint sets and let $A = \bigcup_{n=1}^{\infty} A_n$. Then

$$\nu(A) = \int f \chi_A d\mu = \int \sum_{n=1}^{\infty} f \chi_{A_n} d\mu = \sum_{n=1}^{\infty} \int_{A_n} f d\mu = \sum_{n=1}^{\infty} \nu(A_n).$$

QED

Lemma 3.10.

Let $f \in L^+$. Then

$$\int f d\mu = 0 \iff f = 0 \mu\text{-ae.}$$

Proof. (\Leftarrow) Suppose $f = 0 \mu\text{-ae.}$ Let $\varphi : X \rightarrow [0, \infty]$ be simple with $\varphi \leq f$, say

$$\varphi = \sum_{i=1}^n a_i \chi_{A_i}.$$

Then $\varphi = 0$ ae. This means each $a_i > 0$ implies $\mu(A_i) = 0$. Thus

$$\int \varphi d\mu = 0$$

so that

$$\int f d\mu = 0.$$

(\Rightarrow) Suppose $\int f d\mu = 0$. Let

$$A = \{x \in X : f(x) > 0\}$$

and let

$$A_n = \left\{x \in X : f(x) \geq \frac{1}{n}\right\}, \quad \forall n \in \mathbb{N}.$$

Then

$$A_1 \subseteq A_2 \subseteq \dots$$

with

$$\bigcup_{n=1}^{\infty} A_n = A.$$

Therefore

$$\mu(A) = \lim_{n \rightarrow \infty} \mu(A_n)$$

and

$$0 = \int f d\mu \geq \int \frac{1}{n} \chi_{A_n} d\mu = \frac{1}{n} \mu(A_n),$$

so that each $\mu(A_n) = 0$. Thus $\mu(A) = 0$, as required.

QED

Proposition 3.11.

Let $f \in L^+$ and let $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$. Then

$$\int_{A \cup B} f d\mu = \int_A f d\mu + \int_B f d\mu.$$

Proof. Note that

$$\int_{A \cup B} f d\mu = \int f(\chi_A + \chi_B) d\mu = \int f \chi_A d\mu + \int f \chi_B d\mu = \int_A f d\mu + \int_B f d\mu.$$

QED

Proposition 3.12.

Let $f \in L^+$ and let $A \in \mathcal{A}$ with $\mu(A) = 0$. Then

$$\int_A f d\mu = 0.$$

Proof. Note that $f\chi_A = 0$ μ -ae.

QED

Proposition 3.13.

Let $(f_n)_{n=1}^{\infty} \in L^{+\mathbb{N}}$ be such that

$$f_n \leq f_{n+1} \text{ } \mu\text{-ae}, \quad \forall n \in \mathbb{N}$$

and let $f \in L^{+\mathbb{N}}$ be such that

$$\lim_{n \rightarrow \infty} f_n = f \text{ pointwise } \mu\text{-ae.}$$

Then

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu.$$

Proof. Let

$$A_n = \{x \in X : f_n(x) > f_{n+1}(x)\}$$

and let

$$A_0 = \left\{x \in X : \lim_{n \rightarrow \infty} f_n(x) \neq f(x)\right\}.$$

Then $\mu(A_n) = 0$ for all $n \in \mathbb{N} \cup \{0\}$. Let $A = \bigcup_{n=0}^{\infty} A_n$, so that $\mu(A) = 0$ as well. We have

$$f_n \chi_{X \setminus A} \leq f_{n+1} \chi_{X \setminus A}, \quad \forall n \in \mathbb{N}$$

and

$$f_n \chi_{X \setminus A} \rightarrow f \chi_{X \setminus A} \text{ pointwise.}$$

By the MCT,

$$\int_{X \setminus A} f_n d\mu \rightarrow \int_{X \setminus A} f d\mu.$$

The result then follows from Proposition 3.11 and 3.12.

QED

Theorem 3.14. Fatou's Lemma

Let $(f_n)_{n=1}^{\infty} \in L^{+\mathbb{N}}$. Then

$$\int \liminf_{n \rightarrow \infty} f_n d\mu \leq \liminf_{n \rightarrow \infty} \int f_n d\mu.$$

Proof. Let

$$g_n = \inf_{k \geq n} f_k.$$

Then $(g_n)_{n=1}^{\infty}$ is an increasing sequence in L^+ such that

$$\lim_{n \rightarrow \infty} g_n = \liminf_{n \rightarrow \infty} f_n$$

pointwise. By the monotone convergence theorem,

$$\int \liminf_{n \rightarrow \infty} f_n d\mu = \int \lim_{n \rightarrow \infty} g_n d\mu = \lim_{n \rightarrow \infty} \int g_n d\mu = \liminf_{n \rightarrow \infty} \int g_n d\mu \leq \liminf_{n \rightarrow \infty} \int f_n d\mu.$$

QED

Corollary 3.14.1.

Let $(f_n)_{n=1}^{\infty} \in L^{+\mathbb{N}}$ such that $f_n \rightarrow f$ pointwise for some $f \in L^+$. Then

$$\int f d\mu \leq \liminf_{n \rightarrow \infty} \int f_n d\mu.$$

3. General Integration

Def'n 3.5. **Integrable** Complex-valued Function

Let $f: X \rightarrow \mathbb{C}$ be measurable. We say f is *integrable* if

$$\int |f| d\mu < \infty.$$

In case $f: X \rightarrow \mathbb{R}$ is integrable, we consider the *positive part* f^+ and *negative part* f^- of f defined as

$$\begin{aligned} f^+ &= \max \{f, 0\}, \\ f^- &= -\min \{f, 0\}. \end{aligned}$$

Note that both f^+, f^- are nonnegative and we define the *integral* of f , denoted as $\int f d\mu$, by

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu.^1$$

Let $f: X \rightarrow \mathbb{C}$ be integrable. Then we define the *integral* of f , denoted as $\int f d\mu$, by

$$\int f d\mu = \int \operatorname{Re} \circ f d\mu + i \int \operatorname{Im} \circ f d\mu.^2$$

In case $f: X \rightarrow \mathbb{C}$ is measurable, we define

$$\|f\|_1 = \int |f| d\mu.$$

¹Note that $f = f^+ - f^-$ and $|f| = f^+ + f^-$. Hence $f^+, f^- \leq |f|$ so that both $\int f^+ d\mu, \int f^- d\mu$ are finite.

²Observe that $|\operatorname{Re} \circ f|, |\operatorname{Im} \circ f| \leq |f|$, so that $\operatorname{Re} \circ f, \operatorname{Im} \circ f$ are integrable.

Notation 3.6. $L^1(X, \mathcal{A}, \mu)$

We define

$$L^1(X, \mathcal{A}, \mu) = \{f: X \rightarrow \mathbb{C} : f \text{ is measurable and } \|f\|_1 < \infty\}.$$

We shall write L^1 when (X, \mathcal{A}, μ) are understood.

We state few results without proof.

Proposition 3.15. Linearity

Let $f, g \in L^1$ and $\alpha \in \mathbb{C}$. Then $\alpha f + g \in L^1$ with

$$\int \alpha f + g d\mu = \alpha \int f d\mu + \int g d\mu.$$

Proposition 3.16. Monotonicity

Let $f, g \in L^1$ be real-valued functions. If $f \leq g$, then

$$\int f d\mu \leq \int g d\mu.$$

Def'n 3.7. **Integral** over a Measurable Set

Let $f \in L^1$. For $A \in \mathcal{A}$, we define the *integral* of f over A , denoted as $\int_A f d\mu$, by

$$\int_A f d\mu = \int f \chi_A d\mu.$$

Proposition 3.17.

Let $f \in L^1$ and let $A, B \in \mathcal{A}$ be disjoint. Then

$$\int_{A \cup B} f d\mu = \int_A f d\mu + \int_B f d\mu.$$

The following proposition is surprisingly non-trivial.

Proposition 3.18.

Let $f \in L^1$. Then

$$\left| \int f d\mu \right| \leq \int |f| d\mu.$$

Proof. The case when f is real-valued is trivial:

$$\left| \int f d\mu \right| = \left| \int f^+ d\mu - \int f^- d\mu \right| \leq \int f^+ d\mu + \int f^- d\mu = \int f^+ + f^- d\mu = \int |f| d\mu.$$

Let

$$z = \int f d\mu.$$

Write

$$z = r e^{i\theta}$$

in polar form, so that $r = |z|$. Therefore,

$$\left| \int f d\mu \right| = r = e^{-i\theta} z = \int e^{-i\theta} f d\mu = \operatorname{Re} \int e^{-i\theta} f d\mu = \int \underbrace{\operatorname{Re} \circ e^{-i\theta} f}_{=g} d\mu \leq \int |g| d\mu \leq \int |f| d\mu.$$

QED

Theorem 3.19. Lebesgue Dominated Convergence Theorem (LDCT)

Let $(f_n)_{n=1}^\infty \in (L^1)^\mathbb{N}$ and let $g \in L^1$. If $f_n \rightarrow f$ pointwise for some $f: X \rightarrow \mathbb{C}$ and $|f_n| \leq g$ for all $n \in \mathbb{N}$, then $f \in L^1$ with

$$\int \lim_{n \rightarrow \infty} f_n d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu.$$

Proof. We are going to only prove the case where f, g, f_n are real-valued.

Since $|f| \leq g$ by taking limits as $n \rightarrow \infty$,

$$\int |f| d\mu \leq \int g d\mu < \infty.$$

Hence $f \in L^1$. Then

$$\int g + f d\mu \stackrel{\text{Fatou}}{\leq} \liminf_{n \rightarrow \infty} \int g + f_n d\mu = \int g d\mu + \liminf_{n \rightarrow \infty} \int f_n d\mu.$$

Similarly,

$$\int g - f d\mu \stackrel{\text{Fatou}}{\leq} \liminf_{n \rightarrow \infty} \int g - f_n d\mu = \int g d\mu - \limsup_{n \rightarrow \infty} \int f_n d\mu.$$

Since $\int g d\mu < \infty$,

$$\int f d\mu \leq \liminf_{n \rightarrow \infty} \int f_n d\mu$$

and

$$-\int f d\mu \leq -\limsup_{n \rightarrow \infty} \int f_n d\mu.$$

Therefore,

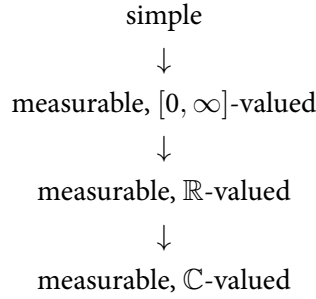
$$\limsup_{n \rightarrow \infty} \int f_n d\mu \leq \int f d\mu \leq \liminf_{n \rightarrow \infty} \int f_n d\mu,$$

which means

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu.$$

QED

In our progression of theory of integration, we proceeded in the order



So what has been missing are the measurable functions which take extended real values. We are going to address this problem quickly.

4. Spaces of Integrable Functions

Proposition 3.20.

$L^1(X, \mathcal{A}, \mu)$ is a Banach space.

Here are some ideas for the proof.

Suppose that V is a normed linear space and let $(a_n)_{n=1}^\infty \in V^n$ be Cauchy. Then we know

there is a subsequence $(a_{n_k})_{k=1}^\infty$ such that $a_{n_k} \rightarrow a \in V \implies a_n \rightarrow a$.

Let $(f_n)_{n=1}^\infty \in L^1(X, \mathcal{A}, \mu)^\mathbb{N}$ be Cauchy. Then

$$|\operatorname{Re} \circ f_n - \operatorname{Re} \circ f_m|^2 \leq |\operatorname{Re} \circ f_n - \operatorname{Re} \circ f_m|^2 + |\operatorname{Im} \circ f_n - \operatorname{Im} \circ f_m|^2 = |f_n - f_m|^2$$

so that

$$|\operatorname{Re} \circ f_n - \operatorname{Re} \circ f_m| \leq |f_n - f_m|.$$

Hence by monotonicity,

$$\|\operatorname{Re} \circ f_n - \operatorname{Re} \circ f_m\|_1 \leq \|f_n - f_m\|_1,$$

which means $(\operatorname{Re} \circ f_n)_{n=1}^\infty$ is Cauchy. Similarly, $(\operatorname{Im} \circ f_n)_{n=1}^\infty$ is also Cauchy.

Proof of Proposition 3.20

Let $(f_n)_{n=1}^\infty \in L^1(X, \mathcal{A}, \mu)$ be Cauchy. Assume each f_n is real-valued without loss of generality. For all $k \in \mathbb{N}$, there is $n_k \in \mathbb{N}$ such that

$$\|f_n - f_m\|_1 < \frac{1}{2^k}, \quad \forall n, m \geq n_k.$$

Without loss of generality assume $(n_k)_{k=1}^\infty$ is increasing. Let

$$\hat{g} = |f_{n_1}| + \sum_{k=1}^\infty |f_{n_{k+1}} - f_{n_k}|.$$

By the MCT,

$$\int \hat{g} d\mu = \int |f_{n_1}| d\mu + \sum_{k=1}^\infty \int |f_{n_{k+1}} - f_{n_k}| d\mu = \|f_{n_1}\|_1 + \sum_{k=1}^\infty \|f_{n_{k+1}} - f_{n_k}\|_1 = \|f_{n_1}\|_1 + 1 < \infty.$$

This means \hat{g} is finite almost everywhere – that is, there is $N \in \mathcal{A}$ such that $\hat{g}|_{X \setminus N}$ is finite and $\mu(N) = 0$. Hence define $g : X \rightarrow \mathbb{R}$ by

$$g(x) = \begin{cases} \hat{g}(x) & \text{if } x \in X \setminus N, \\ 0 & \text{if } x \in N \end{cases}, \quad \forall x \in X.$$

Let $f : X \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} f_{n_1}(x) + \sum_{k=1}^\infty f_{n_{k+1}}(x) - f_{n_k}(x) & \text{if } x \in X \setminus N, \\ 0 & \text{if } x \in N \end{cases}, \quad \forall x \in X.$$

Then $f_{n_k} \rightarrow f$ pointwise almost everywhere and we have that $|f| \leq g$. Then by the LDCT,

$$f \in L^1(X, \mathcal{A}, \mu).$$

Moreover,

$$|f_{n_k}| \leq |f_{n_1}| + \sum_{j=1}^{k-1} |f_{n_{j+1}} - f_{n_j}| \stackrel{\text{ae}}{\leq} g, \quad \forall k \in \mathbb{N}.$$

Finally,

$$|f - f_{n_k}| \leq 2g, \quad \forall k \in \mathbb{N},$$

so by the LDCT,

$$\|f - f_{n_k}\|_1 = \int |f - f_{n_k}| d\mu \rightarrow 0.$$

QED

IV. Product Measures

1. Product Measures

Def'n 4.1. **Measurable Rectangle**

Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be measure spaces. For every $A \in \mathcal{A}, B \in \mathcal{B}$, we call $A \times B$ a *measurable rectangle*.

Lemma 4.1.

Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be measure spaces and let $\{A_k \times B_k\}_{k=1}^{\infty}$ be a collection of measurable rectangles that are pairwise disjoint. Also assume that

$$\bigcup_{k=1}^{\infty} A_k \times B_k = A \times B$$

for some $A \in \mathcal{A}, B \in \mathcal{B}$. Then

$$\mu(A) \nu(B) = \sum_{k=1}^{\infty} \mu(A_k) \nu(B_k).^1$$

¹We are using the convention $0\infty = 0$.

Proof. Fix $x \in A$. For all $y \in B$, there exists a unique $k \in \mathbb{N}$ such that $(x, y) \in A_k \times B_k$. Hence

$$B = \bigcup_{k \in \mathbb{N}: x \in A_k} B_k$$

This means

$$\nu(B) = \sum_{k \in \mathbb{N}: x \in A_k} \nu(B_k),$$

so that

$$\nu(B) \chi_A(x) = \sum_{k=1}^{\infty} \nu(B_k) \chi_{A_k}(x), \quad \forall x \in X.$$

By MCT,

$$\nu(B) \mu(A) = \sum_{k=1}^{\infty} \sum_{k=1}^{\infty} \nu(B_k) \mu(A_k).$$

QED

Let

$$\mathcal{R} = \left\{ \bigcup_{k=1}^n A_k \times B_k : n \geq 0, \forall k \in \{1, \dots, n\} [A_k \in \mathcal{A}, B_k \in \mathcal{B}] \right\}.$$

Proposition 4.2.

Let

$$\lambda : \mathcal{R} \rightarrow [0, \infty]$$

$$\bigcup_{k=1}^n A_k \times B_k \mapsto \sum_{k=1}^n \mu(A_k) \nu(B_k).$$

Then λ is a premeasure.

By Caratheodory, there is a complete measure

$$(X \times Y, \overline{\mathcal{A} \times \mathcal{B}}, \mu \times \nu)$$

on $X \times Y$ such that

$$\mathcal{A} \times \mathcal{B} \subseteq \overline{\mathcal{A} \times \mathcal{B}} = \{A \times B \in \mathcal{A} \times \mathcal{B} : A \times B \text{ is } \lambda^* \text{-measurable}\}.$$

and

$$(\mu \times \nu)(A \times B) = \mu(A) \nu(B), \quad \forall A \in \mathcal{A}, B \in \mathcal{B}.$$

Def'n 4.2. **Product Measure**

Consider the above setting. We call $\mu \times \nu$ the *product measure* on $\mathcal{A} \times \mathcal{B}$.

2. Product Integration

Theorem 4.3. Fubini

Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be complete measure spaces. If $f \in L^1(X \times Y, \overline{\mathcal{A} \times \mathcal{B}}, \mu \times \nu)$, then

(a) For all $x \in X$, let

$$\begin{aligned} f_x : Y &\rightarrow \mathbb{F} \\ y &\mapsto f(x, y). \end{aligned}$$

Then $f_x \in L^1(Y, \mathcal{B}, \nu)$ for almost all x .

(b) For all $y \in Y$, let

$$\begin{aligned} f'_y : X &\rightarrow \mathbb{F} \\ x &\mapsto f(x, y). \end{aligned}$$

Then $f'_y \in L^1(X, \mathcal{A}, \mu)$ for almost all y .

(c) Let

$$\begin{aligned} F : X &\rightarrow \mathbb{F} \\ x &\mapsto \int f_x d\nu. \end{aligned}$$

Then $F \in L^1(X, \mathcal{A}, \mu)$.

(d) Let

$$\begin{aligned} G : Y &\rightarrow \mathbb{F} \\ y &\mapsto \int f'_y d\mu. \end{aligned}$$

Then $G \in L^1(Y, \mathcal{B}, \nu)$.

(e) We have

$$\int_{X \times Y} f d(\mu \times \nu) = \int_X \int_Y f(x, y) d\nu d\mu = \int_Y \int_X f(x, y) d\mu d\nu.$$

Given $E \subseteq X \times Y$, let us write

$$E_x = \{y \in Y : (x, y) \in E\}, \quad \forall x \in X$$

and

$$E^y = \{x \in X : (x, y) \in E\}, \quad \forall y \in Y.$$

Lemma 4.4.

Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be measure spaces and let

$$\mathcal{R} = \left\{ \bigcup_{k=1}^n A_k \times B_k : n \geq 0, \forall k \in \{1, \dots, n\} [A_k \in \mathcal{A}, B_k \in \mathcal{B}] \right\}.$$

Let $E \in \mathcal{R}_{\sigma\delta}$ with $(\mu \times \nu)(E) < \infty$. Then

- (a) $g : X \rightarrow \mathbb{R}$ by $g(x) = \nu(E_x)$ for all $x \in X$ is μ -measurable;
- (b) $g \in L^+ \cap L^1$; and
- (c) $\int g d\mu = (\mu \times \nu)(E)$.

Proof.

Case 1. Suppose $E = A \times B$ for some $A \in \mathcal{A}, B \in \mathcal{B}$.

Then

$$E_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases} \in \mathcal{B}, \quad \forall x \in X$$

Now

$$g(x) = \nu(E_x) = \nu(B) \chi_A(x), \quad \forall x \in X$$

so that g is a nonnegative measurable function, with

$$\int g d\mu = \int \nu(B) \chi_A d\mu = \nu(B) \mu(A) = (\mu \times \nu)(E) < \infty,$$

as needed.

(End of Case 1)

Case 2. Consider $E = \bigcup_{i=1}^{\infty} A_i \times B_i$ for some $A_1, \dots \in \mathcal{A}, B_1, \dots \in \mathcal{B}$.

Without loss of generality, we may assume that the union is disjoint, since intersection of rectangles is still a rectangle.

Define $g_i = \nu(B_i) \chi_{A_i}$ for all $i \in \mathbb{N}$. Then

$$g = \sum_{i=1}^{\infty} g_i$$

so that g is μ -measurable. Moreover, every $E_x = \bigcup_{i=1}^{\infty} (A_i \times B_i)_x$ is measurable.

Then by the MCT,

$$\int g d\mu = \sum_{i=1}^{\infty} \int g_i d\mu = \sum_{i=1}^{\infty} \mu(A_i) \nu(B_i) = \sum_{i=1}^{\infty} (\mu \times \nu)(A_i \times B_i) = (\mu \times \nu)(E) < \infty.$$

(End of Case 2)

Case 3. Consider $E = \bigcap_{n=1}^{\infty} E_n$, where each $E_n \in \mathcal{R}_{\sigma}$.

Without loss of generality, we may assume

$$E_1 \supseteq E_2 \supseteq \dots$$

Moreover, we may also assume that

$$(\mu \times \nu)(E_1) < \infty,$$

since $(\mu \times \nu)(E) < \infty$.

Then we have that

$$E_x = \bigcap_{n=1}^{\infty} (E_n)_x$$

and

$$(E_1)_x \supseteq (E_2)_x \supseteq \cdots,$$

so

$$\lim_{n \rightarrow \infty} \nu((E_n)_x) = \nu(E)$$

and

$$\lim_{n \rightarrow \infty} (\mu \times \nu)(E_n) = (\mu \times \nu)(E).$$

Let

$$\begin{aligned} g_n : X &\rightarrow \mathbb{R} \\ x &\mapsto \nu((E_n)_x), \end{aligned} \quad \forall n \in \mathbb{N}.$$

Then $0 \leq g$ and $g_n \searrow g$ pointwise with

$$\int g_1 d\nu = (\mu \times \nu)(E_1) < \infty,$$

so by the LDCT,

$$\int g d\mu = \lim_{n \rightarrow \infty} \int g_n d\mu = \lim_{n \rightarrow \infty} (\mu \times \nu)(E_n) = (\mu \times \nu)(E).$$

(End of Case 3)

QED

Theorem 4.5. Tonelli

Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be complete measure spaces and suppose $\mu \times \nu$ is σ -finite. If $f \in L^+(X \times Y, \overline{\mathcal{A} \times \mathcal{B}}, \mu \times \nu)$, then

- (a) $f_x, f_y \in L^+$ almost everywhere;
- (b) for all $y \in Y$

$$F : X \rightarrow Y$$

$$x \mapsto \int_Y f_x d\nu$$

is integrable; and

- (c)

$$\int_{X \times Y} f d(\mu \times \nu) = \int_X \int_Y f d\nu d\mu = \int_Y \int_X f d\mu d\nu.$$

Proof. Since $\nu \times \mu$ is σ -finite, let $\{C_n\}_{n=1}^\infty \subseteq \overline{\mathcal{A} \times \mathcal{B}}$ be such that

$$X \times Y = \bigcup_{n=1}^\infty C_n$$

with

$$(\mu \times \nu)(C_n) < \infty, \quad \forall n \in \mathbb{N}.$$

Without loss of generality, we assume

$$C_1 \subseteq C_2 \subseteq \dots$$

by replacing C_n with $C_1 \cup \dots \cup C_n$.

Let

$$f_n = \min(f, n) \chi_{C_n}, \quad \forall n \in \mathbb{N}.$$

Then note that $f_n \rightarrow f$ pointwise where $(f_n)_{n=1}^\infty$ is an increasing sequence of measurable functions. Hence

$$\int f d(\mu \times \nu) \stackrel{\text{MCT}}{=} \lim_{n \rightarrow \infty} \int f_n d(\mu \times \nu) \stackrel{\text{Fub}}{=} \lim_{n \rightarrow \infty} \int_X \underbrace{\int_Y f_n d\nu}_{=F_n} d\mu.$$

Note that $F_n \nearrow F$ pointwise, so by the MCT,

$$\lim_{n \rightarrow \infty} \int_X F_n(x) d\mu = \int_X F(x) d\mu.$$

Thus

$$\int f d(\mu \times \nu) = \int_X \int_Y f d\nu d\mu.$$

QED

V. Differentiation

1. Introduction

We ask the following questions.

- (a) Is there a Lebesgue-measure-theoretic fundamental theorem of calculus?
- (b) Is there a measure theoretic differentiation?
- (c) Given integrable $f : X \rightarrow \mathbb{R}$, to what extent is

$$F : X \rightarrow \mathbb{R}$$

$$x \mapsto C + \int_a^x f dm$$

differentiable?

We are going to consider functions of the form

$$f : [a, b] \rightarrow \mathbb{R}.$$

By considering f^+, f^- , we first assume $f \geq 0$. In this way, we see that F (in (c)) is increasing.

Def'n 5.1. **Upper Derivative, Lower Derivative** of a Real-valued Function

Let $f : [a, b] \rightarrow \mathbb{R}$. We define the

- (a) *upper derivative from the right* of f , denoted as $\overline{D}_r f$, by

$$\overline{D}_r f(x) = \limsup_{h \downarrow 0} \frac{f(x+h) - f(x)}{h}, \quad \forall x \in [a, b];$$

- (b) *upper derivative from the left* of f , denoted as $\overline{D}_l f$, by

$$\overline{D}_l f(x) = \limsup_{h \downarrow 0} \frac{f(x) - f(x-h)}{h}, \quad \forall x \in [a, b];$$

- (c) *lower derivative from the right* of f , denoted as $\underline{D}_r f$, by

$$\underline{D}_r f(x) = \liminf_{h \downarrow 0} \frac{f(x+h) - f(x)}{h}, \quad \forall x \in [a, b];$$

and

- (d) *lower derivative from the left* of f , denoted as $\underline{D}_l f$, by

$$\underline{D}_l f(x) = \liminf_{h \downarrow 0} \frac{f(x) - f(x-h)}{h}, \quad \forall x \in [a, b].$$

Def'n 5.2. **Differentiable** Function

We say $f : [a, b] \rightarrow \mathbb{R}$ is *differentiable* if

$$\overline{D}_r f(x) = \overline{D}_l f(x) = \underline{D}_r f(x) = \underline{D}_l f(x) \in \mathbb{R}, \quad \forall x \in [a, b].$$

In case $f : [a, b] \rightarrow \mathbb{R}$ is differentiable in Def'n 5.2 sense, then all four quantities in Def'n 5.1 are equal to

$$f' : [a, b] \rightarrow \mathbb{R}$$

$$x \mapsto \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Def'n 5.3. **Degenerate** Interval

We say an interval is *degenerate* if it is \emptyset or a singleton.

Def'n 5.4. **Vitali Covering** of a Set

Let $E \subseteq \mathbb{R}$. We say a collection of non-degenerate intervals \mathcal{C} is a *Vitali covering* of E if

$$\forall x \in E, \varepsilon > 0 \exists I \in \mathcal{C} [x \in I, m(I) < \varepsilon].$$

Theorem 5.1. Vitali Covering Lemma

Let $E \subseteq \mathbb{R}$ be such that

$$m^*(E) < \infty$$

and let \mathcal{C} be a Vitali covering of E . Then for every $\varepsilon > 0$, there exist disjoint $I_1, \dots, I_N \in \mathcal{C}$ such that

$$m^*\left(E \setminus \bigcup_{n=1}^N I_n\right) < \varepsilon.$$

Proof. Fix $\varepsilon > 0$.

Recall that when $x \in \mathbb{R}$ and $C \subseteq \mathbb{R}$ is closed,

$$d(x, C) = \inf_{c \in C} |x - c|$$

is well-defined, and

$$x \in C \iff d(x, C) = 0.$$

Fix open $U \supseteq E$ with $m(U) < \infty$ and let

$$\mathcal{C}' = \{I \in \mathcal{C} : I \subseteq U\}.$$

Claim 1. \mathcal{C}' is a Vitali covering of E .

Let $x \in E$ and

$$\delta = d(x, \mathbb{R} \setminus U),$$

then for any $I \in \mathcal{C}$ such that $x \in I$ and $m(I) < \delta$, $I \subseteq U$, so that $I \in \mathcal{C}'$.

(End of Claim 1)

Let $I_1 \in \mathcal{C}$. For every $k > 1$, define $I_k \in \mathcal{C}'$ such that I_1, \dots, I_k are pairwise disjoint and

$$m(I_k) > \frac{\alpha_k}{2},$$

where

$$\alpha_k = \sup \{m(I) : I \in \mathcal{C}', I \text{ is disjoint from } I_1, \dots, I_{k-1}\}.$$

If this construction halts, then we are done; we have covered E by intervals, except possibly at finitely many points. Hence assume that the construction does not halt and we have countably many disjoint intervals $I_1, I_2, \dots \in \mathcal{C}'$.

Now,

$$m\left(\bigcup_{k=1}^{\infty} I_k\right) = \sum_{k=1}^{\infty} m(I_k) \leq m(U) < \infty.$$

We may find $N \in \mathbb{N}$ such that

$$\sum_{k=N+1}^{\infty} m(I_k) < \frac{\varepsilon}{5}.$$

Claim 2. $I_1, \dots, I_N \in \mathcal{C}$ are disjoint with

$$m^* \left(E \setminus \bigcup_{k=1}^N I_k \right) < \varepsilon.$$

Let

$$X = E \setminus \bigcup_{k=1}^N \overline{I_k}.$$

If $x \in X$, let

$$\delta = d \left(x, \bigcup_{k=1}^N \overline{I_k} \right).$$

Since \mathcal{C}' is a Vitali covering of E , we may find $I \in \mathcal{C}'$ such that $x \in I$ and $m(I) < \delta$. Hence I is disjoint from $\bigcup_{k=1}^N I_k$. This means

$$m(I) \leq \alpha_{N+1}.$$

Pick $K > N$ such that

$$\alpha_{K+1} < m(I) \leq \alpha_K.$$

Note that such $K > N$ exists, since $\sum_{k=1}^{\infty} \frac{\alpha_k}{2} \leq \sum_{k=1}^{\infty} m(I_k) < \infty$, which means $\lim_{k \rightarrow \infty} \alpha_k = 0$. But this means I is not disjoint from $\bigcup_{k=1}^K I_k$. Hence let $j \leq K$ be such that

$$I \cap I_j \neq \emptyset.$$

Then

$$m(I_j) > \frac{\alpha_j}{2} \geq \frac{\alpha_K}{2} \geq \frac{m(I)}{2}.$$

Now, let $z \in I_j$ be the midpoint of I_j . Then

$$|x - z| \leq m(I) + \frac{1}{2}m(I_j) \leq 2m(I_j) + \frac{1}{2}m(I_j) = \frac{5}{2}m(I_j).$$

Let J_j be the closed interval with the same midpoint z as I_j and

$$m(J_j) = 5m(I_j).$$

This means $|x - z| = \frac{1}{2}m(J_j)$, so that $x \in J_j$. This means

$$X \subseteq \bigcup_{j=N+1}^{\infty} J_j.$$

Hence

$$m^* \left(E \setminus \bigcup_{k=1}^N I_k \right) = m^*(X) \leq \sum_{j=N+1}^{\infty} m(J_j) = 5 \sum_{j=N+1}^{\infty} m(I_j) < 5 \frac{\varepsilon}{5} = \varepsilon.$$

(End of Claim 2)

QED

Theorem 5.2.

Let $f: [a, b] \rightarrow \mathbb{R}$ be increasing. Then

- (a) f is continuous except on a countable set;
- (b) f is differentiable except on a set of measure zero; and
- (c) the derivative f' of f is L^1 and

$$\int_a^b f' dm \leq f(b) - f(a).$$

¹Since f is differentiable ae, we may define f' in usual way for points at where f is differentiable and set $f'(x) = 0$ for every x where f is not differentiable.

Proof of (a). Extend f to \mathbb{R} by $f(x) = f(a)$ for $x < a$ and $f(x) = f(b)$ for $x > b$. For all $c \in [a, b]$,

$$\lim_{x \uparrow c} f(x) = \sup_{x < c} f(x) \leq f(c) \leq \inf_{x > c} f(x) = \lim_{x \downarrow c} f(x).$$

Hence f is continuous at c unless f has a jump of length

$$j(c) = \lim_{x \downarrow c} f(x) - \lim_{x \uparrow c} f(x).$$

But note that

$$\sum_{c \in [a, b]} j(c) \leq f(b) - f(a).$$

This means for every $n \in \mathbb{N}$, the number of jumps length at least $\frac{1}{n}$ is finite, so there are countably many jumps.

Proof of (b). Clearly we have

$$\underline{D}_r f \leq \overline{D}_r f$$

and

$$\underline{D} f \leq \overline{D} f.$$

Claim 1. *We have*

$$\overline{D} f \leq \underline{D}_r f$$

almost everywhere.

For $u < v$ in \mathbb{Q} , let

$$E_{u,v} = \{x \in [a, b] : \underline{D}_r f(x) < u < v < \overline{D} f(x)\}.$$

Let

$$E = \bigcup_{u,v \in \mathbb{Q}: u < v} E_{u,v}. \quad [5.1]$$

Then by the density of rationals, $E = \{x \in [a, b] : \underline{D}_r f(x) < \overline{D} f(x)\}$. Hence it remains to show

$$m^*(E) = 0.$$

By [5.1], it suffices to show that

$$m^*(E_{u,v})$$

for all $u < v$ in \mathbb{Q} . Hence fix $u < v$ in \mathbb{Q} and say $m^*(E_{u,v}) = s$. Let $\varepsilon > 0$ be given and find an open $E_{u,v} \subseteq U$ such that

$$m(U) < s + \varepsilon$$

by the definition of outer measure. Consider

$$\mathcal{C} = \left\{ [x, x+h] \subseteq U : h > 0, \frac{f(x+h) - f(x)}{h} < u \right\}.$$

For $x \in E_{u,v}$, we have

$$\underline{D}_r f(x) < u$$

so that

$$\lim_{\delta \downarrow 0} \inf_{h \in (0, \delta)} \frac{f(x+h) - f(x)}{h} < u.$$

This means \mathcal{C} has arbitrarily small intervals of the form $[x, x+h]$, where $x \in E_{u,v}$. Hence \mathcal{C} is a Vitali covering for $E_{u,v}$. By the Vitali covering lemma, we have disjoint

$$I_1 = [x_1, x_1 + h_1], \dots, I_N = [x_N, x_N + h_N] \in \mathcal{C}$$

such that

$$m^* \left(E_{u,v} \setminus \bigcup_{j=1}^N I_j \right) < \varepsilon.$$

Therefore,

$$s - \varepsilon < \sum_{j=1}^n m(I_j) = \sum_{j=1}^N h_j < m(U) < s + \varepsilon.$$

Note

$$s = m^*(E_{u,v}) = m^* \left(E_{u,v} \cap \left(\bigcup_{j=1}^N I_j \right) \right) + m^* \left(E_{u,v} \setminus \bigcup_{j=1}^N I_j \right) < m^* \left(E_{u,v} \cap \left(\bigcup_{j=1}^N I_j \right) \right) + \varepsilon.$$

by Caratheodory's criterion. This means

$$m^* \left(E_{u,v} \cap \left(\bigcup_{j=1}^N I_j \right) \right) > s - \varepsilon.$$

Let

$$F = E_{u,v} \cap \left(\bigcup_{j=1}^N (x_j, x_j + h_j) \right) \subseteq \bigcup_{j=1}^N (x_j, x_j + h_j) = V.$$

As before,

$$C' = \left\{ [x - k, x] \subseteq V : k > 0, \frac{f(x) - f(x - k)}{k} > v \right\}$$

is a Vitali cover for F . Again by the Vitali covering lemma, we find

$$J_1 = [y_1 - k_1, y_1], \dots, J_M = [y_M - k_M, y_M] \in C'$$

disjoint such that

$$m^* \left(F \setminus \bigcup_{i=1}^M J_i \right) < \varepsilon.$$

Then

$$\sum_{i=1}^M K_i = \sum_{i=1}^M m(J_i) > m^*(F) - \varepsilon = m^* \left(E_{u,v} \cap \left(\bigcup_{j=1}^N (x_j, x_j + h_j) \right) \right) - \varepsilon > s - 2\varepsilon.$$

Note that

$$J_i \subseteq \bigcup_{j=1}^N I_j$$

for all $i \in \{1, \dots, M\}$. Hence

$$(s - 2\varepsilon) v < \sum_{i=1}^M vk_i < \sum_{i=1}^M (f(y_i), f(y_i - k_i)) \leq \sum_{j=1}^N f(x_j + h_j) - f(x_j) \leq \sum_{j=1}^N uh_j < u(s + \varepsilon).$$

So for all $\varepsilon > 0$,

$$v(s - 2\varepsilon) < u(s + \varepsilon),$$

and by letting $\varepsilon \rightarrow 0$,

$$vs \leq us.$$

But $u < v$, so we conclude that

$$s = 0.$$

(End of Claim 1)

In a similar fashion,

$$\overline{D_i}f \leq \underline{D_i}f$$

almost everywhere.

Proof of (c). Consider

$$g_n : [a, b] \rightarrow \mathbb{R}$$

$$x \mapsto \frac{f\left(x + \frac{1}{n}\right) - f(x)}{\frac{1}{n}}, \quad \forall n \in \mathbb{N}.$$

Since f is monotone, f is measurable, so that each g_n is measurable. Also,

$$g_n(x) \rightarrow f'(x)$$

almost everywhere. Therefore, f' is measurable with $f' \geq 0$, since each $g_n \geq 0$. Then, by Fatou's lemma,

$$\begin{aligned} \int_a^b f' dm &\leq \liminf_{n \rightarrow \infty} \int_a^b g_n dm = \liminf_{n \rightarrow \infty} n \int_a^b f\left(\cdot + \frac{1}{n}\right) dm - n \int_a^b f dm = \liminf_{n \rightarrow \infty} \int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f dm - n \int_a^b f dm \\ &= \liminf_{n \rightarrow \infty} n \int_b^{b+\frac{1}{n}} f dm - n \int_a^{a+\frac{1}{n}} f dm \leq f(b) - f(a). \end{aligned}$$

QED

2. Bounded Variation and Absolute Continuity

Def'n 5.5. Bounded Variation

We say $f : [a, b] \rightarrow \mathbb{R}$ is of **bounded variation** if the **variation** of f ,

$$V_a^b(f) = \sup \left\{ \sum_{k=1}^n |f(x_k) - f(x_{k-1})| : n \in \mathbb{N}, a = x_0 < x_1 < \cdots < x_n = b \right\},$$

is finite.

Example 5.1.

$\chi_{Q \cap [0,1]} : [0, 1] \rightarrow \mathbb{R}$ is not of bounded variation.

Example 5.2.

If $f : [a, b] \rightarrow \mathbb{R}$ is increasing, then for $a = x_0 < x_1 < \cdots < x_n = b$,

$$V_a^b(f) = \sum_{k=1}^n (f(x_k) - f(x_{k-1})) = f(b) - f(a).$$

Proposition 5.3.

Let $f : [a, b] \rightarrow \mathbb{R}$. Then

$$f \text{ is of bounded variation} \iff f = g - h \text{ for some increasing } g, h.$$

Proof. (\Leftarrow) Suppose $f = g - h$ for some increasing g, h . Then for any partition $a = x_0 < x_1 < \dots < x_n = b$,

$$\sum_{k=1}^n |f(x_k) - f(x_{k-1})| \leq \sum_{k=1}^n |g(x_k) - g(x_{k-1})| + \sum_{k=1}^n |h(x_k) - h(x_{k-1})| = g(b) - g(a) + h(b) - h(a) < \infty.$$

(\Rightarrow) Suppose f is of bounded variation. Define

$$\begin{aligned} g : [a, b] &\rightarrow \mathbb{R} \\ x &\mapsto V_a^x(f). \end{aligned}$$

Then g is increasing. Let $h = g - f$. For $x < y$,

$$h(y) - h(x) = V_a^y(f) - f(y) - V_a^x(f) + f(x) = V_x^y(f) - (f(y) - f(x)) \geq |f(y) - f(x)| - (f(y) - f(x)) \geq 0.$$

QED

Corollary 5.3.1.

Let $f : [a, b] \rightarrow \mathbb{R}$ be of bounded variation. Then

- (a) f is continuous except on a countable set;
- (b) f is differentiable except on a set of measure zero; and
- (c) the derivative f' of f is L^1 and

$$\int_a^b f' dm \leq f(b) - f(a).$$

Corollary 5.3.2.

If $f : [a, b] \rightarrow \mathbb{R}$ is L^1 , then

$$\begin{aligned} F : [a, b] &\rightarrow \mathbb{R} \\ x &\mapsto \int_a^x f dm \end{aligned}$$

is of BV.

Def'n 5.6. Absolutely Continuous Function

We say $f : [a, b] \rightarrow \mathbb{R}$ is **absolutely continuous** if for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $(x_1, y_1), \dots, (x_n, y_n) \subseteq [a, b]$ are disjoint with

$$\sum_{k=1}^n y_k - x_k < \delta,$$

then $\sum_{k=1}^n |f(y_k) - f(x_k)| < \varepsilon$.

Proposition 5.4.

Let $f \in L^1(X, \mathcal{A}, \mu)$. For all $\varepsilon > 0$, there is $\delta > 0$ such that for any $A \in \mathcal{A}$ with $\mu(A) < \delta$, we have

$$\int_A |f| d\mu < \varepsilon.$$

Proof. Let $\varepsilon > 0$. We may find a simple nonnegative function $\varphi \leq |f|$ such that

$$\int |f| d\mu < \int \varphi d\mu + \frac{\varepsilon}{2}.$$

Note that, for all $A \in \mathcal{A}$,

$$\int_A |f| - \varphi d\mu \leq \int |f| - \varphi d\mu < \frac{\varepsilon}{2},$$

so that

$$\int_A |f| d\mu < \int_A \varphi d\mu + \frac{\varepsilon}{2}.$$

Say $\varphi \leq M$ for some $M \geq 0$. Take $\delta = \frac{\varepsilon}{2M}$ and suppose $A \in \mathcal{A}$ with $\mu(A) < \delta$. Then

$$\int_A |f| d\mu < \int_A \varphi d\mu + \frac{\varepsilon}{2} \leq M\mu(A) + \frac{\varepsilon}{2} < \varepsilon.$$

QED

Corollary 5.4.1.

Let $f: [a, b] \rightarrow \mathbb{R}$ be L^1 . Then

$$F: [a, b] \rightarrow \mathbb{R}$$

$$x \mapsto \int_{[a, x]} f dm$$

is absolutely continuous.

Proof. Let $\varepsilon > 0$ be given and let $\delta > 0$ be such that

$$\mu(A) < \delta \implies \int_A |f| dm < \varepsilon.$$

Let $(x_1, y_1), \dots, (x_n, y_n) \subseteq [a, b]$ be disjoint with

$$\sum_{k=1}^n m((x_k, y_k)) < \delta.$$

Let $A = \bigcup_{k=1}^n (x_k, y_k)$. Then $m(A) < \delta$, so that $\int_A |f| < \varepsilon$. Thus,

$$\sum_{k=1}^n |F(y_k) - F(x_k)| = \sum_{k=1}^n \left| \int_{x_k}^{y_k} f dm \right| \leq \sum_{k=1}^n \int_{x_k}^{y_k} |f| dm = \int_A |f| dm < \varepsilon.$$

QED

Proposition 5.5.

Let $f: [a, b] \rightarrow \mathbb{R}$. If f is absolutely continuous, then f is of bounded variation.

Proof. Let $\varepsilon = 1$ and let $\delta > 0$ be such that whenever $(x_1, y_1), \dots, (x_n, y_n) \subseteq [a, b]$ are disjoint with $\sum_{k=1}^n y_k - x_k < \delta$, then $\sum_{k=1}^n |f(y_k) - f(x_k)| < \varepsilon$ by definition of absolute continuity. Write

$$[a, b] = \bigcup_{j=1}^p [a_{j-1}, a_j]$$

such that $a_j - a_{j-1} < \delta$. For any partition $a_{j-1} = x_0 < x_1 < \dots < x_m = a_j$, we have

$$\sum_{s=1}^m x_s - x_{s-1} < \delta.$$

Hence

$$\sum_{s=1}^m |f(x_s) - f(x_{s-1})| < 1,$$

so that

$$V_{a_{j-1}}^{a_j}(f) \leq 1 \implies V_a^b(f) = \sum_{j=1}^p V_{a_{j-1}}^{a_j}(f) \leq p.$$

Thus f is of bounded variation.

QED

Example 5.3. Cantor's Function

Let $f: [0, 1] \rightarrow \mathbb{R}$ be the *Cantor's function*. We know that f is an increasing continuous function that is continuous on each of the intervals $(\frac{1}{3}, \frac{2}{3})$, $(\frac{1}{9}, \frac{2}{9})$, \dots , so that

$$f' = 0 \text{ on } [0, 1] \setminus C,$$

where C is the *Cantor set*. Since $m(C) = 0$, f is differentiable everywhere. But

$$\int_0^1 f' dm = 0 < 1 = f(1) - f(0).$$

Since f is increasing, f is of bounded variation. However, f is not absolutely continuous. Indeed, if x_j, y_j for $1 \leq j \leq 2^n$ are the endpoints of the intervals remaining at n th stage of the construction of the Cantor set, then

$$\sum_{j=1}^{2^n} y_j - x_j = \left(\frac{2}{3}\right)^n \rightarrow 0$$

but

$$\sum_{j=1}^{2^n} |f(y_j) - f(x_j)| = f(1) - f(0) = 1.$$

Proposition 5.6.

Let $f: [a, b] \rightarrow \mathbb{R}$ be L^1 . If

$$F: [a, b] \rightarrow \mathbb{R} \\ x \mapsto \int_a^x f dm$$

is increasing, then $f \geq 0$ almost everywhere.

Proof. Let

$$E = \{x \in [a, b] : f(x) < 0\}$$

and let

$$E_n = \left\{x \in [a, b] : f(x) < \frac{-1}{n}\right\}, \quad \forall n \in \mathbb{N},$$

which means $E = \bigcup_{n=1}^{\infty} E_n$.

Suppose for contradiction $m(E) > 0$ so that there is $n \in \mathbb{N}$ such that $m(E_n) > 0$. Let

$$\varepsilon = \frac{m(E_n)}{2n}$$

and let $\delta > 0$ be such that

$$m(A) < \delta \implies \int_A |f| dm < \varepsilon.$$

By regularity of the Lebesgue measure, there is open $U \supseteq E_n$ such that

$$m(U \setminus E_n) < \delta.$$

Since any open subset of \mathbb{R} can be written as a disjoint union of open sets, write

$$U = \bigcup_{k=1}^{\infty} (x_k, y_k).$$

Then

$$0 \leq \sum_{k=1}^{\infty} F(y_k) - F(x_k) = \int_U f dm = \int_{U \setminus E_n} f dm + \int_{E_n} f dm < \varepsilon - \frac{m(E_n)}{n} = -\frac{m(E_n)}{2n},$$

which is a contradiction.

Thus we conclude $m(E) = 0$, as required.

QED

Corollary 5.6.1.

Let $f: [a, b] \rightarrow \mathbb{R}$ be L^1 and let

$$F: [a, b] \rightarrow \mathbb{R}$$

$$x \mapsto \int_a^x f dm.$$

If $F(x) = 0$ for all $x \in [a, b]$, then $f = 0$ almost everywhere.

Theorem 5.7. Lebesgue Differentiation Theorem

Let $f: [a, b] \rightarrow \mathbb{R}$ be L^1 and let

$$F: [a, b] \rightarrow \mathbb{R}$$

$$x \mapsto C + \int_a^x f dm$$

for some $C \in \mathbb{R}$. Then $F' = f$ almost everywhere.

Proof. Since F is of bounded variation, F' exists almost everywhere and is L^1 . For convenience, extend

$$f(x) = 0, \quad \forall x > b$$

so that

$$F(x) = F(b), \quad \forall x > b.$$

Also, $(g_n)_{n=1}^{\infty}$ by

$$g_n(x) = n \left(F\left(x + \frac{1}{n}\right) - F(x) \right), \quad \forall n \in \mathbb{N}, x \geq a$$

converges to F' pointwise almost everywhere.

Case 1. $|f| \leq M$ for some $M \geq 0$.

Then

$$g_n(x) = n \int_x^{x+\frac{1}{n}} f dm \implies |g_n(x)| \leq n \int_x^{x+\frac{1}{n}} |f| dm \leq \frac{1}{n} M = M, \quad \forall n \in \mathbb{N}, x \geq a.$$

But $\int_a^b M dm < \infty$, so we are at a position to apply the Lebesgue dominated convergence theorem. That is, for $c \in [a, b]$,

$$\begin{aligned} \int_a^c F' dm &= \lim_{n \rightarrow \infty} \int_a^c g_n dm = \lim_{n \rightarrow \infty} n \underbrace{\int_a^c F\left(x + \frac{1}{n}\right) - F(x) dx}_{\text{Riemann integral}} = \lim_{n \rightarrow \infty} n \int_{a+\frac{1}{n}}^{c+\frac{1}{n}} F(x) dx - n \int_a^c F(x) dx \\ &= \lim_{n \rightarrow \infty} n \int_c^{c+\frac{1}{n}} F(x) dx - n \int_a^{a+\frac{1}{n}} F(x) dx \stackrel{\text{FTC}}{=} F(c) - F(a) = \int_a^c f dm. \end{aligned}$$

Note that we can replace Lebesgue integral by the corresponding Riemann integral since F is (absolutely) continuous.

Hence

$$\int_a^c F' - f dm = 0, \quad \forall c \in [a, b] \implies F' - f = 0 \text{ almost everywhere}$$

by Corollary 5.6.1.

(End of Case 1)

Case 2. $f \geq 0$.

Let

$$f_n = \min(f, n), \quad \forall n \in \mathbb{N},$$

so that each $|f_n| < n$. Hence Case 1 applies to each f_n . Then, for almost every $x \in [a, b]$,

$$F(x) = \int_a^x f_n dm + \int_a^x f - f_n dm \implies F'(x) = f_n(x) + \frac{d}{dx} \int_a^x f - f_n dm \geq f(x).$$

For all $c \in [a, b]$, since F is of bounded variation and $F' \geq f_n$ almost everywhere for all $n \in \mathbb{N}$ implies $F' \geq f$ almost everywhere,

$$\int_a^c F' dm \leq F(c) - F(a) = \int_a^c f dm \leq \int_a^c F' dm \implies \int_a^c f dm = \int_a^c F' dm.$$

Hence $F' - f = 0$ almost everywhere.

(End of Case 2)

For the general case, consider f^+, f^- and use Case 2.

QED

Lemma 5.8.

Let $f: [a, b] \rightarrow \mathbb{R}$ be absolutely continuous. If $f' = 0$ almost everywhere, then f is constant.

Proof. Let $c \in (a, b]$ and let $\varepsilon > 0$ be given. Take $\delta > 0$ as per the definition of absolute continuity. Consider

$$E = \{x \in (a, c) : f'(x) = 0\},$$

which is measurable since f' is a pointwise limit of measurable functions (or we can simply invoke completeness of Lebesgue measure), so that

$$m([a, c] \setminus E) = 0.$$

Define

$$\mathcal{C} = \{[x, x+h] \subseteq (a, c) : x \in E, h > 0, |f(x+h) - f(x)| < \varepsilon h\}.$$

We see that \mathcal{C} is a Vitali covering for E . So by the Vitali covering lemma, we may find disjoint $I_1, \dots, I_n \in \mathcal{C}$ such that

$$m\left(E \setminus \bigcup_{i=1}^n I_i\right) < \delta.$$

Since $m([a, c] \setminus E) = 0$,

$$m\left([a, c] \setminus \bigcup_{i=1}^n I_i\right) < \delta$$

as well. Say

$$I_i = [a_i, b_i], \quad \forall i \in \{1, \dots, n\}$$

with

$$a < a_1 < b_1 < a_2 < b_2 < \dots < a_n < b_n < c.$$

Therefore,

$$\begin{aligned}
|f(c) - f(a)| &\leq \sum_{i=1}^n |f(b_i) - f(a_i)| + |f(a_1) - f(a)| + |f(c) - f(b_n)| + \sum_{i=1}^{n-1} |f(a_{i+1}) - f(b_i)| \\
&< \sum_{i=1}^n |f(b_i) - f(a_i)| + \varepsilon && \text{since } m\left([a, c] \setminus \bigcup_{i=1}^n I_i\right) < \delta \\
&< \sum_{i=1}^n \varepsilon (b_i - a_i) + \varepsilon && \text{by definition of } \mathcal{C} \\
&\leq \varepsilon (c - a) + \varepsilon.
\end{aligned}$$

Since our choice of $\varepsilon > 0$ was arbitrary, it follows $f(a) = f(c)$.

QED

Theorem 5.9.

Let $F : [a, b] \rightarrow \mathbb{R}$. The following are equivalent.

(a) There is $f : L^1([a, b])$ such that

$$F(x) = C + \int_a^x f dm, \quad \forall x \in [a, b].$$

(b) F is absolutely continuous.

(c) F is differentiable almost everywhere with $F' \in L^1([a, b])$ and

$$F(x) = F(a) + \int_a^x F' dm, \quad \forall x \in [a, b].$$

Proof. (c) \implies (a) is trivial and (a) \implies (b) is proven in Corollary 5.4.1.

For (b) \implies (c), assume F is absolutely continuous. This means F is of bounded variation, so F' exists almost everywhere with $F' \in L^1([a, b])$. Consider

$$\begin{aligned}
G : [a, b] &\rightarrow \mathbb{R} \\
x &\mapsto \int_a^x F' dm.
\end{aligned}$$

Then by the Lebesgue differentiation theorem, $G' = F'$ almost everywhere. Now $G - F$ is absolutely continuous as a sum of two absolutely continuous function. This means $(G - F)' = G' - F' = 0$ almost everywhere, so that $G - F$ is constant, say $G = F + C$. That is,

$$F(x) = C + \int_a^x F' dm, \quad \forall x \in [a, b].$$

But by noticing

$$F(a) = C + \int_a^a F' dm = C,$$

we conclude

$$F(x) = F(a) + \int_a^x F' dm, \quad \forall x \in [a, b].$$

QED

VI. Measure Decomposition

1. Signed Measure

Def'n 6.1. **Signed Measure** on a Measurable Space

Let (X, \mathcal{A}) be a measurable space. A **signed measure** $\nu : \mathcal{A} \rightarrow [-\infty, \infty]$ on (X, \mathcal{A}) such that

- (a) $\nu(\emptyset) = 0$;
- (b) for all countable collection of disjoint sets $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$, $\nu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \nu(A_n)$; and
- (c) ν takes on at most one of the values $-\infty, \infty$.

Note (c) in Def'n 6.1 is essential; for, if we have disjoint $A, B \in \mathcal{A}$ with $\nu(A) = \infty$, $\nu(B) = -\infty$, then $\nu(A \cup B)$ would be a problem.

Proposition 6.1.

Suppose ν is a signed measure on (X, \mathcal{A}) . Suppose

$$\left| \nu \left(\bigcup_{n=1}^{\infty} A_n \right) \right| < \infty.$$

Then $\sum_{n=1}^{\infty} \nu(A_n)$ converges absolutely.

Proof. Suppose $\sum_{n=1}^{\infty} \nu(A_n)$ converges conditionally. Then the subseries of positive terms and negative terms diverges to $\infty, -\infty$, respectively. But this means, by taking A to be the union of A_n 's with positive measures and B to be the union of A_n 's with negative measures, we see that $\nu(A) = \infty$, $\nu(B) = -\infty$, which is a contradiction.

QED

Proposition 6.2. Example of Signed Measures

Let $f \in L^1(X, \mathcal{A}, \mu)$ be real-valued and define

$$\begin{aligned} \nu : \mathcal{A} &\rightarrow [-\infty, \infty] \\ A &\mapsto \int_A f d\mu \end{aligned}$$

Then ν is a signed measure.

Proof. Clearly $\nu(\emptyset) = 0$. Since f is L^1 , note that $|\nu(A)| \leq \int_A |f| d\mu < \infty$, so that ν takes neither ∞ nor $-\infty$. It remains to check countable additivity.

Let $A_1, A_2, \dots \in \mathcal{A}$ be disjoint and let $A = \bigcup_{n=1}^{\infty} A_n$. Let

$$B_n = \bigcup_{k=1}^n A_k, \quad \forall n \in \mathbb{N}.$$

Then $f\chi_{B_n} \rightarrow f\chi_A$ pointwise and $|f\chi_{B_n}| \leq |f|$, where $|f|$ is L^1 . Hence by the LDCT,

$$\int_{B_n} f d\mu \rightarrow \int_A f d\mu.$$

This precisely means

$$\sum_{k=1}^n \nu(A_k) = \sum_{k=1}^n \int_{A_k} f d\mu = \int_{B_n} f d\mu \rightarrow \int_A f d\mu = \nu(A),$$

as needed.

QED

Example 6.1.

Let $f \in L^+(X, \mathcal{A}, \mu)$ and let $g \in L^1(X, \mathcal{A}, \mu) \cap L^+(X, \mathcal{A}, \mu)$, where both f, g are real-valued. Then

$$\begin{aligned} \nu : \mathcal{A} &\rightarrow [-\infty, \infty] \\ A &\mapsto \int_A f d\mu - \int_A g d\mu \end{aligned}$$

is a signed measure, with possibly $\nu(A) = \infty$.

Def'n 6.2. Null Set, Positive Set, Positive Set for a Signed Measure

Let (X, \mathcal{A}) be a measurable space and let ν be a signed measure on (X, \mathcal{A}) . We say $A \in \mathcal{A}$ is

- (a) a **null set** for ν if for all $B \in \mathcal{A}$ with $B \subseteq A$, we have $\nu(B) = 0$;
- (b) a **positive set** for ν if $\nu(B) \geq 0$ for all $B \in \mathcal{A}$ with $B \subseteq A$; and
- (c) a **negative set** for ν if $\nu(B) \leq 0$ for all $B \in \mathcal{A}$ with $B \subseteq A$.

Theorem 6.3. Hahn Decomposition Theorem

Let (X, \mathcal{A}) be a measurable space and let ν be a signed measure on (X, \mathcal{A}) . Then there exists positive $P \in \mathcal{A}$ and negative $N \in \mathcal{A}$ such that

$$X = P \cup N.$$

If $X = P' \cup N'$ is another such decomposition, then $P \triangle P', N \triangle N'$ are null.

Postponed

Lemma 6.3.1.

Let (X, \mathcal{A}) be a measurable space and let ν be a signed measure on (X, \mathcal{A}) . If $A \in \mathcal{A}$ is such that $0 < \nu(A) < \infty$, then there is positive $P \subseteq A$ such that $\nu(P) > 0$.

Proof. If A is positive, take $P = A$ and we are done.

Suppose A is not positive, so there is a subset of A with a negative signed measure. So take measurable $B_1 \subseteq A$ such that

$$\nu(B_1) \leq \frac{1}{2} \inf \left\{ \nu(B) : B \in \mathcal{A}, B \subseteq A \setminus \bigcup_{k=1}^{n-1} B_k \right\}.$$

Recursively, choose

$$B_n \subseteq A \setminus \bigcup_{k=1}^{n-1} B_k$$

so that

$$\nu(B_n) \leq \frac{1}{2} \left\{ \nu(B) : B \in \mathcal{A}, B \subseteq A \setminus \bigcup_{k=1}^{n-1} B_k \right\}.$$

We remark that, if we cannot find such a B_n at n th recursive step, then every measurable subset of $A \setminus \bigcup_{k=1}^{n-1} B_k$ has a positive signed measure. Moreover,

$$\nu \left(A \setminus \bigcup_{k=1}^{n-1} B_k \right) = \underbrace{\nu(A)}_{>0} - \underbrace{\sum_{k=1}^{n-1} \nu(B_k)}_{<0} > 0,$$

so that $A \setminus \bigcup_{k=1}^{n-1} B_k \subseteq A$ is a positive set we were looking for.

Hence suppose the recursive process continues so that we have B_1, B_2, \dots . Take

$$P = A \setminus \bigcup_{k=1}^{\infty} B_k.$$

As before,

$$A = P \cup \bigcup_{k=1}^{\infty} B_k.$$

Since $|v(A)| < \infty$, by Proposition 6.1, $v(P) < \infty$.

Claim 1. *P is positive.*

Suppose there is measurable $B \subseteq P$ such that $v(B) < 0$. Since $\sum_{k=1}^{\infty} v(B_k)$ converges, $v(B_k) \rightarrow 0$. Hence we may take $n \in \mathbb{N}$ such that

$$v(B) < 2v(B_n).$$

But

$$2v(B_n) \leq \inf \left\{ v(C) : C \in \mathcal{A}, C \subseteq \setminus \bigcup_{k=1}^{n-1} B_k \right\} \leq v(B),$$

which is a contradiction.

(End of Claim 1)

QED

Lemma 6.3.2.

If $A_1, A_2, \dots \in \mathcal{A}$ are positive, then $\bigcup_{n=1}^{\infty} A_n$ is positive.

Proof. Let $B \subseteq \bigcup_{n=1}^{\infty} A_n$ and let

$$B_n = B \cap \left(A_n \setminus \bigcup_{k=1}^{n-1} A_k \right).$$

Then $B = \bigcup_{n=1}^{\infty} B_n$ where each $B_n \subseteq A_n$. But each A_n is positive, so that $v(B_n) \geq 0$. Thus

$$v(B) = \sum_{n=1}^{\infty} v(B_n) \geq 0.$$

QED

Proof of Theorem 6.3

We may assume v does not take on the value of ∞ (otherwise, consider $-v$). Let

$$M = \sup \{ v(A) : A \text{ is positive} \}.$$

Note that there is at least one positive set in \mathcal{A} : namely \emptyset . We may find positive $A_1, A_2, \dots \in \mathcal{A}$ such that

$$\mu(A_n) \rightarrow M.$$

By Lemma 6.3.2,

$$P = \bigcup_{n=1}^{\infty} A_n$$

is positive. Also,

$$\mu(P) = v(A_n) + v(P \setminus A_n) \geq v(A_n), \quad \forall n \in \mathbb{N},$$

which means $M \leq v(P)$. But P is positive, so $v(P) \leq M$, so that

$$v(P) = M.$$

Since v only takes finite values, it follows $M < \infty$ as well.

Let

$$N = X \setminus P.$$

Claim 1. N is negative.

For contradiction, suppose there is $E \in \mathcal{A}$ such that $E \subseteq N$ and $\nu(E) > 0$. By Lemma 6.3.1, there is a positive subset $A \subseteq E$ such that $\nu(A) > 0$. But then $P \cup A$ is a disjoint union of positive sets, so that $P \cup A$ is positive and

$$\nu(P \cup A) = \nu(P) + \nu(A) = M + \nu(A) > M,$$

since $M < \infty$, which is a contradiction.

(End of Claim 1)

Suppose

$$X = P' \cup N'$$

similarly. Then $P \setminus P' = N' \setminus N$ and $P' \setminus P = N \setminus N'$. Note that the sets are null, since they are simultaneously positive and negative. It follows that

$$P \Delta P' = (P \setminus P') \cup (P' \setminus P) = (N' \setminus N) \cup (N \setminus N') = N \Delta N'$$

is also null, as a union of null sets.

QED

Example 6.2.

Let $f \in L^1(X, \mathcal{A}, \mu)$ be real-valued and let

$$\begin{aligned} \nu : \mathcal{A} &\rightarrow [-\infty, \infty] \\ A &\mapsto \int_A f d\mu \end{aligned}.$$

Let

$$\begin{aligned} P &= \{x \in X : f(x) \geq 0\} \\ N &= \{x \in X : f(x) < 0\}. \end{aligned}$$

Then, for all $A \subseteq P$,

$$\nu(A) = \int_A f d\mu \geq 0$$

and similarly, for all $B \subseteq N$,

$$\nu(B) = \int_B f d\mu \leq 0.$$

Thus $P \cup N$ is a Hahn decomposition of X .

Note that

$$\begin{aligned} \nu^+ : \mathcal{A} &\rightarrow [0, \infty] \\ A &\mapsto \nu(A \cap P) \end{aligned}$$

Then ν^+ is measure on (X, \mathcal{A}) , with

$$\nu^+(A) = \int_{A \cap P} f d\mu = \int_A f \chi_P d\mu = \int_A f^+ d\mu, \quad \forall A \in \mathcal{A}.$$

Similarly,

$$\begin{aligned} \nu^- : \mathcal{A} &\rightarrow [0, \infty] \\ A &\mapsto -\nu(A \cap N) \end{aligned}$$

is a measure on (X, \mathcal{A}) with

$$\nu^-(A) = \int_A f^- d\mu, \quad \forall A \in \mathcal{A}.$$

But then

$$\nu(A) = \int_A f d\mu = \int_A f^+ d\mu - \int_A f^- d\mu = \nu^+(A) - \nu^-(A), \quad \forall A \in \mathcal{A},$$

so that $\nu = \nu^+ - \nu^-$. That is, we *decomposed* a signed measure into its positive and negative parts.

Def'n 6.3. **Mutually Singular Signed Measures**

Suppose (X, \mathcal{A}) is a measurable space and let μ, ν be signed measures. We say μ, ν are *mutually singular*, denoted as $\mu \perp \nu$, if $X = A \cup B$ such that A is ν -null and B is μ -null.

Consider the setting of Def'n 6.3. Given $C \in \mathcal{A}$,

$$C = (C \cap A) \cup (C \cap B).$$

This means

$$\mu(C) = \mu(C \cap A)$$

and similarly

$$\nu(C) = \nu(C \cap B).$$

As we can see, ν^+, ν^- from Example 6.2 are mutually singular, which is of interest of the next theorem.

Theorem 6.4. Jordan Decomposition Theorem

Let (X, \mathcal{A}) be a measurable space and let ν be a signed measure on (X, \mathcal{A}) . Then there exists a unique pair (ν^+, ν^-) of mutually singular measures such that

$$\nu = \nu^+ - \nu^-.$$

Proof. Let $X = P \cup N$ be a Hahn decomposition with respect to ν . Consider

$$\begin{aligned} \nu^+ : \mathcal{A} &\rightarrow [0, \infty] \\ A &\mapsto \nu(A \cap P) \\ \nu^- : \mathcal{A} &\rightarrow [0, \infty] \\ A &\mapsto -\nu(A \cap N) \end{aligned}.$$

By construction, ν^+, ν^- are mutually singular measures such that $\nu = \nu^+ - \nu^-$. Indeed, given $A \subseteq N$,

$$\nu^+(A) = \nu(A \cap P) \geq \nu(N \cap P) = \nu(\emptyset) = 0$$

and similarly, given any $A \subseteq P$, $\nu^-(A) = 0$. We also have that

$$\mu(A) = \mu((A \cap P) \cup (A \cap N)) = \mu(A \cap P) + \mu(A \cap N) = \mu^+(A) - \mu^-(A).$$

For uniqueness, suppose $\nu = \mu^+ - \mu^-$, where μ^+, μ^- are mutually singular measures; say $X = P' \cup N'$ such that P' is μ^- -null and N' is μ^+ -null. For $A \in \mathcal{A}$, $A \subseteq P'$,

$$\nu(A) = \mu^+(A) - \mu^-(A) = \mu^+(A) \geq 0,$$

so that P' is positive with respect to ν . Similarly, N' is negative with respect to ν . By Hahn decomposition, $P \triangle P' = N \triangle N'$ is null. Therefore, for all $A \in \mathcal{A}$,

$$\mu^+(A) = \mu^+(A \cap P') = \nu(A \cap P') = \nu(A \cap P) = \nu^+(A),$$

and similarly, $\mu^-(A) = \nu^-(A)$. Thus $\nu^+ = \mu^+, \nu^- = \mu^-$, as required.

QED

2. Decomposing Measures

Proposition 6.5.

Suppose ν is a signed measure with the Jordan decomposition $\nu = \nu^+ - \nu^-$. The following are equivalent.

- (a) A is ν -null.
- (b) A is ν^+, ν^- -null.
- (c) A is $|\nu|$ -null.

Proof. We first observe that

$$|\nu| = \nu^+ + \nu^-.$$

(a) \implies (b) Suppose $B \subseteq A$ and let $X = P \cup N$ be a Hahn decomposition of X . Then $\nu^+(B) = \nu(B \cap P) = 0$ since $B \cap P \subseteq B \subseteq A$. Similarly, $\nu^-(B) = \nu(B \cap N) = 0$.

(b) \implies (c) Clearly, given $B \subseteq A$,

$$|\nu(B)| = \nu^+(B) + \nu^-(B) = 0 + 0 = 0.$$

(c) \implies (a) Suppose $B \subseteq A$. Then

$$\nu^+(B) + \nu^-(B) = 0,$$

where both ν^+, ν^- are measures, so that

$$\nu(B) = \nu^+(B) - \nu^-(B) = 0.$$

QED

Def'n 6.4. **Absolutely Continuous** Signed Measure with respect to a Measure

Let ν be a signed measure and let μ be a measure on a measurable space (X, \mathcal{A}) . We say ν is *absolutely continuous* with respect to μ , denoted as $\nu \ll \mu$, if for all $A \in \mathcal{A}$,

$$\mu(A) = 0 \implies \nu(A) = 0.$$

Note that we are using the term *absolute continuity* again. The following exercise shows where this is coming from.

Exercise 6.3.

Let ν be a finite signed measure and let μ be a measure on a measurable space (X, \mathcal{A}) . Then

$$\nu \ll \mu \iff \forall \varepsilon > 0 \exists \delta > 0 \forall A \in \mathcal{A} [\mu(A) < \delta \implies |\nu(A)| < \varepsilon].$$

In particular, Proposition 5.4 is a special case of the above exercise, with ν defined as $\nu(A) = \int_A |f| d\mu$ for some $f \in L^1(X, \mathcal{A}, \mu)$.

Theorem 6.6. Radon-Nikodym Theorem

Let ν, μ be σ -finite measures on a measurable space (X, \mathcal{A}) . If $\nu \ll \mu$, then there exists $f \in L^1(X, \mathcal{A}, \mu)$ such that

$$\nu(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A}.$$

Moreover, f is uniquely determined μ -almost everywhere.

We will only prove the case when ν, μ are *finite*. The σ -finite case is left as an easy exercise.

Proof of Existence. For each $r \in \mathbb{Q}, r > 0$, let $X = P_r \cup N_r$ be a Hahn decomposition with respect to $\nu - r\mu$. Set $P_0 = X, N_0 = \emptyset$. Consider $f: X \rightarrow \mathbb{R}$ by

$$f(x) = \sup \{r \in \mathbb{Q} : x \in P_r\}, \quad \forall x \in X.$$

For $t > 0$,

$$f^{-1}((t, \infty]) = \bigcup_{r \in \mathbb{Q}: r > t} P_r \in \mathcal{A},$$

as a countable union of measurable subsets. Moreover, $f^{-1}([0, \infty]) = X$, so that $f \in L^+(X, \mathcal{A}, \mu)$.

Suppose $0 < r < s$ in \mathbb{Q} . Then P_s is positive for $\nu - s\mu$ and so is positive for $\nu - r\mu$. This means

$$(\nu - r\mu)(N_r \cap P_s) = 0,$$

so that

$$\nu(N_r \cap P_s) = r\mu(N_r \cap P_s).$$

On the other hand, N_r is negative for $\nu - r\mu$ but $r < s$, so that N_r is negative for $\nu - s\mu$. This means

$$\nu(N_r \cap P_s) = s\mu(N_r \cap P_s)$$

as well, where $s \neq r$. Hence it follows that

$$\mu(N_r \cap P_s) = 0.$$

It follows

$$\mu\left(N_r \cap \bigcup_{s \in \mathbb{Q}: s > r} P_s\right) = 0.$$

Hence

$$f|_{N_r} \leq r \mu\text{-almost everywhere,}$$

so that

$$\mu(f^{-1}((r, \infty])) \leq \mu(P_r).$$

Now,

$$\begin{aligned} (\nu - r\mu)(P_r) \geq 0 &\implies \nu(P_r) \geq r\nu(P_r) \\ &\implies \nu(P_r) \leq \frac{1}{r}\nu(P_r) \leq \frac{1}{r}\nu(X). \end{aligned}$$

Taking $r \rightarrow \infty$,

$$\mu(f^{-1}((r, \infty])) = \nu(P_r) \leq \frac{1}{r}\nu(X) \rightarrow 0.$$

This means

$$\nu(f^{-1}(\{\infty\})) = 0,$$

which means f is finite almost everywhere.

Let $E \in \mathcal{A}$ and fix $N \in \mathbb{N}$. Consider

$$E_k = E \cap P_{\frac{k}{N}} \cap N_{\frac{k+1}{N}}, \quad \forall k \in \mathbb{N} \cup \{0\}.$$

Let

$$E_\infty = E \setminus \bigcup_{k=1}^{\infty} E_k.$$

We proceed to show that $\mu(E_\infty) = 0$. If $E_\infty = \emptyset$, we are done. Otherwise, fix $x \in E_\infty$. Since $P_0 = X$, $x \in P_0$. If there is $k \geq 0$ such that $x \in P_{\frac{k}{N}}$, $x \notin P_{\frac{k+1}{N}}$, then $x \in N_{\frac{k+1}{N}}$. But this means $x \in E_k$, which contradicts $x \in E_\infty$. It follows that $x \in P_{\frac{k}{N}}$ for all $k \geq 0$, so that

$$E_\infty \subseteq \bigcap_{k \in \mathbb{N} \cup \{0\}} P_{\frac{k}{N}}.$$

Hence,

$$\mu(E_\infty) \leq \mu\left(P_{\frac{k}{N}}\right) \leq \frac{N}{k}\mu(X) \rightarrow 0,$$

so that $\mu(E_\infty) = 0$ as well. It follows

$$\nu(E_\infty) = 0$$

by the absolute continuity of ν with respect to μ .

Now,

$$\begin{aligned} \left(\nu - \frac{k}{N}\mu\right)(E_k) &\geq 0 \\ \left(\nu - \frac{k+1}{N}\mu\right)(E_k) &\leq 0 \end{aligned}$$

since $E_k \subseteq P_{\frac{k}{N}} \cap N_{\frac{k+1}{N}}$ where $P_{\frac{k}{N}}$ is positive for $\nu - \frac{k}{N}\mu$ and $N_{\frac{k+1}{N}}$ is negative for $\nu - \frac{k+1}{N}\mu$. This implies

$$\frac{k}{N}\mu(E_k) \leq \nu(E_k) \leq \frac{k+1}{N}\mu(E_k). \quad [6.1]$$

Moreover, for $x \in E_k$,

$$\frac{k}{N} \leq f(x)$$

by definition and

$$f(x) \leq \frac{k+1}{N} \mu\text{-almost everywhere,}$$

by considering $f(x) \leq f|_{N_{\frac{k+1}{N}}}(x)$ and that $f_{N_r} \leq r$ μ -almost everywhere for $r \in \mathbb{Q}$. Hence

$$\frac{k}{N}x_{E_k} \leq f\chi_{E_k} \leq \frac{k+1}{N}\chi_{E_k}$$

μ -almost everywhere, so that

$$\frac{k}{N}\mu(E_k) \leq \int_{E_k} f d\mu \leq \frac{k+1}{N}\mu(E_k). \quad [6.2]$$

Summing over $k \geq 0$, we obtain that

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{k}{N}\mu(E_k) &\leq \sum_{k=0}^{\infty} \nu(E_k) = \underbrace{E_\infty}_{=0} + \sum_{k=0}^{\infty} \nu(E_k) = \nu(E) \\ &\leq \sum_{k=0}^{\infty} \frac{k+1}{N}\mu(E_k) = \sum_{k=1}^{\infty} \frac{k}{N}\mu(E_k) + \sum_{k=0}^{\infty} \frac{1}{N}\mu(E_k) = \sum_{k=0}^{\infty} \frac{k}{N}\mu(E_k) + \frac{\mu(E)}{N} \end{aligned}$$

from [6.1]. In a similar way, we obtain

$$\sum_{k=0}^{\infty} \frac{k}{N}\mu(E_k) \leq \int_E f d\mu \leq \sum_{k=0}^{\infty} \frac{k}{N}\mu(E_k) + \frac{\mu(E)}{N}.$$

It follows that

$$\left| \nu(E) - \int_E f d\mu \right| \leq \frac{\mu(E)}{N} \leq \frac{\mu(X)}{N} \rightarrow 0.$$

It follows that $\int_E f d\mu = \nu(E)$.

Proof of Uniqueness upto μ -almost Everywhere. Let $f, g \in L^+(X, \mathcal{A}, \mu)$ be such that

$$\nu(A) = \int_A f d\mu = \int_A g d\mu, \quad \forall A \in \mathcal{A}.$$

Consider $B = \{x \in X : f(x) > g(x)\}$ and

$$B_n = \left\{ x \in X : f(x) \geq g(x) + \frac{1}{n} \right\}, \quad \forall n \in \mathbb{N}.$$

Suppose for contradiction that $\mu(B) > 0$. This means there is $n \in \mathbb{N}$ such that

$$\mu(B_n) > 0.$$

Therefore, for such $n \in \mathbb{N}$,

$$\nu(B_n) = \int_{B_n} f d\mu \geq \int_{B_n} g + \frac{1}{n} d\mu = \int_{B_n} g d\mu + \underbrace{\frac{\mu(B_n)}{n}}_{>0} > \int_{B_n} g d\mu = \nu(B_n),$$

which is a contradiction.

This means $\mu(B) = 0$, which implies

$$f \leq g \text{ } \mu\text{-almost everywhere.}$$

By symmetry, $g \leq f$ μ -almost everywhere, so that

$$f = g \text{ } \mu\text{-almost everywhere,}$$

as required.

QED

Observe that absolute continuity is necessary for the Radon-Nikodym theorem. For instance, if $f \in L^+(X, \mathcal{A}, \mu)$, then

$$\begin{aligned} \nu : \mathcal{A} &\rightarrow [0, \infty] \\ A &\mapsto \int_A f d\mu \end{aligned}$$

is such that

$$\mu(A) \implies \nu(A) = \int_A f d\mu = 0,$$

so that $\nu \ll \mu$.

The following example demonstrates that the σ -finite assumption is also necessary.

Example 6.4.

Let $X = [0, 1]$, $\mathcal{A} = \text{Bor}([0, 1])$ and let m_c be the counting measure on (X, \mathcal{A}) . This means $m \ll m_c$, where m is the Lebesgue measure on (X, \mathcal{A}) . Observe that m_c is not σ -finite.

Suppose for contradiction that there is $f \in L^+(X, \mathcal{A}, m_c)$ such that

$$m(A) = \int_A f dm_c.$$

Then for all $a \in [0, 1]$,

$$0 = m(\{a\}) = \int_{\{a\}} f dm_c = f(a) m_c(\{a\}) = f(a)$$

which means

$$m([0, 1]) = \int 0 dm_c = 0,$$

which is a contradiction.

Corollary 6.6.1.

Let μ, ν be measure and signed measure, respectively, on a measurable space (X, \mathcal{A}) . If $|\nu|, \mu$ are σ -finite and $|\nu| \ll \mu$, then there exists $f = g - h$ with at least one of g, h is in $L^1(X, \mathcal{A}, \mu)$ and

$$\nu(A) = \int_A f d\mu, \quad \forall A \in \mathcal{A}.$$

Proof. We utilize the following claim.

Claim 1. *There exists $p : X \rightarrow \mathbb{R}$ with $|p(x)| = 1$ for all $x \in X$ such that*

$$\nu(A) = \int_A p d|\nu|.$$

Let $X = P \cup N$ be a Hahn decomposition of X with respect to ν and let

$$p = \chi_P - \chi_N.$$

Then, with the Jordan decomposition

$$\nu = \nu^+ - \nu^-$$

for ν , we have

$$\begin{aligned} \int_A \chi_P - \chi_N d|\nu| &= \int_A p d\nu^+ + \int_A p d\nu^- = \int_{A \cap P} p d\nu^+ + \int_{A \cap N} p d\nu^- = \int_{A \cap P} 1 d\nu^+ + \int_{A \cap N} -1 d\nu^- \\ &= \int_{A \cap P} 1 d\nu^+ - \int_{A \cap N} 1 d\nu^- = \nu^+(A \cap P) - \nu^-(A \cap N) = \nu^+(A) - \nu^-(A) = \nu(A). \end{aligned}$$

(End of Claim 1)

Since $\nu \ll \mu$, we have $|\nu| \ll \mu$. So by the Radon-Nikodym theorem, there exists $q \in L^+(X, \mathcal{A}, \mu)$ such that

$$|\nu|(A) = \int_A q d\mu.$$

Then, for $A \in \mathcal{A}$,

$$\nu(A) = \nu^+(A) - \nu^-(A) = \nu(A \cap P) + \nu(A \cap N) = |\nu|(A \cap P) - |\nu|(A \cap N) = \int_{A \cap P} q d\mu - \int_{A \cap N} q d\mu = \int_A pq d\mu,$$

so by letting $f = pq$, we have

$$\int_A f d\mu = \nu(A).$$

But

$$f = pq = q(\chi_P - \chi_N) = q\chi_P - q\chi_N,$$

so let $g = q\chi_P, h = q\chi_N$. Since signed measure cannot take both $-\infty, \infty$, it follows that

$$\int_P q d\mu < \infty \text{ or } \int_N q d\mu < \infty,$$

which means one of g, h is L^1 .

QED

Theorem 6.7. Lebesgue Decomposition Theorem

Let ν, μ be σ -finite measures on (X, \mathcal{A}) . Then there exists a unique decomposition

$$\nu = \nu_a + \nu_s$$

such that $\nu_a \ll \mu$ and $\nu_s \perp \mu$.

Proof. Consider

$$\lambda = \mu + \nu.$$

Then λ is a measure and $\nu, \mu \ll \lambda$. By the Radon-Nikodym theorem, there exists $f, g \in L^+(X, \mathcal{A}, \lambda)$ such that

$$\mu(A) = \int_A f d\lambda, \nu(A) = \int_A g d\lambda, \quad \forall A \in \mathcal{A}.$$

Let

$$A = f^{-1}((0, \infty]), B = f^{-1}(\{0\}).$$

Define $\nu_a, \nu_s : \mathcal{A} \rightarrow [0, \infty]$ by

$$\nu_a(E) = \nu(E \cap A), \nu_s(E) = \nu(E \cap B), \quad \forall E \in \mathcal{A}.$$

Clearly $\nu = \nu_a + \nu_s$.

Claim 1. $\nu_s \perp \mu$.

Consider $X = A \cup B$.

If $C \subseteq A$, then

$$\nu_s(C) = \nu(C \cap B) = \nu(\emptyset) = 0.$$

Hence A is ν_s -null. On the other hand, given $C \subseteq B$,

$$\mu(C) = \int_C f d\lambda = \int_C 0 d\lambda = 0.$$

Hence B is μ -null.

(End of Claim 1)

Claim 2. $\nu_a \ll \mu$.

Suppose $E \in \mathcal{A}$ with $\mu(E) = 0$. Then

$$\int f \chi_E d\lambda = \int_E f d\lambda = 0.$$

Since $f \in L^+(X, \mathcal{A}, \lambda)$, it follows that $f\chi_E$ is a measurable nonnegative function, so that

$$f\chi_E = 0 \text{ } \lambda\text{-almost everywhere.}$$

Hence

$$\nu_a(E) = \nu(E \cap A) = \lambda(E \cap A) = 0.$$

(End of Claim 2)

Proof of uniqueness is left as an exercise.

QED

VII. L^p Spaces

Fix a measure space (X, \mathcal{A}, μ) .

1. L^p Spaces

Given measurable $f: X \rightarrow \mathbb{R}$, let

$$[f] = \{g \in \mathbb{R}^X : g = f \mu\text{-almost everywhere}\}.$$

Def'n 7.1. $L^p(X, \mathcal{A}, \mu)$

Given $p \in [1, \infty)$, we define

$$L^p(X, \mathcal{A}, \mu) = \{[f] : f \in \mathbb{R}^X, f \text{ is measurable}, |f|^p \in L^1(X, \mathcal{A}, \mu)\}.$$

We define

$$L^\infty(X, \mathcal{A}, \mu) = \{[f] : f \in \mathbb{R}^X, f \text{ is measurable}, \sup \{t \geq 0 : \mu(\{x \in X : |f(x)| > t\}) > 0\} < \infty\}.$$

For convenience, we are going to *treat* equivalence classes $[f]$ as functions f .

Example 7.1. $\mathbb{R}^{\mathbb{N}}$

Consider $(\mathbb{N}, \mathcal{P}(\mathbb{N}), m_c)$, where m_c is the counting measure on $(X, \mathcal{P}(\mathbb{N}))$. Then given $f: \mathbb{N} \rightarrow \mathbb{R}$, f is measurable and

$$\int f d\mu = \sum_{n=1}^{\infty} f(n).$$

Hence for $p \in [1, \infty)$,

$$f \in L^p \iff \int |f|^p d\mu < \infty \iff \sum_{n=1}^{\infty} |f(n)|^p < \infty \iff f \in \mathbb{R}^{\mathbb{N}}.$$

Proposition 7.1.

Let $p \in [1, \infty]$. Then $(L^p, \|\cdot\|_p)$ is a Banach space, where

$$\|f\|_p = \left(\int |f|^p d\mu \right)^{\frac{1}{p}}, \quad \forall f \in L^p$$

for $p \in [1, \infty)$ and

$$\|f\|_\infty = \sup \{t \geq 0 : \mu(\{x \in X : |f(x)| > t\}) > 0\}, \quad \forall f \in L^\infty.$$

Proposition 7.2.

Let (X, \mathcal{A}, μ) be a measure space.

(a) For $p \in [1, \infty)$, the set of simple functions of finite support is dense in $L^p(X, \mathcal{A}, \mu)$.

(b) The set of simple functions is dense in $L^\infty(X, \mathcal{A}, \mu)$.

Proof of (a). Let $f \in L^p$ and let $(\varphi_n)_{n=1}^\infty$ be an increasing sequence of simple functions converging pointwise to f . Then

$$|\varphi_n|^p \leq |f|^p, \quad \forall n \in \mathbb{N},$$

so that $\varphi_n \in L^p$. This means, for a value a which φ_n takes, $\varphi_n^{-1}(a)$ have finite measure. So $(\varphi_n)_{n=1}^{\infty}$ is a sequence of simple functions of finite support. It remains to show $\varphi_n \rightarrow f$ in $\|\cdot\|_p$.

But $|\varphi_n - f| \leq |\varphi_n| + |f| \leq 2|f|$, so that

$$|\varphi_n - f| \leq 2^p |f|^p$$

for all $n \in \mathbb{N}$. Hence by the LDCT,

$$\int |\varphi_n - f|^p d\mu \rightarrow 0,$$

as required.

Proof of (b). Exercise.

QED

Recall 7.2. **Dual Space** of a Normed Linear Space

Let V be a normed linear space over \mathbb{K} . The **dual space** of V , denoted as V^* , is defined as

$$V^* = \{T : V \rightarrow \mathbb{K} : T \text{ is linear and continuous}\}.$$

Recall the following results for normed linear spaces.

Proposition 7.3.

Let $(V, \|\cdot\|)$ be a normed linear space and let $\varphi : V \rightarrow \mathbb{K}$ be a linear functional. The following are equivalent.

- (a) φ is continuous.
- (b) φ is continuous at 0.
- (c) φ is bounded.

Proposition 7.4.

Let $(V, \|\cdot\|)$ be a normed linear space. Then $(V^*, \|\cdot\|)$ is a Banach space.

Theorem 7.5. Holder's Inequality

Let (X, \mathcal{A}, μ) be a measure space and let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ or $p = 1, q = \infty$. If $f \in L^p(X, \mathcal{A}, \mu)$, $g \in L^q(X, \mathcal{A}, \mu)$, then $fg \in L^1$ and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Example 7.2.

Let (X, \mathcal{A}, μ) be a finite measure space and let $p < r$ in $[1, \infty)$.

Claim 1. $L^\infty \subseteq L^r$.

Let $f \in L^\infty$. Then $|f| \leq M$ almost everywhere for some $M \geq 0$. This means $\int |f|^r d\mu \leq \int M^r d\mu = M^r \mu(X) < \infty$.

(End of Claim 1)

Claim 2. $L^r \subseteq L^p$.

For $f \in L^r$, $\int |f|^r d\mu < \infty$, so that $f^{\frac{r}{p}} \in L^p$. Let s be the Holder conjugate of $\frac{r}{p}$. Then

$$\|f\|_p^p = \| |f|^p \cdot 1 \|_1 \leq \| |f|^p \|_{\frac{r}{p}} \|1\|_s = \|f\|_r^{\frac{p}{r}} \mu(X) < \infty.$$

(End of Claim 2)

It turns out there are no containment relations for $L^p(\mathbb{R}, \mathcal{M}, m)$, where m is the Lebesgue measure and \mathcal{M} is the collection of Lebesgue measurable sets. On the other hand,

$$L^p \subseteq L^r$$

for $p < r$ in $[1, \infty]$.

Theorem 7.6. Riesz Representation Theorem for L^p

Let (X, \mathcal{A}, μ) be a σ -finite measure space and let $p \in [1, \infty)$. Let q be the Holder conjugate for p . Then

$$\begin{aligned} \varphi : L^q &\rightarrow (L^p)^* \\ g &\mapsto \Phi_g \end{aligned}$$

is an isometric isomorphism, where

$$\Phi_g(f) = \int fg d\mu, \quad \forall g \in L^q, f \in L^p.$$

Proof. Claim 1. For $g \in L^q$,

$$\|\Phi_g\| = \|g\|_q.$$

We consider the case where $p \in (1, \infty)$ only.

For $f \in L^p$, by Holder's inequality,

$$|\Phi_g(f)| = \left| \int fg d\mu \right| \leq \int |fg| d\mu = \|fg\|_1 \leq \|f\|_p \|g\|_q,$$

so that

$$\|\Phi_g\| \leq \|g\|_q.$$

Since the case $g = 0$ is trivial, assume $g \neq 0$ and let

$$f = \frac{|g|^{q-1} \operatorname{sgn}(g)}{\|g\|_q^{q-1}}.$$

Note $p(q-1) = pq\left(1 - \frac{1}{q}\right) = q$, so that

$$|f|^p = \frac{|g|^q}{\|g\|_q^q},$$

which means

$$\|f\|_p^p = \int |f|^p d\mu = \frac{1}{\|g\|_q^q} \int |g|^q d\mu = 1.$$

Moreover,

$$|\Phi_g(f)| = \left| \int fg d\mu \right| = \left| \int \frac{|g|^q}{\|g\|_q^{q-1}} d\mu \right| = \|g\|_q.$$

Thus $\|\Phi_g\| = \|g\|_q$, as required.

(End of Claim 1)

Claim 2. If $g : X \rightarrow \mathbb{R}$ is measurable with

$$\left| \int \psi g d\mu \right| \leq M \|\psi\|_p,$$

for all simple ψ with finite support, then $g \in L^q$ and $\|g\|_q \leq M$.

We first consider the case where $p, q \in (1, \infty)$.

Let $(\psi_n)_{n=1}^\infty$ be a sequence of simple functions such that $\psi_n \rightarrow g$ pointwise and

$$|\psi_n| \leq |\psi_{n+1}| \leq |g|, \quad \forall n \in \mathbb{N}.$$

Since X is σ -finite, write

$$X = \bigcup_{n=1}^\infty X_n,$$

where each $\mu(X_n) < \infty$ and $X_1 \subseteq X_2 \subseteq \dots$. Let

$$\zeta_n = \psi_n \chi_{X_n},$$

which is a simple function with a finite support. Then

$$|\zeta_n| \leq |\zeta_{n+1}| \leq |g|, \quad \forall n \in \mathbb{N}$$

and $\zeta_n \rightarrow g$ pointwise. Define

$$f_n = \frac{|\zeta_n|^{q-1} \operatorname{sgn}(g)}{\|\zeta_n\|_q^{q-1}}$$

Then each f_n is simple with finite support and $\|f_n\|_p = 1$, just as in Claim 1. Then

$$M \geq \sup_{n \in \mathbb{N}} \left| \int f_n dg d\mu \right| = \sup_{n \in \mathbb{N}} \int \frac{|\zeta_n|^{q-1} |g|}{\|\zeta_n\|_q^{q-1}} d\mu \geq \sup_{n \in \mathbb{N}} \int \frac{|\zeta_n|^q}{\|\zeta_n\|_q^{q-1}} d\mu = \sup_{n \in \mathbb{N}} \|\zeta_n\|_q.$$

Now, $|\zeta_n|^q \leq |g|^q$, $(|\zeta_n^q|)_{n=1}^\infty$ is increasing, and $|\zeta_n|^q \rightarrow |g|^q$ pointwise, so by the monotone convergence theorem,

$$\sup_{n \in \mathbb{N}} \|\zeta_n\|_q = \lim_{n \rightarrow \infty} \|\zeta_n\|_q = \|g\|_q.$$

Thus

$$M \geq \|g\|_q,$$

as required.

Now suppose $p = 1, q = \infty$. Let $\varepsilon > 0$ be given and consider

$$A = \{x : |g(x)| \geq M + \varepsilon\}.$$

Since we want to show $\|g\|_\infty \leq M$, suppose $\mu(A) > 0$ for contradiction. Since X is σ -finite, we may find $B \subseteq A$ such that

$$0 < \mu(B) < \infty.$$

Take

$$f = \frac{1}{\mu(B)} \operatorname{sgn}(g) \chi_B$$

so that f is simple and $\|f\|_1 = 1$. Then

$$\int fg d\mu = \frac{1}{\mu(B)} \int |g| \chi_B d\mu = \frac{1}{\mu(B)} \int_B |g| d\mu \geq \frac{1}{\mu(B)} \int_B M + \varepsilon d\mu = M + \varepsilon > M = M \|f\|_1,$$

which is a contradiction.

Since the choice of ε was arbitrary, it follows M is an essential bound for $|g|$, so that

(End of Claim 2)

We now turn to the proof of the Riesz representation theorem. We consider two cases.

Case 1. $\mu(X) < \infty$.

Let $\Phi \in L^p(X, \mathcal{A}, \mu)^*$, where we desire to find $g \in L^q$ such that $\Phi = \Phi_g$. Consider

$$\begin{aligned} v : \mathcal{A} &\mapsto \mathbb{R} \\ A &\mapsto \Phi(\chi_A). \end{aligned}$$

Note that

$$v(\emptyset) = \Phi(\chi_\emptyset) = \Phi(0) = 0.$$

Let $\{A_n\}_{n=1}^\infty \subseteq \mathcal{A}$ and let $A = \bigcup_{n=1}^\infty A_n$. Then

$$\left\| \chi_A - \sum_{n=1}^N \chi_{A_n} \right\|_p^p = \left\| \sum_{n=N+1}^\infty \chi_{A_n} \right\|_p^p = \left(\left(\int \left(\sum_{n=N+1}^\infty \chi_{A_n} \right)^p d\mu \right)^{\frac{1}{p}} \right)^p = \int \sum_{n=N+1}^\infty \chi_{A_n} d\mu = \mu \left(\bigcup_{n=N+1}^\infty A_n \right) = \sum_{n=N+1}^\infty \mu(A_n).$$

Since $\mu(X) < \infty$, it follows $\mu(A) = \sum_{n=1}^\infty \mu(A_n) < \infty$, so that

$$\sum_{n=N+1}^\infty \mu(A_n) \rightarrow 0.$$

Hence $\chi_A = \sum_{n=1}^\infty \chi_{A_n} \in L^p$. By continuity of Φ ,

$$v(A) = \Phi(\chi_A) = \sum_{n=1}^\infty \Phi(\chi_{A_n}) = \sum_{n=1}^\infty v(A_n).$$

Hence v is a measure.

If $\mu(A) = 0$, then $\chi_A = 0$ μ -almost everywhere, so that

$$v(A) = \Phi(0) = 0.$$

This means $v \ll \mu$, so by the Radon-Nikodym theorem, there is $g \in L^1$ such that

$$v(A) = \int_A g d\mu.$$

Note that g is L^1 since the measure space is finite. Take a simple function

$$\psi = \sum_{k=1}^n a_k \chi_{A_k}.$$

Then

$$\Phi(\psi) = \sum_{k=1}^n a_k \chi_{A_k} = \sum_{k=1}^n a_k v(A_k) = \int \psi dv.$$

Also,

$$\sum_{k=1}^n a_k v(A_k) = \sum_{k=1}^n a_k \int_{A_k} g d\mu = \sum_{k=1}^n \int a_k \chi_{A_k} g d\mu = \int \psi g d\mu.$$

That is,

$$\Phi(\psi) = \int \psi dv = \int \psi g d\mu = \Phi_g(\psi).$$

Hence

$$\left| \int \psi g d\mu \right| = |\Phi(\psi)| \leq \|\Phi\| \|\psi\|_p.$$

By taking $M = \|\Phi\|$, we see that $g \in L^q$ with $\|g\|_q \leq M$. Then Φ, Φ_g are continuous functions that coincide on a dense subset of L^p , so that $\Phi = \Phi_g$.

(End of Case 1)

We now consider the general case, where X is assumed to be σ -finite. Write $X = \bigcup_{n=1}^{\infty} X_n$ so that each $\mu(X_n) < \infty$ and

$$X_1 \subseteq X_2 \subseteq \cdots.$$

We may identify $L^r(X_n, \mathcal{A} \cap \mathcal{P}(X_n), \mu)$ as a subset of $L^r(X, \mathcal{A}, \mu)$.

Let $\Phi \in L^p(X, \mathcal{A}, \mu)^*$. For every $n \in \mathbb{N}$, there exists a unique $g_n \in L^q(X_n)$ such that

$$\Phi|_{X_n} = \Phi_{g_n}$$

by Case 1. Moreover,

$$\|g_n\|_q = \|\Phi_{g_n}\| \leq \|\Phi\|.$$

By uniqueness, there is a unique $g : X \rightarrow \mathbb{R}$ such that for all $n \in \mathbb{N}$, $g|_{X_n} = g_n$. Since $X_1 \subseteq X_2 \subseteq \cdots$, $g_n \rightarrow g$ pointwise, which means g is measurable.

Note that, since X_n 's are nested, $(|g_n|^q)_{n=1}^{\infty}$ is an increasing sequence converging pointwise to $|g|^q$, so that

$$\|g_n\|_q \rightarrow \|g\|_q$$

by the monotone convergence theorem. It follows that

$$\|g\|_q \leq \|\Phi\| < \infty,$$

so that $g \in L^q(X, \mathcal{A}, \mu)$.

If $f \in L^p(X, \mathcal{A}, \mu)$, we have

$$|f\chi_{X_n} - f|^p \leq (2|f|)^p = 2^p |f|^p.$$

By the Lebesgue dominated convergence theorem,

$$f\chi_{X_n} \rightarrow f \text{ in } L^p.$$

Hence, by continuity of Φ ,

$$\Phi(f) = \lim_{n \rightarrow \infty} \Phi(f\chi_{X_n}) = \lim_{n \rightarrow \infty} \int (f\chi_{X_n}) g d\mu = \lim_{n \rightarrow \infty} \int_{X_n} fg d\mu = \int fg d\mu = \Phi_g(f),$$

where the second last equality follows from the Lebesgue dominated convergence theorem.

QED

Example 7.3.

Consider $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ with

$$\begin{aligned} \mu(\emptyset) &= 0 \\ \mu(A) &= \infty, \quad \forall A \neq \emptyset. \end{aligned}$$

Then observe that, for $f : \mathbb{N} \rightarrow \mathbb{R}$, if there is $n \in \mathbb{N}$ such that $f(n) \neq 0$,

$$\int |f| d\mu \geq \int_{\{n\}} |f| d\mu = \infty,$$

so that $L^1 = \{0\}$.

But we have $L^\infty = l^\infty$, so that

$$(L^1)^* \neq L^\infty.$$

Theorem 7.7. Riesz Representation Theorem II

Let (X, \mathcal{A}, μ) and let $p, q \in (1, \infty)$ be Holder conjugates. Then $g \mapsto \Phi_g$ is an isometric isomorphism from L^q to $(L^p)^*$.

Proof Idea. Use $M = \sup \left\{ \|g_E\|_q : E \subseteq X \text{ is } \sigma\text{-finite} \right\}$.

QED