I. Measures

1. Motivation

Let X be a set and let $A \subseteq X$. We aim to develop a *meaningful* theory of integration that is

$$\int_A f,$$

where $f : X \to \mathbb{R}$.

There are a bunch of natural question that come out here.

(a) *Which A are appropirate?*

(b) *Which f are appropirate?*

(c) *What does $\int_A f$ even mean?*

Moreover, we want the following:

$$\mu(A) = \int_A 1$$

to be some meaningful idea of size/volume/measure. Some μ's do this better than others. Here are some properties we want μ to satisfy:

(a) $\mu(\emptyset) = 0$.

(b) $\mu(A \mathbin{\dot\cup} B) = \mu(A) + \mu(B)$.

(c) $\mu(A \cup B) \leq \mu(A) + \mu(B)$.

(d) $A \subseteq B \implies \mu(A) \leq \mu(B)$.

(e) $\mu(X) \in [0, \infty]$.

(f) $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$.

(g) $\mu\left(\dot\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$.

Let's take a step back. If we are going to achieve those things, we want some basics. Let D be the domain of μ – the *nonprecise measure function* handed on us. We need:

(a) $\emptyset \in D$; and

(b) if $A_1, A_2, \ldots \in D$, then $\bigcup_{n=1}^{\infty} A_n \in D$.

2. σ-algebras

Def'n 1.1. σ-**algebra** of Subsets of X

Let X be a set and let $\mathcal{A} \subseteq \mathcal{P}(X)$. We say $\mathcal{A}$ is an ***algebra***[1] of subsets of X if

(a) $\emptyset \in \mathcal{A}$;

(b) $A \in \mathcal{A}$ implies $X \setminus A \in \mathcal{A}$; and *closure under complements*

(c) $A, B \in \mathcal{A}$ implies $A \cup B \in \mathcal{A}$. *closure under finite union*

Moreover, we say $\mathcal{A}$ is a ***σ-algebra*** if it satisfies in addition

$$\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A} \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}\,.$$

That is, $\mathcal{A}$ is *closed under countable unions*.

[1]The word *algebra* comes from boolean algebra, one of the most universal objects in abstract math.

Question 1.1.
Are all algebra a σ-algebra?

Answer. To answer this question, we should think about:

what is preserved for finite sets but not infinite sets?

The easiest answer is *finiteness*. Let X be an infinite set and let

$$\mathcal{A} = \{A \subseteq X : A \text{ is finite or } X \setminus A \text{ is finite}\}.$$

Then $\mathcal{A}$ is an algebra but not a σ-algebra.

QED

Let $\mathcal{A} \subseteq \mathcal{P}$ be an algebra. Then, as a corollary to Def'n 1.1,

(a) $A, B \in \mathcal{A}$ implies $X \setminus A, X \setminus B \in \mathcal{A}$, so that $A \cap B = X \setminus ((X \setminus A) \cup (X \setminus B)) \in \mathcal{A}$; *closure under closure*

(b) $X = X \setminus \emptyset \in \mathcal{A}$;

(c) $A, B \in \mathcal{A}$ implies $A \setminus B = A \cap (X \setminus B) \in \mathcal{A}$; and *closure under set difference*

(d) $A, B \in \mathcal{A}$ implies $A \triangle B \in \mathcal{A}$. *closure under symmetric set difference*

Moreover, if $\mathcal{A}$ is a σ-algebra, then (a) holds with countable number of sets.

Proposition 1.1. Generating σ-algebra from a Collection of Subsets
Let X be a set and let $\mathcal{E} \subseteq \mathcal{P}(X)$. Then

$$\langle \mathcal{E} \rangle = \bigcap \{\mathcal{A} \supseteq \mathcal{E} : \mathcal{A} \text{ is a } \sigma\text{-algebra}\}$$

is a σ-algebra.

Exercise

Def'n 1.2. σ-algebra **Generated** by $\mathcal{E}$
Consider Proposition 1.1. We call $\langle \mathcal{E} \rangle$ the σ-algebra ***generated*** by $\mathcal{E}$.

Def'n 1.3. **Borel σ-algebra** of a Topological Space
Let (X, τ) be a topological space. Then

$$\mathrm{Bor}(X) = \langle \tau \rangle$$

is called the ***Borel σ-algebra*** of (X, τ).

We call elements of $\mathrm{Bor}(X)$ the ***Borel sets***.

Def'n 1.4. **Measurable Space**
Let X be a set and let $\mathcal{A}$ be a σ-algebra of X. Then we call $(X, \mathcal{A})$ a ***measurable space***.

The elements of $\mathcal{A}$ are called the ***measurable sets***.

3. Measures

In this course, we often work in the extend real numbers $\mathbb{R} \cup \{-\infty, \infty\} = [-\infty, \infty]$. Here are things that we assume.

Assumption 1. Assumptions about Extended Real Numbers
For all $a \in \mathbb{R}$,

(a) $a + \infty = \infty$;

(b) $a - \infty = -\infty$;

(c) $\infty + \infty = \infty$; and

(d) $-\infty - \infty = -\infty$.

However, we leave the following expressions to be *undefined*:

(a) $\infty - \infty$;

(b) $\frac{\infty}{\infty}$; and

(c) 0∞.

Def'n 1.5. **Measure** on a Measurable Space

Let $(X, \mathcal{A})$ be a measurable space. A ***measure*** on $(X, \mathcal{A})$[1] is a function $\mu : \mathcal{A} \to [0, \infty]$ such that

(a) $\mu(\emptyset) = 0$; and

(b) we have

$$\mu\left(\dot{\bigcup_{n\in\mathbb{N}}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

for every $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ with $A_n \cap A_m$ for $n \neq m$. *countable additivity*

In case μ is a measure on $(X, \mathcal{A})$, we call $(X, \mathcal{A}, \mu)$ a ***measure space***.

[1] Or, ***measure*** on X if we are lazy.

Example 1.2. Examples of Measures

Let X be a set.

(a) $\mu(A) = 0$ for all $A \in \mathcal{P}(X)$ is a measure on $(X, \mathcal{P}(X))$. *zero measure*

(b) $\mu(\emptyset) = 0, \mu(A) = \infty$ for all $A \in \mathcal{P}(X) \setminus \{\emptyset\}$ is a measure on $(X, \mathcal{P}(X))$.

(c) $\mu(A) = |A|$ (where $|A| = \infty$ if A is infinite) is a measure on $(X, \mathcal{P}(X))$. *counting measure*

(d) Fix $x \in X$ and define

$$\mu(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

for all $A \in \mathcal{P}(X)$. Then μ is a measure on $(X, \mathcal{P}(X))$. *point-mass measure*

Proposition 1.2.

Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) For all $A, B \in \mathcal{A}$ and $A \subseteq B$, $\mu(A) \leq \mu(B)$. *monotonicity*

(b) For all $A, B \in \mathcal{A}$ with $A \subseteq B$ and $\mu(A) < \infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$. *excision*

(c) If $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$, then $\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu(A_n)$. *countable subadditivity*

Proof.

(a) Consider $B \setminus A$, which is measurable since $\mathcal{A}$ is closed under set difference. Hence we have

$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A).$$

(b) We have

$$\mu(A) + \mu(B \setminus A) = \mu(B)$$

as seen in (a). Since $\mu(A) < \infty$, we can freely subtract $\mu(A)$ from both sides to obtain that $\mu(B \setminus A) = \mu(B) - \mu(A)$.

(c) Let $B_1 = A_1$ and let $B_n = A_n \setminus \bigcup_{m=1}^{n-1} A_m$ for all $n \geq 2$. Then each B_n is measurable with $B_n \subseteq A_n$ and we have

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \mu\left(\dot{\bigcup_{n\in\mathbb{N}}} B_n\right) = \sum_{n\in\mathbb{N}} \mu(B_n) \leq \sum_{n\in\mathbb{N}} \mu(A_n).$$

QED

Proposition 1.3. Continuity of Measure

Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) Let $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be an ascending chain. That is,

$$A_1 \subseteq A_2 \subseteq \cdots .$$

Then

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n). \qquad \textit{continuity from below}$$

(b) Let $\{B_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be a decending chain with $\mu(B_1) < \infty$. That is,

$$B_1 \supseteq B_2 \supseteq \cdots .$$

Then

$$\mu\left(\bigcap_{n\in\mathbb{N}} B_n\right) = \lim_{n\to\infty} \mu(B_n). \qquad \textit{continuity from above}$$

Proof.

(a) Let $C_1 = A_1$ and let $C_n = A_n \setminus A_{n-1} = A_n \setminus \bigcup_{m=1}^{n-1} A_m$ for all $n \geq 2$, where the last equality follows from the ascending chain condition.

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \mu\left(\dot{\bigcup_{n\in\mathbb{N}}} C_n\right) = \sum_{n\in\mathbb{N}} \mu(C_n) = \lim_{N\to\infty} \sum_{n=1}^{N} \mu(C_n) = \lim_{N\to\infty} \mu\left(\bigcup_{n=1}^{N} C_n\right) = \lim_{N\to\infty} \mu(A_N).$$

(b) Let $D_n = B_1 \setminus B_n$ for all $n \in \mathbb{N}$, so that $\{D_n\}_{n\in\mathbb{N}}$ is an ascending chain. Then

$$B_1 \setminus \bigcap_{n\in\mathbb{N}} B_n = \bigcup_{n\in\mathbb{N}} D_n,$$

so that

$$\mu\left(B_1 \setminus \bigcap_{n\in\mathbb{N}} B_n\right) = \mu\left(\bigcup_{n\in\mathbb{N}} D_n\right) = \lim_{n\to\infty} \mu(D_n) = \lim_{n\to\infty} \mu(B_1) - \mu(B_n) = \mu(B_1) - \lim_{n\to\infty} \mu(B_n).$$

The result then follows from excision property of μ.

QED

Def'n 1.6. **Finite**, **Probability**, **σ-finite**, **Semifinite**, **Complete** Measure

Let $(X, \mathcal{A}, \mu)$ be a measure space. We say μ is

(a) ***finite*** if $\mu(X) < \infty$;

(b) a ***probability*** measure if $\mu(X) = 1$;

(c) ***σ-finite*** if

$$X = \bigcup_{n=1}^{\infty} A_n$$

for some $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ with $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$;

(d) ***semifinite*** if

$$\forall A \in \mathcal{A}\left[\mu(A) \neq 0 \implies B \in \mathcal{A}\left[B \subseteq A, 0 < \mu(B) < \infty\right]\right];$$

and

(e) ***complete*** if

$$\forall A \in \mathcal{A}\left[\mu(A) = 0 \implies \forall B \subseteq A\left[B \in \mathcal{A}\right]\right].$$

Example 1.3. An Example of Non-complete Measure

Let $X = \{a, b\}, \mathcal{A} = \{\emptyset, \{a, b\}\}, \mu = 0$. Then μ is not complete, as $\{a\} \in \mathcal{A}$.

The goal of this section is:

given a measure space $(X, \mu, \mathcal{A})$, if μ is not complete, we extend $\mathcal{A}$ and μ so that the result is complete.

A natural way of doing this is throw every subsets of measure-zero sets into $\mathcal{A}$.

Proposition 1.4. Completion of a Measure Space

Let $(X, \mu, \mathcal{A})$ be a measure space. Let

$$\overline{\mathcal{A}} = \{A \cup F : A \in \mathcal{A}, \exists N \in \mathcal{A}\, [F \subseteq N, \mu(N) = 0]\}$$

and define

$$\begin{aligned} \overline{\mu} &: \overline{\mathcal{A}} \to [0, \infty] \\ A \cup F &\mapsto \mu(A) \end{aligned}.$$

Then

(a) $\overline{\mathcal{A}}$ is a σ-algebra;

(b) $\overline{\mu}$ is a measure;

(c) $\overline{\mu}|_{\mathcal{A}} = \mu$; and

(d) $\overline{\mu}$ is complete.

Proof.

(a) Note that $\emptyset = \emptyset \cup \emptyset$ with $\emptyset \subseteq \emptyset$ where $\mu(\emptyset) = 0$. Hence $\emptyset \in \overline{\mathcal{A}}$.

Let $E = A \cup F$ with $A \in \mathcal{A}, F \subseteq N \in \mathcal{A}$ where $\mu(N) = 0$. Then

$$X \setminus E = \underbrace{X \setminus (A \cup N)}_{\in \mathcal{A}} \cup \underbrace{(N \setminus (A \cup F))}_{\subseteq N} \in \overline{\mathcal{A}}.$$

Let $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ with $E_n = A_n \cup F_n$ where $F_n \subseteq N_n$ for some $n \in \mathbb{N}$. Then

$$\bigcup_{n=1}^{\infty} E_n = \left(\bigcup_{n=1}^{\infty} A_n\right) \cup \left(\bigcup_{n=1}^{\infty} F_n\right).$$

But $\bigcup_{n=1}^{\infty} F_n \subseteq \bigcup_{n=1}^{\infty} N_n$ with $\mu\left(\bigcup_{n=1}^{\infty} N_n\right) \leq \sum_{n=1}^{\infty} \mu(N_n) = 0$. Thus $\bigcup_{n=1}^{\infty} E_n \in \overline{\mathcal{A}}$.

(b) We first check that $\overline{\mu}$ is well-defined. Let

$$E = A_1 \cup F_1 = A_2 \cup F_2$$

for some $A_1, A_2 \in \mathcal{A}$ and $F_1 \subseteq N_1, F_2 \subseteq N_2$ with $\mu(N_1) = \mu(N_2) = 0$.

Then note that

$$A_1 \cap A_2 \subseteq A_i \subseteq E \subseteq (A_1 \cup F_1) \cap (A_2 \cup F_2) \subseteq (A_1 \cap A_2) \cup N_1 \cup N_2.$$

Hence

$$\mu(A_1 \cap A_2) \leq \mu(A_i) \leq \mu(E_1 \cap E_2).$$

This means $\mu(A_i) = \mu(A_1 \cap A_2)$, sot hat $\mu(E_1) = \mu(E_2)$.

Thus $\overline{\mu}$ is well-defined.

To show $\overline{\mu}$ is a measure, note that
$$\overline{\mu}(\emptyset) = \overline{\mu}(\emptyset \cup \emptyset) = \mu(\emptyset) = 0.$$
Say we have a collection of disjoint sets in $\overline{\mathcal{A}}$, $\{E_n\}_{n=1}^{\infty} \subseteq \overline{\mathcal{A}}$, with
$$E_n = A_n \cup F_n$$
for some $E_n \subseteq N_n$ with $\mu(N_n) = 0$. Then
$$\dot{\bigcup_{n=1}^{\infty}} E_n = \left(\dot{\bigcup_{n=1}^{\infty}} A_n\right) \cup \underbrace{\left(\dot{\bigcup_{n=1}^{\infty}} F_n\right)}_{\subseteq \bigcup_{n=1}^{\infty} N_n}.$$
Thus
$$\overline{\mu}\left(\dot{\bigcup_{n=1}^{\infty}} E_n\right) = \mu\left(\dot{\bigcup_{n=1}^{\infty}} A_n\right) = \sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} \overline{\mu}(A_n),$$
so $\overline{\mu}$ is a measure.

(c) Given $A \in \mathcal{A}$, $A = A \,\dot\cup\, \emptyset$, so that $\overline{\mu}(A) = \mu(A)$.

(d) Let $A \subseteq B \in \overline{A}$ with $\overline{\mu}(B) = 0$. We are going to show $A \in \overline{\mathcal{A}}$.

We can write
$$B = E \cup F$$
for some $F \subseteq N \in \mathcal{A}$ with $\mu(N) = 0$. Then
$$\overline{\mu}(B) = \mu(E) = 0.$$
Since $A \subseteq B \subseteq E \cup N$ with $\mu(E \cup N) = 0$ (complete this).

QED

Def'n 1.7. **Completion** of a Measure Space

Let $(X, \mu, \mathcal{A})$ be a measure space. We call $\left(X, \overline{\mu}, \overline{\mathcal{A}}\right)$ the ***completion*** of $(X, \mu, \mathcal{A})$.

5. Construction of Measures

Def'n 1.8. **Outer Measure** on a Set

Let X be a nonempty set. An ***outer measure*** on X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that

(a) $\mu^*(\emptyset) = 0$;

(b) $A \subseteq B$ implies $\mu^*(A) \leq \mu^*(B)$; and *monotonicity*

(c) $\{A_n\}_{n=1}^{\infty} \mathcal{P}(X)$ implies $\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$. *countable subadditivity*

The idea is that

outer measures are naive approaches to measure **every** *subset of X.*

We start with $\mathcal{E} \subseteq \mathcal{P}(X)$ which are *easy* to measure. We use the outer measure μ^* and $\mathcal{E}$ to construct a measure.

Proposition 1.5. Construction of an Outer Measure

Suppose $\{\emptyset, X\} \subseteq \mathcal{E} \subseteq \mathcal{P}(X)$ and $\mu : \mathcal{E} \to [0, \infty]$ satisfies $\mu(\emptyset) = 0$. For $A \subseteq X$, define
$$\mu^*(A) = \inf\left\{\sum_{n=1}^{\infty} \mu(E_n) : \{E_n\}_{n=1}^{\infty} \subseteq \mathcal{E}, A \subseteq \bigcup_{n=1}^{\infty} E_n\right\}.$$
Then μ^* is an outer measure on X.

Example 1.4. Lebesgue Outer Measure

Let $X = \mathbb{R}, \mathcal{E} = \{(a,b) : -\infty < a < b < \infty\} \cup \{\emptyset, X\}$. Define

$$\mu((a,b)) = b - a, \mu(X) = \infty.$$

Then μ^* as said in Proposition 1.5 is called the ***Lebesgue outer measure***.

Proposition 1.6.

Suppose $\{\emptyset, X\} \subseteq \mathcal{E} \subseteq X$ and let $\mu : \mathcal{E} \to [0, \infty]$. If $\mu(\emptyset) = 0$, then $\mu^* : \mathcal{P}(X) \to [0, \infty]$ by

$$\mu^*(A) = \inf\left\{\sum_{n=1}^{\infty} \mu(A_n) : \{A_n\}_{n=1}^{\infty} \subseteq \mathcal{E}, A \subseteq \bigcup_{n=1}^{\infty} A_n\right\}, \qquad \forall A \in \mathcal{E}.$$

is an outer measure.

Proof. We verify few things.

(a) Note that $\emptyset \subseteq \bigcup_{n=1}^{\infty} \emptyset$ and so $0 \leq \mu^*(\emptyset) \leq \sum_{n=1}^{\infty} \mu(\emptyset) = 0$.

(b) Say $A \subseteq B \subseteq X$. Then

$$\left\{\sum_{n=1}^{\infty} \mu(A_n) : \forall n \in \mathbb{N}[A_n \in \mathcal{E}], A \subseteq \bigcup_{n=1}^{\infty} A_n\right\} \supseteq \left\{\sum_{n=1}^{\infty} \mu(A_n) : \forall n \in \mathbb{N}[A_n \in \mathcal{E}], B \subseteq \bigcup_{n=1}^{\infty} A_n\right\}$$

by definition. By taking infimum, we see that

$$\mu^*(A) \leq \mu^*(B).$$

(c) Say $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(X)$ and consider $\bigcup_{n=1}^{\infty} A_n$. We claim that

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

We may assume $\sum_{n=1}^{\infty} \mu^*(A_n) < \infty$.

Let $\varepsilon > 0$ be given. For every A_i, we may find $\{E_{i,j}\}_{j=1}^{\infty} \subseteq \mathcal{E}$ such that

$$A_i \subseteq \bigcup_{n=1}^{\infty} E_{i,j}$$

and

$$\sum_{j=1}^{\infty} \mu(E_{i,j}) < \mu^*(A_i) + \frac{\varepsilon}{2^i}$$

We then have

$$\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i,j=1}^{\infty} E_{i,j}.$$

Hence

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \overset{\inf}{\leq} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu(E_{i,j}) \leq \sum_{i=1}^{\infty} \mu^*(A_i) + \frac{\varepsilon}{2^i} = \left(\sum_{i=1}^{\infty} \mu^*(A_i)\right) + \varepsilon.$$

Since ε is an arbitary positive number, we see that μ^* is countably subadditive.

QED

Def'n 1.9. μ^***-measurable** Set

Let μ^* be an outer measure on X. We say $A \subseteq X$ is μ^****-measurable*** if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap (X \setminus A))$$

for all $E \subseteq X$.

Let $A, E \subseteq X$.

(a) Note

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap (X \setminus A)).$$

Hence it suffices to prove the reverse inequality to show that A is μ^*-measurable.

(b) As a corollary to (a), we may assume $\mu^*(E) < \infty$ when proving A is μ^*-measurable.

(c) When $A = \emptyset$,

$$\mu^*(E \cap \emptyset) + \mu^*(E \cap (X \setminus \emptyset)) = 0 + \mu^*(E) = \mu^*(E).$$

Thus $\emptyset$ is μ^*-measurable.

(d) If A is μ^*-measurable, then $X \setminus A$ is also μ^*-measurable. This is direct from the definition of μ^*-measurability.

Theorem 1.7. Caratheodory

Let μ^* be an outer measure on X. Then the collection of μ^*-measurable subsets of X,

$$\mathcal{A} = \{A \subseteq X : A \text{ is } \mu^*\text{-measurable}\},$$

is a σ-algebra.

Moreover, $\mu = \mu^*|_{\mathcal{A}}$ is a complete measure on $(X, \mathcal{A})$.

Proof. Let $A, B \in \mathcal{A}$ and let $E \subseteq X$. Then

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap (X \setminus A) \cap B) + \mu^*(E \cap (X \setminus A) \cap (X \setminus B)) && \text{since } A, B \text{ are } \mu^*\text{-measurable} \\ &\geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (X \setminus (A \cup B))). && \text{by subadditivity of } \mu^* \text{ and de Morgan's Law} \end{aligned}$$

Since we know the other direction of the above inequality, we see that $A \cup B \in \mathcal{A}$. Inductively, $\mathcal{A}$ is closed under finite union, which means $\mathcal{A}$ is an algebra on X (we know $\emptyset \in \mathcal{A}$).

Now assume $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$. For any $E \subseteq X$,

$$\mu^*(E \cap (A \cupdot B)) = \mu^*(E \cap (A \cupdot B) \cap A) + \mu^*(E \cap (A \cupdot B) \cap (X \setminus A)) = \mu^*(E \cap A) + \mu^*(E \cap B).$$

By taking $E = X$, we see that

$$\mu^*(A \cupdot B) = \mu^*(A) + \mu^*(B),$$

so that μ^* is finitely additive.

Assume $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$, let $B_n = \bigcup_{k=1}^{n} A_k$, and let $A'_n = A_1 \setminus \bigcup_{k=1}^{n-1} A_k$ for all $n \in \mathbb{N}$. Since $\mathcal{A}$ is an algebra, each $A'_n, B_n \in \mathcal{A}$. Then $B_n = \bigcupdot_{n=1}^{\infty} A'_k$ and $B = \bigcup_{n=1}^{\infty} A_n = \bigcupdot_{n=1}^{\infty} A'_n$. For any $E \subseteq X$,

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap B_n) + \mu^*(E \cap (X \setminus B_n)) \\ &\geq \mu^*(E \cap B_n) + \mu^*(E \cap (X \setminus B)) && \text{by monotonicity of } \mu^* \\ &= \sum_{k=1}^{n} \mu^*\left(E \cap A'_k\right) + \mu^*(E \cap (X \setminus B)) \\ &\geq \lim_{n\to\infty} \sum_{k=1}^{n} \mu^*\left(E \cap A'_k\right) + \mu^*(E \cap (X \setminus B)) \\ &\geq \mu^*(E \cap B) + \mu^*(E \cap (X \setminus B)) \\ &\geq \mu^*(E). && \text{by subadditivity of } \mu^* \end{aligned}$$

This means $\mu^*(E) = \mu^*(E \cap B) + \mu^*(E \cap (X \setminus B))$, so $\bigcup_{n=1}^{\infty} A_n = B \in \mathcal{A}$. Hence $\mathcal{A}$ is a σ-algebra.

Assume $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ is a collection of disjoint sets in $\mathcal{A}$. By taking $A_n' = A_n$ for all $n \in \mathbb{N}$ and $E = B$, we see that

$$\mu^*(B) \geq \sum_{n=1}^{\infty} \mu^*(B \cap A_n) + \underbrace{\mu^*(B \cap (X \setminus B))}_{=0} \geq \mu^*(B) \implies \mu^*(B) = \sum_{n=1}^{\infty} \mu^*(B \cap A_n)$$

from the series of inequalities we used for proving closure of $\mathcal{A}$ under countable union.

We now show that μ is complete. Let $A \subseteq X$ with $\mu^*(A) = 0$. For any $E \subseteq X$,

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap (X \setminus A)) \leq \underbrace{\mu^*(A)}_{=0} + \mu^*(E).$$

This means every set A with $\mu^*(A) = 0$ is measurable. But given any $B \in \mathcal{A}$ with $\mu(B) = 0$, we have

$$0 \leq \mu^*(A) \leq \mu^*(B) = \mu(B) = 0, \qquad \forall A \subseteq B,$$

so that $\mu^*(A) = 0$ and that A is measurable.

QED

We can construct a measure as follows. Given $\mathcal{E} \subseteq \mathcal{P}(X)$ with $\{\emptyset, X\} \subseteq \mathcal{E}$ and $\mu : \mathcal{E} \to [0, \infty]$, we let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be an outer measure as defind in Proposition 1.6.

In general, $\mathcal{A} = \{A \subseteq X : A \text{ is } \mu^*\text{-measurable}\}$ and $\mu^*|_{\mathcal{A}}$ are very different from $\mathcal{E}, \mu$. To resolve this, we introduce the following notion.

Def'n 1.10. **Premeasure** on an Algebra of Subsets

Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra of subsets of X. We say $\mu : \mathcal{A} \to [0, \infty]$ is a ***premeasure*** on $\mathcal{A}$ if

(a) $\mu(\emptyset) = 0$; and

(b) for any $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ with $\dot{\bigcup}_{n=1}^{\infty} A_n \in \mathcal{A}$, we have

$$\mu\left(\dot{\bigcup_{n=1}^{\infty}} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Theorem 1.8. Constructing Measure from Premeasure I

Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra and let $\mu : \mathcal{A} \to [0, \infty]$ be a premeasure on $\mathcal{A}$. Let μ^* be the outer measure constructed with $\mathcal{A}$:

$$\mu^*(A) = \inf\left\{\sum_{n=1}^{\infty} \mu(A_n) : \{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}, A \subseteq \bigcup_{n=1}^{\infty} A_n\right\}, \qquad \forall A \in \mathcal{P}(X).$$

Then

(a) $\mu^*|_{\mathcal{A}} = \mu$; and

(b) every $A \in \mathcal{A}$ is μ^*-measurable.

Proof.

(a) We show $\mu^*|_{\mathcal{A}} = \mu$. Let $E \in \mathcal{A}$. Say

$$E \subseteq \bigcup_{n=1}^{\infty} A_n$$

where each $A_n \in \mathcal{A}$. Then by taking $A_n' = A_n \setminus \bigcup_{k=1}^{n-1} A_k$,

$$E = \bigcup_{n=1}^{\infty} (A_n \cap E) = \dot{\bigcup_{n=1}^{\infty}} (A_n' \cap E).$$

But each $A_n' \cap E \in \mathcal{A}$, so that

$$\mu(E) = \sum_{n=1}^{\infty} \mu\left(A_n' \cap E\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

by the monotonicity of μ.[1] Therefore, $\mu(E) \leq \mu^*(E)$ by taking infimum.

On the other hand, by letting $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ with $A_1 = E, A_2 = A_3 = \cdots = \emptyset$, we see that $\mu^*(E) \geq \mu(E)$. Hence $\mu^*|_{\mathcal{A}} = \mu$.

(b) Let $A \in \mathcal{A}$. We show A is μ^*-measurable. Let $E \subseteq X$ and let $\varepsilon > 0$ be given. We may find $\{B_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ such that $E \subseteq \bigcup_{n=1}^{\infty} B_n$ and

$$\sum_{n=1}^{\infty} \mu(B_n) < \mu^*(E) + \varepsilon.$$

Then,

$$\begin{aligned}
\mu^*(E) + \varepsilon &\geq \sum_{n=1}^{\infty} \mu(B_n) \\
&= \sum_{n=1}^{\infty} \mu(B_n \cap A) + \mu(B_n \cap (X \setminus A)) \\
&= \sum_{n=1}^{\infty} \mu^*(B_n \cap A) + \sum_{n=1}^{\infty} \mu^*(B_n \cap (X \setminus A)) && \text{by (a)} \\
&\geq \mu^*\left(\left(\bigcup_{n=1}^{\infty} B_n\right) \cap A\right) + \mu^*\left(\left(\bigcup_{n=1}^{\infty} B_n\right) \cap (X \setminus A)\right) && \text{by subadditivity of } \mu^* \\
&\geq \mu^*(E \cap A) + \mu^*(E \cap (X \setminus A)). && \text{by monotonicity of } \mu^* \text{ since } E \subseteq \bigcup_{n=1}^{\infty} B_n
\end{aligned}$$

[1] It suffices to note that premeasures are finitely additive, which implies monotonicity.

QED

Theorem 1.9. Constructing Measure from Premeasure II

Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra and let μ^* be as in Theorem 1.8. Let $\mathcal{B} = \sigma(\mathcal{A})$. Then

(a) $\overline{\mu} = \mu^*|_{\mathcal{B}}$ is a complete measure with $\overline{\mu}|_{\mathcal{A}} = \mu$.

(b) Let ν be another measure on $\mathcal{B}$ with $\nu|_{\mathcal{A}} = \mu$. Then $\nu \leq \overline{\mu}$. That is,

$$\nu(A) \leq \overline{\mu}(A), \qquad \forall A \in \mathcal{B}.$$

(c) For any $E \in \mathcal{B}$, if $\overline{\mu}(E) < \infty$, then $\nu(E) = \overline{\mu}(E)$.

(d) If μ is σ-finite,[1] then $\overline{\mu} = \nu$.

[1] We say a premeasure is *σ-finite* if $X = \bigcup_{n=1}^{\infty} A_n$ for some $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ with $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$.

Proof.

(a) Let

$$\mathcal{C} = \{A \subseteq \mathcal{P}(X) : A \text{ is } \sigma\text{-measurable}\},$$

which is a σ-algebra. Then by Theorem 1.8, $\mathcal{A} \subseteq \mathcal{C}$, and so $\mathcal{B} \subseteq \mathcal{C}$ by minimality of $\mathcal{B}$. Therefore,

$$\overline{\mu} = \mu^*|_{\mathcal{B}}$$

is the restriction of $\mu^*|_{\mathcal{C}}$ to $\mathcal{B}$. Since $\mu^*|_{\mathcal{C}}$ is a complete measure on $(X, \mathcal{C})$, it follows $\overline{\mu} = \mu^*|_{\mathcal{B}}$ is a complete measure on $(X, \mathcal{B})$. Since $\mu^*|_{\mathcal{A}} = \mu$, $\overline{\mu}|_{\mathcal{A}} = \mu$ as well.

(b) Let $A \in \mathcal{B}$ and let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ be such that $A \subseteq \bigcup_{n=1}^{\infty} A_n$. Since ν is a measure extending μ,

$$\nu(A) \leq \nu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \nu(A_n) \overset{\nu|_{\mathcal{A}}=\mu}{=} \sum_{n=1}^{\infty} \mu(A_n).$$

By recalling that μ^* is defined as the *greatest* lower bound, it follows

$$\nu(A) \leq \mu^*(A) = \overline{\mu}(A).$$

(c) Let $A \in \mathcal{B}$ with $\overline{\mu}(A) < \infty$. Let $\varepsilon > 0$ be given. We may find $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ such that $A \subseteq \bigcup_{n=1}^{\infty} A_n$ and

$$\sum_{n=1}^{\infty} \mu(A_n) < \overline{\mu}(A) + \varepsilon.$$

Let $B = \bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$. Note that

$$\nu(B) = \nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{k\to\infty} \nu\left(\bigcup_{n=1}^{k} A_n\right) = \lim_{k\to\infty} \overline{\mu}\left(\bigcup_{n=1}^{k} A_n\right) = \overline{\mu}\left(\bigcup_{n=1}^{\infty} A_n\right) = \overline{\mu}(B).$$

Moreover

$$\overline{\mu}(B) \leq \sum_{n=1}^{\infty} \mu(A_n) < \overline{\mu}(A) + \varepsilon < \infty.$$

It follows

$$\overline{\mu}(B \setminus A) < \varepsilon,$$

so that

$$\overline{\mu}(A) \leq \overline{\mu}(B) = \nu(B) = \nu(A) + \nu(B \setminus A) \leq \nu(A) + \overline{\mu}(B \setminus A) < \nu(A) < \varepsilon.$$

Since ε was given arbitrarily, we have $\overline{\nu}(A) \leq \nu(A)$. Since the reverse inequality is given in (b), we thus conclude $\overline{\mu}(A) = \nu(A)$.

(d) Say $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ is such that $X = \bigcup_{n=1}^{\infty} A_n$ with $\mu(A_n) < \infty$. Write $A_n' = A_n \setminus \bigcup_{m=1}^{n-1} A_m$ so that

$$X = \dot{\bigcup_{n=1}^{\infty}} A_n'.$$

Therefore,

$$\overline{\mu}(A) = \overline{\mu}\left(\dot{\bigcup_{n=1}^{\infty}} \left(A \cap A_n'\right)\right) = \sum_{n=1}^{\infty} \overline{\mu}\left(A \cap A_n'\right) = \sum_{n=1}^{\infty} \nu\left(A \cap A_n'\right) = \nu(A).$$

QED

6. Lebesgue-Stieltjes Measures on $\mathbb{R}$

Suppose we have a measure space $(\mathbb{R}, \mathrm{Bor}(\mathbb{R}), \mu)$, where we are working with the usual topology on $\mathbb{R}$. We further assume that

for all compact $K \subseteq \mathbb{R}$, $\mu(K) < \infty$.

We consider

$$F : \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \begin{cases} \mu([0, x]) & x \geq 0 \\ -\mu((x, 0)) & x < 0 \end{cases}.$$

Then by definition, F is increasing.

Let $(x_n)_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ be a decreasing sequence with $x_n \to x \in \mathbb{R}$. In case $x \geq 0$,

$$F(x) = \mu([0,x]) = \mu\left(\bigcap_{n=1}^{\infty} [0, x_n]\right) = \lim_{n\to\infty} \mu([0,x_n]) = \lim_{n\to\infty} F(x_n),$$

where we are using the compactness assumption to use the continuity from above. Hence F is ***right-continuous*** on $[0,\infty)$.

Exercise 1.5.

Show that F is right-continuous on $(-\infty, 0)$. That is, when $x < 0$,

$$F(x) = \lim_{n\to\infty} F(x_n).$$

Example 1.6.

Consider the point-mass measure

$$\mu_0 : \mathrm{Bor}(\mathbb{R}) \to [0,\infty]$$
$$A \mapsto \begin{cases} 0 & \text{if } 0 \notin A \\ 1 & \text{if } 0 \in A \end{cases}$$

and the measure space $(\mathbb{R}, \mathrm{Bor}(\mathbb{R}), \mu_0)$.

Then note that,

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases},$$

which is right-continuous but not left-continuous.

The goal of this section is, then:

given an increasing right-continuous $F : \mathbb{R} \to \mathbb{R}$, we make a measure μ_F on $(\mathbb{R}, \mathrm{Bor}(\mathbb{R}))$.

That is, we are doing the converse of the motivation for this section.

The idea is to start with

$$\mu_F((a,b]) = F(b) - F(a), \qquad \forall a, b \in \mathbb{R}, a < b.$$

Let $\mathcal{A}$ be the set of finite unions of half-open intervals of the form $(a,b]$, where $a \in [-\infty, \infty), b \in (-\infty, \infty]$ (we note that when $b = \infty$, we are taking (a,∞) instead of $(a,\infty]$, since we are working with subsets of $\mathbb{R}$).

We note that

$$\mathbb{R} \setminus (a,b] = (-\infty, a] \cup (b, \infty) \in \mathcal{A}$$

so that $\mathcal{A}$ is an algebra.

In addition, we insist

(a) $F(\infty) = \lim_{x\to\infty} F(x)$ and $F(-\infty) = \lim_{x\to-\infty} F(x)$; and

(b) $\mu_F\left(\bigcup_{k=1}^{n} (a_k, b_k]\right) = \sum_{k=1}^{n} F(b_k) - F(a_k)$.

In this way we get a fuction $\mu_F : \mathcal{A} \to [0,\infty]$.

Fact 1.10.

μ_F is a premeasure on $(\mathbb{R}, \mathcal{A})$.

Theorem 1.11.

Consider the above setting. There is a complete measure space $(\mathbb{R}, \mathcal{B}, \overline{\mu_F})$ such that

(a) $\overline{\mu_F}|_{\mathcal{A}} = \mu_F$; and

(b) $\mathrm{Bor}(\mathbb{R}) \subseteq \mathcal{B}$.

Proof. Consider μ_F^* be the outer measure constructed as in Theorem 1.8 and let $\mathcal{B}$ be the σ-algebra of μ_F^*-measurable sets. We set $\overline{\mu_F} = \mu_F^*|_{\mathcal{B}}$. By Theorem 1.8, we know that $(\mathbb{R}, \mathcal{B}, \overline{\mu_F})$ is complete and $\overline{\mu_F}|_{\mathcal{A}} = \mu_F$.

By Theorem 1.8 again, $\mathcal{A} \subseteq \mathcal{B}$ (which was implicit in restricting $\overline{\mu_F}$ to $\mathcal{A}$). In particular, half-open intervals are $\mathcal{B}$, so that

$$(a, b) = \bigcup_{n=1}^{\infty} \left(a, b - \frac{1}{n}\right] \in \mathcal{B}$$

for all $a < b$ in $\mathbb{R}$. Since $\mathcal{B}$ has every open intervals, which generate the Borel σ-algebra on $\mathbb{R}$, it follows $\mathrm{Bor}\,(\mathbb{R}) \subseteq \mathcal{B}$.

QED

Theorem 1.12.

When $F(x) = x$ for all $x \in \mathbb{R}$, then

(a) $\overline{\mu_F}$ is the Lebesgue measure; and

(b) $\mathcal{B}$ is the set of Lebesgue measurable sets.

Def'n 1.11. **Lebesgue-Steltjes Measure**

Any measure of the form $\overline{\mu_F}$ is called a ***Lebesgue-Steltjes measure***.

Theorem 1.13. Regularity of Lebesgue-Steltjes Measures

Let $(\mathbb{R}, \mathcal{B}, \overline{\mu_F})$ as above and let $A \subseteq \mathbb{R}$. The following are equivalent.

(a) $A \in \mathcal{B}$ (i.e. A is μ_F^*-measurable).

(b) For all $\varepsilon > 0$, there is open $U \subseteq \mathbb{R}$ such that $A \subseteq U$ and $\mu_F^*(U \setminus A) < \varepsilon$.

(c) For all $\varepsilon > 0$, there is closed $C \subseteq \mathbb{R}$ such that $C \subseteq A$ and $\mu_F^*(A \setminus C) < \varepsilon$.

(d) There exists a G_δ-set[1] such that $A \subseteq G$ and $\mu_F^*(G \setminus A) = 0$.

(e) There exists a F_σ-set[2] such that $F \subseteq A$ and $\mu_F^*(A \setminus F) = 0$.

---

[1] A set is G_δ if it is a countable intersection of open sets.
[2] A set is F_σ if it is a countable union of closed sets.

Proof. (1) $\Longrightarrow$ (2) Assume $A \in \mathcal{B}$ and let $\varepsilon > 0$ be given.

Case 1. *Suppose A is bounded.*

Then $A \subseteq (a, b]$ and $\overline{\mu_F}(A) \leq F(b) - F(a) < \infty$. We may find $\{(a_n, b_n]\}_{n=1}^{\infty}$ such that

$$B = \bigcup_{n=1}^{\infty} (a_n, b_n]$$

contains A and

$$\overline{\mu_F}(B) < \overline{\mu_F}(A) + \frac{\varepsilon}{2}.$$

Now, choose $c_n > b_n$ such that

$$F(c_n) < F(b_n) + \frac{\varepsilon}{2^{n+1}}$$

by the right-continuity of F. Let $U = \bigcup_{n=1}^{\infty} (a_n, c_n)$. Since $A \in \mathcal{B}$, we have

$$\overline{\mu_F}(B) = \overline{\mu_F}(A) + \overline{\mu_F}(B \setminus A)$$

by Caratheodory measurability condition (Def'n 1.9). So by excision,

$$\overline{\mu_F}(B \setminus A) = \overline{\mu_F}(B) - \overline{\mu_F}(A) < \frac{\varepsilon}{2}.$$

Hence

$$\overline{\mu_F}\left(U \setminus A\right) \leq \overline{\mu_F}\left(U \setminus B\right) + \overline{\mu_F}\left(B \setminus A\right) < \overline{\mu_F}\left(\bigcup_{n=1}^{\infty}\left(b_n, c_n\right)\right) + \frac{\varepsilon}{2} \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} + \frac{\varepsilon}{2} = \varepsilon.$$

(End of Case 1)

Case 2. Let $A \in \mathcal{B}$ and consider $A_n = A \cap [-n, n]$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ be given and choose open U_n such that $A_n \subseteq U_n$ and

$$\mu_F^*\left(U_n \setminus A_n\right) < \frac{\varepsilon}{2^n}$$

for all $n \in \mathbb{N}$. Consider $U = \bigcup_{n=1}^{\infty} U_n$. Then $A = \bigcup_{n=1}^{\infty} A_n \subseteq U$ and

$$\mu_F^*\left(U \setminus A\right) \leq \mu_F^*\left(\bigcup_{n=1}^{\infty}\left(U_n \setminus A_n\right)\right) \leq \sum_{n=1}^{\infty} \mu_F^*\left(U_n \setminus A_n\right) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

(End of Case 2)

(2) $\Longrightarrow$ (4) For every $n \in \mathbb{N}$, find open $U_n \subseteq \mathbb{R}$ containing A such that

$$\mu_F^*\left(U_n \setminus A\right) < \frac{1}{n}.$$

Take

$$G = \bigcap_{n=1}^{\infty} U_n.$$

Then $A \subseteq G$ and

$$\mu_F^*\left(G \setminus A\right) \leq \mu_F^*\left(U_n \setminus A\right) < \frac{1}{n}$$

for all $n \in \mathbb{N}$. Thus $\mu_F^*\left(G \setminus A\right) = 0$.

(4) $\Longrightarrow$ (1) Take a G_δ-set $G \subseteq \mathbb{R}$ containing A with $\mu^*\left(G \setminus A\right) = 0$. In particular, we have that $G \setminus A \in \mathcal{B}$.[1] Since every open set is in $\mathcal{B}$ and $\mathcal{B}$ is closed under countable intersection, $G \in \mathcal{B}$ as a countable intersection of open sets, and

$$A = G \setminus \left(G \setminus A\right) \in \mathcal{B}.$$

(1) $\Longrightarrow$ (3) Let $A \in \mathcal{B}$ and let $\varepsilon > 0$. Since $X \setminus A \in \mathcal{B}$, we may find open $U \supseteq X \setminus A$ such that

$$\mu_F^*\left(U \setminus \left(X \setminus A\right)\right) < \varepsilon.$$

Letting $C = X \setminus U$, $C \subseteq A$ and

$$\mu_F^*\left(A \setminus C\right) = \mu_F^*\left(U \setminus \left(X \setminus A\right)\right) < \varepsilon.$$

(3) $\Longrightarrow$ (5) Choose $C_n \subseteq A$ such that

$$\mu_F^*\left(A \setminus C_n\right) < \frac{1}{n}$$

for all $n \in \mathbb{N}$ and let

$$K = \bigcup_{n=1}^{\infty} C_n.$$

(5) $\Longrightarrow$ (1) Let K be a F_σ-set contained in A with $\mu_F^*\left(A \setminus F\right) = 0$. Then we observe that $A = \left(A \setminus F\right) \cup F \in \mathcal{B}$.

[1] See the proof of Theorem 1.7, Caratheodory theorem.

QED

II. Measurable Functions

1. Measurable Functions

Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be measurable spaces. We care about functions $f : X \to Y$ which relay information about the measurable spaces.

Def'n 2.1. **Measurable** Function

Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be measurable spaces. We say $f : X \to Y$ is ***measurable*** if

$$f^{-1}(B) = \{x \in X : f(x) \in B\} \in \mathcal{A}, \qquad \forall B \in \mathcal{B}.$$

Before we proceed, here is a convention that we are going to use. Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$ and let $(X, \mathcal{A})$. We say

$$f : X \to Y \text{ is measurable} \iff f \text{ is measurable with respect to } (X, \mathcal{A}), (\mathbb{F}, \mathrm{Bor}(\mathbb{F})).$$

By Assignment 1, we see that

$$f : X \to Y \text{ is measurable} \iff \text{for all open } B, f^{-1}(B) \in \mathcal{A},$$

since $\mathrm{Bor}(\mathbb{F})$ is generated by open subsets of $\mathbb{F}$. In case $\mathbb{F} = \mathbb{R}$, we can replace B with open interval, since every open subset of $\mathbb{R}$ is a countable union of open intervals.

Recall the following trick for analysis. Let $a < b$ in $\mathbb{R}$. Then

$$\begin{aligned}
(a, b] &= \bigcap_{n=1}^{\infty} \left(a, b + \frac{1}{n}\right) \\
(a, b) &= \bigcup_{n=1}^{\infty} \left(a, b - \frac{1}{n}\right] \\
[a, b] &= \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right). \\
(a, \infty) &= \bigcup_{n=1}^{\infty} (a, a + n) \\
(a, b] &= (-\infty, b] \cap (a, \infty) \\
&\vdots
\end{aligned}$$

That is, all interval types independently generate $\mathrm{Bor}(\mathbb{R})$.

Proposition 2.1.

Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$. The following are equivalent.

(a) f is measurable.

(b) For all $\alpha \in \mathbb{R}$, $f^{-1}((\alpha, \infty)) \in \mathcal{A}$.

(c) For all $\alpha \in \mathbb{R}$, $f^{-1}([\alpha, \infty)) \in \mathcal{A}$.

(d) For all $\alpha \in \mathbb{R}$, $f^{-1}((-\infty, \alpha)) \in \mathcal{A}$.

(e) For all $\alpha \in \mathbb{R}$, $f^{-1}((-\infty, \alpha]) \in \mathcal{A}$.

Proposition 2.2.

Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{C}$. The following are equivalent. Then

$$f \text{ is measurable} \iff \operatorname{Re} \circ f \text{ and } \operatorname{Im} \circ f \text{ are measurable.}$$

Proof Sketch. ($\Longleftarrow$) Every open $U \subseteq \mathbb{C}$ can be written as a countable union of open rectangles $(a, b) \times (c, d)$. Then

$$f^{-1}\left((a, b) \times (c, d)\right) = (\operatorname{Re} \circ f)^{-1}\left((a, b)\right) \cap (\operatorname{Im} \circ f)^{-1}\left((c, d)\right).$$

($\Longrightarrow$) Note that

$$(\operatorname{Re} \circ f)^{-1}\left((a, b)\right) = f^{-1}(V)$$

where

$$V = \{x + iy : a < x < b\}.$$

Similarly,

$$(\operatorname{Im} \circ f)^{-1}\left((c, d)\right) = f^{-1}(H)$$

where

$$H = \{x + iy : c < y < d\}.$$

QED

Proposition 2.3.

Let (X, τ) be a topological space. If $f : X \to \mathbb{F}$ is continuous, then f is measurable.

Proof. It suffices to check that $f^{-1}(U) \in \operatorname{Bor}(X)$ for all open $U \subseteq \mathbb{F}$, which is guaranteed by the continuity of f.

QED

Proposition 2.4.

Let $(X, \mathcal{A})$ be a measurable space and let $f, g : X \to \mathbb{F}$ be measurable.

(a) For any $\lambda \in \mathbb{F}$, $\lambda f + g$ is measurable.

(b) fg is measurable.

(c) If $g(x) \neq 0$ for all $x \in X$, then $\frac{1}{g}$ is measurable.

Proof. By considering Proposition 2.2, we assume $\mathbb{F} = \mathbb{R}$.

(a) Suppose $\lambda > 0$. Then given $\alpha \in \mathbb{R}$,

$$(\lambda f)^{-1}\left((\alpha, \infty)\right) = \{x \in X : \lambda f(x) > \alpha\} = \left\{x \in X : f(x) > \frac{\alpha}{\lambda}\right\} = f^{-1}\left(\left(\frac{\alpha}{\lambda}, \infty\right)\right),$$

which is measurable.

In case $\lambda < 0$,

$$(\lambda f)^{-1}\left((\alpha, \infty)\right) = f^{-1}\left(\left(-\infty, \frac{\alpha}{\lambda}\right)\right)$$

is measurable.

When $\lambda = 0$, λf is the constant 0 function, which is trivially measurable.

Let $\alpha \in \mathbb{R}$. Then

$$\begin{aligned}(f + g)^{-1}\left((\alpha, \infty)\right) &= \{x \in X : f(x) + g(x) > \alpha\} = \{x \in X : f(x) > \alpha - g(x)\} \\ &= \bigcup_{q \in \mathbb{Q}} \left(\{x \in X : f(x) > q\} \cap \{x \in X : g(x) > \alpha - q\}\right) = \bigcup_{q \in \mathbb{Q}} \left(f^{-1}\left((q, \infty)\right) \cap g^{-1}(\alpha - q, \infty)\right),\end{aligned}$$

which is measurable as a countable union of measurable sets.

(b) Note

$$(f+g)^2 = f^2 + 2fg + g^2.$$

Hence it suffices to show that f^2 is measurable. Let $\alpha \in \mathbb{R}$.

Suppose $\alpha \geq 0$. Then

$$\begin{aligned} f^{-1}((\alpha,\infty)) = \left\{x \in X : f(x)^2 > \alpha\right\} &= \left\{x \in X : f(x) > \sqrt{\alpha}\right\} \cup \left\{x \in X : f(x) < -\sqrt{\alpha}\right\} \\ &= f^{-1}\left(\left(\sqrt{\alpha},\infty\right)\right) \cup f^{-1}\left(\left(-\infty,-\sqrt{\alpha}\right)\right) \end{aligned}$$

is a union of measurable of measurable sets.

If $\alpha < \infty$, then

$$\left(f^2\right)^{-1}((\alpha,\infty)) = \left\{x \in X : f(x)^2 > \alpha\right\} = X$$

is measurable.

(c) Let $\alpha \in \mathbb{R}$. Suppose $\alpha > 0$. Then

$$\begin{aligned} \left(\frac{1}{g}\right)^{-1}((-\infty,\alpha)) = \left\{x \in X : \frac{1}{g(x)} < \alpha\right\} &= \left\{x \in X : g(x) > \frac{1}{\alpha}\right\} \cup \left\{x \in X : g(x) < 0\right\} \\ &= g^{-1}\left(\left(\frac{1}{\alpha},\infty\right)\right) \cup g^{-1}((-\infty,0)). \end{aligned}$$

The cases where $\alpha < 0, \alpha = 0$ are similar.

QED

Notation 2.2. $\overline{\mathbb{R}}$

We write $\overline{\mathbb{R}}$ to denote

$$\overline{\mathbb{R}} = [-\infty,\infty].$$

Def'n 2.3. **Borel σ-algebra** of Subsets of $\overline{\mathbb{R}}$

We define the ***Borel σ-algebra*** of subsets of $\overline{\mathbb{R}}$, denoted as $\mathrm{Bor}\left(\overline{\mathbb{R}}\right)$, by

$$\mathrm{Bor}\left(\overline{\mathbb{R}}\right) = \left\{A \subseteq \overline{\mathbb{R}} : A \cap \mathbb{R} \in \mathrm{Bor}(\mathbb{R})\right\}.$$

To show that $\mathrm{Bor}\left(\overline{\mathbb{R}}\right)$ is *really Borel*, we consider the following metric on $\overline{\mathbb{R}}$. Define

$$\begin{aligned} d : \overline{\mathbb{R}}^2 &\to [0,\infty) \\ (x,y) &\mapsto \left|\arctan(x) - \arctan(y)\right|, \end{aligned}$$

where $\arctan(-\infty) = -\frac{\pi}{2}$, $\arctan(\infty) = \frac{\pi}{2}$.

Exercise 2.1.

Show that $\mathrm{Bor}\left(\overline{\mathbb{R}}\right)$ is generated by the open subsets of $\left(\overline{\mathbb{R}}, d\right)$.

$\mathrm{Bor}\left(\overline{\mathbb{R}}\right)$ is (independently) generated by intervals of the form $(\alpha,\infty]$, $[-\infty,\alpha)$.

Proposition 2.5.

Let $(f_n)_{\mathbb{R}}^{\infty}$ be a sequence of measurable functions from X to $\mathbb{R}$.

(a) $\sup_{n\in\mathbb{N}} f_n$ is measurable.

(b) $\inf_{n\in\mathbb{N}} f_n$ is measurable.

(c) $\limsup_{n\in\mathbb{N}} f_n$ is measurable.

(d) $\liminf_{n\in\mathbb{N}} f_n$ is measurable.

Proof.

(a) Note that, given $\alpha \in \mathbb{R}$,

$$\left(\sup_{n\in\mathbb{N}} f_n\right)^{-1}((\alpha,\infty]) = \left\{x \in X : \sup_{n\in\mathbb{N}} f_n(x) > \alpha\right\} = \bigcup_{n\in\mathbb{N}} \{x \in X : f_n(x) > \alpha\} = \bigcup_{n\in\mathbb{N}} f_n^{-1}((\alpha,\infty)).$$

(b) It suffices to note that $\inf_{n\in\mathbb{N}} f_n = -\sup_{n\in\mathbb{N}}(-f_n)$.

(c) Recall that

$$\limsup_{n\in\mathbb{N}} f_n = \lim_{n\to\infty} \sup_{k\geq n} f_k = \inf_{n\in\mathbb{N}} \sup_{k\geq n} f_k.$$

Hence by (a), (b), $\limsup_{n\in\mathbb{N}} f_n$ is measurable.

(d) Similar to (c),

$$\liminf_{n\in\mathbb{N}} f_n = \sup_{n\in\mathbb{N}} \inf_{k\geq n} f_k.$$

Hence $\liminf_{n\in\mathbb{N}} f_n$ is measurable.

QED

Corollary 2.5.1.

Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions from X to $\mathbb{R}$. If $f_n \to x$ pointwise, then f is measurable.

Proof. Note that

$$f_n \to x \iff \liminf_{n\in\mathbb{N}} f_n = \limsup_{n\in\mathbb{N}} f_n = \lim_{n\to\infty} f_n.$$

QED

Let $(X, \mathcal{A})$ be a measurable space. Then given measurable $f : X \to \mathbb{F}$ and continuous $g : \mathbb{F} \to \mathbb{F}$, $g \circ f$ is measurable, as for any open $U \subseteq \mathbb{F}$,

$$(g \circ f)^{-1}(U) = f^{-1}\left(g^{-1}(U)\right),$$

where $g^{-1}(U)$ is open.

In particular, this gives alternative proofs that f^2, $\frac{1}{f}$, $\mathrm{Re} \circ f$, $\mathrm{Im} \circ f$ are measurable. Moreover, $|f|$ is measurable.

Def'n 2.4. **μ-almost Everywhere** Predicate

Let $(X, \mathcal{A}, \mu)$ be a measure space and let P be a predicate on X. We say P is true ***μ-almost everywhere*** (or ***μ-ae***) if there exists $N \in \mathcal{A}$ with $\mu(N) = 0$ such that $P(x)$ is true for all $x \in X \setminus N$.

Note that the definition of μ-almost everywhere does not say that

$$N = \{x \in X : P(x) \text{ is false}\}$$

is measurable. But in case μ is complete, N is measurable with $\mu(N) = 0$.

Proposition 2.6.

Let $(X, \mathcal{A}, \mu)$ be a complete measure space and let $f : X \to \mathbb{F}$ be measurable. Suppose that $g : X \to \mathbb{F}$ is such that $f = g$ μ-ae. Then g is measurable.

Proof. Let $N \in \mathcal{A}$ be such that $\mu(N) = 0$ with $f = g$ on $X \setminus N$. Then given any measurable $U \subseteq \mathbb{R}$,

$$g^{-1}(U) = \left(g^{-1}(U) \cap N\right) \cup \left(g^{-1}(U) \setminus N\right).$$

Note that $g^{-1}(U) \cap N \subseteq N$ so has measure 0, which means $g^{-1}(U) \cap N \in \mathcal{A}$ by the completeness of μ. Moreover, $f = g$ on $X \setminus N$ so that $g^{-1}(U) \setminus N = f^{-1}(U) \setminus N$, which is measurable. Thus $g^{-1}(U)$ is measurable, as required.

QED

Def'n 2.5. **Characteristic Function** of a Subset
Let X be a set and let $A \subseteq X$. The ***characteristic function*** of A, denoted as χ_A, is defined as

$$\begin{aligned} \chi_A &: X \to \mathbb{R} \\ x &\mapsto \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}. \end{aligned}$$

Note that, given $A \subseteq X$,

$$\chi_A \text{ is measurable} \iff A \text{ is measurable.}$$

Def'n 2.6. **Simple** Function
Let $(X, \mathcal{A})$ be a measurable space. We say $\varphi : X \to \mathbb{F}$ is ***simple*** if

$$\varphi = \sum_{k=1}^{n} a_k \chi_{A_k}$$

where $a_1, \ldots, a_n \in \mathbb{F}$ and $A_1, \ldots, A_n \in \mathcal{A}$ are pairwise disjoint.

Let $(X, \mathcal{A})$ be a measurable space and let $\varphi : X \to \mathbb{F}$. Then

$$\varphi \text{ is simple} \iff \varphi \text{ is measurable and } \varphi(X) \text{ is finite.}$$

To see the reverse direction, suppose φ is measurable and $\varphi(X)$ is finite, say

$$\varphi(X) = \{a_k\}_{k=1}^{n}.$$

Then each $A_k = \varphi^{-1}(\{a_k\})$ is measurable and $\varphi = \sum_{k=1}^{n} a_k \chi_{a_k}$.

The goal of this subsection is to show

$$f : X \to \mathbb{R} \text{ is measurable} \iff f \text{ is a pointwise limit of simple functions.}$$

Proposition 2.7.
Let $(X, \mathcal{A})$ be a measurable space and let $f : X \to \mathbb{R}$ be measurable and bounded. Then for all $\varepsilon > 0$, there are simple $\varphi_\varepsilon, \psi_\varepsilon : X \to \mathbb{R}$ such that

(a) $\varphi_\varepsilon \leq f \leq \psi_\varepsilon$; and
(b) $0 \leq \psi_\varepsilon - \varphi_\varepsilon < \varepsilon$.

Proof. Let $\varepsilon > 0$. Say $f(X) \subseteq [a, b)$. Let $y_0, \ldots, y_n$ be given such that

$$a = y_0 < y_1 < \cdots < y_n = b,$$

where each $y_k - y_{k-1} < \varepsilon$. Let $I_k = [y_{k-1}, y_k)$. Then each $A_k = f^{-1}(I_k)$ is measurable. Define

$$\varphi = \sum_{k=1}^{n} y_{k-1} \chi_{A_k}, \psi = \sum_{k=1}^{n} y_k \chi_{A_k}.$$

Then for any $x \in X$, we have $x \in I_k$ for some k, so that $\varphi(x) = y_{k-1} \leq f(x) \leq y_k = \psi(x)$.
Moreover,

$$0 < \psi(x) - \varphi(x) = y_k - y_{k-1} < \varepsilon.$$

QED

Theorem 2.8. Simple Approximation

Let $(X, \mathcal{A})$ be a measure space and let $f : X \to \mathbb{R}$. Then

$$f \text{ is measurable} \iff \text{there are simple } \varphi_1, \varphi_2, \ldots : X \to \mathbb{R} \text{ with } \varphi_n \to f \text{ pointwise and } \left|\varphi_n\right| \leq f \text{ for all } n \in \mathbb{N}.$$

Proof. $(\Longleftarrow)$ Recall that pointwise limit of measurable functions is measurable, where each φ_n is measurable.

$(\Longrightarrow)$ We split into few cases.

Case 1. *Suppose $f \geq 0$.*

Let

$$A_n = \{x \in X : f(x) \leq n\}.$$

Note that

$$\mathcal{A}' = \{B \cap A_n : B \in \mathcal{A}\}$$

is a σ-algebra of subsets of A_n. Then $(A_n, \mathcal{A}')$ is a measurable space and $f|_{A_n}$ is measurable, since

$$(f|_{A_n})^{-1}(U) = f^{-1}(U) \cap A_n \in \mathcal{A}'$$

for all measurable $U \subseteq \mathbb{R}$. Moreover, by definition $f|_{A_n}$ is bounded.

Hence by Proposition 2.7, we can find simple $\varphi_m, \psi_m : A_n \to \mathbb{R}$, $m \in \mathbb{N}$, such that

$$0 \leq \varphi_m \leq f \leq \psi_m$$

and

$$0 \leq \psi_m - \varphi_m < \frac{1}{m}$$

for all $m \in \mathbb{N}$ on A_n.

Extend $\varphi_m(x) = n$ for all $x \in X \setminus A_n$, so that $\varphi_m \leq f$ on X.

Now fix $x \in X$. Then $x \in A_N$ for some N, and so

$$0 \leq f(x) - \varphi_N(x) \leq \psi_N(x) - \varphi_N(x) < \frac{1}{N}.$$

This means given any $\varepsilon > 0$ we can take $N' > N$ so that $\frac{1}{N'} < \varepsilon$, which means for all $m \geq N'$,

$$0 \leq f(x) - \varphi_m(x) < \frac{1}{N'} < \varepsilon.$$

Thus $\varphi_m \to f$ pointwise.

(End of Case 1)

Case 2. *Consider the general case on f. That is, we only assume that f is measurable.*

Let

$$A = \{x \in X : f(x) \geq 0\} \in \mathcal{A}$$
$$B = \{x \in X : f(x) < 0\} \in \mathcal{A}$$

and let $g = f\chi_A$, $h = -f\chi_B$, so that both $g, h \geq 0$. By Case 1, there exist $(\varphi_n)_{n=1}^{\infty}, (\psi_n)_{n=1}^{\infty}$ such that $\varphi_n \nearrow g$ and $\psi_n \nearrow h$ pointwise as $n \to \infty$. Then $f = g - h$ so that $\varphi_n - \psi_n \to g - h = f$ pointwise. Moreover,

$$\left|\varphi_n - \psi_n\right| \leq \left|\varphi_n\right| + \left|\psi_n\right| = \varphi_n + \psi_n \leq g + h = |f|.$$

(End of Case 2)

QED

Note that in the proof, we know that, given a fixed $n \in \mathbb{N}$, we have

$$0 \leq f - \varphi_m \leq \frac{1}{m}$$

on A_n. That is,

$$0 \leq f(x) - \varphi_m(x) \leq \frac{1}{m}, \qquad \forall x \in A_n,$$

so that $\varphi_m \to f$ *uniformly* as $m \to \infty$ on A_n.

Suppose that $f \geq 0$ is measurable and that

$$0 \leq \varphi_n \leq f, \qquad \forall n \in \mathbb{N}$$

with $\varphi_n \to f$ pointwise. Then by taking $\psi_n = \max\{\varphi_1, \ldots, \varphi_n\}$, φ_n is still simple. Then

$$0 \leq \psi_n \leq f, \qquad \forall n \in \mathbb{N}$$

as well, so that $\psi_n \nearrow f$ pointwise as $n \to \infty$.

3. Two Theorems

We are going to prove two useful theorems in measure theory in this subsection.

Lemma 2.9.

Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $(f_n)_{n=1}^{\infty} \in \left(\mathbb{R}^X\right)^{\mathbb{N}}$ be a sequence of measurable functions such that $f_n \to f$ pointwise for some measurable $f: X \to \mathbb{R}$. Then for every $\alpha, \beta > 0$, there exist $B \in \mathcal{A}$, $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \alpha, \qquad \forall x \in B, n \geq N$$

and

$$\mu(X \setminus B) < \beta.$$

Proof Sketch. Let

$$A_n = \{x \in X : \forall k \geq n\, [f_k(x) - f(x) < \alpha]\}, \qquad \forall n \in \mathbb{N}.$$

Then

$$A_n = \bigcap_{k \geq n} |f_k - f|^{-1}((-\infty, \alpha)),$$

which is measurable. Since $f_n \to f$ pointwise, we have

$$X = \bigcup_{n=1}^{\infty} A_n.$$

We also have an increasing chain

$$A_1 \subseteq A_2 \subseteq \cdots,$$

so that

$$\lim_{n \to \infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu(X) < \infty$$

by the continuity from below. Hence we may find $N \in \mathbb{N}$ such that

$$\mu(X) - \mu(A_n) < \beta, \qquad \forall n \geq N.$$

Since $\mu(X) < \infty$, each $\mu(A_n) < \infty$ as well, so that

$$\mu(X \setminus A_n) < \beta, \qquad \forall n \geq N.$$

By taking $B = A_N$, we are done.

QED

Theorem 2.10. Egoroff

Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $(f_n)_{n=1}^{\infty} \in \left(\mathbb{R}^X\right)^{\mathbb{N}}$ be a sequence of measurable functions such that $f_n \to f$ pointwise for some measurable $f : X \to \mathbb{R}$. Then for all $\varepsilon > 0$ there exists $A \in \mathcal{A}$ such that

(a) $f_n \to f$ uniformly on A; and

(b) $\mu(X \setminus A) < \varepsilon$.

Proof. Let $\varepsilon > 0$ be given. For all $n \in \mathbb{N}$, we may find $A_n \in \mathcal{A}$ and $N_n \in \mathbb{N}$ such that

$$\forall x \in A_n, k \geq N_n \left[|f_k(x) - f(x)| < \frac{1}{n}\right]$$

and

$$\mu(X \setminus A_n) < \frac{\varepsilon}{2^n}.$$

Let

$$A = \bigcap_{n=1}^{\infty} A_n.$$

Given any $\varepsilon' > 0$, by taking $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon'$, we have, for all $k \geq N_n$ and $x \in A$,

$$|f_k(x) - f(x)| < \frac{1}{n} < \varepsilon'.$$

Hence $f_k \to f$ uniformly on A. Finally,

$$\mu(X \setminus A) = \mu\left(\bigcup_{n=1}^{\infty} (X \setminus A_n)\right) \leq \sum_{n=1}^{\infty} \mu(X \setminus A_n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

QED

Let m be the Lebesgue measure on $\mathbb{R}$ and let $A \subseteq \mathbb{R}$ with $m(A) < \infty$. Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions from A to $\mathbb{R}$ that converges to $f : A \to \mathbb{R}$. Then by Egoroff's theorem, for every $\varepsilon > 0$, there is $B \subseteq A$ such that

$$f_n \to f \text{ uniformly on } B$$

and

$$m(A \setminus B) < \frac{\varepsilon}{2}.$$

Then we can find a closed subset $C \subseteq B$ with

$$m(B \setminus C) < \frac{\varepsilon}{2}$$

by the regularity of Lebesgue measure. Then

$$f_n \to f \text{ uniformly on } C$$

and

$$m(A \setminus C) = m(A \setminus B) + m(B \setminus C) < \varepsilon.$$

Hence for the Lebesgue measure (in fact, any Lebesgue-Steltjes measure), we can assume that $f_n \to f$ uniformly on a closed set with arbitrarily small difference.

Lemma 2.11.

Let $A \subseteq \mathbb{R}$ be Lebesgue measurable and let $\varphi : A \to \mathbb{R}$ be Lebesgue-simple. Then for all $\varepsilon > 0$, there exists closed $C \subseteq \mathbb{R}$ and a continuous $g : \mathbb{R} \to \mathbb{R}$ such that

(a) $C \subseteq A$;

(b) $\varphi = g$ on C; and

(c) $m(A \setminus C) < \varepsilon$.

Proof. Write

$$\varphi = \sum_{i=1}^{n} a_i \chi_{A_i},$$

where each $a_i \neq 0$ and $A_i = \varphi^{-1}(\{a_i\})$. Let $A_0 = \varphi^{-1}(\{0\})$. We also insist that $a_i \neq a_j$ for $i \neq j$. Then

$$A = \dot{\bigcup_{i=0}^{n}} A_i.$$

Let $\varepsilon > 0$ be given. For each i, let C_i be a closed such that $C_i \subseteq A_i$ and

$$m(A_i \setminus C_i) < \frac{\varepsilon}{n+1}$$

by regularity of Lebesgue measure. Let

$$C = \dot{\bigcup_{i=0}^{n}} C_i,$$

which is closed. Since φ is continuous on each C_i and $C_i \cap C_j = \emptyset$, φ is continuous on C. Then there is continuous $g : \mathbb{R} \to \mathbb{R}$ that extends $\varphi : C \to \mathbb{R}$. Finally,

$$m(A \setminus C) = m\left(\dot{\bigcup_{i=0}^{n}} A_i \setminus C_i\right) = \sum_{i=0}^{n} m(A_i \setminus C_i) < \varepsilon.$$

QED

Theorem 2.12. Lusin

Let $f : A \to \mathbb{R}$ be Lebesgue measurable. Then for all $\varepsilon > 0$, there exists continuous $g : \mathbb{R} \to \mathbb{R}$ and closed $C \subseteq \mathbb{R}$ such that

(a) $C \subseteq A$;

(b) $f = g$ on C; and

(c) $m(A \setminus C) < \varepsilon$.

Proof. We split the proof into two cases. Let $\varepsilon > 0$ be given.

Case 1. *Suppose $m(A) < \infty$.*

Let $(\varphi_n)_{n=1}^{\infty}$ be a sequence of simple functions such that $\varphi_n \to f$ pointwise by simple approximation. For each $n \in \mathbb{N}$, let $C_n \subseteq \mathbb{R}$ be closed and $g_n : \mathbb{R} \to \mathbb{R}$ be continuous such that $\varphi_n = g_n$ on C_n and

$$m(A \setminus C_n) < \frac{\varepsilon}{2^{n+1}}.$$

By Egoroff, let C_0 be the closed set such that

$$\varphi_n \to f \text{ uniformly on } C_0$$

and

$$m(A \setminus C_0) < \frac{\varepsilon}{2}.$$

Let

$$C = \bigcap_{n=0}^{\infty} C_n.$$

Then,

$$g_n = \varphi_n \to f \text{ uniformly on } C.$$

In particular, f is continous on C. This means we can extend $f|_C$ to continuous $g : \mathbb{R} \to \mathbb{R}$. Finally,

$$m(A \setminus C) = m\left(A \setminus \bigcap_{n=0}^{\infty} C_n\right) = m\left(\bigcup_{n=0}^{\infty} (A \setminus C_n)\right) \leq m(A \setminus C_0) + \sum_{n=1}^{\infty} m(A \setminus C_n) < \varepsilon.$$

(End of Case 1)

Case 2. *Suppose $m(A) < \infty$.*

This is left as an exercise.

(End of Case 2)

QED

III. Integration

1. Nonnegative Measurable Functions

Def'n 3.1. **Integral** of a Nonnegative Simple Function
Let $(X, \mathcal{A}, \mu)$ be a measure space and let

$$\varphi = \sum_{i=1}^{n} a_i \chi_{A_i} : X \to [0, \infty]$$

be simple. We define the ***integral*** of φ, denoted as $\int \varphi d\mu$, by

$$\int \varphi d\mu = \sum_{i=1}^{n} a_i \mu(A_i).$$ [1]

[1] For this, we use the convention $0\infty = \infty 0 = 0$.

Proposition 3.1.
Let $\varphi : X \to [0, \infty]$ be simple. Then $\int \varphi d\mu$ is well-defined.

Proof Sketch. Say

$$\varphi = \sum_{i=1}^{n} a_i \chi_{E_i} = \sum_{j=1}^{m} b_j \chi_{F_j}.$$

Suppose that $\varphi(X) = \{c_1, \dots, c_p\}$ and let

$$A_k = \varphi^{-1}(\{c_k\}), \qquad \forall k \in \{1, \dots, p\}.$$

Then

$$\sum_{i=1}^{n} a_i \mu(E_i) = \sum_{k=1}^{p} c_k \sum_{i:a_i=c_k} \mu(E_i) = \sum_{k=1}^{p} c_k \mu\left(\dot{\bigcup_{i:a_i=c_k}} E_i\right) = \sum_{k=1}^{p} c_k \mu(A_k).$$

By symmetry, $\sum_{j=1}^{m} b_j \chi_{F_j} = \sum_{k=1}^{p} c_k \mu(A_k)$. Thus $\int \varphi d\mu$ is well-defined.

QED

Proposition 3.2.
Let $\varphi, \psi : X \to [0, \infty]$ be simple.

(a) If $\alpha \geq 0$, then

$$\int \alpha\varphi d\mu = \alpha \int \varphi d\mu.$$

(b)

$$\int \varphi + \psi d\mu = \int \varphi d\mu + \int \psi d\mu.$$

(c) $\varphi \leq \psi \implies \int \varphi d\mu \leq \int \psi d\mu$.

Proof. Write

$$\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}, \psi = \sum_{j=1}^{m} b_j \chi_{F_j}$$

and let $a_0 = b_0 = 0$, with $E_0 = X \setminus \bigcup_{i=1}^n E_i, F_0 = X \setminus \bigcup_{j=1}^m F_j$. This means

$$\varphi = \sum_{i=0}^n a_i \chi_{E_i}, \psi = \sum_{j=0}^m b_j \chi_{F_j}$$

as well.

(a) Note that

$$\int \alpha\varphi d\mu = \sum_{i=1}^n \alpha a_i \mu\left(A_i\right) = \alpha \sum_{i=1}^n a_i \mu\left(A_i\right) = \alpha \int \varphi d\mu.$$

(b) For all $i \in \{0, \ldots, n\}, j \in \{0, \ldots, n\}$, let

$$A_{i,j} = E_i \cap F_j.$$

Then it follows that

$$\varphi = \sum_{i=0}^n \sum_{j=0}^m a_i \chi_{A_{i,j}}$$

and

$$\psi = \sum_{j=0}^m \sum_{i=0}^n b_j \chi_{A_{i,j}}.$$

Thus

$$\int \varphi + \psi d\mu = \sum_{i=0}^n \sum_{j=0}^m \left(a_i + b_j\right) \mu\left(A_{i,j}\right) = \sum_{i=0}^n \sum_{j=0}^m a_i \mu\left(A_{i,j}\right) + \sum_{j=0}^m \sum_{i=0}^n b_j \mu\left(A_{i,j}\right) = \int \varphi d\mu + \int \psi d\mu.$$

(c) Given $i \in \{0, \ldots, n\}, j \in \{0, \ldots, m\}$, if $A_{i,j} \neq \emptyset$, then $a_i \leq b_j$. Otherwise, $\mu\left(A_{i,j}\right) = 0$. This means

$$a_i \mu\left(A_{i,j}\right) \leq b_j \mu\left(A_{i,j}\right), \qquad \forall i \in \{0, \ldots, n\}, j \in \{0, \ldots, m\},$$

so that

$$\int \varphi d\mu = \sum_{i=0}^n \sum_{j=0}^m a_i \mu\left(A_{i,j}\right) \leq \sum_{j=0}^m \sum_{i=0}^n b_j \mu\left(A_{i,j}\right) = \int \psi d\mu.$$

QED

Def'n 3.2. **Integral** of a Nonnegative Simple Function over a Measurable Subset

Let $\varphi : X \to [0, \infty]$ be simple and let $A \in \mathcal{A}$. We define the ***integral*** of φ over A, denoted as $\int_A \varphi d\mu$, by

$$\int_A \varphi d\mu = \int \varphi \chi_A d\mu.$$

Proposition 3.3.

Let $\varphi : X \to [0, \infty]$ be simple. Define $\nu : \mathcal{A} \to [0, \infty]$ by

$$\nu(A) = \int_A \varphi d\mu.$$

Then ν is a measure on $(X, \mathcal{A})$.

Proof. Write

$$\varphi = \sum_{i=1}^n a_i \chi_{E_i}.$$

We have

$$\nu(\emptyset) = \int \chi_\emptyset \varphi d\mu = 0.$$

Let $\{A_m\}_{m=1}^\infty \subseteq \mathcal{A}$ be a collection of disjoint sets and $A = \dot\bigcup_{m=1}^\infty A_m$. Then

$$\begin{aligned}\nu(A) &= \int_A \varphi d\mu = \int \varphi\chi_A d\mu = \int \sum_{i=1}^n a_i \chi_{E_i}\chi_A d\mu = \int \sum_{i=1}^n a_i \chi_{E_i\cap A} d\mu = \sum_{i=1}^n a_i \mu(E_i \cap A) = \sum_{i=1}^n a_i \mu\left(\dot\bigcup_{m=1}^\infty (E_i \cap A_m)\right) \\ &= \sum_{i=1}^n a_i \sum_{m=1}^\infty \mu(E_i \cap A_m) = \sum_{m=1}^\infty \sum_{i=1}^n a_i \mu(E_i \cap A_m) = \sum_{m=1}^\infty \int_{A_m} \varphi d\mu = \sum_{m=1}^\infty \nu(A_m).\end{aligned}$$

QED

Notation 3.3. $\mathrm{L}^+(X, \mathcal{A}, \mu)$

We write $\mathrm{L}^+(X, \mathcal{A}, \mu)$, or simply L^+ when $(X, \mathcal{A}, \mu)$ is understood, to mean

$$\mathrm{L}^+(X, \mathcal{A}, \mu) = \{f : X \to [0,\infty] : f \text{ is measurable}\}.$$

Def'n 3.4. **Integral** of a L^+-function

Let $f \in \mathrm{L}^+$. We define the ***integral*** of f, denoted as $\int f d\mu$, by

$$\int f d\mu = \sup\left\{\int \varphi d\mu : \varphi : [0,\infty] \to X, \varphi \leq f, \varphi \text{ is simple}\right\}.$$

If $A \in \mathcal{A}$, we define the ***integral*** of f over A, denoted as $\int_A f d\mu$, by

$$\int_A f d\mu = \int f \chi_A d\mu.$$

Proposition 3.4.

Let $f, g \in \mathrm{L}^+$.

(a) If $\alpha \geq 0$, then

$$\int \alpha f d\mu = \alpha \int f d\mu.$$

(b) If $f \leq g$, then

$$\int f d\mu \leq \int g d\mu.$$

Proof.

(a) This is trivial when $\alpha = 0$. For $\alpha > 0$,

$$\begin{aligned}\{\varphi : X \to [0,\infty] : \varphi \leq \alpha f, \varphi \text{ is simple}\} &= \left\{\varphi : X \to [0,\infty] : \frac{1}{\alpha}\varphi \leq f, \varphi \text{ is simple}\right\} \\ &= \{\alpha\psi : \psi : X \to [0,\infty], \psi \leq f, \psi \text{ is simple}\}.\end{aligned}$$

By taking sup, we have the desired equality.

(b) It suffices to note

$$\{\varphi : X \to [0,\infty] : \varphi \leq f, \varphi \text{ is simple}\} \subseteq \{\psi : X \to [0,\infty] : \psi \leq g, \psi \text{ is simple}\}.$$

QED

We are leaving (a one-liner!) proof of $\int f + g d\mu = \int f d\mu + \int g d\mu$ for later.

Lemma 3.5.

Let $\varphi : X \to [0,\infty]$ be simple and let $(A_n)_{n=1}^{\infty} \in \mathcal{A}^{\mathbb{N}}$ be an ascending chain with $X = \bigcup_{n=1}^{\infty} A_n$. Then

$$\lim_{n\to\infty} \int_{A_n} \varphi d\mu = \int \varphi d\mu.$$

Proof. Recall that $\nu : \mathcal{A} \to [0,\infty]$ by

$$\nu(A) = \int_A \varphi d\mu, \qquad \forall A \in \mathcal{A}$$

is a measure. Hence by the continuity from below,

$$\lim_{n\to\infty} \int_{A_n} \varphi d\mu = \lim_{n\to\infty} \nu(A_n) = \nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \nu(X) = \int \varphi d\mu.$$

QED

Theorem 3.6. Monotone Convergence Theorem (MCT)

Let $(f_n)_{n=1}^{\infty} \in \mathrm{L}^{+\mathbb{N}}$ be an increasing sequence and define $f \in \mathrm{L}^+$ by

$$f(x) = \lim_{n\to\infty} f_n(x), \qquad \forall x \in X.$$

Then

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

Proof. For every $x \in X$, $(f_n(x))_{n=1}^{\infty}$ is an increasing sequence. Hence by the MCT for sequences, $\lim_{n\to\infty} f_n(x)$ converges in $[0,\infty]$. Define

$$f(x) = \lim_{n\to\infty} f_n(x), \qquad \forall x \in X.$$

In fact, MCT for sequences tells us that

$$f(x) = \sup_{n\in\mathbb{N}} f_n(x), \qquad \forall x \in X,$$

so that

$$f_1 \leq f_2 \leq \cdots \leq f.$$

This means

$$\int f_1 d\mu \leq \int f_2 d\mu \leq \cdots \leq \int f d\mu$$

using monotonicity of integral, so that

$$\lim_{n\to\infty} \int f_n d\mu = \sup_{n\in\mathbb{N}} \int f_n d\mu \leq \int f d\mu.$$

Let $\varphi : X \to [0,\infty]$ be a simple function with $\varphi \leq f$. Let $\varepsilon \in (0,1)$ and let

$$A_n = \{x \in X : (1-\varepsilon)\varphi(x) \leq f_n(x)\}, \qquad \forall n \in \mathbb{N}.$$

Then

$$A_1 \subseteq A_2 \subseteq \cdots$$

and

$$X = \bigcup_{n=1}^{\infty} A_n,$$

since $f_n(x) \to f(x)$ means there must be $N \in \mathbb{N}$ such that $(1-\varepsilon)\varphi(x) \le f_n(x)$, as $(1-\varepsilon)\varphi(x) < \varphi(x) \le f(x)$. This means

$$(1-\varepsilon)\int \varphi d\mu = \int (1-\varepsilon)\varphi d\mu = \lim_{n\to\infty}\int_{A_n}(1-\varepsilon)\varphi d\mu \le \lim_{n\to\infty}\int_{A_n} f_n d\mu \le \lim_{n\to\infty}\int f_n d\mu.$$

Since the choice of ε was arbitrary, we conclude

$$\int \varphi d\mu \le \lim_{n\to\infty}\int f_n d\mu.$$

But $\int f d\mu$ is the supremum of such φ, so it follows that

$$\int f d\mu \le \lim_{n\to\infty}\int f_n d\mu,$$

as required.

QED

Proposition 3.7.
Let $f, g \in \mathrm{L}^+$. Then

$$\int f + g d\mu = \int f d\mu + \int g d\mu.$$

Proof. By simple approximation, we can find increasing sequence of simple functions $(\varphi_n)_{n=1}^{\infty}, (\psi_n)_{n=1}^{\infty}$ such that $\varphi_n \nearrow f, \psi_n \nearrow g$ pointwise. Thus by the MCT,

$$\int f + g d\mu = \lim_{n\to\infty}\int \varphi_n + \psi_n d\mu = \lim_{n\to\infty}\int \varphi_n d\mu + \int \psi_n d\mu = \int f d\mu + \int g d\mu.$$

QED

Proposition 3.8.
Let $(f_n)_{n=1}^{\infty} \in \mathrm{L}^{+\mathbb{N}}$. Then

$$\int \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty}\int f_n d\mu.$$

Proof. Note that $\left(\sum_{n=1}^{k} f_n\right)_{k=1}^{\infty} \in \mathrm{L}^{+\mathbb{N}}$ is increasing, so that

$$\int \sum_{n=1}^{\infty} f_n d\mu = \int \lim_{k\to\infty}\sum_{n=1}^{k} f_n d\mu = \lim_{k\to\infty}\int \sum_{n=1}^{k} f_n d\mu = \lim_{k\to\infty}\int \sum_{n=1}^{k} f_n d\mu = \sum_{n=1}^{\infty}\int f_n d\mu.$$

QED

Proposition 3.9.
Let $f \in \mathrm{L}^+$. Then

$$\nu : \mathcal{A} \to [0,\infty]$$
$$A \mapsto \int_A f d\mu$$

is a measure.

Proof. Clearly $\nu(\emptyset) = \int_{\emptyset} f d\mu = 0$.
Write $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ be a collection of disjoint sets and let $A = \dot{\bigcup}_{n=1}^{\infty} A_n$. Then

$$\nu(A) = \int f\chi_A d\mu = \int \sum_{n=1}^{\infty} f\chi_{A_n} d\mu = \sum_{n=1}^{\infty} \int_{A_n} f d\mu = \sum_{n=1}^{\infty} \nu(A).$$

QED

Lemma 3.10.
Let $f \in \mathrm{L}^+$. Then

$$\int f d\mu = 0 \iff f = 0 \ \mu\text{-ae.}$$

Proof. ($\Longleftarrow$) Suppose $f = 0$ μ-ae. Let $\varphi : X \to [0, \infty]$ be simple with $\varphi \leq f$, say

$$\varphi = \sum_{i=1}^{n} a_i \chi_{A_i}.$$

Then $\varphi = 0$ ae. This means each $a_i > 0$ implies $\mu(A_i) = 0$. Thus

$$\int \varphi d\mu = 0$$

so that

$$\int f d\mu = 0.$$

($\Longrightarrow$) Suppose $\int f d\mu = 0$. Let

$$A = \{x \in X : f(x) > 0\}$$

and let

$$A_n = \left\{ x \in X : f(x) \geq \frac{1}{n} \right\}, \qquad \forall n \in \mathbb{N}.$$

Then

$$A_1 \subseteq A_2 \subseteq \cdots$$

with

$$\bigcup_{n=1}^{\infty} A_n = A.$$

Therefore

$$\mu(A) = \lim_{n \to \infty} \mu(A_n)$$

and

$$0 = \int f d\mu \geq \int \frac{1}{n} \chi_{A_n} d\mu = \frac{1}{n} \mu(A_n),$$

so that each $\mu(A_n) = 0$. Thus $\mu(A) = 0$, as required.

QED

Proposition 3.11.
Let $f \in \mathrm{L}^+$ and let $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$. Then

$$\int_{A \cup B} f d\mu = \int_A f d\mu + \int_B f d\mu.$$

Proof. Note that

$$\int_{A \cup B} f d\mu = \int f(\chi_A + \chi_B)\, d\mu = \int f \chi_A d\mu + \int f \chi_B d\mu = \int_A f d\mu + \int_B f d\mu.$$

QED

Proposition 3.12.

Let $f \in \mathrm{L}^{+}$ and let $A \in \mathcal{A}$ with $\mu(A) = 0$. Then

$$\int_A f d\mu = 0.$$

Proof. Note that $f\chi_A = 0$ μ-ae.

QED

Proposition 3.13.

Let $(f_n)_{n=1}^{\infty} \in \mathrm{L}^{+\mathbb{N}}$ be such that

$$f_n \leq f_{n+1} \ \mu\text{-ae}, \qquad \forall n \in \mathbb{N}$$

and let $f \in \mathrm{L}^{+\mathbb{N}}$ be such that

$$\lim_{n\to\infty} f_n = f \text{ pointwise } \mu\text{-ae.}$$

Then

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

Proof. Let

$$A_n = \{x \in X : f_n(x) > f_{n+1}(x)\}$$

and let

$$A_0 = \left\{x \in X : \lim_{n\to\infty} f_n(x) \neq f(x)\right\}.$$

Then $\mu(A_n) = 0$ for all $n \in \mathbb{N} \cup \{0\}$. Let $A = \bigcup_{n=0}^{\infty} A_n$, so that $\mu(A) = 0$ as well. We have

$$f_n \chi_{X\setminus A} \leq f_{n+1}\chi_{X\setminus A}, \qquad \forall n \in \mathbb{N}$$

and

$$f_n \chi_{X\setminus A} \to f\chi_{X\setminus A} \text{ pointwise.}$$

By the MCT,

$$\int_{X\setminus A} f_n d\mu \to \int_{X\setminus A} f d\mu.$$

The result then follows from Proposition 3.11 and 3.12.

QED

Theorem 3.14. Fatou's Lemma

Let $(f_n)_{n=1}^{\infty} \in \mathrm{L}^{+\mathbb{N}}$. Then

$$\int \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int f_n d\mu.$$

Proof. Let

$$g_n = \inf_{k\geq n} f_k.$$

Then $(g_n)_{n=1}^{\infty}$ is an increasing sequence in L^+ such that

$$\lim_{n\to\infty} g_n = \liminf_{n\to\infty} f_n$$

pointwise. By the monotone convergence theorem,

$$\int \liminf_{n\to\infty} f_n d\mu = \int \lim_{n\to\infty} g_n d\mu = \lim_{n\to\infty} \int g_n d\mu = \liminf_{n\to\infty} \int g_n d\mu \leq \liminf_{n\to\infty} \int f_n d\mu.$$

QED

Corollary 3.14.1.

Let $(f_n)_{n=1}^{\infty} \in \mathrm{L}^{+\mathbb{N}}$ such that $f_n \to f$ pointwise for some $f \in \mathrm{L}^+$. Then

$$\int f d\mu \leq \liminf_{n\to\infty} \int f_n d\mu.$$

Def'n 3.5. **Integrable** Complex-valued Function
Let $f : X \to \mathbb{C}$ be measurable. We say f is ***integrable*** if

$$\int |f| \, d\mu < \infty.$$

In case $f : X \to \mathbb{R}$ is integrable, we consider the ***positive part*** f^+ and ***negative part*** f^- of f defined as

$$\begin{aligned} f^+ &= \max\{f, 0\}, \\ f^- &= -\min\{f, 0\}. \end{aligned}$$

Note that both f^+, f^- are nonnegative and we define the ***integral*** of f, denoted as $\int f d\mu$, by

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu.$$[1]

Let $f : X \to \mathbb{C}$ be integrable. Then we define the ***integral*** of f, denoted as $\int f d\mu$, by

$$\int f d\mu = \int \operatorname{Re} \circ f d\mu + i \int \operatorname{Im} \circ f d\mu.$$[2]

In case $f : X \to \mathbb{C}$ is measurable, we define

$$\|f\|_1 = \int |f| \, d\mu.$$

[1] Note that $f = f^+ - f^-$ and $|f| = f^+ + f^-$. Hence $f^+, f^- \leq |f|$ so that both $\int f^+ d\mu, \int f^- d\mu$ are finite.
[2] Observe that $|\operatorname{Re} \circ f|, |\operatorname{Im} \circ f| \leq |f|$, so that $\operatorname{Re} \circ f, \operatorname{Im} \circ f$ are integrable.

Notation 3.6. $L^1(X, \mathcal{A}, \mu)$
We define

$$L^1(X, \mathcal{A}, \mu) = \{f : X \to \mathbb{C} : f \text{ is measurable and } \|f\|_1 < \infty\}.$$

We shall write L^1 when $(X, \mathcal{A}, \mu)$ are understood.

We state few results without proof.

Proposition 3.15. Linearity
Let $f, g \in L^1$ and $\alpha \in \mathbb{C}$. Then $\alpha f + g \in L^1$ with

$$\int \alpha f + g d\mu = \alpha \int f d\mu + \int g d\mu.$$

Proposition 3.16. Monotonicity
Let $f, g \in L^1$ be real-valued functions. If $f \leq g$, then

$$\int f d\mu \leq \int g d\mu.$$

Def'n 3.7. **Integral** over a Measurable Set

Let $f \in L^1$. For $A \in \mathcal{A}$, we define the ***integral*** of f over A, denoted as $\int_A f d\mu$, by

$$\int_A f d\mu = \int f \chi_A d\mu.$$

Proposition 3.17.

Let $f \in L^1$ and let $A, B \in \mathcal{A}$ be disjoint. Then

$$\int_{A \cup B} f d\mu = \int_A f d\mu + \int_B f d\mu.$$

The following proposition is surprisingly non-trivial.

Proposition 3.18.

Let $f \in L^1$. Then

$$\left| \int f d\mu \right| \leq \int |f| \, d\mu.$$

Proof. The case when f is real-valued is trivial:

$$\left| \int f d\mu \right| = \left| \int f^+ d\mu - \int f^- d\mu \right| \leq \int f^+ d\mu + \int f^- d\mu = \int f^+ + f^- d\mu = \int |f| \, d\mu.$$

Let

$$z = \int f d\mu.$$

Write

$$z = re^{i\theta}$$

in polar form, so that $r = |z|$. Therefore,

$$\left| \int f d\mu \right| = r = e^{-i\theta} z = \int e^{-i\theta} f d\mu = \operatorname{Re} \int e^{-i\theta} f d\mu = \int \underbrace{\operatorname{Re} \circ e^{-i\theta} f}_{=g} d\mu \leq \int |g| \, d\mu \leq \int |f| \, d\mu.$$

QED

Theorem 3.19. Lebesgue Dominated Convergence Theorem (LDCT)

Let $(f_n)_{n=1}^{\infty} \in \left(L^1\right)^{\mathbb{N}}$ and let $g \in L^1$. If $f_n \to f$ pointwise for some $f : X \to \mathbb{C}$ and $|f_n| \leq g$ for all $n \in \mathbb{N}$, then $f \in L^1$ with

$$\int \lim_{n \to \infty} f_n d\mu = \lim_{n \to \infty} \int f_n d\mu.$$

Proof. We are going to only prove the case where f, g, f_n are real-valued.

Since $|f| \leq g$ by taking limits as $n \to \infty$,

$$\int |f| \, d\mu \leq \int g d\mu < \infty.$$

Hence $f \in L^1$. Then

$$\int g + f d\mu \overset{\text{Fatou}}{\leq} \liminf_{n \to \infty} \int g + f_n d\mu = \int g d\mu + \liminf_{n \to \infty} \int f_n d\mu.$$

Similarly,

$$\int g - f d\mu \overset{\text{Fatou}}{\leq} \liminf_{n\to\infty} \int g - f_n d\mu = \int g d\mu - \limsup_{n\to\infty} \int f_n d\mu.$$

Since $\int g d\mu < \infty$,

$$\int f d\mu \leq \liminf_{n\to\infty} \int f_n d\mu$$

and

$$-\int f d\mu \leq -\limsup_{n\to\infty} \int f_n d\mu.$$

Therefore,

$$\limsup_{n\to\infty} \int f_n d\mu \leq \int f d\mu \leq \liminf_{n\to\infty} \int f_n,$$

which means

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

QED

In our progression of theory of integration, we proceeded in the order

$$\begin{gathered} \text{simple} \\ \downarrow \\ \text{measurable, } [0,\infty]\text{-valued} \\ \downarrow \\ \text{measurable, } \mathbb{R}\text{-valued} \\ \downarrow \\ \text{measurable, } \mathbb{C}\text{-valued} \end{gathered}$$

So what has been missing are the measurable functions which take extended real values. We are going to address this problem quickly.

4. Spaces of Integrable Functions

Proposition 3.20.
$\mathrm{L}^1(X, \mathcal{A}, \mu)$ is a Banach space.

Here are some ideas for the proof.
Suppose that V is a normed linear space and let $(a_n)_{n=1}^{\infty} \in V^n$ be Cauchy. Then we know

$$\text{there is a subsequence } (a_{n_k})_{k=1}^{\infty} \text{ such that } a_{n_k} \to a \in V \implies a_n \to a.$$

Let $(f_n)_{n=1}^{\infty} \in \mathrm{L}^1(X, \mathcal{A}, \mu)^{\mathbb{N}}$ be Cauchy. Then

$$|\mathrm{Re}\circ f_n - \mathrm{Re}\circ f_m|^2 \leq |\mathrm{Re}\circ f_n - \mathrm{Re}\circ f_m|^2 + |\mathrm{Im}\circ f_n - \mathrm{Im}\circ f_m| = |f_n - f_m|^2$$

so that

$$|\mathrm{Re}\circ f_n - \mathrm{Re}\circ f_m| \leq |f_n - f_m|.$$

Hence by monotonicity,

$$\|\mathrm{Re}\circ f_n - \mathrm{Re}\circ f_m\|_1 \leq \|f_n - f_m\|_1,$$

which means $(\mathrm{Re}\circ f_n)_{n=1}^{\infty}$ is Cauchy. Similarly, $(\mathrm{Im}\circ f_n)_{n=1}^{\infty}$ is also Cauchy.

Proof of Proposition 3.20

Let $(f_n)_{n=1}^{\infty} \in \mathrm{L}^1(X, \mathcal{A}, \mu)$ be Cauchy. Assume each f_n is real-valued without loss of generality. For all $k \in \mathbb{N}$, there is $n_k \in \mathbb{N}$ such that

$$\|f_n - f_m\|_1 < \frac{1}{2^k}, \qquad \forall n, m \geq n_k.$$

Without loss of generality assume $(n_k)_{k=1}^{\infty}$ is increasing. Let

$$\hat{g} = |f_{n_1}| + \sum_{k=1}^{\infty} \left|f_{n_{k+1}} - f_{n_k}\right|.$$

By the MCT,

$$\int \hat{g} d\mu = \int |f_{n_1}| \, d\mu + \sum_{k=1}^{\infty} \int \left|f_{n_{k+1}} - f_{n_k}\right| d\mu = \|f_{n_1}\|_1 + \sum_{k=1}^{\infty} \left\|f_{n_{k+1}} - f_{n_k}\right\|_1 = \|f_{n_1}\|_1 + 1 < \infty.$$

This means $\hat{g}$ is finite almost everywhere – that is, there is $N \in \mathcal{A}$ such that $\hat{g}|_{X \setminus N}$ is finite and $\mu(N) = 0$. Hence define $g : X \to \mathbb{R}$ by

$$g(x) = \begin{cases} \hat{g}(x) & \text{if } x \in X \setminus N \\ 0 & \text{if } x \in N \end{cases}, \qquad \forall x \in X.$$

Let $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} f_{n_1}(x) + \sum_{k=1}^{\infty} f_{n_{k+1}}(x) - f_{n_k}(x) & \text{if } x \in X \setminus N \\ 0 & \text{if } x \in N \end{cases}, \qquad \forall x \in X.$$

Then $f_{n_k} \to f$ pointwise almost everywhere and we have that $|f| \leq g$. Then by the LDCT,

$$f \in \mathrm{L}^1(X, \mathcal{A}, \mu).$$

Moreover,

$$|f_{n_k}| \leq |f_{n_1}| + \sum_{j=1}^{k-1} \left|f_{n_{j+1}} - f_{n_j}\right| \overset{\text{ae}}{\leq} g, \qquad \forall k \in \mathbb{N}.$$

Finally,

$$|f - f_{n_k}| \leq 2g, \qquad \forall k \in \mathbb{N},$$

so by the LDCT,

$$\|f - f_{n_k}\|_1 = \int |f - f_{n_k}| \, d\mu \to 0.$$

QED

IV. Product Measures

1. Product Measures

Def'n 4.1. **Measurable Rectangle**
Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be measure spaces. For every $A \in \mathcal{A}, B \in \mathcal{B}$, we call $A \times B$ a ***measurable rectangle***.

Lemma 4.1.
Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be measure spaces and let $\{A_k \times B_k\}_{k=1}^{\infty}$ be a collection of measurable rectangles that are pairwise disjoint. Also assume that

$$\dot{\bigcup_{k=1}^{\infty}} A_k \times B_k = A \times B$$

for some $A \in \mathcal{A}, B \in \mathcal{B}$. Then

$$\mu(A)\,\nu(B) = \sum_{k=1}^{\infty} \mu(A_k)\,\nu(B_k).$$ [1]

[1] We are using the convention $0\infty = 0$.

Proof. Fix $x \in A$. For all $y \in B$, there exists a unique $k \in \mathbb{N}$ such that $(x, y) \in A_k \times B_k$. Hence

$$B = \dot{\bigcup_{k \in \mathbb{N} : x \in A_k}} B_k$$

This means

$$\mu(B) = \sum_{k \in \mathbb{N} : x \in A_k} \mu(B_k),$$

so that

$$\nu(B)\,\chi_A(x) = \sum_{k=1}^{\infty} \nu(B_k)\,\chi_{A_k}, \qquad \forall x \in X.$$

By MCT,

$$\nu(B)\,\mu(A) - \sum_{k=1}^{\infty} \sum_{k=1}^{\infty} \nu(B_k)\,\mu(A_k).$$

QED

Let

$$\mathcal{R} = \left\{ \dot{\bigcup_{k=1}^{n}} A_k \times B_k : n \geq 0, \forall k \in \{1, \ldots, n\}\,[A_k \in \mathcal{A}, B_k \in \mathcal{B}] \right\}.$$

Proposition 4.2.
Let

$$\begin{aligned} \lambda : \mathcal{R} &\to [0, \infty] \\ \dot{\bigcup_{k=1}^{n}} A_k \times B_k &\mapsto \sum_{k=1}^{n} \mu(A_k)\,\nu(B_k) \end{aligned}.$$

Then λ is a premeasure.

By Caratheodory, there is a complete measure

$$\left(X \times Y, \overline{\mathcal{A} \times \mathcal{B}}, \mu \times \nu\right)$$

on $X \times Y$ such that

$$\mathcal{A} \times \mathcal{B} \subseteq \overline{\mathcal{A} \times \mathcal{B}} = \{A \times B \in \mathcal{A} \times \mathcal{B} : A \times B \text{ is } \lambda^*\text{-measurable}\}.$$

and

$$(\mu \times \nu)(A \times B) = \mu(A)\,\nu(B), \qquad \forall A \in \mathcal{A}, B \in \mathcal{B}.$$

Def'n 4.2. **Product Measure**
Consider the above setting. We call $\mu \times \nu$ the ***product measure*** on $\mathcal{A} \times \mathcal{B}$.

2. Product Integration

Theorem 4.3. Fubini

Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be complete measure spaces. If $f \in \mathrm{L}^1\left(X \times Y, \overline{\mathcal{A} \times \mathcal{B}}, \mu \times \nu\right)$, then

(a) For all $x \in X$, let

$$\begin{aligned} f_x : Y &\to \mathbb{F} \\ y &\mapsto f(x, y) \end{aligned}.$$

Then $f_x \in \mathrm{L}^1(Y, \mathcal{B}, \nu)$ for almost all x.

(b) For all $y \in Y$, let

$$\begin{aligned} f^y : X &\to \mathbb{F} \\ x &\mapsto f(x, y) \end{aligned}.$$

Then $f^y \in \mathrm{L}^1(X, \mathcal{A}, \mu)$ for almost all y.

(c) Let

$$\begin{aligned} F : X &\to \mathbb{F} \\ x &\mapsto \int f_x d\nu \end{aligned}.$$

Then $F \in \mathrm{L}^1(X, \mathcal{A}, \nu)$.

(d) Let

$$\begin{aligned} G : X &\to \mathbb{F} \\ y &\mapsto \int f^y d\mu \end{aligned}.$$

Then $G \in \mathrm{L}^1(X, \mathcal{A}, \nu)$.

(e) We have

$$\int_{X \times Y} fd(\mu \times \nu) = \int_X \int_Y f(x, y)\, d\nu d\mu = \int_Y \int_X f(x, y)\, d\mu d\nu.$$

Given $E \subseteq X \times Y$, let us write write

$$E_x = \{y \in Y : (x, y) \in E\}, \qquad \forall x \in X$$

and

$$E^y = \{x \in X : (x, y) \in E\}, \qquad \forall y \in Y.$$

Lemma 4.4.

Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be measure spaces and let

$$\mathcal{R} = \left\{ \dot{\bigcup_{k=1}^{n}} A_k \times B_k : n \geq 0, \forall k \in \{1, \ldots, n\} \left[A_k \in \mathcal{A}, B_k \in \mathcal{B} \right] \right\}.$$

Let $E \in \mathcal{R}_{\sigma\delta}$ with $(\mu \times \nu)(E) < \infty$. Then

(a) $g : X \to \mathbb{R}$ by $g(x) = \nu(E_x)$ for all $x \in X$ is μ-measurable;

(b) $g \in \mathrm{L}^+ \cap \mathrm{L}^1$; and

(c) $\int g d\mu = (\mu \times \nu)(E)$.

Proof.

Case 1. *Suppose $E = A \times B$ for some $A \in \mathcal{A}, B \in \mathcal{B}$.*

Then

$$E_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases} \in \mathcal{B}, \qquad \forall x \in X$$

Now

$$g(x) = \nu(E_x) = \nu(B)\, \chi_A(x), \qquad \forall x \in X$$

so that g is a nonnegative measurable function, with

$$\int g d\mu = \int \nu(B)\, \chi_A d\mu = \nu(B)\, \mu(A) = (\mu \times \nu)(E) < \infty,$$

as needed.

(End of Case 1)

Case 2. *Consider $E = \bigcup_{i=1}^{\infty} A_i \times B_i$ for some $A_1, \ldots \in \mathcal{A}, B_1, \ldots \in \mathcal{B}$.*

Without loss of generality, we may assume that the union is disjoint, since intersection of rectangles is still a rectangle.

Define $g_i = \nu(B_i)\, \chi_{A_i}$ for all $i \in \mathbb{N}$. Then

$$g = \sum_{i=1}^{\infty} g_i$$

so that g is μ-measurable. Moreover, every $E_x = \dot{\bigcup}_{i=1}^{\infty} (A_i \times B_i)_x$ is measurable.

Then by the MCT,

$$\int g d\mu = \sum_{i=1}^{\infty} \int g_i d\mu = \sum_{i=1}^{\infty} \mu(A_i)\, \nu(B_i) = \sum_{i=1}^{\infty} (\mu \times \nu)(A_i \times B_i) = (\mu \times \nu)(E) < \infty.$$

(End of Case 2)

Case 3. *Consider $E = \bigcap_{n=1}^{\infty} E_n$, where each $E_n \in \mathcal{R}_\sigma$.*

Without loss of generality, we may assume

$$E_1 \supseteq E_2 \supseteq \cdots.$$

Moreover, we may also assume that

$$(\mu \times \nu)(E_1) < \infty,$$

since $(\mu \times \nu)(E) < \infty$.

Then we have that

$$E_x = \bigcap_{n=1}^{\infty} (E_n)_x$$

and

$$(E_1)_x \supseteq (E_2)_x \supseteq \cdots ,$$

so

$$\lim_{n\to\infty} \nu\left((E_n)_x\right) = \nu(E)$$

and

$$\lim_{n\to\infty} (\mu \times \nu)(E_n) = (\mu \times \nu)(E) .$$

Let

$$\begin{aligned} g_n : X &\to \mathbb{R} \\ x &\mapsto \nu\left((E_n)_x\right) \end{aligned}, \qquad \forall n \in \mathbb{N} .$$

Then $0 \geq g$ and $g_n \searrow g$ pointwise with

$$\int g_1 d\nu = (\mu \times \nu)(E_1) < \infty,$$

so by the LDCT,

$$\int g d\mu = \lim_{n\to\infty} \int g_n d\mu = \lim_{n\to\infty} (\mu \times \nu)(E_n) = (\mu \times \nu)(E) .$$

(End of Case 3)

QED

Theorem 4.5. Tonelli

Let $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$ be complete measure spaces and suppose $\mu \times \nu$ is σ-finite. If $f \in L^+\left(X \times Y, \overline{\mathcal{A} \times \mathcal{B}}, \mu \times \nu\right)$, then

(a) $f_x, f^y \in L^+$ almost everywhere;

(b) for all $y \in Y$

$$\begin{aligned} F : X &\to Y \\ x &\mapsto \int_Y f_x d\nu \end{aligned}$$

is integrable; and

(c)

$$\int_{X \times Y} f d(\mu \times \nu) = \int_X \int_Y f d\nu d\mu = \int_Y \int_X f d\mu d\nu.$$

Proof. Since $\nu \times \mu$ is σ-finite, let $\{C_n\}_{n=1}^{\infty} \subseteq \overline{\mathcal{A} \times \mathcal{B}}$ be such that

$$X \times Y = \bigcup_{n=1}^{\infty} C_n$$

with

$$(\mu \times \nu)(C_n) < \infty, \qquad \forall n \in \mathbb{N}.$$

Without loss of generality, we assme

$$C_1 \subseteq C_2 \subseteq \cdots$$

by replacing C_n with $C_1 \cup \cdots \cup C_n$.

Let

$$f_n = \min(f, n)\, \chi_{C_n}, \qquad \forall n \in \mathbb{N}.$$

Then note that $f_n \to f$ pointwise where $(f_n)_{n=1}^{\infty}$ is an increasing sequence of measurable functions. Hence

$$\int f d(\mu \times \nu) \overset{\text{MCT}}{=} \lim_{n \to \infty} \int f_n d(\mu \times \nu) \overset{\text{Fub}}{=} \lim_{n \to \infty} \int_X \underbrace{\int_Y f_n d\nu}_{=F_n} d\mu.$$

Note that $F_n \nearrow F$ pointwise, so by the MCT,

$$\lim_{n \to \infty} \int_X F_n(x)\, d\mu = \int_X F(x)\, d\mu.$$

Thus

$$\int f d(\mu \times \nu) = \int_X \int_Y f d\nu d\mu.$$

QED

V. Differentiation

1. Introduction

We ask the following questions.

(a) Is there a Lebesgue-measure-theoretic fundamental theorem of calculus?

(b) Is there a measure theoretic differentiation?

(c) Given integrable $f: X \to \mathbb{R}$, to what extent is

$$\begin{aligned} F: X &\to \mathbb{R} \\ x &\mapsto C + \int_a^x f dm \end{aligned}$$

differentiable?

We are going to consider functions of the form

$$f: [a, b] \to \mathbb{R}.$$

By considering f^+, f^-, we first assume $f \geq 0$. In this way, we see that F (in (c)) is increasing.

Def'n 5.1. **Upper Derivative**, **Lower Derivative** of a Real-valued Function
Let $f: [a, b] \to \mathbb{R}$. We define the

(a) ***upper derivative from the right*** of f, denoted as $\overline{D_r}f$, by

$$\overline{D_r}f(x) = \limsup_{h \downarrow 0} \frac{f(x+h) - f(x)}{h}, \qquad \forall x \in [a, b];$$

(b) ***upper derivative from the left*** of f, denoted as $\overline{D_l}f$, by

$$\overline{D_l}f(x) = \limsup_{h \downarrow 0} \frac{f(x) - f(x-h)}{h}, \qquad \forall x \in [a, b];$$

(c) ***lower derivative from the right*** of f, denoted as $\underline{D_r}f$, by

$$\underline{D_r}f(x) = \liminf_{h \downarrow 0} \frac{f(x+h) - f(x)}{h}, \qquad \forall x \in [a, b];$$

and

(d) ***lower derivative from the left*** of f, denoted as $\underline{D_l}f$, by

$$\underline{D_l}f(x) = \liminf_{h \downarrow 0} \frac{f(x) - f(x-h)}{h}, \qquad \forall x \in [a, b].$$

Def'n 5.2. **Differentiable** Function
We say $f: [a, b] \to \mathbb{R}$ is ***differentiable*** if

$$\overline{D_r}f(x) = \overline{D_l}f(x) = \underline{D_r}f(x) = \underline{D_l}f(x) \in \mathbb{R}, \qquad \forall x \in [a, b].$$

In case $f: [a, b] \to \mathbb{R}$ is differentiable in Def'n 5.2 sense, then all four quantities in Def'n 5.1 are equal to

$$\begin{aligned} f': [a, b] &\to \mathbb{R} \\ x &\mapsto \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \end{aligned}$$

Def'n 5.3. **Degenerate** Interval
We say an interval is ***degenerate*** if it is $\emptyset$ or a singleton.

Def'n 5.4. **Vitali Covering** of a Set
Let $E \subseteq \mathbb{R}$. We say a collection of non-degenerate intervals $\mathcal{C}$ is a ***Vitali covering*** of E if

$$\forall x \in E, \varepsilon > 0 \exists I \in \mathcal{C}\left[x \in I, m\left(I\right) < \varepsilon\right].$$

Theorem 5.1. Vitali Covering Lemma
Let $E \subseteq \mathbb{R}$ be such that

$$m^*\left(E\right) < \infty$$

and let $\mathcal{C}$ be a Vitali covering of E. Then for every $\varepsilon > 0$, there exist disjoint $I_1, \ldots, I_N \in \mathcal{C}$ such that

$$m^*\left(E \setminus \bigcup_{n=1}^{N} I_n\right) < \varepsilon.$$

Proof. Fix $\varepsilon > 0$.

Recall that when $x \in \mathbb{R}$ and $C \subseteq \mathbb{R}$ is closed,

$$d\left(x, C\right) = \inf_{c \in C} |x - c|$$

is well-defined, and

$$x \in C \iff d\left(x, C\right) = 0.$$

Fix open $U \supseteq E$ with $m\left(U\right) < \infty$ and let

$$\mathcal{C}' = \left\{I \in \mathcal{C} : I \subseteq U\right\}.$$

Claim 1. *$\mathcal{C}'$ is a Vitali covering of E.*

Let $x \in E$ and

$$\delta = d\left(x, \mathbb{R} \setminus U\right),$$

then for any $I \in \mathcal{C}$ such that $x \in I$ and $m\left(I\right) < \delta$, $I \subseteq U$, so that $I \in \mathcal{C}'$.

(End of Claim 1)

Let $I_1 \in \mathcal{C}$. For every $k > 1$, define $I_k \in \mathcal{C}'$ such that $I_1, \ldots, I_k$ are pairwise disjoint and

$$m\left(I_k\right) > \frac{\alpha_k}{2},$$

where

$$\alpha_k = \sup\left\{m\left(I\right) : I \in \mathcal{C}', I \text{ is disjoint from } I_1, \ldots, I_{k-1}\right\}.$$

If this construction halts, then we are done; we have covered E by intervals, except possibly at finitely many points. Hence assume that the construction does not halt and we have countably many disjoint intervals $I_1, I_2, \ldots \in \mathcal{C}'$.

Now,

$$m\left(\dot{\bigcup_{k=1}^{\infty}} I_k\right) = \sum_{k=1}^{\infty} m\left(I_k\right) \leq m\left(U\right) < \infty.$$

We may find $N \in \mathbb{N}$ such that such that

$$\sum_{k=N+1}^{\infty} m\left(I_k\right) < \frac{\varepsilon}{5}.$$

Claim 2. *$I_1, \ldots, I_N \in \mathcal{C}$ are disjoint with*

$$m^*\left(E \setminus \bigcup_{k=1}^{N} I_k\right) < \varepsilon.$$

Let

$$X = E \setminus \bigcup_{k=1}^{N} \overline{I_k}.$$

If $x \in X$, let

$$\delta = d\left(x, \bigcup_{k=1}^{N} \overline{I_k}\right).$$

Since $\mathcal{C}'$ is a Vitali covering of E, we may find $I \in \mathcal{C}'$ such that $x \in I$ and $m(I) < \delta$. Hence I is disjoint from $\bigcup_{k=1}^{N} I_k$. This means

$$m(I) \leq \alpha_{N+1}.$$

Pick $K > N$ such that

$$\alpha_{K+1} < m(I) \leq \alpha_K.$$

Note that such $K > N$ exists, since $\sum_{k=1}^{\infty} \frac{\alpha_k}{2} \leq \sum_{k=1}^{\infty} m(I_k) < \infty$, which means $\lim_{k\to\infty} \alpha_k = 0$. But this means I is not disjoint from $\bigcup_{k=1}^{K} I_k$. Hence let $j \leq K$ be such that

$$I \cap I_j \neq \emptyset.$$

Then

$$m\left(I_j\right) > \frac{\alpha_j}{2} \geq \frac{\alpha_K}{2} \geq \frac{m(I)}{2}.$$

Now, let $z \in I_j$ be the midpoint of I_j. Then

$$|x - z| \leq m(I) + \frac{1}{2} m\left(I_j\right) \leq 2m\left(I_j\right) + \frac{1}{2} m\left(I_j\right) = \frac{5}{2} m\left(I_j\right).$$

Let J_j be the closed interval with the same midpoint z as I_j and

$$m\left(J_j\right) = 5m\left(I_j\right).$$

This means $|x - z| = \frac{1}{2} m\left(J_j\right)$, so that $x \in J_j$. This means

$$X \subseteq \bigcup_{j=N+1}^{\infty} J_j.$$

Hence

$$m^*\left(E \setminus \bigcup_{k=1}^{N} I_k\right) = m^*(X) \leq \sum_{j=N+1}^{\infty} m\left(J_j\right) = 5 \sum_{j=N+1}^{\infty} m\left(I_j\right) < 5\frac{\varepsilon}{5} = \varepsilon.$$

(End of Claim 2)

QED

Theorem 5.2.

Let $f : [a, b] \to \mathbb{R}$ be increasing. Then

(a) f is continuous except on a countable set;

(b) f is differentiable except on a set of measure zero; and

(c) the derivative f'[1] of f is L^1 and

$$\int_a^b f' dm \leq f(b) - f(a).$$

[1] Since f is differentiable ae, we may define f' in usual way for points at where f is differentiable and set $f'(x) = 0$ for every x where f is not differentiable.

Proof of (a). Extend f to $\mathbb{R}$ by $f(x) = f(a)$ for $x < a$ and $f(x) = f(b)$ for $x > b$. For all $c \in [a, b]$,

$$\lim_{x \uparrow c} f(x) = \sup_{x<c} f(x) \leq f(c) \leq \inf_{x>c} f(x) = \lim_{x \downarrow c} f(x) .$$

Hence f is continuous at c unless f has a jump of length

$$j(c) = \lim_{x \downarrow c} f(x) - \lim_{x \uparrow c} f(x) .$$

But note that

$$\sum_{c \in [a,b]} j(c) \leq f(b) - f(a) .$$

This means for every $n \in \mathbb{N}$, the number of jumps length at least $\frac{1}{n}$ is finite, so there are countably many jumps.

Proof of (b). Clearly we have

$$\underline{D_r} f \leq \overline{D_r} f$$

and

$$\underline{D_l} f \leq \overline{D_l} f.$$

Claim 1. *We have*

$$\overline{D_l} f \leq \underline{D_r} f$$

almost everywhere.

For $u < v$ in $\mathbb{Q}$, let

$$E_{u,v} = \left\{ x \in [a, b] : \underline{D_r} f(x) < u < v < \overline{D_l} f(x) \right\} .$$

Let

$$E = \bigcup_{u,v \in \mathbb{Q} : u<v} E_{u,v}. \qquad [5.1]$$

Then by the density of rationals, $E = \left\{ x \in [a, b] : \underline{D_r} f(x) < \overline{D_l} f(x) \right\}$. Hence it remains to show

$$m^* (E) = 0.$$

By [5.1], it suffices to show that

$$m^* (E_{u,v})$$

for all $u < v$ in $\mathbb{Q}$. Hence fix $u < v$ in $\mathbb{Q}$ and say $m^* (E_{u,v}) = s$. Let $\varepsilon > 0$ be given and find an open $E_{u,v} \subseteq U$ such that

$$m(U) < s + \varepsilon$$

by the definition of outer measure. Consider

$$\mathcal{C} = \left\{ [x, x + h] \subseteq U : h > 0, \frac{f(x + h) - f(x)}{h} < u \right\} .$$

For $x \in E_{u,v}$, we have

$$\underline{D_r} f(x) < u$$

so that

$$\lim_{\delta \downarrow 0} \inf_{h \in (0,\delta)} \frac{f(x + h) - f(x)}{h} < u.$$

This means $\mathcal{C}$ has arbitrarily small intervals of the form $[x, x + h]$, where $x \in E_{u,v}$. Hence $\mathcal{C}$ is a Vitali covering for $E_{u,v}$. By the Vitali covering lemma, we have disjoint

$$I_1 = [x_1, x_1 + h_1] , \ldots , I_N = [x_N, x_N + h_N] \in \mathcal{C}$$

such that

$$m^*\left(E_{u,v}\setminus\bigcup_{j=1}^{N}I_j\right)<\varepsilon.$$

Therefore,

$$s-\varepsilon<\sum_{j=1}^{n}m\left(I_j\right)=\sum_{j=1}^{N}h_j<m\left(U\right)<s+\varepsilon.$$

Note

$$s=m^*\left(E_{u,v}\right)=m^*\left(E_{u,v}\cap\left(\bigcup_{j=1}^{N}I_j\right)\right)+m^*\left(E_{u,v}\setminus\bigcup_{j=1}^{N}I_j\right)<m^*\left(E_{u,v}\cap\left(\bigcup_{j=1}^{N}I_j\right)\right)+\varepsilon.$$

by Caratheodory's criterion. This means

$$m^*\left(E_{u,v}\cap\left(\bigcup_{j=1}^{N}I_j\right)\right)>s-\varepsilon.$$

Let

$$F=E_{u,v}\cap\left(\bigcup_{j=1}^{N}\left(x_j,x_j+h_j\right)\right)\subseteq\bigcup_{j=1}^{N}\left(x_j,x_j+h_j\right)=V.$$

As before,

$$C'=\left\{[x-k,x]\subseteq V:k>0,\frac{f(x)-f(x-k)}{k}>v\right\}$$

is a Vitali cover for F. Again by the Vitali covering lemma, we find

$$J_1=[y_1-k_1,y_1]\,,\ldots,J_M=[y_M-k_M,y_M]\in\mathcal{C}'$$

disjoint such that

$$m^*\left(F\setminus\bigcup_{i=1}^{M}J_i\right)<\varepsilon.$$

Then

$$\sum_{i=1}^{M}K_i=\sum_{i=1}^{N}m\left(J_i\right)>m^*\left(F\right)-\varepsilon=m^*\left(E_{u,v}\cap\left(\bigcup_{j=1}^{N}\left(x_j,x_j+h_j\right)\right)\right)-\varepsilon>s-2\varepsilon.$$

Note that

$$J_i\subseteq\bigcup_{j=1}^{N}I_j$$

for all $i\in\{1,\ldots,M\}$. Hence

$$(s-2\varepsilon)\,v<\sum_{i=1}^{M}vk_i<\sum_{i=1}^{M}\left(f\left(y_i\right),f(y_i-k_i)\right)\leq\sum_{j=1}^{N}f\left(x_j+h_j\right)-f\left(x_j\right)\leq\sum_{j=1}^{N}uh_j<u\left(s+\varepsilon\right).$$

So for all $\varepsilon>0$,

$$v\left(s-2\varepsilon\right)<u\left(s+\varepsilon\right),$$

and by letting $\varepsilon\to 0$,

$$vs\leq us.$$

But $u < v$, so we conclude that

$$s = 0.$$

(End of Claim 1)

In a similar fashion,

$$\overline{D_r}f \leq \underline{D_l}f$$

almost everywhere.

Proof of (c). Consider

$$\begin{aligned} g_n : [a,b] &\to \mathbb{R} \\ x &\mapsto \frac{f\left(x+\frac{1}{n}\right)-f(x)}{\frac{1}{n}}, \end{aligned} \qquad \forall n \in \mathbb{N}.$$

Since f is monotone, f is measurable, so that each g_n is measurable. Also,

$$g_n(x) \to f'(x)$$

almost everywhere. Therefore, f' is measurable with $f' \geq 0$, since each $g_n \geq 0$. Then, by Fatou's lemma,

$$\begin{aligned} \int_a^b f' dm &\leq \liminf_{n\to\infty} \int_a^b g_n dm = \liminf_{n\to\infty} n \int_a^b f\left(\cdot + \frac{1}{n}\right) dm - n\int_a^b f dm = \liminf_{n\to\infty} \int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f dm - n \int_a^b f dm \\ &= \liminf_{n\to\infty} n \int_b^{b+\frac{1}{n}} f dm - n \int_a^{a+\frac{1}{n}} f dm \leq f(b) - f(a). \end{aligned}$$

QED

2. Bounded Variation and Absolute Continuity

Def'n 5.5. **Bounded Variation**

We say $f : [a,b] \to \mathbb{R}$ is of ***bounded variation*** if the ***variation*** of f,

$$V_a^b(f) = \sup\left\{\sum_{k=1}^n |f(x_k) - f(x_{k-1})| : n \in \mathbb{N}, a = x_0 < x_1 < \cdots < x_n = b\right\},$$

is finite.

Example 5.1.

$\chi_{\mathbb{Q}\cap[0,1]} : [0,1] \to \mathbb{R}$ is not of bounded variation.

Example 5.2.

If $f : [a,b] \to \mathbb{R}$ is increasing, then for $a = x_0 < x_1 < \cdots < x_n = b$,

$$V_a^b(f) = \sum_{k=1}^n (f(x_k) - f(x_{k-1})) = f(b) - f(a).$$

Proposition 5.3.

Let $f : [a,b] \to \mathbb{R}$. Then

$$f \text{ is of bounded variation} \iff f = g - h \text{ for some increasing } g, h.$$

Proof. ($\Longleftarrow$) Suppose $f = g - h$ for some increasing g, h. Then for any partition $a = x_0 < x_1 < \cdots < x_n = b$,

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \le \sum_{k=1}^{n} |g(x_k) - g(x_{k-1})| + \sum_{k=1}^{n} |h(x_k) - h(x_{k-1})| = g(b) - g(a) + h(b) - h(a) < \infty.$$

($\Longrightarrow$) Suppose f is of bounded variation. Define

$$\begin{aligned} g : [a, b] &\to \mathbb{R} \\ x &\mapsto V_a^x(f) \end{aligned}.$$

Then g is increasing. Let $h = g - f$. For $x < y$,

$$h(y) - h(x) = V_a^y(f) - f(y) - V_a^x f(x) + f(x) = V_x^y(f) - (f(y) - f(x)) \ge |f(y) - f(x)| - (f(y) - f(x)) \ge 0.$$

QED

Corollary 5.3.1.

Let $f : [a, b] \to \mathbb{R}$ be of bounded variation. Then

(a) f is continuous except on a countable set;

(b) f is differentiable except on a set of measure zero; and

(c) the derivative f' of f is L^1 and

$$\int_a^b f' dm \le f(b) - f(a).$$

Corollary 5.3.2.

If $f : [a, b] \to \mathbb{R}$ is L^1, then

$$\begin{aligned} F : [a, b] &\to \mathbb{R} \\ x &\mapsto \int_a^x f dm \end{aligned}$$

is of BV.

Def'n 5.6. **Absolutely Continuous** Function

We say $f : [a, b] \to \mathbb{R}$ is ***absolutely continuous*** if for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $(x_1, y_1), \ldots, (x_n, y_n) \subseteq [a, b]$ are disjoint with

$$\sum_{k=1}^{n} y_k - x_k < \delta,$$

then $\sum_{k=1}^{n} |f(y_k) - f(x_k)| < \varepsilon$.

Proposition 5.4.

Let $f \in L^1(X, \mathcal{A}, \mu)$. For all $\varepsilon > 0$, there is $\delta > 0$ such that for any $A \in \mathcal{A}$ with $\mu(A) < \delta$, we have

$$\int_A |f| \, d\mu < \varepsilon.$$

Proof. Let $\varepsilon > 0$. We may find a simple nonnegative function $\varphi \le |f|$ such that

$$\int |f| \, d\mu < \int \varphi d\mu + \frac{\varepsilon}{2}.$$

Note that, for all $A \in \mathcal{A}$,

$$\int_A |f| - \varphi d\mu \leq \int |f| - \varphi d\mu < \frac{\varepsilon}{2},$$

so that

$$\int_A |f| \, d\mu < \int_A \varphi d\mu + \frac{\varepsilon}{2}.$$

Say $\varphi \leq M$ for some $M \geq 0$. Take $\delta = \frac{\varepsilon}{2M}$ and suppose $A \in \mathcal{A}$ with $\mu(A) < \delta$. Then

$$\int_A |f| \, d\mu < \int_A \varphi d\mu + \frac{\varepsilon}{2} \leq M\mu(A) + \frac{\varepsilon}{2} < \varepsilon.$$

QED

Corollary 5.4.1.

Let $f : [a, b] \to \mathbb{R}$ be L^1. Then

$$\begin{aligned} F : [a, b] &\to \mathbb{R} \\ x &\mapsto \int_{[a,x]} f dm \end{aligned}$$

is absolutely continuous.

Proof. Let $\varepsilon > 0$ be given and let $\delta > 0$ be such that

$$\mu(A) < \delta \implies \int_A |f| \, dm < \varepsilon.$$

Let $(x_1, y_1), \ldots, (x_n, y_n) \subseteq [a, b]$ be disjoint with

$$\sum_{k=1}^{n} m((x_k, y_k)) < \delta.$$

Let $A = \dot\bigcup_{k=1}^{n} (x_k, y_k)$. Then $m(A) < \delta$, so that $\int_A |f| < \varepsilon$. Thus,

$$\sum_{k=1}^{n} |F(y_k) - F(x_k)| = \sum_{k=1}^{n} \left| \int_{x_k}^{y_k} f dm \right| \leq \sum_{k=1}^{n} \int_{x_k}^{y_k} |f| \, dm = \int_A |f| \, dm < \varepsilon.$$

QED

Proposition 5.5.

Let $f : [a, b] \to \mathbb{R}$. If f is absolutely continuous, then f is of bounded variation.

Proof. Let $\varepsilon = 1$ and let $\delta > 0$ be such that whenever $(x_1, y_1), \ldots, (x_n, y_n) \subseteq [a, b]$ are disjoint with $\sum_{k=1}^{n} y_k - x_k < \delta$, then $\sum_{k=1}^{n} |f(y_k) - f(x_k)| < \varepsilon$ by definition of absolute continuity. Write

$$[a, b] = \bigcup_{j=1}^{p} [a_{j-1}, a_j]$$

such that $a_j - a_{j-1} < \delta$. For any partition $a_{j-1} = x_0 < x_1 < \cdots < x_m = a_j$, we have

$$\sum_{s=1}^{m} x_s - x_{s-1} < \delta.$$

Hence

$$\sum_{s=1}^{m} |f(x_s) - f(x_{s-1})| < 1,$$

so that

$$V_{a_{j-1}}^{a_j}(f) \leq 1 \implies V_a^b(f) = \sum_{j=1}^{p} V_{a_{j-1}}^{a_j}(f) \leq p.$$

Thus f is of bounded variation.

QED

Example 5.3. Cantor's Function

Let $f:[0,1] \to \mathbb{R}$ be the *Cantor's function*. We know that f is an increasing continuous function that is continuous on each of the intervals $\left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{1}{9}, \frac{2}{9}\right), \ldots$, so that

$$f' = 0 \text{ on } [0,1] \setminus C,$$

where C is the *Cantor set*. Since $m(C) = 0$, f is differentiable everywhere. But

$$\int_0^1 f' dm = 0 < 1 = f(1) - f(0).$$

Since f is increasing, f is of bounded variation. However, f is not absolutely continuous. Indeed, if x_j, y_j for $1 \leq j \leq 2^n$ are the endpoitns of the intervals remaining at nth stage of the construction of the Cantor set, then

$$\sum_{j=1}^{2^n} y_j - x_j = \left(\frac{2}{3}\right)^n \to 0$$

but

$$\sum_{j=1}^{2^n} \left|f(y_j) - f(x_j)\right| = f(1) - f(0) = 1.$$

Proposition 5.6.

Let $f:[a,b] \to \mathbb{R}$ be L^1. If

$$\begin{aligned} F : [a,b] &\to \mathbb{R} \\ x &\mapsto \int_a^x f dm \end{aligned}$$

is increasing, then $f \geq 0$ almost everywhere.

Proof. Let

$$E = \{x \in [a,b] : f(x) < 0\}$$

and let

$$E_n = \left\{x \in [a,b] : f(x) < \frac{-1}{n}\right\}, \qquad \forall n \in \mathbb{N},$$

which means $E = \bigcup_{n=1}^{\infty} E_n$.

Suppose for contradiction $m(E) > 0$ so that there is $n \in \mathbb{N}$ such that $m(E_n) > 0$. Let

$$\varepsilon = \frac{m(E_n)}{2n}$$

and let $\delta > 0$ be such that

$$m(A) < \delta \implies \int_A |f| \, dm < \varepsilon.$$

By regularity of the Lebesgue measure, there is open $U \supseteq E_n$ such that

$$m(U \setminus E_n) < \delta.$$

Since any open subset of $\mathbb{R}$ can be written as a disjoint union of open sets, write

$$U = \bigsqcup_{k=1}^{\infty} (x_k, y_k)\,.$$

Then

$$0 \leq \sum_{k=1}^{\infty} F(y_k) - F(x_k) = \int_U f dm = \int_{U \setminus E_n} f dm + \int_{E_n} f dm < \varepsilon - \frac{m(E_n)}{n} = -\frac{m(E_n)}{2n},$$

which is a contradiction.

Thus we conclude $m(E) = 0$, as required.

QED

Corollary 5.6.1.
Let $f : [a, b] \to \mathbb{R}$ be L^1 and let

$$\begin{aligned} F : [a, b] &\to \mathbb{R} \\ x &\mapsto \int_a^x f dm \end{aligned}.$$

If $F(x) = 0$ for all $x \in [a, b]$, then $f = 0$ almost everywhere.

Theorem 5.7. Lebesgue Differentiation Theorem
Let $f : [a, b] \to \mathbb{R}$ be L^1 and let

$$\begin{aligned} F : [a, b] &\to \mathbb{R} \\ x &\mapsto C + \int_a^x f dm \end{aligned}$$

for some $C \in \mathbb{R}$. Then $F' = f$ almost everywhere.

Proof. Since F is of bounded variation, F' exists almost everywhere and is L^1. For convenience, extend

$$f(x) = 0, \qquad \forall x > b$$

so that

$$F(x) = F(b)\,, \qquad \forall x > b.$$

Also, $(g_n)_{n=1}^{\infty}$ by

$$g_n(x) = n\left(F\left(x + \frac{1}{n}\right) - F(x)\right), \qquad \forall n \in \mathbb{N}, x \geq a$$

converges to F' pointwise almost everywhere.

Case 1. *$|f| \leq M$ for some $M \geq 0$.*

Then

$$g_n(x) = n \int_x^{x+\frac{1}{n}} f dm \implies |g_n(x)| \leq n \int_x^{x+\frac{1}{n}} |f|\, dm \leq n \frac{1}{n} M = M, \qquad \forall n \in \mathbb{N}, x \geq a.$$

But $\int_a^b M dm < \infty$, so we are at a position to apply the Lebesgue dominated convergence theorem. That is, for $c \in [a, b]$,

$$\begin{aligned} \int_a^c F' dm &= \lim_{n\to\infty} \int_a^c g_n dm = \lim_{n\to\infty} n \underbrace{\int_a^c F\left(x + \frac{1}{n}\right) - F(x)\, dx}_{\text{Riemann integral}} = \lim_{n\to\infty} n \int_{a+\frac{1}{n}}^{c+\frac{1}{n}} F(x)\, dx - n \int_a^c F(x)\, dx \\ &= \lim_{n\to\infty} n \int_c^{c+\frac{1}{n}} F(x)\, dx - n \int_a^{a+\frac{1}{n}} F(x)\, dx \overset{\text{FTC}}{=} F(c) - F(a) = \int_a^c f dm. \end{aligned}$$

Note that we can replace Lebesgue integral by the corresponding Riemann integral since F is (absolutely) continuous.

Hence

$$\int_a^c F' - fdm = 0, \qquad \forall c \in [a,b] \implies F' - f = 0 \text{ almost everywhere}$$

by Corollary 5.6.1.

(End of Case 1)

Case 2. $f \geq 0$.

Let

$$f_n = \min(f, n), \qquad \forall n \in \mathbb{N},$$

so that each $|f_n| < n$. Hence Case 1 applies to each f_n. Then, for almost every $x \in [a,b]$,

$$F(x) = \int_a^x f_n dm + \int_a^x f - f_n dm \implies F'(x) = f_n(x) + \frac{d}{dx}\int_a^x f - f_n dm \geq f(x).$$

For all $c \in [a,b]$, since F is of bounded variation and $F' \geq f_n$ almost everywhere for all $n \in \mathbb{N}$ implies $F' \geq f$ almost everwyere,

$$\int_a^c F' dm \leq F(c) - F(a) = \int_a^c fdm \leq \int_a^c F' dm \implies \int_a^c fdm = \int_a^c F' dm.$$

Hence $F' - f = 0$ almost everywhere.

(End of Case 2)

For the general case, consider f^+, f^- and use Case 2.

QED

Lemma 5.8.

Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous. If $f' = 0$ almost everywhere, then f is constant.

Proof. Let $c \in (a,b]$ and let $\varepsilon > 0$ be given. Take $\delta > 0$ as per the definition of absolute continuity. Consider

$$E = \left\{x \in (a,c) : f'(x) = 0\right\},$$

which is measurable since f' is a pointwise limit of measurable functions (or we can simply invoke completeness of Lebesgue measure), so that

$$m([a,c] \setminus E) = 0.$$

Define

$$\mathcal{C} = \{[x, x+h] \subseteq (a,c) : x \in E, h > 0, |f(x+h) - f(x)| < \varepsilon h\}.$$

We see that $\mathcal{C}$ is a Vitali covering for E. So by the Vitali covering lemma, we may find disjoint $I_1, \ldots, I_n \in \mathcal{C}$ such that

$$m\left(E \setminus \dot{\bigcup_{i=1}^{n}} I_i\right) < \delta.$$

Since $m([a,c] \setminus E) = 0$,

$$m\left([a,c] \setminus \dot{\bigcup_{i=1}^{n}} I_i\right) < \delta$$

as well. Say

$$I_i = [a_i, b_i], \qquad \forall i \in \{1, \ldots, n\}$$

with

$$a < a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n < c.$$

Therefore,

$$\begin{aligned}
|f(c)-f(a)| &\leq \sum_{i=1}^{n}|f(b_i)-f(a_i)|+|f(a_1)-f(a)|+|f(c)-f(b_n)|+\sum_{i=1}^{n-1}|f(a_{i+1})-f(b_i)| \\
&< \sum_{i=1}^{n}|f(b_i)-f(a_i)|+\varepsilon && \text{since } m\left([a,c]\setminus\dot{\bigcup_{i=1}^{n}}I_i\right)<\delta \\
&< \sum_{i=1}^{n}\varepsilon\left(b_i-a_i\right)+\varepsilon && \text{by definition of } \mathcal{C} \\
&\leq \varepsilon\left(c-a\right)+\varepsilon.
\end{aligned}$$

Since our choice of $\varepsilon>0$ was arbitrary, it follows $f(a)=f(c)$.

QED

Theorem 5.9.

Let $F:[a,b]\to\mathbb{R}$. The following are equivalent.

(a) There is $f:\mathrm{L}^1([a,b])$ such that

$$F(x)=C+\int_a^x f dm, \qquad \forall x\in[a,b].$$

(b) F is absolutely continuous.

(c) F is differentiable almost everywhere with $F'\in\mathrm{L}^1([a,b])$ and

$$F(x)=F(a)+\int_a^x F' dm, \qquad \forall x\in[a,b].$$

Proof. (c) $\implies$ (a) is trivial and (a) $\implies$ (b) is proven in Corollary 5.4.1.

For (b) $\implies$ (c), assume F is absolutely continuous. This means F is of bounded variation, so F' exists almost everywhere with $F'\in\mathrm{L}^1([a,b])$. Consider

$$\begin{aligned}
G:[a,b]&\to\mathbb{R}\\
x&\mapsto\int_a^x F' dm.
\end{aligned}$$

Then by the Lebesgue differentiation theorem, $G'=F'$ almost everywhere. Now $G-F$ is absolutely continuous as a sum of two absolutely continuous function. This means $(G-F)'=G'-F'=0$ almost everywhere, so that $G-F$ is constant, say $G=F+C$. That is,

$$F(x)=C+\int_a^x F' dm, \qquad \forall x\in[a,b].$$

But by noticing

$$F(a)=C+\int_a^a F' dm=C,$$

we conclude

$$F(x)=F(a)+\int_a^x F' dm, \qquad \forall x\in[a,b].$$

QED

VI. Measure Decomposition

1. Signed Measure

Def'n 6.1. **Signed Measure** on a Measurable Space
Let $(X, \mathcal{A})$ be a measurable space. A ***signed measure*** $\nu : \mathcal{A} \to [-\infty, \infty]$ on $(X, \mathcal{A})$ such that

(a) $\nu(\emptyset) = 0$;

(b) for all countable collection of disjoint sets $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$, $\nu\left(\dot{\bigcup}_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n)$; and

(c) ν takes on at most one of the values $-\infty, \infty$.

Note (c) in Def'n 6.1 is essential; for, if we have disjoint $A, B \in \mathcal{A}$ with $\nu(A) = \infty$, $\nu(B) = -\infty$, then $\nu(A \cup B)$ would be a problem.

Proposition 6.1.
Suppose ν is a signed measure on $(X, \mathcal{A})$. Suppose

$$\left| \nu\left(\dot{\bigcup_{n=1}^{\infty}} A_n \right) \right| < \infty.$$

Then $\sum_{n=1}^{\infty} \nu(A_n)$ converges absolutely.

Proof. Suppose $\sum_{n=1}^{\infty} \nu(A_n)$ converges conditionally. Then the subseries of positive terms and negative terms diverges to $\infty, -\infty$, respectively. But this means, by taking A to be the union of A_n's with positive measures and B to be the union of A_n's with negative measures, we see that $\nu(A) = \infty$, $\nu(B) = -\infty$, which is a contradiction.

QED

Proposition 6.2. Example of Signed Measures
Let $f \in \mathrm{L}^1(X, \mathcal{A}, \mu)$ be real-valued and define

$$\begin{aligned} \nu : \mathcal{A} &\to [-\infty, \infty] \\ A &\mapsto \int_A f d\mu \end{aligned}.$$

Then ν is a signed measure.

Proof. Clearly $\nu(\emptyset) = 0$. Since f is L^1, note that $|\nu(A)| \leq \int_A |f| \, d\mu < \infty$, so that ν takes neither ∞ nor $-\infty$. It remains to check countable additivity.

Let $A_1, A_2, \ldots \in \mathcal{A}$ be disjoint and let $A = \dot{\bigcup}_{n=1}^{\infty} A_n$. Let

$$B_n = \dot{\bigcup_{k=1}^{n}} A_k, \qquad \forall n \in \mathbb{N}.$$

Then $f\chi_{B_n} \to f\chi_A$ pointwise and $\left| f\chi_{B_n} \right| \leq |f|$, where $|f|$ is L^1. Hence by the LDCT,

$$\int_{B_n} f d\mu \to \int_A f d\mu.$$

This precisely means

$$\sum_{k=1}^{n} \nu(A_k) = \sum_{k=1}^{n} \int_{A_k} f d\mu = \int_{B_n} f d\mu \to \nu(A),$$

as needed.

QED

Example 6.1.

Let $f \in L^+(X, \mathcal{A}, \mu)$ and let $g \in L^1(X, \mathcal{A}, \mu) \cap L^+(X, \mathcal{A}, \mu)$, where both f, g are real-valued. Then

$$\begin{aligned} \nu : \mathcal{A} &\to [-\infty, \infty] \\ A &\mapsto \int_A f d\mu - \int_A g d\mu \end{aligned}$$

is a signed measure, with possibly $\nu(A) = \infty$.

Def'n 6.2. **Null Set**, **Positive Set**, **Positive Set** for a Signed Measure

Let $(X, \mathcal{A})$ be a measurable space and let ν be a signed measure on $(X, \mathcal{A})$. We say $A \in \mathcal{A}$ is

(a) a ***null set*** for ν if for all $B \in \mathcal{A}$ with $B \subseteq A$, we have $\nu(B) = 0$;

(b) a ***positive set*** for ν if $\nu(B) \geq 0$ for all $B \in \mathcal{A}$ with $B \subseteq A$; and

(c) a ***negative set*** for ν if $\nu(B) \leq 0$ for all $B \in \mathcal{A}$ with $B \subseteq A$.

Theorem 6.3. Hahn Decomposition Theorem

Let $(X, \mathcal{A})$ be a measurable space and let ν be a signed measure on $(X, \mathcal{A})$. Then there exists positive $P \in \mathcal{A}$ and negative $N \in \mathcal{A}$ such that

$$X = P \mathbin{\dot\cup} N.$$

If $X = P' \mathbin{\dot\cup} N'$ is another such decomposition, then $P \triangle P', N \triangle N'$ are null.

Postponed

Lemma 6.3.1.

Let $(X, \mathcal{A})$ be a measurable space and let ν be a signed measure on $(X, \mathcal{A})$. If $A \in \mathcal{A}$ is such that $0 < \nu(A) < \infty$, then there is positive $P \subseteq A$ such that $\nu(P) > 0$.

Proof. If A is positive, take $P = A$ and we are done.

Suppose A is not positive, so there is a subset of A with a negative signed measure. So take measurable $B_1 \subseteq A$ such that

$$\nu(B_1) \leq \frac{1}{2} \inf \left\{ \nu(B) : B \in \mathcal{A}, B \subseteq A \setminus \dot{\bigcup_{k=1}^{n-1}} B_k \right\}.$$

Recursively, choose

$$B_n \subseteq A \setminus \dot{\bigcup_{k=1}^{n-1}} B_k$$

so that

$$\nu(B_n) \leq \frac{1}{2} \left\{ \nu(B) : B \in \mathcal{A}, B \subseteq A \setminus \dot{\bigcup_{k=1}^{n-1}} B_k \right\}.$$

We remark that, if we cannot find such a B_n at nth recursive step, then every measurable subset of $A \setminus \dot\bigcup_{k=1}^{n-1} B_k$ has a positive signed measure. Moreover,

$$\nu\left(A \setminus \dot{\bigcup_{k=1}^{n-1}} B_k \right) = \underbrace{\nu(A)}_{>0} - \underbrace{\sum_{k=1}^{n-1} \nu(B_k)}_{<0} > 0,$$

so that $A \dot\bigcup_{k=1}^{n-1} B_k \subseteq A$ is a positive set we were looking for.

Hence suppose the recursive process continues so that we have $B_1, B_2, \ldots$. Take

$$P = A \setminus \dot{\bigcup_{k=1}^{\infty}} B_k.$$

As before,

$$A = P \cup \dot{\bigcup_{k=1}^{\infty}} B_k.$$

Since $|\nu(A)| < \infty$, by Proposition 6.1, $\nu(P) < \infty$.

Claim 1. *P is positive.*

Suppose there is measurable $B \subseteq P$ such that $\nu(B) < 0$. Since $\sum_{k=1}^{\infty} \nu(B_k)$ converges, $\nu(B_k) \to 0$. Hence we may take $n \in \mathbb{N}$ such that

$$\nu(B) < 2\nu(B_n).$$

But

$$2\nu(B_n) \leq \inf\left\{\nu(C) : C \in \mathcal{A}, C \subseteq \setminus \dot{\bigcup_{k=1}^{n-1}} B_k\right\} \leq \nu(B),$$

which is a contradiction.

(End of Claim 1)

QED

Lemma 6.3.2.

If $A_1, A_2, \ldots \in \mathcal{A}$ are positive, then $\bigcup_{n=1}^{\infty} A_n$ is positive.

Proof. Let $B \subseteq \bigcup_{n=1}^{\infty} A_n$ and let

$$B_n = B \cap \left(A_n \setminus \bigcup_{k=1}^{n-1} A_k\right).$$

Then $B = \dot{\bigcup}_{n=1}^{\infty} B_n$ where each $B_n \subseteq A_n$. But each A_n is positive, so that $\nu(B_n) \geq 0$. Thus

$$\nu(B) = \sum_{n=1}^{\infty} \nu(B_n) \geq 0.$$

QED

Proof of Theorem 6.3

We may assume ν does not take on the value of ∞ (otherwise, consider $-\nu$). Let

$$M = \sup\{\nu(A) : A \text{ is positive}\}.$$

Note that there is at least one positive set in $\mathcal{A}$: namely $\emptyset$. We may find positive $A_1, A_2, \ldots \in \mathcal{A}$ such that

$$\mu(A_n) \to M.$$

By Lemma 6.3.2,

$$P = \bigcup_{n=1}^{\infty} A_n$$

is positive. Also,

$$\mu(P) = \nu(A_n) + \nu(P \setminus A_n) \geq \nu(A_n), \qquad \forall n \in \mathbb{N},$$

which means $M \leq \nu(P)$. But P is positive, so $\nu(P) \leq M$, so that

$$\nu(P) = M.$$

Since ν only takes finite values, it follows $M < \infty$ as well.

Let

$$N = X \setminus P.$$

Claim 1. *N is negative.*

For contradiction, suppose there is $E \in \mathcal{A}$ such that $E \subseteq N$ and $\nu(E) > 0$. By Lemma 6.3.1, there is a positive subset $A \subseteq E$ such that $\nu(A) > 0$. But then $P \cup A$ is a disjoint union of positive sets, so that $P \cup A$ is positive and

$$\nu(P \cup A) = \nu(P) + \nu(A) = M + \nu(A) > M,$$

since $M < \infty$, which is a contradiction.

(End of Claim 1)

Suppose

$$X = P' \cup N'$$

similarly. Then $P \setminus P' = N' \setminus N$ and $P' \setminus P = N \setminus N'$. Note that the sets are null, since they are simultaneously positive and negative. It follows that

$$P \triangle P' = \left(P \setminus P'\right) \cup \left(P' \setminus P\right) = \left(N' \setminus N\right) \cup \left(N \setminus N'\right) = N \triangle N'$$

is also null, as a union of null sets.

QED

Example 6.2.

Let $f \in \mathrm{L}^1(X, \mathcal{A}, \mu)$ be real-valued and let

$$\begin{aligned} \nu : \mathcal{A} &\to [-\infty, \infty] \\ A &\mapsto \int_A f d\mu \end{aligned}.$$

Let

$$\begin{aligned} P &= \{x \in X : f(x) \geq 0\} \\ N &= \{x \in X : f(x) < 0\} \end{aligned}.$$

Then, for all $A \subseteq P$,

$$\nu(A) = \int_A f d\mu \geq 0$$

and similarly, for all $B \subseteq N$,

$$\nu(B) = \int_B f d\mu \leq 0.$$

Thus $P \cup N$ is a Hahn decomposition of X.

Note that

$$\begin{aligned} \nu^+ : \mathcal{A} &\to [0, \infty] \\ A &\mapsto \nu(A \cap P) \end{aligned}.$$

Then ν^+ is measure on $(X, \mathcal{A})$, with

$$\nu^+(A) = \int_{A \cap P} f d\mu = \int_A f \chi_P d\mu = \int_A f^+ d\mu, \qquad \forall A \in \mathcal{A}.$$

Similarly,

$$\begin{aligned} \nu^- : \mathcal{A} &\to [0, \infty] \\ A &\mapsto -\nu(A \cap N) \end{aligned}$$

is a measure on $(X, \mathcal{A})$ with

$$\nu^-(A) = \int_A f^- d\mu, \qquad \forall A \in \mathcal{A}.$$

But then

$$\nu(A) = \int_A f d\mu = \int_A f^+ d\mu - \int_A f^- d\mu = \nu^+(A) - \nu^-(A), \qquad \forall A \in \mathcal{A},$$

so that $\nu = \nu^+ - \nu^-$. That is, we *decomposed* a signed measure into its positive and negative parts.

Def'n 6.3. **Mutually Singular** Signed Measures

Suppose $(X, \mathcal{A})$ is a measurable space and let μ, ν be signed measures. We say μ, ν are ***mutually singular***, denoted as $\mu \perp \nu$, if $X = A \cupdot B$ such that A is ν-null and B is μ-null.

Consider the setting of Def'n 6.3. Given $C \in \mathcal{A}$,

$$C = (C \cap A) \cupdot (C \cap B).$$

This means

$$\mu(C) = \mu(C \cap A)$$

and similarly

$$\nu(C) = \nu(C \cap B).$$

As we can see, ν^+, ν^- from Example 6.2 are mutually singular, which is of interest of the next theorem.

Theorem 6.4. Jordan Decomposition Theorem

Let $(X, \mathcal{A})$ be a measurable spcae and let ν be a signed measure on $(X, \mathcal{A})$. Then there exists a unique pair (ν^+, ν^-) of mutually singular measures such that

$$\nu = \nu^+ - \nu^-.$$

Proof. Let $X = P \cupdot N$ be a Hahn decomposition with respect to ν. Consider

$$\begin{aligned} \nu^+ &: \mathcal{A} \to [0, \infty] \\ &\quad A \mapsto \nu(A \cap P) \\ \nu^- &: \mathcal{A} \to [0, \infty] \\ &\quad A \mapsto -\nu(A \cap N) \end{aligned}.$$

By construction, ν^+, ν^- are mutually singular measures such that $\nu = \nu^+ - \nu^-$. Indeed, given $A \subseteq N$,

$$\nu^+(A) = \nu(A \cap P) \geq \nu(N \cap P) = \nu(\emptyset) = 0$$

and similarly, given any $A \subseteq P$, $\nu^-(A) = 0$. We also have that

$$\mu(A) = \mu((A \cap P) \cupdot (A \cap N)) = \mu(A \cap P) + \mu(A \cap N) = \mu^+(A) - \mu^-(A).$$

For uniqueness, suppose $\nu = \mu^+ - \mu^-$, where μ^+, μ^- are mutually singular measures; say $X = P' \cupdot N'$ such that P' is μ^--null and N' is μ^+-null. For $A \in \mathcal{A}, A \subseteq P'$,

$$\nu(A) = \mu^+(A) - \mu^-(A) = \mu^+(A) \geq 0,$$

so that P' is positive with respect to ν. Similarly, N' is negative with respect to ν. By Hahn decomposition, $P \triangle P' = N \triangle N'$ is null. Therefore, for all $A \in \mathcal{A}$,

$$\mu^+(A) = \mu^+\left(A \cap P'\right) = \nu\left(A \cap P'\right) = \nu(A \cap P) = \nu^+(A),$$

and similarly, $\mu^-(A) = \nu^+(A)$. Thus $\nu^+ = \mu^+, \nu^- = \mu^-$, as required.

QED

Proposition 6.5.
Suppose ν is a signed measure with the Jordan decomposition $\nu = \nu^+ - \nu^-$. The following are equivalent.
(a) A is ν-null.
(b) A is ν^+, ν^--null.
(c) A is $|\nu|$-null.

Proof. We first observe that
$$|\nu| = \nu^+ - \nu^-.$$
(a) $\implies$ (b) Suppose $B \subseteq A$ and let $X = P \cup N$ be a Hahn decomposition of X. Then $\nu^+(B) = \nu(B \cap P) = 0$ since $B \cap P \subseteq B \subseteq A$. Similarly, $\nu^-(B) = \nu(B \cap N) = 0$.

(b) $\implies$ (c) Clearly, given $B \subseteq A$,
$$|\nu(B)| = \nu^+(B) + \nu^-(B) = 0 + 0 = 0.$$
(c) $\implies$ (a) Suppose $B \subseteq A$. Then
$$\nu^+(B) + \nu^-(B) = 0,$$
where both ν^+, ν^- are measures, so that
$$\nu(B) = \nu^+(B) - \nu^-(B) = 0.$$

QED

Def'n 6.4. **Absolutely Continuous** Signed Measure with respect to a Measure
Let ν be a signed measure and let μ be a measure on a measurable space $(X, \mathcal{A})$. We say ν is ***absolutely continuous*** with respect to μ, denoted as $\nu \ll \mu$, if for all $A \in \mathcal{A}$,
$$\mu(A) = 0 \implies \nu(A) = 0.$$

Note that we are using the term *absolute continuity* again. The following exercise shows where this is coming from.

Exercise 6.3.
Let ν be a finite signed measure and let μ be a measure on a measurable space $(X, \mathcal{A})$. Then
$$\nu \ll \mu \iff \forall \varepsilon > 0 \exists \delta > 0 \forall A \in \mathcal{A}\left[\mu(A) < \varepsilon \implies |\nu(A)| < \varepsilon\right].$$

In particular, Proposition 5.4 is a special case of the above exercise, with ν defined as $\nu(A) = \int_A |f|\, d\mu$ for some $f \in L^1(X, \mathcal{A}, \mu)$.

Theorem 6.6. Radon-Nikodym Theorem
Let ν, μ be σ-finite measures on a measurable space $(X, \mathcal{A})$. If $\nu \ll \mu$, then there exists $f \in L^+(X, \mathcal{A}, \mu)$ such that
$$\nu(A) = \int_A f d\mu, \qquad \forall A \in \mathcal{A}.$$
Moreover, f is uniquely determined μ-almost everywhere.

We will only prove the case when ν, μ are *finite*. The σ-finite case is left as an easy exercise.

Proof of Existence. For each $r \in \mathbb{Q}, r > 0$, let $X = P_r \cup N_r$ be a Hahn decomposition with respect to $\nu - r\mu$. Set $P_0 = X, N_0 = \emptyset$. Consider $f : X \to \mathbb{R}$ by
$$f(x) = \sup\{r \in \mathbb{Q} : x \in P_r\}, \qquad \forall x \in X.$$

For $t > 0$,

$$f^{-1}((t,\infty]) = \bigcup_{r\in\mathbb{Q}:r>t} P_r \in \mathcal{A},$$

as a countable union of measurable subsets. Moreover, $f^{-1}([0,\infty]) = X$, so that $f \in \mathrm{L}^+(X,\mathcal{A},\mu)$.

Suppose $0 < r < s$ in $\mathbb{Q}$. Then P_s is positive for $\nu - s\mu$ and so is positive for $\nu - r\mu$. This means

$$(\nu - r\mu)(N_r \cap P_s) = 0,$$

so that

$$\nu(N_r \cap P_s) = r\mu(N_r \cap P_s).$$

On the other hand, N_r is negative for $\nu - r\mu$ but $r < s$, so that N_r is negative for $\nu - s\mu$. This means

$$\nu(N_r \cap P_s) = s\mu(N_r \cap P_s)$$

as well, where $s \neq r$. Hence it follows that

$$\mu(N_r \cap P_s) = 0.$$

It follows

$$\mu\left(N_r \cap \bigcup_{s\in\mathbb{Q}:s>r} P_s\right) = 0.$$

Hence

$$f|_{N_r} \leq r \ \mu\text{-almost everywhere},$$

so that

$$\mu\left(f^{-1}((r,\infty])\right) \leq \mu(P_r).$$

Now,

$$\begin{aligned}(\nu - r\mu)(P_r) \geq 0 &\implies \nu(P_r) \geq r\nu(P_r)\\ &\implies \nu(P_r) \leq \frac{1}{r}\nu(P_r) \leq \frac{1}{r}\nu(X).\end{aligned}$$

Taking $r \to \infty$,

$$\mu\left(f^{-1}((r,\infty])\right) = \nu(P_r) \leq \frac{1}{r}\nu(X) \to 0.$$

This means

$$\nu\left(f^{-1}(\{\infty\})\right) = 0,$$

which means f is finite almost everywhere.

Let $E \in \mathcal{A}$ and fix $N \in \mathbb{N}$. Consider

$$E_k = E \cap P_{\frac{k}{N}} \cap N_{\frac{k+1}{N}}, \qquad \forall k \in \mathbb{N} \cup \{0\}.$$

Let

$$E_\infty = E \setminus \bigcup_{k=1}^{\infty} E_k.$$

We proceed to show that $\mu(E_\infty) = 0$. If $E_\infty = \emptyset$, we are done. Otherwise, fix $x \in E_\infty$. Since $P_0 = X$, $x \in P_0$. If there is $k \geq 0$ such that $x \in P_{\frac{k}{N}}, x \notin P_{\frac{k+1}{N}}$, then $x \in N_{\frac{k+1}{N}}$. But this means $x \in E_k$, which contradicts $x \in E_\infty$. It follows that $x \in P_{\frac{k}{N}}$ for all $k \geq 0$, so that

$$E_\infty \subseteq \bigcap_{k\in\mathbb{N}\cup\{0\}} P_{\frac{k}{N}}.$$

Hence,

$$\mu\left(E_{\infty}\right) \leq \mu\left(P_{\frac{k}{N}}\right) \leq \frac{N}{k} \mu(X) \rightarrow 0,$$

so that $\mu\left(E_{\infty}\right)=0$ as well. It follows

$$\nu\left(E_{\infty}\right)=0$$

by the absolute continuity of ν with respect to μ.

Now,

$$\left(\nu-\frac{k}{N} \mu\right)\left(E_k\right) \geq 0$$

$$\left(\nu-\frac{k+1}{N} \mu\right)\left(E_k\right) \leq 0$$

since $E_k \subseteq P_{\frac{k}{N}} \cap N_{\frac{k+1}{N}}$ where $P_{\frac{k}{N}}$ is positive for $\nu-\frac{k}{N} \mu$ and $N_{\frac{k+1}{N}}$ is negative for $\nu-\frac{k+1}{N}$. This implies

$$\frac{k}{N} \mu\left(E_k\right) \leq \nu\left(E_k\right) \leq \frac{k+1}{N} \mu\left(E_k\right). \qquad [6.1]$$

Moreover, for $x \in E_k$,

$$\frac{k}{N} \leq f(x)$$

by definition and

$$f(x) \leq \frac{k+1}{N} \ \mu\text{-almost everywhere,}$$

by considering $f(x) \leq f|_{N_{\frac{k+1}{N}}}(x)$ and that $f_{N_r} \leq r$ μ-almost everywhere for $r \in \mathbb{Q}$. Hence

$$\frac{k}{N} x_{E_k} \leq f \chi_{E_k} \leq \frac{k+1}{N} \chi_{E_k}$$

μ-almost everywhere, so that

$$\frac{k}{N} \mu\left(E_k\right) \leq \int_{E_k} f d\mu \leq \frac{k+1}{N} \mu\left(E_k\right). \qquad [6.2]$$

Summing over $k \geq 0$, we obtain that

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{k}{N} \mu\left(E_k\right) \leq \sum_{k=0}^{\infty} \nu\left(E_k\right) & =\underbrace{E_{\infty}}_{=0}+\sum_{k=0}^{\infty} \nu\left(E_k\right)=\nu(E) \\
& \leq \sum_{k=0}^{\infty} \frac{k+1}{N} \mu\left(E_k\right)=\sum_{k=1}^{\infty} \frac{k}{N} \mu\left(E_k\right)+\sum_{k=0}^{\infty} \frac{1}{N} \mu\left(E_k\right)=\sum_{k=0}^{\infty} \frac{k}{N} \mu\left(E_k\right)+\frac{\mu(E)}{N}
\end{aligned}$$

from [6.1]. In a similar way, we obtain

$$\sum_{k=0}^{\infty} \frac{k}{N} \mu\left(E_k\right) \leq \int_E f d\mu \leq \sum_{k=0}^{\infty} \frac{k}{N} \mu\left(E_k\right)+\frac{\mu(E)}{N}.$$

It follows that

$$\left|\nu(E)-\int_E f d\mu\right| \leq \frac{\mu(E)}{N} \leq \frac{\mu(X)}{N} \rightarrow 0.$$

It follows that $\int_E f d\mu=\nu(E)$.

Proof of Uniqueness upto μ-almost Everywhere. Let $f, g \in \mathrm{L}^{+}(X, \mathcal{A}, \mu)$ be such that

$$\nu(A)=\int_A f d\mu=\int_A g d\mu, \qquad \forall A \in \mathcal{A}.$$

Consider $B = \{x \in X : f(x) > g(x)\}$ and

$$B_n = \left\{x \in X : f(x) \geq g(x) + \frac{1}{n}\right\}, \qquad \forall n \in \mathbb{N}.$$

Suppose for contradiction that $\mu(B) > 0$. This means there is $n \in \mathbb{N}$ such that

$$\mu(B_n) > 0.$$

Therefore, for such $n \in \mathbb{N}$,

$$\nu(B_n) = \int_{B_n} f d\mu \geq \int_{B_n} g + \frac{1}{n} d\mu = \int_{B_n} g d\mu + \underbrace{\frac{\mu(B_n)}{n}}_{>0} > \int_{B_n} g d\mu = \nu(B_n),$$

which is a contradiction.

This means $\mu(B) = 0$, which implies

$$f \leq g \ \mu\text{-almost everywhere.}$$

By symmetry, $g \leq f$ μ-almost everywhere, so that

$$f = g \ \mu\text{-almost everywhere,}$$

as required.

QED

Observe that absolute continuity is necessary for the Radon-Nikodym theorem. For instance, if $f \in \mathrm{L}^+(X, \mathcal{A}, \mu)$, then

$$\begin{aligned} \nu : \mathcal{A} &\to [0, \infty] \\ A &\mapsto \int_A f d\mu \end{aligned}$$

is such that

$$\mu(A) \implies \nu(A) = \int_A f d\mu = 0,$$

so that $\nu \ll \mu$.

The following example demonstrates that the σ-finite assumption is also necessary.

Example 6.4.

Let $X = [0,1]$, $\mathcal{A} = \mathrm{Bor}([0,1])$ and let m_c be the counting measure on $(X, \mathcal{A})$. This means $m \ll m_c$, where m is the Lebesgue measure on $(X, \mathcal{A})$. Observe that m_c is not σ-finite.

Suppose for contradiction that there is $f \in \mathrm{L}^+(X, \mathcal{A}, m_c)$ such that

$$m(A) = \int_A f dm_c.$$

Then for all $a \in [0,1]$,

$$0 = m(\{a\}) = \int_{\{a\}} f dm_c = f(a)\, m_c(\{a\}) = f(a)$$

which means

$$m([0,1]) = \int 0 dm_c = 0,$$

which is a contradiction.

Corollary 6.6.1.

Let μ, ν be measure and signed measure, respectively, on a measurable space $(X, \mathcal{A})$. If $|\nu|, \mu$ are σ-finite and $|\nu| \ll \mu$, then there exists $f = g - h$ with at least one of g, h is in $\mathrm{L}^1(X, \mathcal{A}, \mu)$ and

$$\nu(A) = \int_A f d\mu, \qquad \forall A \in \mathcal{A}.$$

Proof. We utilize the following claim.

Claim 1. *There exists $p : X \to \mathbb{R}$ with $|p(x)| = 1$ for all $x \in X$ such that*

$$\nu(A) = \int_A p d|\nu|.$$

Let $X = P \cup N$ be a Hahn decomposition of X with respect to ν and let

$$p = \chi_P - \chi_N.$$

Then, with the Jordan decomposition

$$\nu = \nu^+ - \nu^-$$

for ν, we have

$$\begin{aligned}\int_A \chi_P - \chi_N d|\nu| &= \int_A p d\nu^+ + \int_A p d\nu^- = \int_{A\cap P} p d\nu^+ + \int_{A\cap N} p d\nu^- = \int_{A\cap P} 1 d\nu^+ + \int_{A\cap N} -1 d\nu^- \\ &= \int_{A\cap P} 1 d\nu^+ - \int_{A\cap N} 1 d\nu^- = \nu^+(A\cap P) - \nu^-(A\cap N) = \nu^+(A) - \nu^-(A) = \nu(A).\end{aligned}$$

(End of Claim 1)

Since $\nu \ll \mu$, we have $|\nu| \ll \mu$. So by the Radon-Nikodym theorem, there exists $q \in \mathrm{L}^+(X, \mathcal{A}, \mu)$ such that

$$|\nu|(A) = \int_A q d\mu.$$

Then, for $A \in \mathcal{A}$,

$$\nu(A) = \nu^+(A) - \nu^-(A) = \nu(A\cap P) + \nu(A\cap N) = |\nu|(A\cap P) - |\nu|(A\cap N) = \int_{A\cap P} q d\mu - \int_{A\cap N} d\mu = \int_A pq d\mu,$$

so by letting $f = pq$, we have

$$\int_A f d\mu = \nu(A).$$

But

$$f = pq = q(\chi_P - \chi_N) = q\chi_P - q\chi_N,$$

so let $g = q\chi_P, h = q\chi_N$. Since signed measure cannot take both $-\infty, \infty$, it follows that

$$\int_P q d\mu < \infty \text{ or } \int_N p d\mu < \infty,$$

which means one of g, h is L^1.

QED

Theorem 6.7. Lebesgue Decomposition Theorem

Let ν, μ be σ-finite measures on $(X, \mathcal{A})$. Then there exists a unique decomposition

$$\nu = \nu_a + \nu_s$$

such that $\nu_a \ll \mu$ and $\nu_s \perp \mu$.

Proof. Consider

$$\lambda = \mu + \nu.$$

Then λ is a measure and $\nu, \mu \ll \lambda$. By the Radon-Nikodym theorem, there exists $f, g \in \mathrm{L}^+(X, \mathcal{A}, \lambda)$ such that

$$\mu(A) = \int_A f d\lambda, \nu(A) = \int_A g d\lambda, \qquad \forall A \in \mathcal{A}.$$

Let

$$A = f^{-1}((0, \infty]), B = f^{-1}(\{0\}).$$

Define $\nu_a, \nu_s : \mathcal{A} \to [0, \infty]$ by

$$\nu_a(E) = \nu(E \cap A), \nu_s(E) = \nu(E \cap B), \qquad \forall E \in \mathcal{A}.$$

Clearly $\nu = \nu_a + \nu_s$.

Claim 1. $\nu_s \perp \mu$.

Consider $X = A \uplus B$.

If $C \subseteq A$, then

$$\nu_s(C) = \nu(C \cap B) = \nu(\emptyset) = 0.$$

Hence A is ν_s-null. On the other hand, given $C \subseteq B$,

$$\mu(C) = \int_C f d\lambda = \int_C 0 d\lambda = 0.$$

Hence B is μ-null.

(End of Claim 1)

Claim 2. $\nu_a \ll \mu$.

Suppose $E \in \mathcal{A}$ with $\mu(E) = 0$. Then

$$\int f\chi_E d\lambda = \int_E f d\lambda = 0.$$

Since $f \in \mathrm{L}^+(X, \mathcal{A}, \lambda)$, it follows that $f\chi_E$ is a measurable nonnegative function, so that

$$f\chi_E = 0 \ \lambda\text{-almost everywhere.}$$

Hence

$$\nu_a(E) = \nu(E \cap A) = \lambda(E \cap A) = 0.$$

(End of Claim 2)

Proof of uniqueness is left as an exercise.

QED

VII. L^p Spaces

Fix a measure space $(X, \mathcal{A}, \mu)$.

1. L^p Spaces

Given measurable $f : X \to \mathbb{R}$, let

$$[f] = \left\{ g \in \mathbb{R}^X : g = f \ \mu\text{-almost everywhere} \right\}.$$

Def'n 7.1. $\mathrm{L}^p(X, \mathcal{A}, \mu)$
Given $p \in [1, \infty)$, we define

$$\mathrm{L}^p(X, \mathcal{A}, \mu) = \left\{ [f] : f \in \mathbb{R}^X, f \text{ is measurable}, |f|^p \in \mathrm{L}^1(X, \mathcal{A}, \mu) \right\}.$$

We define

$$\mathrm{L}^\infty(X, \mathcal{A}, \mu) = \left\{ [f] : f \in \mathbb{R}^X, f \text{ is measurable}, \sup\{t \geq 0 : \mu(\{x \in X : |f(x)| > t\}) > 0\} < \infty \right\}.$$

For convenience, we are going to *treat* equivalence classes $[f]$ as functions f.

Example 7.1. l^p

Consider $(\mathbb{N}, \mathcal{P}(\mathbb{N}), m_c)$, where m_c is the counting measure on $(X, \mathcal{P}(\mathbb{N}))$. Then given $f : \mathbb{N} \to \mathbb{R}$, f is measurable and

$$\int f d\mu = \sum_{n=1}^{\infty} f(n).$$

Hence for $p \in [1, \infty)$,

$$f \in \mathrm{L}^p \iff \int |f|^p \, d\mu < \infty \iff \sum_{n=1}^{\infty} |f(n)|^p << \infty \iff f \in l^p.$$

Proposition 7.1.

Let $p \in [1, \infty]$. Then $\left(\mathrm{L}^p, \|\cdot\|_p\right)$ is a Banach space, where

$$\|f\|_p = \left(\int |f|^p \, d\mu \right)^{\frac{1}{p}}, \qquad \forall f \in \mathrm{L}^p$$

for $p \in [1, \infty)$ and

$$\|f\|_\infty = \sup\{t \geq 0 : \mu(\{x \in X : |f(x)| > t\}) > 0\}, \qquad \forall f \in \mathrm{L}^\infty.$$

Proposition 7.2.

Let $(X, \mathcal{A}, \mathcal{U})$ be a measure space.

(a) For $p \in [1, \infty)$, the set of simple functions of finite support is dense in $\mathrm{L}^p(X, \mathcal{A}, \mu)$.

(b) The set of simple functions is dense in $\mathrm{L}^\infty(X, \mathcal{A}, \mu)$.

Proof of (a). Let $f \in \mathrm{L}^p$ and let $(\varphi_n)_{n=1}^{\infty}$ be an increasing sequence of simple functions converging pointwise to f. Then

$$|\varphi_n|^p \leq |f|^p, \qquad \forall n \in \mathbb{N},$$

so that $\varphi_n \in \mathrm{L}^p$. This means, for a value a which φ_n takes, $\varphi_n^{-1}(a)$ have finite measure. So $(\varphi_n)_{n=1}^{\infty}$ is a sequence of simple functions of finite support. It remains to show $\varphi_n \to f$ in $\|\cdot\|_p$.

But $|\varphi_n - f| \leq |\varphi_n| + |f| \leq 2|f|$, so that

$$|\varphi_n - f| \leq 2^p |f|^p$$

for all $n \in \mathbb{N}$. Hence by the LDCT,

$$\int |\varphi_n - f|^p \, d\mu \to 0,$$

as required.

Proof of (b). Exercise.

QED

Recall 7.2. **Dual Space** of a Normed Linear Space
Let V be a normed linear space over $\mathbb{K}$. The ***dual space*** of V, denoted as V^*, is defined as

$$V^* = \{T : V \to \mathbb{K} : T \text{ is linear and continuous}\}.$$

Recall the following results for normed linear spaces.

Proposition 7.3.
Let $(V, \|\cdot\|)$ be a normed linear space and let $\varphi : V \to \mathbb{K}$ be a linear functional. The following are equivalent.

(a) φ is continuous.

(b) φ is continuous at 0.

(c) φ is bounded.

Proposition 7.4.
Let $(V, \|\cdot\|)$ be a normed linear space. Then $(V^*, \|\cdot\|)$ is a Banach space.

Theorem 7.5. Holder's Inequality
Let $(X, \mathcal{A}, \mu)$ be a measure space and let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ or $p = 1, q = \infty$. If $f \in \mathrm{L}^p(X, \mathcal{A}, \mu)$, $g \in \mathrm{L}^q(X, \mathcal{A}, \mu)$, then $fg \in \mathrm{L}^1$ and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Example 7.2.
Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $p < r$ in $[1, \infty)$.

Claim 1. $\mathrm{L}^\infty \subseteq \mathrm{L}^r$.
Let $f \in \mathrm{L}^\infty$. Then $|f| \leq M$ almost everywhere for some $M \geq 0$. This means $\int |f|^r \, d\mu \leq \int M^r d\mu = M^r \mu(X) < \infty$.
(End of Claim 1)

Claim 2. $\mathrm{L}^r \subseteq \mathrm{L}^p$.
For $f \in \mathrm{L}^r$, $\int |f|^r \, d\mu < \infty$, so that $f^{\frac{r}{p}} \in \mathrm{L}^p$. Let s be the Holder conjugate of $\frac{r}{p}$. Then

$$\|f\|_p^p = \left\| |f|^p \cdot 1 \right\|_1 \leq \left\| |f|^p \right\|_{\frac{r}{p}} \|1\|_s = \|f\|_r^{\frac{p}{r}} \mu(X) < \infty.$$

(End of Claim 2)

It turns out there are no containment relations for $L^p(\mathbb{R}, \mathcal{M}, m)$, where m is the Lebesgue measure and $\mathcal{M}$ is the collection of Lebesgue measurable sets. On the other hand,

$$l^p \subseteq l^r$$

for $p < r$ in $[1, \infty]$.

Theorem 7.6. Riesz Representation Theorem for L^p

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and let $p \in [1, \infty)$. Let q be the Holder conjugate for p. Then

$$\begin{aligned} \varphi : L^q &\to (L^p)^* \\ g &\mapsto \Phi_g \end{aligned}$$

is an isometric isomorphism, where

$$\Phi_g(f) = \int fg d\mu, \qquad \forall g \in L^q, f \in L^p.$$

Proof. Claim 1. *For $g \in L^q$,*

$$\left\|\Phi_g\right\| = \|g\|_q.$$

We consider the case where $p \in (1, \infty)$ only.

For $f \in L^p$, by Holder's inequality,

$$\left|\Phi_g(f)\right| = \left|\int fg d\mu\right| \le \int |fg|\, d\mu = \|fg\|_1 \le \|f\|_p \|g\|_q,$$

so that

$$\left\|\Phi_g\right\| \le \|g\|_q.$$

Since the case $g = 0$ is trivial, assume $g \ne 0$ and let

$$f = \frac{|g|^{q-1}\operatorname{sgn}(g)}{\|g\|_q^{q-1}}.$$

Note $p(q-1) = pq\left(1 - \frac{1}{q}\right) = q$, so that

$$|f|^p = \frac{|g|^q}{\|g\|_q^q},$$

which means

$$\|f\|_p^p = \int |f|^p\, d\mu = \frac{1}{\|g\|_q^q}\int |g|^q\, d\mu = 1.$$

Moreover,

$$\left|\Phi_g(f)\right| = \left|\int fg d\mu\right| = \left|\int \frac{|g|^q}{\|g\|_q^{q-1}}\right| = \|g\|_q.$$

Thus $\left\|\Phi_g\right\| = \|g\|_q$, as required.

(End of Claim 1)

Claim 2. *If $g : X \to \mathbb{R}$ is measurable with*

$$\left|\int \psi g \mu\right| \le M \|\psi\|_p,$$

for all simple ψ with finite support, then $g \in L^q$ and $\|g\|_q \le M$.

We first consider the case where $p, q \in (1, \infty)$.

Let $(\psi_n)_{n=1}^{\infty}$ be a sequence of simple functions such that $\psi_n \to g$ pointwise and

$$|\psi_n| \leq |\psi_{n+1}| \leq |g|, \qquad \forall n \in \mathbb{N}.$$

Since X is σ-finite, write

$$X = \bigcup_{n=1}^{\infty} X_n,$$

where each $\mu(X_n) < \infty$ and $X_1 \subseteq X_2 \subseteq \cdots$. Let

$$\zeta_n = \psi_n \chi_{X_n},$$

which is a simple function with a finite support. Then

$$|\zeta_n| \leq |\zeta_{n+1}| \leq |g|, \qquad \forall n \in \mathbb{N}$$

and $\zeta_n \to g$ pointwise. Define

$$f_n = \frac{|\zeta_n|^{q-1} \operatorname{sgn}(g)}{\|\zeta_n\|_q^{q-1}}$$

Then each f_n is simple with finite support and $\|f_n\|_p = 1$, just as in Claim 1. Then

$$M \geq \sup_{n\in\mathbb{N}} \left| \int f_n d g d\mu \right| = \sup_{n\in\mathbb{N}} \int \frac{|\zeta_n|^{q-1} |g|}{\|\zeta_n\|_q^{q-1}} d\mu \geq \sup_{n\in\mathbb{N}} \int \frac{|\zeta_n|^q}{\|\zeta_n\|_q^{q-1}} d\mu = \sup_{n\in\mathbb{N}} \|\zeta_n\|_q.$$

Now, $|\zeta_n|^q \leq |g|^q$, $(|\zeta_n^q|)_{n=1}^{\infty}$ is increasing, and $|\zeta_n|^q \to |g|^q$ pointwise, so by the monotone convergence theorem,

$$\sup_{n\in\mathbb{N}} \|\zeta_n\|_q = \lim_{n\to\infty} \|\zeta_n\|_q = \|g\|_q.$$

Thus

$$M \geq \|g\|_q,$$

as required.

Now suppose $p = 1, q = \infty$. Let $\varepsilon > 0$ be given and consider

$$A = \{x : |g(x)| \geq M + \varepsilon\}.$$

Since we want to show $\|g\|_\infty \leq M$, suppose $\mu(A) > 0$ for contradiction. Since X is σ-finite, we may find $B \subseteq A$ such that

$$0 < \mu(B) < \infty.$$

Take

$$f = \frac{1}{\mu(B)} \operatorname{sgn}(g) \chi_B$$

so that f is simple and $\|f\|_1 = 1$. Then

$$\int f g d\mu = \frac{1}{\mu(B)} \int |g| \chi_B d\mu = \frac{1}{\mu(B)} \int_B |g| d\mu \geq \frac{1}{\mu(B)} \int_B M + \varepsilon d\mu = M + \varepsilon > M = M \|f\|_1,$$

which is a contradiction.

Since the choice of ε was arbitrary, it follows M is an essential bound for $|g|$, so that

(End of Claim 2)

We now turn to the proof of the Riesz representation theorem. We consider two cases.

Case 1. $\mu(X) < \infty$.

Let $\Phi \in \mathrm{L}^p(X, \mathcal{A}, \mu)^*$, where we desire to find $g \in \mathrm{L}^q$ such that $\Phi = \Phi_g$. Consider

$$\begin{aligned} \nu : \mathcal{A} &\mapsto \mathbb{R} \\ A &\mapsto \Phi(\chi_A) \end{aligned}.$$

Note that

$$\nu(\emptyset) = \Phi(\chi_\emptyset) = \Phi(0) = 0.$$

Let $\{A_n\}_{n=1}^\infty \subseteq \mathcal{A}$ and let $A = \bigcup_{n=1}^\infty A_n$. Then

$$\left\| \chi_A - \sum_{n=1}^N \chi_{A_n} \right\|_p^p = \left\| \sum_{n=N+1}^\infty \chi_{A_n} \right\|_p^p = \left(\left(\int \left(\sum_{n=N+1}^\infty \chi_{A_n} \right)^p d\mu \right)^{\frac{1}{p}} \right)^p = \int \sum_{n=N+1}^\infty \chi_{A_n} d\mu = \mu\left(\bigcup_{n=N+1}^\infty A_n \right) = \sum_{n=N+1}^\infty \mu(A_n).$$

Since $\mu(X) < \infty$, it follows $\mu(A) = \sum_{n=1}^\infty \mu(A_n) < \infty$, so that

$$\sum_{n=N+1}^\infty \mu(A_n) \to 0.$$

Hence $\chi_A = \sum_{n=1}^\infty \chi_{A_n} \in \mathrm{L}^p$. By continuity of Φ,

$$\nu(A) = \Phi(\chi_A) = \sum_{n=1}^\infty \Phi(\chi_{A_n}) = \sum_{n=1}^\infty \nu(A_n).$$

Hence ν is a measure.

If $\mu(A) = 0$, then $\chi_A = 0$ μ-almost everywhere, so that

$$\nu(A) = \Phi(0) = 0.$$

This means $\nu \ll \mu$, so by the Radon-Nikodym theorem, there is $g \in \mathrm{L}^1$ such that

$$\nu(A) = \int_A g d\mu.$$

Note that g is L^1 since the measure space is finite. Take a simple function

$$\psi = \sum_{k=1}^n a_k \chi_{A_k}.$$

Then

$$\Phi(\psi) = \sum_{k=1}^n a_k \chi_{A_k} = \sum_{k=1}^n a_k \nu(A_k) = \int \psi d\nu.$$

Also,

$$\sum_{k=1}^n a_k \nu(A_k) = \sum_{k=1}^n a_k \int_{A_k} g d\mu = \sum_{k=1}^n \int a_k \chi_{A_k} g d\mu = \int \psi g d\mu.$$

That is,

$$\Phi(\psi) = \int \psi d\nu = \int \psi g d\mu = \Phi_g(\psi).$$

Hence

$$\left| \int \psi g d\mu \right| = |\Phi(\psi)| \le \|\Phi\| \, \|\psi\|_p.$$

By taking $M = \|\Phi\|$, we see that $g \in \mathrm{L}^q$ with $\|g\|_q \le M$. Then Φ, Φ_g are continuous functions that coincide on a dense subset of L^p, so that $\Phi = \Phi_g$.

(End of Case 1)

We now consider the general case, where X is assumed to be σ-finite. Write $X = \bigcup_{n=1}^{\infty} X_n$ so that each $\mu(X_n) < \infty$ and

$$X_1 \subseteq X_2 \subseteq \cdots .$$

We may identify $\mathrm{L}^r(X_n, \mathcal{A} \cap \mathcal{P}(X_n), \mu)$ as a subset of $\mathrm{L}^r(X, \mathcal{A}, \mu)$.

Let $\Phi \in \mathrm{L}^p(X, \mathcal{A}, \mu)^*$. For every $n \in \mathbb{N}$, there exists a unique $g_n \in \mathrm{L}^q(X_n)$ such that

$$\Phi|_{X_n} = \Phi_{g_n}$$

by Case 1. Moreover,

$$\|g_n\|_q = \left\|\Phi_{g_n}\right\| \leq \|\Phi\| .$$

By uniqueness, there is a unique $g : X \to \mathbb{R}$ such that for all $n \in \mathbb{N}$, $g|_{X_n} = g_n$. Since $X_1 \subseteq X_2 \subseteq \cdots$, $g_n \to g$ pointwise, which means g is measurable.

Note that, since X_n's are nested, $(|g_n|^q)_{n=1}^{\infty}$ is an increasing sequence converging pointwise to $|g|^q$, so that

$$\|g_n\|_q \to \|g\|_q$$

by the monotone convergence theorem. It follows that

$$\|g\|_q \leq \|\Phi\| < \infty,$$

so that $g \in \mathrm{L}^q(X, \mathcal{A}, \mu)$.

If $f \in \mathrm{L}^p(X, \mathcal{A}, \mu)$, we have

$$\left|f\chi_{X_n} - f\right|^p \leq (2|f|)^p = 2^p |f|^p .$$

By the Lebesgue dominated convergence theorem,

$$f\chi_{X_n} \to f \text{ in } \mathrm{L}^p.$$

Hence, by continuity of Φ,

$$\Phi(f) = \lim_{n\to\infty} \Phi\left(f\chi_{X_n}\right) = \lim_{n\to\infty} \int \left(f\chi_{X_n}\right) g d\mu = \lim_{n\to\infty} \int_{X_n} f g_n d\mu = \int f g d\mu = \Phi_g(f),$$

where the second last equality follows from the Lebesgue dominated convergence theorem.

QED

Example 7.3.

Consider $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ with

$$\mu(\emptyset) = 0$$
$$\mu(A) = \infty, \qquad \forall A \neq \emptyset$$

Then observe that, for $f : \mathbb{N} \to \mathbb{R}$, if there is $n \in \mathbb{N}$ such that $f(n) \neq 0$,

$$\int |f|\, d\mu \geq \int_{\{n\}} |f|\, d\mu = \infty,$$

so that $\mathrm{L}^1 = \{0\}$.

But we have $\mathrm{L}^\infty = l^\infty$, so that

$$\left(\mathrm{L}^1\right)^* \neq \mathrm{L}^\infty .$$

Theorem 7.7. Riesz Representation Theorem II

Let $(X, \mathcal{A}, \mu)$ and let $p, q \in (1, \infty)$ be Holder conjugates. Then $g \mapsto \Phi_g$ is an isometric isomorphism from L^q to $(\mathrm{L}^p)^*$.

Proof Idea. Use $M = \sup\left\{\|g_E\|_q : E \subseteq X \text{ is } \sigma\text{-finite}\right\}$.

QED