1.
Linear Programming

Gaussian Elimination

(1.1)
Gaussian Elimination

Consider the following system

$$\underbrace{\begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & -1 & 2 & -2 \\ -1 & \frac{1}{2} & -1 & 1 \end{bmatrix}}_{=A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{=x} = \underbrace{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}}_{=b} \qquad [1.1]$$

which we desire to solve by using *Gaussian elimination*. In short, what Gaussian elimination does is to rewrite the system via linear combination of rows to make it easy to find a solution or to show there is no solution. For instance, [1.1] can be solved as follows.

(a) $\mathrm{Row}_2(A|b) - 2\,\mathrm{Row}_1(A|b)$ and $\mathrm{Row}_3(A|b) + \mathrm{Row}_1(A|b)$ gives

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & -\frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix},$$

say $[A'|b']$.

(b) $\mathrm{Row}_3(A'|b') + \frac{1}{2}\mathrm{Row}_2(A'|b')$ gives

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \frac{5}{2} \end{bmatrix},$$

which means the system has no solution.

Now consider more general setting of $Ax = b$:

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \qquad [1.2]$$

where we desire to see when [1.2] fails to have a solution, say $\bar{x}$. Since Gaussian elimination solves systems by taking linear combinations of rows, we write

$$\sum_{i=1}^{m} y_i \begin{bmatrix} A_{i1} & A_{i2} & \cdots & A_{in} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \sum_{i=1}^{m} y_i b_i \qquad [1.3]$$

(i.e. multiply row $i \in \{1, \ldots, m\}$ by $y_i \in \mathbb{R}$ and add rows). Letting

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix},$$

we can write [1.3] as

$$\left(y^T A\right) x = y^T b. \tag{1.4}$$

[1.4] is what the following theorem is about.

Theorem 1.1. *Given $A \in M_{m\times n}(\mathbb{R}), b \in M_{m\times 1}(\mathbb{R})$, the system*

$$Ax = b$$

has a solution if and only if $y^T b = 0$ for every $y \in M_{m\times 1}(\mathbb{R})$ such that $y^T A = 0$.

Lemma 1.1.1. *Each equation obtained on Gaussian elimination is a linear combination of the original equations $Ax = b$.*

Proof. This result is clear from the definition of Gaussian elimination. ∎

Lemma 1.1.2. *Suppose that there exists nonzero $y \in M_{m\times 1}(\mathbb{R})$ such that*

$$y^T A = 0, y^T b \neq 0. \tag{1.5}$$

Then $Ax = b$ has no solution.

Proof. Suppose that the system has a solution, say $\bar{x} \in M_{n\times 1}(\mathbb{R})$,

$$A\bar{x} = b. \tag{1.6}$$

Then we can multiply both sides of [1.6] by any $y^T \in M_{1\times m}(\mathbb{R})$ and still obtain an equality, and in particular by y^T such that [1.5] holds. But this means that

$$y^T b = y^T A\bar{x} = \left(y^T A\right)\bar{x} = 0\bar{x} = 0,$$

which contradicts with [1.5]. Thus there is no solution of the system. ∎

Proof of Theorem 1.1.

- ($\Longrightarrow$) This direction is provided by Lemma 1.1.2.
- ($\Longleftarrow$) Fix any $y \in M_{m\times 1}(\mathbb{R})$ such that $y^T A = 0$, where we desire to show that $y^T b = 0$. By Lemma 1.1.1, this means that when we apply Gaussian elimination to $Ax = b$, we are not going to get an equation of the form

  $$0^T x = \delta$$

  with $\delta \neq 0$, which means Gaussian elimination is going to find a solution of the system. ∎

Corollary 1.1.3. *$Ax = b$ has a solution if and only if $y^T A = 0, y^T b = 1$[a] has no solution.*

[a] We can replace 1 by any nonzero scalar, since we can scale y with any nonzero scalar that we would like to use.

Def'n 1.1. **Decision Problem**
A ***decision problem*** is a problem with yes, no answer.

(EX 1.2) Does $Ax = b$ have a solution?

(1.3) Formally speaking, given a set X and a subset $Y \subseteq X$, a decision problem asks the following question: given $x \in X$, $x \in L$? In other words, we identify this problem by L. For instance, (EX 1.2) can be described by

$$X = \{(A,b) : A \in M_{m\times n}(\mathbb{R}), b \in M_{m\times 1}(\mathbb{R})\}$$

and

$$L = \{(A,b) \in X : \exists \bar{x}[A\bar{x} = b]\}.$$

Def'n 1.2. **Algorithm**
An ***algorithm*** is a list of instructions to solve a problem.

Def'n 1.3. **Running Time**
The ***running time*** of an algorithm[a] is the number of elementary bit operations.[b]

[a] or an instance of an algorithm with a given input
[b] ***Elementary bit operations*** refer to addition, subtraction, multiplication, division, composition.

Def'n 1.4. **Size** of a Problem
The ***size*** of a problem is the number of bits to write it down.

(EX 1.4) Example of Sizes

(a) $n \in \mathbb{Z}$:

$$\operatorname{size}(n) = \underbrace{1}_{\text{sign}} + \underbrace{\lceil \log_2(|n|+1) \rceil}_{\substack{\text{write } n \text{ in binary and} \\ \text{round up to an integer}}}.$$

(b) $r = \frac{p}{q} \in \mathbb{Q}$, where $p \in \mathbb{Z}, q \in \mathbb{N}$, where p,q are coprime:

$$\operatorname{size}(r) = 1 + \lceil \log_2(|p|+1) \rceil + \lceil \log_2(q+1) \rceil.$$

(c) $A \in M_{m\times n}(\mathbb{Q})$:

$$\operatorname{size}(A) = mn + \sum_{i=1}^{m} \sum_{j=1}^{n} \operatorname{size}(A_{ij}).$$

(d) $Ax = b$, where A, b are rational:

$$\operatorname{size}(Ax = b) = \operatorname{size}[A|b].$$

Def'n 1.5. **Polynomial-time (P-time, P)** Algorithm
An algorithm is called ***polynomial-time (P-time, P)*** if there exists a polynomial $f \in \mathbb{R}[x]$ such that for a problem of size n the running time of the algorithm is at most $f(n)$.

Def'n 1.6. **P**
The class of decision problems solvable in polynomial time is denoted by ***P***.

Def'n 1.7. **NP**
The class of decision problems L such that, for any $l \in L$, there exists a proof of the fact that $l \in L$, of length polynomially bounded by the size of l, is denoted by ***NP***.

Def'n 1.8. **CO-NP**
The class of decision problems whose complement[a] is NP is denoted by ***CO-NP***.

[a] The ***complement*** of a decision problem L is $L^C = X \setminus L$, where X is the universe set. See (1.3).

(1.5)
NP and CO-NP

(a) What it means for a decision problem L to be NP is that, if $x \in X$ is such that $x \in L$, then we can verify the fact that $x \in L$ in a polynomial time.

(b) On the other hand, if a decision problem L is CO-NP, then given any $x \in X$ such that $x \notin L$, we can verify the fact that $x \notin L$ in a polynomial time.

A problem is said to have a *good characterization* if it is in NP∩CO-NP.

Def'n 1.9. **Big Oh Notation**
We say that the running time of an algorithm is $O(g(n))$ if there exists a function g such that the running time is bounded above by Kg for some $K \in \mathbb{R}$.

Proposition 1.2. *Let $A \in M_{m\times n}(\mathbb{Q})$ and let $b \in M_{m\times 1}(\mathbb{Q})$. Then the problem "does $Ax = b$ have a solution?" has a good characterization.*

Lemma 1.2.1. *Let $M \in M_{n\times n}(\mathbb{Q})$. Then*
$$\det(M) < 2\,\mathrm{size}(M).$$

Lemma 1.2.2. *If a rational system $Ax = b$ has a solution, then there exists a solution of size polynomially bounded by the sizes of A and b.*

Theorem 1.3.
Rational Linear System is of Class P

Gaussian elimination is a P-time algorithm for (EX 1.6).

Corollary 1.3.1. *The following have P-time algorithms.*

(a) Given $A \in M_{m\times n}(\mathbb{Q})$, what is $\mathrm{rank}(A)$?

(b) Given $A \in M_{n\times n}(\mathbb{Q})$, what is $\det(A)$?

(c) Given $M \in M_{n\times n}(\mathbb{Q})$, what is M^{-1}?

Linear Diophantine Equations

(1.6) We describe some results from the theory of linear diophantine equations.

Herminte Normal Form

Def'n 1.10. Let $A \in M_{m\times n}(\mathbb{R})$. We say A is in ***Hermite normal form*** if A is of the form

$$A = \begin{bmatrix} B & 0 \end{bmatrix}$$

where $B \in M_{n\times n}(\mathbb{R})$ is an invertible, lower triangular, and nonnegative matrix such that each row of B has a unique maximum entry, located on the main diagonal of B.

Elementary Unimodular Column Operations

Def'n 1.11. The following operations on a matrix are called ***elementary unimodular column operations***:

(a) exchaning two columns;

(b) multiplying a column by a unit (i.e. $1, -1$ for $\mathbb{R}$); and

(c) adding an integral multiple of one column to another column.

Theorem 1.4. Hermite Normal Form

Let $A \in M_{m\times n}(\mathbb{Q})$. If A is of full row rank, then A can be brought into Hermite normal form by a series of elementary unimodular column operations.

Proof. Let $A \in M_{m\times n}(\mathbb{Q})$ be of full row rank (this implies $m \le n$). Without loss of generality, assume that A is integral, since if A is not integral, then we can find $k \in \mathbb{Z}$ such that kA is integral and find the Hermite normal form of kA, and then divide the Hermite normal form of kA by k to find the Hermite normal form of A. Clearly we can transform A to the form

$$\begin{bmatrix} b & 0 \\ C & D \end{bmatrix}$$

for some $b \in \mathbb{N}$ and C, D of appropirate size by elementary unimodular column operations. So assume that we have transformed A to the form

$$\begin{bmatrix} B & 0 \\ C & D \end{bmatrix}$$

by elementary unimodular column operations, where $B \in M_{k\times k}(\mathbb{N})$ is lower triangular with $k < m$. Then it suffices to show that we can expand the lower triangular matrix part, while maintaining the properties described in Def'n 1.10. Now perform elementary unimodular column operations on D such that the sum $\sum_{i=1}^{n-k} D_{1i}$ of the entries of the first row of D is minimum and $D_{11} \ge D_{12} \ge \cdots \ge D_{1(n-k)} \ge 0$. Note that $D_{11} > 0$, since D has a full row rank. Moreover, if we assume that $D_{12} > 0$ for contradiction, then we see that

$$0 \le (D_{11} - D_{12}) + D_{12} + \cdots < D_{11} + D_{12} + \cdots,$$

a contradiction. Therefore $D_{12}, \ldots, D_{i(n-k)} = 0$, so we obtained a larger lower triangular matrix, and we can add integral multiple of $(k+1)$th column of D (i.e. 1st column of D) such that the entries $C_{11}, \ldots, C_{1k}$ are positive and smaller than D_{11}. ∎

Corollary 1.4.1. Solution of Linear Diophantine System

Let $A \in M_{m\times n}(\mathbb{Q})$ and let $b \in M_{m\times 1}(\mathbb{Q})$. Then $Ax = b$ has an integral solution $x \in M_{n\times 1}(\mathbb{Z})$ if and only if $yb \in \mathbb{Z}$ for all $y \in M_{m\times 1}(\mathbb{Q})$ such that yA is integral.

Proof.

- ($\implies$) Assume that $Ax = b$ for some $x \in M_{n\times 1}(\mathbb{Z})$. Then given any $y \in M_{m\times 1}(\mathbb{Q})$ such that yA is integral,
$$yb = yAx \in \mathbb{Z},$$
since both yA, x are integral.
- ($\impliedby$) Assume that given any $y \in M_{m\times 1}(\mathbb{Q})$ such that yA is integral, $yb \in \mathbb{Z}$. Then we have a solution to $Ax = b$, since otherwise we can find $y \in M_{m\times 1}(\mathbb{Q})$ such that $yA = 0$ but yb is some nonintegral rational number by Corollary 1.13, contradicting our hypothesis. Therefore, A is in full row rank, so we may transform A to its Hermite normal form
$$\begin{bmatrix} B & 0 \end{bmatrix}. \tag{1.7}$$
Note that the statement of the corollary is invariant under elementary unimodular column operations, so we may work with $\begin{bmatrix} B & 0 \end{bmatrix}$ instead of A. Since
$$B^{-1}\begin{bmatrix} B & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix},$$
it follows that
$$B^{-1}b$$
is integral. Since
$$\begin{bmatrix} B & 0 \end{bmatrix}\begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix} = b,$$
$x = \begin{bmatrix} B^{-1} \\ b \end{bmatrix}$ is an integral solution for $Ax = b$. ∎

Def'n 1.12. **Unimodular** Matrix

Let $A \in M_{n\times n}(\mathbb{R})$. We say A is ***unimodular*** if U is integral and $\det(A)$ is a unit.

(1.7) Unimodular Matrix

Similar to how Gauss-Jordan elemination is done by multiplying elementary matrices to left, Hermite normal form process is done by multiplying unimodular matrices to right.

Corollary 1.4.2.

Let $A \in M_{m\times n}(\mathbb{Q})$ be of full row rank. Then there is a unimodular $U \in M_{n\times n}(\mathbb{Z})$ such that AU is the Hermite normal form of A. Moreover, if A is invertible (i.e. $m = n$), then U is unique.

Proof. The existence part is provided by Theorem 1.1. Proof of the uniqueness part is omitted. ∎

System of Linear Inequalities

(1.8)
Fourier-Motzkin Elimination (FME)

Consider the following problem:

System of Linear Inequalities

Given $A \in M_{m\times n}(\mathbb{F}), b \in \mathbb{F}^m$, determine if $Ax \le b$ has a solution, and if so, find one. [1.8]

Note that we are using $\mathbb{F}$ to denote $\mathbb{R}$ or $\mathbb{Q}$ throughout this section. We proceed to solve [1.8] as follows. We may assume that each entry in the first column of A is a unit (i.e. $1,-1$) or 0, since we may scale each row of $Ax = b$ by any nonzero scalar. After reordering rows, we obtain a system of the form

$$\begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ 1 & a_{m'2} & \cdots & a_{m'n} \\ -1 & a_{(m'+1)2} & \cdots & a_{(m'+1)n} \\ \vdots & \vdots & & \vdots \\ -1 & a_{m''2} & \cdots & a_{m''n} \\ 0 & a_{(m''+1)2} & \cdots & a_{(m''+1)n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \le \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}, \tag{1.9}$$

where $m', m'' \le m$ are such that $m' \le m''$ and

$$a_{11} = \cdots = a_{m'1} = 1, a_{(m'+1)1} = \cdots = a_{m''1}, a_{(m''+1)1} = \cdots = a_{m1} = 0.$$

Note that the first $m' + m''$ rows of [1.9] are equivalent to

$$\max_{m'+1\le j\le m''} \underbrace{\begin{bmatrix} a_{j2} & \cdots & a_{jn} \end{bmatrix}}_{=\alpha_j} \underbrace{\begin{bmatrix} x_2 \\ \vdots \\ x_n \end{bmatrix}}_{=\xi} - b_j \le x_1 \le \min_{1\le i\le m'} b_i - \underbrace{\begin{bmatrix} a_{i2} & \cdots & a_{in} \end{bmatrix}}_{=\alpha_i} \underbrace{\begin{bmatrix} x_2 \\ \vdots \\ x_n \end{bmatrix}}, \tag{1.10}$$

which means we can eliminate x_1. Hence [1.10] leads to a system of $n-1$ indeterminates

$$\begin{cases} (\alpha_i + \alpha_j)\,\xi \le b_i + b_j & \forall i \in \{1,\dots,m'\}, j \in \{m'+1,\dots,m''\} \\ \alpha_k \xi \le b_k & \forall k \in \{m''+1,\dots,m\} \end{cases}, \tag{1.11}$$

which is equivalent to [1.9], where $\alpha_k = \begin{bmatrix} a_{k2} & \cdots & a_{kn} \end{bmatrix}$ for all $k \in \{m''+1,\dots,m\}$. In other words, if $\xi = \{x_2,\dots,x_n\}$ is a solution for [1.11], then for any x_1 that satisfies [1.10], $(x_1,\dots,x_n)$ is a solution for the original system [1.9]. In other words, to solve [1.8], we run the above process over and over until we are left with only one indeterminate, and if the system is consistent, then by using [1.10] we can expand our solution. Otherwise, the system $Ax \le b$ is inconsistent, so we halt. FME has a theoretical importance because it allows one to prove *Farkas' lemma*.

Theorem 1.5.
Farkas' Lemma

Let $A \in M_{m\times n}(\mathbb{F})$ and let $b \in \mathbb{F}^m$. Then ther exists $x \in \mathbb{F}^n$ such that $Ax = b$ and $x \ge 0$ if and only if for each $y \in \mathbb{F}^m$ with $y^T A \ge 0$, $y^T b \ge 0$.

Proof.

- ($\implies$)Assume that there exists nonnegative $x \in \mathbb{F}^n$ such that $Ax = b$. Then given any $y \in \mathbb{F}^m$ with $y^T A \geq 0$,
$$y^T b = y^T Ax = \underbrace{\left(y^T A\right)}_{\geq 0} \underbrace{x}_{\geq 0} \geq 0.$$
- ($\impliedby$)Assume that there does not exist nonnegative $x \in \mathbb{F}^n$ such that $Ax = b$. Note that the system $Ax = b, x \geq 0$, is equivalent to
$$\begin{bmatrix} A \\ -A \\ -I \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}, \qquad [1.12]$$
so for convenience sake, denote $H = \begin{bmatrix} A \\ -A \\ -I \end{bmatrix}, h = \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}$. Now we apply FME to [1.12], where we note that, in any step of FME, we are performing any of the following two procedures:

  (a) Divide a row by the absolute value of the leading coefficient, provided that the coefficient is nonzero.

  (b) Take two rows, where one starts with 1 and the other starts with -1, and add them together.

  Observe that both (a), (b) can be expressed as taking nonnegative linear combination of rows, so it follows thatt we can describe FME by a finite sequence of nonnegative matrices, say $\{E_i\}_{i=1}^{n-1}$ of appropirate size, such that each E_i eliminates x_i. This implies that, if
$$\alpha' x_n = \beta' \qquad [1.13]$$
is the final system that we obtain from performing FME on [1.12], then
$$E_{n-1} \cdots E_1 Hx \leq E_{n-1} \cdots E_1 h \qquad [1.14]$$
is equivalent to [1.13], so [1.12]. But note that we can proceed one step further than what FME does: eliminate x_n too. This will give us an ineqality of the form
$$0 \leq \begin{bmatrix} \beta_1 \\ \cdots \\ \beta_N \end{bmatrix}, \qquad [1.15]$$
since we eliminated every indeterminate. This, too, has a nonnegative matrix E_n such that
$$E_n \cdots E_1 Hx \leq E_n \cdots E_1 h \qquad [1.16]$$
is equivalent to [1.15]. For brevity, denote $E = E_n \cdots E_1$ so that we may write
$$EHx \leq Eh. \qquad [1.17]$$
By construction, [1.17] is equivalent to the original system $Ax = b, x \geq 0$, which we asssumed to have no solution. In other words, [1.17] is inconsistent, which means there is a row of E, say $\begin{bmatrix} e_1 & \cdots & e_{2m+n} \end{bmatrix}$, such that
$$\begin{bmatrix} e_1 & \cdots & e_{2m+n} \end{bmatrix} h < 0. \qquad [1.18]$$
Now write
$$v = \begin{bmatrix} e_1 & \cdots & e_m \end{bmatrix}, u = \begin{bmatrix} e_{m+1} & \cdots & e_{2m} \end{bmatrix}, w = \begin{bmatrix} 2_{2m+1} & \cdots & e_{2m+n} \end{bmatrix}$$

for convenience. Then we have

$$(v-u)b = \begin{bmatrix} v & u & w \end{bmatrix} \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix} = \begin{bmatrix} e_1 & \cdots & e_{2n+m} \end{bmatrix} h \overset{[1.18]}{<} 0 \quad [1.19]$$

and

$$(v-u)A = \begin{bmatrix} v & u & w \end{bmatrix} \begin{bmatrix} A \\ -A \\ -I \end{bmatrix} + w \overset{EH=0}{=} w \overset{E\geq 0}{\geq} 0, \quad [1.20]$$

where $E \geq 0$ is by the fact that $E = E_n \cdots E_1$, where $E_1, \ldots, E_n \geq 0$ by construction. Thus by letting $y = (v-u)^T$, we have

$$y^T b \overset{[1.19]}{<} 0, y^T A \overset{[1.20]}{\geq} 0.$$

∎

Def'n 1.13. **Conical Combination, Convex Cone, Polyhedral Cone**

Given $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{F}^n$, a ***conical combination*** of $\mathbf{x}_1, \ldots, \mathbf{x}_m$ is a linear combination

$$\sum_{i=1}^m a_i \mathbf{x}_i$$

with nonnegative coefficients $a_1, \ldots, a_m \geq 0$. A subset $C \subseteq \mathbb{F}^n$ is called a ***convex cone*** (or ***cone***) if

$$a\mathbf{x} + b\mathbf{y} \in C$$

for all $\mathbf{x}, \mathbf{y} \in C$. C is called ***polyhedral*** if there exists $A \in M_{m \times n}(\mathbb{F})$ such that $C = \{\mathbf{x} \in \mathbb{F}^n : A\mathbf{x} \leq 0\}$.

Theorem 1.6. Fundamental Theorem of Linear Inequalities

Let $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{F}^n$. Then given any $\mathbf{y} \in \mathbb{F}^n$, exactly one of the following is true.

(a) $\mathbf{y} \in \operatorname{cone}\{\mathbf{x}_i\}_{i=1}^m$.

(b) There exists $\mathbf{z} \in \mathbb{F}^n$ such that the orthogonal complement $H = \left\{\mathbf{x} \in \mathbb{F}^n : \mathbf{z}^T\mathbf{x} = 0\right\}$ of $\{\mathbf{z}\}$ contains $t-1$ linearly independent vectors from $a_1, \ldots, a_m$, such that $\mathbf{z}^T\mathbf{y} < 0$ and $\sum_{i=1}^m \mathbf{z}^T\mathbf{x}_i \geq 0$, where $t = \operatorname{rank}\{\mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{y}\}$.

(1.9) We derive some important consequences of the fundamental theorem of linear inequalities.

Def'n 1.14. **Half-space**

A subset $H \subseteq \mathbb{F}^n$ of the form

$$H = \left\{\mathbf{x} \in \mathbb{F}^n : \mathbf{v}^T\mathbf{x} \leq 0\right\}$$

for some nonzero $\mathbf{v} \in \mathbb{F}^n$ is called a ***half-space***.

Observe that, given $A \in M_{m \times n}(\mathbb{F})$, we can write

$$A = \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_m^T \end{bmatrix}$$

for some $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{F}^n$. That is, the solution set $\{\mathbf{x} \in \mathbb{F}^n : A\mathbf{x} \leq 0\}$ is an intersection of m half-spaces $H_1, \ldots, H_m$ with

$$H_i = \left\{\mathbf{x} \in \mathbb{F}^n : \mathbf{v}_i^T\mathbf{x} \leq 0\right\}$$

for all $i \in \{1, \dots, m\}$. In other words, every half-space is a polyhedral cone and every polyhedral cone is a finite intersection of half-spaces.

(1.10) Cone Generated by Finite Number of Points

Note that given any $\mathbf{x}_1, \dots, \mathbf{x}_m \in \mathbb{F}^n$, the set

$$\left\{\sum_{i=1}^m a_i\mathbf{x}_i : a_1, \dots, a_m \geq 0\right\}$$

of every conical combination of $\mathbf{x}_1, \dots, \mathbf{x}_m$ is a cone, denoted as $\operatorname{cone}\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$. In fact, $\operatorname{cone}\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ is the minimal cone that has $\mathbf{x}_1, \dots, \mathbf{x}_m$.

Proof. We verify two things.

(a) $\operatorname{cone}\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ is a cone that has $\mathbf{x}_1, \dots, \mathbf{x}_m$.

Proof. Clearly $\mathbf{x}_1, \dots, \mathbf{x}_m \in \operatorname{cone}\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$. Fix $\mathbf{v}, \mathbf{u} \in \operatorname{cone}\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$. This means there are $a_1, \dots, a_m, b_1, \dots, b_m \geq 0$ such that

$$\mathbf{v} = \sum_{i=1}^m a_i\mathbf{x}_i, \mathbf{u} = \sum_{i=1}^m b_i\mathbf{x}_i. \tag{1.21}$$

Then given any $a, b \geq 0$, we have that

$$a\mathbf{v} + b\mathbf{u} = a\sum_{i=1}^m a_i\mathbf{x}_i + b\sum_{i=1}^m b_i\mathbf{x}_i = \sum_{i=1}^m \underbrace{aa_i + bb_i}_{\geq 0}\mathbf{x}_i \in C,$$

so $\operatorname{cone}\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ is a cone. ◁

(b) $\operatorname{cone}\{\mathbf{x}_1, \dots, \mathbf{x}_m\} \subseteq C$ for any cone $C \subseteq \mathbb{F}^n$ with $\mathbf{x}_1, \dots, \mathbf{x}_m \in C$.

Proof. Let

$$\mathbf{v} = \sum_{i=1}^m a_i\mathbf{x}_i \in \operatorname{cone}\{\mathbf{x}_1, \dots, \mathbf{x}_m\} \tag{1.22}$$

for some $a_1, \dots, a_m \geq 0$, where we desire to show that $\mathbf{v} \in C$. But note that if $\sum_{i=1}^k a_i\mathbf{x}_i \in C$ for some $k \in \{1, \dots, m-1\}$, then we have $1, a_{k+1} \geq 0$ and $x_{k+1} \in C$ such that

$$1\sum_{i=1}^k a_k\mathbf{x}_k + a_{k+1}\mathbf{x}_{k+1} \in C.$$

Thus by the induction the result holds. ◁

We call $\operatorname{cone}\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ the cone ***generated by*** $\mathbf{x}_1, \dots, \mathbf{x}_m$.

Def'n 1.15. Finitely Generated Cone
Let $C \subseteq \mathbb{F}^n$ be a cone. If there exist finite number of vectors $\mathbf{x}_1, \dots, \mathbf{x}_m \in \mathbb{F}^n$ such that $C = \operatorname{cone}\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$, then we say C is ***finitely generated***.

The fundamental theorem of linear inequalities can be used to show that the notion of polyhedral cones coincides with the notion of finitely generated cones.

Theorem 1.7. Minkowski-Weyl Theorem

Let $C \subseteq \mathbb{F}^n$ be a cone. Then C is polyhedral if and only if finitely generated.

Proof.

- ($\Longleftarrow$) Assume that C is finitely generated by $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{F}^n$. We may assume that span $\{\mathbf{x}_j\}_{j=1}^m = \mathbb{F}^n$, since we can work in span $\{\mathbf{x}_j\}_{j=1}^m$ otherwise. Now consider a collection $\mathcal{C} = \{H_i\}_{i=1}^l$ of half-spaces of the form
$$H_i = \left\{\mathbf{x} \in \mathbb{F}^n : \mathbf{v}_i^T \mathbf{x} \geq 0\right\} \qquad [1.23]$$
for some $\mathbf{v}_i \in \mathbb{F}^n$ such that the associated hyperplane $\left\{\mathbf{x} \in \mathbb{F}^n : \mathbf{v}_i^T \mathbf{x} = 0\right\}$ is spanned by $n-1$ linearly independent vectors from $\mathbf{x}_1, \ldots, \mathbf{x}_m$. Now the claim is that
$$\operatorname{cone}\left\{\mathbf{x}_j\right\}_{j=1}^m = \bigcap_{i=1}^l H_i. \qquad [1.24]$$
To verify [1.25], observe that
$$\begin{aligned} \mathbf{y} \in \operatorname{cone}\left\{\mathbf{x}_j\right\}_{j=1}^l &\implies \mathbf{y} \text{ is a conical combination of } \mathbf{x}_1, \ldots, \mathbf{x}_m \\ &\overset{[1.24]}{\implies} \forall i \in \{1, \ldots, l\} \left[\mathbf{v}_i^T \mathbf{y} \geq 0\right] \\ &\implies \mathbf{y} \in \left\{\mathbf{x} \in \mathbb{F}^n : \mathbf{v}_1^T \mathbf{x}, \ldots, \mathbf{v}_l^T \mathbf{x} \geq 0\right\} = \bigcap_{i=1}^l H_i. \end{aligned}$$
Conversely,
$$\begin{aligned} \mathbf{y} \notin \operatorname{cone}\left\{\mathbf{x}_j\right\}_{j=1}^l &\implies \exists i \in \{1, \ldots, l\} \left[\mathbf{v}_i^T \mathbf{y} < 0\right] && \text{by Theorem 1.6} \\ &\implies \exists i \in \{1, \ldots, l\} \left[\mathbf{y} \notin H_i\right] \\ &\implies \mathbf{y} \notin \bigcap_{i=1}^l H_i, \end{aligned}$$
so [1.25] is established. But [1.25] means that cone $\left\{\mathbf{x}_j\right\}_{j=1}^l$ is a finite intersection of half-spaces, so by (1.9), cone $\left\{\mathbf{x}_j\right\}_{j=1}^l$ is polyhedral. ◁

- ($\Longrightarrow$) Assume that C is polyhedral, say $C = \{\mathbf{x} \in \mathbb{F}^n : A\mathbf{x} \leq 0\}$. By letting $\mathbf{a}_1^T, \ldots, \mathbf{a}_m^T$ be the rows of A,
$$A = \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix},$$
we can write C as
$$C = \left\{x \in \mathbb{F}^n : \mathbf{a}_1^T \mathbf{x} \leq 0, \ldots, \mathbf{a}_m^T \mathbf{x} \leq 0\right\}. \qquad [1.25]$$
Now, by the reverse direction,
$$\operatorname{cone}\{\mathbf{a}_i\}_{i=1}^m = \left\{x \in \mathbb{F}^n : \mathbf{v}_1^T \mathbf{x} \leq 0, \ldots, \mathbf{v}_l^T \mathbf{x} \leq 0\right\} \qquad [1.26]$$
for some $\mathbf{v}_1, \ldots, \mathbf{v}_l \in \mathbb{F}^n$. Now the claim is
$$C = \operatorname{cone}\{\mathbf{v}_i\}_{i=1}^l. \qquad [1.27]$$
To verify [1.28], first note that, given any $i \in \{1, \ldots, l\}$, we have
$$\mathbf{a}_j^T \mathbf{v}_i = \mathbf{v}_i^T \mathbf{a}_j \overset{[1.27]}{\leq} 0,$$

so $\mathbf{v}_i \in C$ by [1.26]. This means $\text{cone}\{\mathbf{v}_i\}_{i=1}^{l} \subseteq C$. Conversely, suppose that $\mathbf{y} \notin \text{cone}\{\mathbf{v}_i\}_{i=1}^{l}$. Since $\text{cone}\{\mathbf{v}_i\}_{i=1}^{l}$ is polyhedral by the reverse direction, there are $\mathbf{w}_1, \ldots, \mathbf{w}_r \in \mathbb{F}^n$ such that

$$\text{cone}\{\mathbf{v}_i\}_{i=1}^{l} = \left\{\mathbf{x} \in \mathbb{F}^n : \mathbf{w}_1^T\mathbf{x} \leq 0, \ldots, \mathbf{w}_r^T\mathbf{x} \leq 0\right\}, \qquad [1.28]$$

which means there is $j \in \{1, \ldots, r\}$ such that $\mathbf{w}_j^T\mathbf{y} > 0$. Moreover, [1.29] means

$$\mathbf{w}_j^T\mathbf{v}_i \leq 0 \qquad [1.29]$$

for all $i \in \{1, \ldots, l\}$, so by [1.27], $\mathbf{w}_j \in \text{cone}\{\mathbf{a}_i\}_{i=1}^{m} \subseteq C$, which implies

$$\mathbf{w}_j^T\mathbf{x} \leq 0$$

for all $\mathbf{x} \in C$ by [1.26]. But we noted that $\mathbf{w}_j^T\mathbf{y} > 0$, so $\mathbf{y} \notin C$. ∎

Def'n 1.16. **Convex Polyhedron**

Let $P \subseteq \mathbb{F}^n$. If there exists $A \in M_{m\times n}(\mathbb{F})$ and $b \in \mathbb{F}^m$ such that

$$P = \{\mathbf{x} \in \mathbb{F}^n : A\mathbf{x} \leq \mathbf{b}\},$$

then we say P is a ***convex polyhedron*** (or ***polyhedron***).

Def'n 1.17. **Affine Half-space**

An ***affine half-space*** is a subset $S \subseteq \mathbb{F}^n$ of the form

$$S = \left\{\mathbf{x} \in \mathbb{F}^n : \mathbf{w}^T\mathbf{x} \leq c\right\}$$

for some $c \in \mathbb{F}$ and nonzero $\mathbf{w} \in \mathbb{F}^n$.

Def'n 1.18. **Convex Combination, Convex Hull**

Let $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{F}^n$. A ***convex combination*** of $\mathbf{x}_1, \ldots, \mathbf{x}_m$ is a linear combination

$$\sum_{i=1}^{m} a_i\mathbf{x}_i$$

such that $a_1, \ldots, a_m \geq 0$ and $\sum_{i=1}^{m} a_i = 1$. Moreover, given a subset $S \subseteq \mathbb{F}^n$, the ***convex hull*** of S, denoted as $\text{conv}(S)$, is the smallest convex set that contains S.

Proposition 1.8. *Let $S \subseteq \mathbb{F}^n$. Then*

$$\text{conv}(S) = \{\textit{convex combinations of the elements of } S\}.$$

Def'n 1.19. **Polytope**

Given any finite number of points $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{F}^n$, the convex hull $\text{conv}\{\mathbf{x}_i\}_{i=1}^{m}$ is called a ***polytope***.

Theorem 1.9. Decomposition Theorem for Polyhedra

Let $P \subseteq \mathbb{F}^n$. Then P is a polyhedron if and only if there exist a polytope $Q \subseteq \mathbb{F}^n$ and a polyhedral cone $C \subseteq \mathbb{F}^n$ such that $P = Q + C$.

Proof.

- ($\implies$) Assume that P is a polyhedraon, where we desire to find a polygonal cone $C \subseteq \mathbb{F}^n$ and a polytope $Q \subseteq \mathbb{F}^n$ such that $P = Q + C$. Now, let $A \in M_{m\times n}(\mathbb{F})$ and $\mathbf{b} \in \mathbb{F}^m$ such that

$$P = \{\mathbf{x} \in \mathbb{F}^n : A\mathbf{x} \leq \mathbf{b}\}, \tag{1.30}$$

and consider the polyhedral cone

$$D = \left\{ \begin{bmatrix} \mathbf{x} \\ \lambda \end{bmatrix} \in \mathbb{F}^{n+1} : \mathbf{x} \in \mathbb{F}^n, \lambda \geq 0, A\mathbf{x} - \lambda\mathbf{b} \leq 0 \right\}. \tag{1.31}$$

Then by the Minkowski-Weyl theorem, there exists $\mathbf{x}_1, \ldots, \mathbf{x}_l \in \mathbb{F}^n$, $\lambda_1, \ldots, \lambda_l \in \mathbb{F}^n$ such that

$$D = \operatorname{cone}\left\{ \begin{bmatrix} \mathbf{x}_i \\ \lambda_i \end{bmatrix} \right\}_{i=1}^{l}. \tag{1.32}$$

We may assume that each λ_i is 0 or 1, since by [1.32] $\lambda_1, \ldots, \lambda_l \geq 0$, and if $\mathbf{v} \in D$, then given any $c \geq 0$, $c\mathbf{v} \in D$, so we may scale each $\begin{bmatrix} \mathbf{x}_i \\ \lambda_i \end{bmatrix}$ with a positive scalar $c \geq 0$ such that $c\lambda_i = 0$ or $c\lambda_i = 1$. Now let $I = \{i \in \{1, \ldots, l\} : \lambda_i = 1\}$ and define

$$Q = \operatorname{conv}\{\mathbf{x}_i\}_{i\in I}, C = \operatorname{cone}\{\mathbf{x}_i\}_{i\notin I}. \tag{1.33}$$

We claim that $P = Q + C$. To verify this claim, observe that

$$\begin{aligned} \mathbf{v} \in P \iff A\mathbf{v} \leq \mathbf{b} \iff A\mathbf{v} - 1\mathbf{b} \leq 0 \iff \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} \in D \\ \iff \exists \mu_1, \ldots, \mu_l \geq 0 \left[\sum_{i=1}^{l} \mu_i \begin{bmatrix} \mathbf{x}_i \\ \lambda_i \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix} \right] \\ \iff \exists \mu_1, \ldots, \mu_l \geq 0 \left[\sum_{i\in I} \mu_i = 1, \mathbf{v} = \sum_{i\in I} \mu_i \mathbf{x}_i + \sum_{i\notin I} \mu_i \mathbf{x}_i \right] \\ \iff \exists \mathbf{v}_Q \in Q, \mathbf{v}_C \in C\, [\mathbf{v} = \mathbf{v}_Q + \mathbf{v}_C]. \end{aligned}$$

◁

- Assume that we are given a polyhedral cone $C \subseteq \mathbb{F}^n$ and a polytope $Q \subseteq \mathbb{F}^n$, where we desire to show that $P = Q + C$ is a polyhedron. In other words, we desire to find a matrix $A \in M_{m\times n}(\mathbb{F})$ and a vector $\mathbf{b} \in \mathbb{F}^m$ such that

$$P = \{\mathbf{x} \in \mathbb{F}^n : A\mathbf{x} \leq \mathbf{b}\}.$$

Since C is a polyhedral cone, Minkowski-Weyl theorem implies that there exist $\mathbf{x}_1, \ldots, \mathbf{x}_r \in \mathbb{F}^n$ such that

$$C = \operatorname{cone}\{\mathbf{x}_i\}_{i=1}^{r}. \tag{1.34}$$

Moreover, since Q is a polytope, we can find $\mathbf{x}_{r+1}, \ldots, \mathbf{x}_l \in \mathbb{F}^n$ such that

$$Q = \operatorname{conv}\{\mathbf{x}_i\}_{i=r+1}^{l}. \tag{1.35}$$

So $\mathbf{v} \in P$ if and only if

$$\mathbf{v} = \sum_{i=1}^{r} \lambda_i \mathbf{x}_i + \sum_{i=r+1}^{l} \lambda_i \mathbf{x}_i = \sum_{i=1}^{l} \lambda_i \mathbf{x}_i$$

for some $\lambda_1,\ldots,\lambda_l \geq 0$ such that $\sum_{i=r+1}^{l} \lambda_i = 1$. In other words,

$$\begin{bmatrix}\mathbf{v}\\1\end{bmatrix} \in \underbrace{\text{cone}\left\{\begin{bmatrix}\mathbf{x}_1\\0\end{bmatrix},\ldots,\begin{bmatrix}\mathbf{x}_r\\0\end{bmatrix},\begin{bmatrix}\mathbf{x}_{r+1}\\1\end{bmatrix},\ldots,\begin{bmatrix}\mathbf{x}_l\\1\end{bmatrix}\right\}}_{=D} \iff \mathbf{v} \in P. \qquad [1.36]$$

This D is finitely generated, so by the Minkowski-Weyl theorem, there exists $\begin{bmatrix}A & -\mathbf{b}\end{bmatrix} \in M_{m\times(n+1)}(\mathbb{F})$ with $\mathbf{b} \in \mathbb{F}^m$ such that

$$D = \left\{\begin{bmatrix}\mathbf{x}\\1\end{bmatrix} \in \mathbb{F}^{n+1} : \begin{bmatrix}A & -\mathbf{b}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\1\end{bmatrix} \leq 0\right\}. \qquad [1.37]$$

Thus

$$\begin{aligned}\mathbf{v} \in P \overset{[1.37]}{\underset{[1.38]}{\iff}} \begin{bmatrix}\mathbf{v}\\1\end{bmatrix} &\in \left\{\begin{bmatrix}\mathbf{x}\\1\end{bmatrix} \in \mathbb{F}^{n+1} : A\mathbf{x} - \mathbf{b} \leq 0\right\}\\ \iff \mathbf{v} &\in \{\mathbf{x} \in \mathbb{F}^n : A\mathbf{x} \leq \mathbf{b}\},\end{aligned}$$

so P is a polyhedron, as desired. ∎

The decomposition theorem allows the followind definition.

Def'n 1.20. **Polyhedron Generated by** Points and Directions
Let $P \subseteq \mathbb{F}^n$ be a polyhedron. Then by the decomposition theorem, there exist $\mathbf{x}_1,\ldots,\mathbf{x}_r,\mathbf{x}_{r+1},\ldots,\mathbf{x}_l \in \mathbb{F}^n$ such that $P = \text{conv}\{\mathbf{x}_i\}_{i=1}^{r} + \text{cone}\{\mathbf{x}_i\}_{i=r+1}^{l}$. We say P is ***generated by*** the points $\mathbf{x}_1,\ldots,\mathbf{x}_r$ and directions $\mathbf{x}_{r+1},\ldots,\mathbf{x}_l$.

Theorem 1.10. Finite Basis Theorem for Polytopes

Let $Q \subseteq \mathbb{F}^n$. Then Q is a polytope if and only if Q is a bounded polyhedron.

Proof. Let $Q \subseteq \mathbb{F}^n$. We verify the following three things.

(a) If Q is a polytope, then Q is a polyhedron.

Proof. Note that $Q = Q + \varnothing$, where $\varnothing \subseteq \mathbb{F}^n$ is a cone. ◁

(b) If Q is a polytope, then Q is bounded.

Proof. Let $\mathbf{x}_1,\ldots,\mathbf{x}_l \in \mathbb{F}^n$ be such that $Q = \text{conv}\{\mathbf{x}_i\}_{i=1}^{l}$. Then given any $\mathbf{v} \in Q$, we have $\lambda_1,\ldots,\lambda_l \geq 0$ with $\sum_{i=1}^{l} \lambda_i 1$ such that $\sum_{i=1}^{l} \lambda_i \mathbf{v}_i = \mathbf{v}$. So

$$\|\mathbf{v}\| = \left\|\sum_{i=1}^{l} \lambda_i \mathbf{x}_i\right\| \leq \sum_{i=1}^{l} \lambda_i \|\mathbf{x}_i\| \leq \sum_{i=1}^{l} \|x_i\|,$$

so by taking $r = \sum_{i=1}^{l} \|\mathbf{x}_i\|$, $Q \subseteq B(0;r)$. ◁

(c) Every nonempty cone is unbounded.

Proof. Let $C \subseteq \mathbb{F}^n$ be a nonempty cone, so let $\mathbf{v} \in C$. Then given any $r > 0$, we can find $\lambda \geq 0$ such that $\|\lambda\mathbf{v}\| > r$, where $\lambda\mathbf{v} \in C$. So C is not bounded. ◁

(a), (b) provide the forward direction. For the reverse direction, assume that Q is a bounded polyhedron. Then by the decomposition theorem, $Q = Q' + C$ for some polytope $Q' \subseteq \mathbb{F}^n$ and polyhedral cone $C \subseteq \mathbb{F}^n$. Then by the boundedness of Q and (c), $C = \varnothing$, $Q = Q'$, so Q is a polytope. ∎

Linear Programming

(1.11)
Linear Programming

Linear programming (LP) concerns the problem of minimizing or maximizing a linear functional over a polyhedron. For instance,

$$\begin{aligned}&\text{maximize } \mathbf{c}^T\mathbf{x}\\&\text{subject to } A\mathbf{x} \le \mathbf{b}\end{aligned}$$

and

$$\begin{aligned}&\text{minimize } \mathbf{y}^T\mathbf{b}\\&\text{subject to } \mathbf{y}^T A = \mathbf{0}\\&\mathbf{y}^T \ge \mathbf{0}\end{aligned}$$

are linear programming problems. We begin our discussion of linear programming by the duality theorem.

Theorem 1.11.
Duality Theorem for Linear Programming

Let $A \in M_{m\times n}(\mathbb{F})$, let $\mathbf{b} \in \mathbb{F}^m$, and let $\mathbf{c} \in \mathbb{F}^n$. Then

$$\max \underbrace{\left\{\mathbf{c}^T\mathbf{x} : A\mathbf{x} \le \mathbf{b}\right\}}_{=X} = \min \underbrace{\left\{\mathbf{y}^T\mathbf{b} : \mathbf{y} \ge \mathbf{0}, \mathbf{y}^T A = \mathbf{c}^T\right\}}_{=Y} \quad [1.38]$$

provided that X,Y are nonempty.

Proof. Assume that X,Y are nonempty. Note that, if we choose any $\mathbf{x} \in X, \mathbf{y} \in Y$, we have that

$$\underbrace{\mathbf{c}^T}_{=\mathbf{y}^T A} x = \mathbf{y}^T \underbrace{A\mathbf{x}}_{\le \mathbf{b}} \le \mathbf{y}^T\mathbf{b},$$

so X is bounded above and Y is bounded below, where one has

$$\max(X) = \max_{\mathbf{x}\in\mathbb{F}^n : A\mathbf{x}\le\mathbf{b}} \mathbf{c}^T\mathbf{x} \le \min_{\substack{\mathbf{y}\in\mathbb{F}^m:\\ \mathbf{y}\ge\mathbf{0},\mathbf{y}^T A=\mathbf{c}^T}} \mathbf{y}^T\mathbf{b} = \min(Y). \quad [1.39]$$

So it suffices to show that there exists $\mathbf{x} \in X, \mathbf{y} \in Y$ such that $\mathbf{c}^T\mathbf{x} = \mathbf{y}^T\mathbf{b}$. That is, we desire to show that

$$\exists \mathbf{x} \in \mathbb{F}^n, \mathbf{y} \in \mathbb{F}^m \left[\mathbf{y} \ge 0, \begin{bmatrix} A & \mathbf{0} \\ -\mathbf{c}^T & \mathbf{b} \\ \mathbf{0}^T & A^T \\ \mathbf{0}^T & -A^T \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \\ \mathbf{c} \\ -\mathbf{c} \end{bmatrix}\right]. \quad [1.40]$$

By a variant of Farkas' lemma, [1.40] is equivalent to

$$\begin{aligned}&\forall \mathbf{u} \in \mathbb{F}^m, \mathbf{v}, \mathbf{w} \in \mathbb{F}^n, \lambda \in \mathbb{F}\\&\left[\mathbf{u}, \mathbf{v}, \mathbf{w}, \lambda \ge 0, \mathbf{u}^T A - \lambda\mathbf{c} = 0, \lambda\mathbf{b}^T + \mathbf{v}^T A^T - \mathbf{w}^T A^T \ge \mathbf{0} \implies \mathbf{u}^T\mathbf{b} + \mathbf{v}^T\mathbf{c} - \mathbf{w}^T\mathbf{c} \ge \mathbf{0}\right]\end{aligned}. \quad [1.41]$$

To verify [1.41], assume that $\mathbf{u} \in \mathbb{F}^n, \mathbf{v}, \mathbf{w} \in \mathbb{F}^m, \lambda \in \mathbb{F}$ satisfy the hypothesis of [1.41].

(a) If $\lambda > 0$, then

$$\mathbf{u}^T\mathbf{b} = \lambda^{-1}\lambda\mathbf{b}^T\mathbf{u} \ge \lambda^{-1}(\mathbf{w}-\mathbf{v})^T A^T\mathbf{u} = \lambda^{-1}\lambda(\mathbf{w}-\mathbf{v})^T\mathbf{c} = (\mathbf{w}-\mathbf{v})^T\mathbf{c}. \quad [1.42]$$

(b) If $\lambda = 0$, then take any $\mathbf{x}_0 \in X, \mathbf{y}_0 \in Y$, as we assumed that X,Y are nonempty. Then

$$\mathbf{u}^T\mathbf{b} \ge \mathbf{u}A\mathbf{x}_0 = \left(\mathbf{u}A - \underbrace{\lambda\mathbf{c}}_{=0}\right)\mathbf{x}_0 = 0 \ge (\mathbf{w}-\mathbf{v})^T A^T\mathbf{y}_0 = (\mathbf{w}-\mathbf{v})^T\mathbf{c}. \quad [1.43]$$

Both [1.42], [1.43] means $\mathbf{u}^T\mathbf{b} \geq (\mathbf{w}-\mathbf{v})^T\mathbf{c}$, so by rearranging, we obtain

$$\mathbf{u}^T\mathbf{b}+\mathbf{v}^T\mathbf{c}-\mathbf{w}^T\mathbf{c} \geq \mathbf{0},$$

so the conclusion of [1.41] is satisfied, as desired. ∎

(a) Theorem 1.11 has alternative names: *strong duality theorem*, *LP duality theorem*, ... Equation [1.39] is called the *weak duality theorem*.

(b) The regions described by the inequalities are called ***feasible regions***.

(c) In order to establish $\max, \min$ in [1.39] instead of $\sup, \inf$ (which exists by the fact that X is upper bounded and Y is lower bounded), we should ensure that the linear functionals attain their extremum on the associated regions, respectively. This problem is easily resolved, however, since any linear functional is continuous and the regions are closed sets by the fact that the inequalities are not strict.

(d) We call a linear programming problem of the form

$$\begin{aligned} &\text{maximize } \mathbf{c}^T\mathbf{x} \\ &\text{subject to } A\mathbf{x} \leq \mathbf{b} \end{aligned}$$

primal, whose ***dual***

$$\begin{aligned} \text{minimize } & \mathbf{y}^T\mathbf{b} \\ \text{subject to } & \mathbf{y}^T A = \mathbf{0} \\ & \mathbf{y}^T \geq \mathbf{0} \end{aligned}$$

is given by the duality theorem.

(1.12) Equivalent Forms of Linear Programming Problems

The following are equivalent LP problems. In other words, they can be reduced to each other's form (not necessarily with the same variables).

(a) $\max\left\{\mathbf{c}^T\mathbf{x} : A\mathbf{x} \leq \mathbf{b}\right\}$.

(b) $\max\left\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \geq \mathbf{0}, A\mathbf{x} \leq \mathbf{b}\right\}$.

(c) $\max\left\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \geq \mathbf{0}, A\mathbf{x} = \mathbf{b}\right\}$.

(d) $\min\left\{\mathbf{c}^T\mathbf{x} : A\mathbf{x} \geq \mathbf{b}\right\}$.

(e) $\min\left\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \geq \mathbf{0}, A\mathbf{x} \geq \mathbf{b}\right\}$.

(f) $\min\left\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \geq \mathbf{0}, A\mathbf{x} = \mathbf{b}\right\}$.

Theorem 1.12. Affine Farkas Lemma

Let $A \in M_{m\times n}(\mathbb{F})$ and let $\mathbf{b} \in \mathbb{F}^m$ be such that $\underbrace{\left\{\mathbf{x} \in \mathbb{F}^n : A\mathbf{x} \geq \mathbf{b}\right\}}_{=X}$ is nonempty, and suppose that $\mathbf{c} \in \mathbb{F}^n, \delta \in \mathbb{F}$ are such that

$$\forall \mathbf{x} \in X \left[\mathbf{c}^T\mathbf{x} \leq \delta\right].$$

Then there exists $\delta' \leq \delta$ such that $\mathbf{c}^T\mathbf{x} \leq \delta'$ is a nonnegative linear combination of the inequalities in the system $A\mathbf{x} \leq \mathbf{b}$.

Proof. By assumption, we know that

$$\begin{aligned} &\text{maximize } \mathbf{c}^T\mathbf{x} \\ &\text{subject to } A\mathbf{x} \leq \mathbf{b} \end{aligned}$$

has an optimal solution, say $\mathbf{x}_0 \in X$, and has the dual LP

$$\begin{aligned} &\text{minimize } \mathbf{y}^T\mathbf{b} \\ &\text{subject to } \mathbf{y}^T A = \mathbf{c}^T \\ &\qquad\qquad \mathbf{y} \geq \mathbf{0} \end{aligned} \quad . \tag{1.44}$$

We have the following two cases.

(a) Assume that [1.44] is not feasible (i.e. the set $\left\{\mathbf{y} \in \mathbb{F}^m : \mathbf{y}^T A = \mathbf{c}^T, \mathbf{y} \geq 0\right\}$ is empty), for the sake of contradiction. By the Farkas' lemma, there exists a vector $\mathbf{z} \in \mathbb{F}^n$ such that $A\mathbf{z} \leq \mathbf{0}$ and $\mathbf{c}^T\mathbf{z} > 0$. But this means that

$$\mathbf{c}^T\mathbf{x}_0 < \mathbf{c}^T\mathbf{x}_0 + \underbrace{\mathbf{c}^T\mathbf{z}}_{>0} = \mathbf{c}^T\left(\mathbf{x}_0 + \mathbf{z}\right), \tag{1.45}$$

where $\mathbf{x}_0 + \mathbf{z} \in X$, since

$$A\left(\mathbf{x}_0 + \mathbf{z}\right) = \underbrace{A\mathbf{x}_0}_{\geq \mathbf{0}} + \underbrace{A\mathbf{z}}_{\geq \mathbf{0}} \geq \mathbf{0}.$$

But then [1.45] contradicts the maximality of $\mathbf{c}^T\mathbf{x}_0$.

(b) Assume that [1.44] has an optimal solution, say $\mathbf{y}_0 \in \mathbb{F}^n$ (assuming [1.44] has a solution is equivalent to this by the strong duality). Then by the strong duality, there exists $\delta' \in \mathbb{F}$ such that

$$\delta' = \max_{\mathbf{x} \in X} \mathbf{c}^T\mathbf{x} = \min_{\mathbf{y} \in Y} \mathbf{y}^T\mathbf{b}.$$

Then note t hat $\delta' \geq \delta$ clearly, and $\mathbf{c}^T\mathbf{x} \leq \delta'$ is a nonnegative linear combination of the inequalities in $A\mathbf{x} \leq \mathbf{b}$, since

$$\mathbf{c}^T\mathbf{x} = \underbrace{\mathbf{y}_0^T}_{\geq \mathbf{0}} A\mathbf{x}, \delta' = \underbrace{\mathbf{y}_0}_{\geq \mathbf{0}}\mathbf{b}.$$

∎

Proposition 1.13. Characterization of Optimal Solutions

Let $A \in M_{m\times n}(\mathbb{F})$, let $\mathbf{b} \in \mathbb{F}^m$, and let $\mathbf{c} \in \mathbb{F}^n$ be such that

$$X = \left\{\mathbf{x} \in \mathbb{F}^n : A\mathbf{x} \leq \mathbf{b}\right\}, Y = \left\{\mathbf{y} \in \mathbb{F}^n : \mathbf{y} \geq \mathbf{0}, \mathbf{y}^T A = \mathbf{c}^T\right\}$$

are nonempty. Let $\mathbf{x}_0 \in X, \mathbf{y}_0 \in Y$. Then the following are equivalent.

(a) $\mathbf{x}_0, \mathbf{y}_0$ are optimal (with respect to the associated LP problems).

(b) $\mathbf{c}^T\mathbf{x}_0 = \mathbf{y}_0^T\mathbf{b}$.

(c) For every positive component y_i of $\mathbf{y}_0$, the corresponding inequality $\mathbf{a}_i^T\mathbf{x} \leq \mathbf{b}$ holds with an equality. That is, $\mathbf{y}_0^T\left(\mathbf{b} - A\mathbf{x}\right) = 0$. Note that we are writing

$$A = \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}, \mathbf{y}_0 = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}.$$

Proof.

- (a) $\iff$ (b) This is precisely the statement of the strong duality. ◁
- (b) $\iff$ (c) Note that we have the weak duality

$$\mathbf{c}^T\mathbf{x}_0 \leq \mathbf{y}_0^T\mathbf{b}. \tag{1.46}$$

  [1.48] implies the following:

$$\mathbf{c}^T\mathbf{x}_0 \iff \mathbf{y}_0^T A\mathbf{x}_0 = \mathbf{y}^T\mathbf{b} \iff \mathbf{y}^T(A\mathbf{x}_0 - \mathbf{b}) = 0.$$ ∎

The part (b) $\iff$ (c) is known as the *complementary slackness*, and (c) especially is called the *complementary slackness conditions*.

Proposition 1.14. *Let $A \in M_{m\times n}(\mathbb{F})$ and let $\mathbf{b} \in \mathbb{F}^m$. Then for each inequality $\mathbf{a}_i^T\mathbf{x} \leq b_i$ of*

$$\underbrace{\begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}}_{=A} \mathbf{x} \leq \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}},$$

exactly one of the following holds.

(a) There exists an optimal solution $\mathbf{x}_0 \in \mathbb{F}^n$ to the LP problem

$$\begin{aligned} &\text{maximize } \mathbf{c}^T\mathbf{x} \\ &\text{subject to } A\mathbf{x} \leq \mathbf{b} \end{aligned} \tag{1.47}$$

such that $\mathbf{a}_i^T\mathbf{x}_0 < b_i$.

(b) The dual of [1.47] has an optimal solution $\mathbf{y}_0 \in \mathbb{F}^m$ such that the ith entry y_i of $\mathbf{y}_0$ satisfies $y_i > 0$.

Theorem 1.15. Motzkin's Transposition Theorem

Let $A, B \in M_{m\times n}(\mathbb{F})$ and let $\mathbf{b}, \mathbf{c} \in \mathbb{F}^m$. Then there exists $\mathbf{x} \in \mathbb{F}^n$ such that $A\mathbf{x} < \mathbf{b}$ and $B\mathbf{x} \leq \mathbf{c}$ if and only if

(a) for all $\mathbf{y}, \mathbf{z} \in \mathbb{F}^n$, if $\mathbf{y}, \mathbf{z} \geq \mathbf{0}$ and $\mathbf{y}^T A + \mathbf{z}^T B = \mathbf{0}$, then $\mathbf{y}^T\mathbf{b} + \mathbf{z}^T\mathbf{c} \geq 0$; and

(b) for all $\mathbf{y}, \mathbf{z} \in \mathbb{F}^n$, if $\mathbf{y} > \mathbf{0}, \mathbf{z} \geq \mathbf{0}$ and $\mathbf{y}^T A + \mathbf{z}^T B = \mathbf{0}$, then $\mathbf{y}^T\mathbf{b} + \mathbf{z}^T\mathbf{c} > 0$.

Lemma 1.15.1. *Consider the setting of Theorem 1.15. For each $i \in \{1, \ldots, m\}$, there exists $\mathbf{z}_i \in \mathbb{F}^n$ such that*

$$A\mathbf{z}_i \leq \mathbf{b}, B\mathbf{z}_i \leq \mathbf{c}, \mathbf{a}_i^T\mathbf{z}_i < \mathbf{b}$$

if and only if there is no solution to

$$\mathbf{y}, \mathbf{z} \geq \mathbf{0}, \mathbf{y}^T A + \mathbf{z}^T B = -\mathbf{a}_i, \mathbf{y}^T\mathbf{b} + \mathbf{z}^T\mathbf{c} \leq -b_i,$$

where

$$A = \begin{bmatrix} \mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_m^T \end{bmatrix}, b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

Proof. Consider the primal LP

$$\begin{aligned} &\text{maximize } -\mathbf{a}_i^T \mathbf{x} \\ &\text{subject to } \begin{bmatrix} A \\ B \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix} \end{aligned}, \qquad [1.48]$$

whose dual LP is

$$\begin{aligned} &\text{minimize } \begin{bmatrix} \mathbf{y}^T & \mathbf{z}^T \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix} \\ &\text{subject to } \begin{bmatrix} \mathbf{y}^T & \mathbf{z}^T \end{bmatrix} \geq \mathbf{0}^T \\ &\begin{bmatrix} \mathbf{y}^T & \mathbf{z}^T \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = -\mathbf{a}_i \end{aligned}. \qquad [1.49]$$

Note that it suffices to show that there is a feasible solution $\mathbf{z}_i \in \mathbb{F}^n$ for [1.1] such that the inequality $\mathbf{a}_i^T \mathbf{z}_i < b_i$ holds. We consider the following two cases.

(a) Assume that [1.2] does not have a feasible solution. Then by the Farkas' lemma, there is $\mathbf{w} \in \mathbb{F}^n$ such that

$$\begin{bmatrix} A \\ B \end{bmatrix} \mathbf{w} \leq \mathbf{0}, -\mathbf{a}_i^T \mathbf{w} > 0. \qquad [1.50]$$

This means [1.1] is not bounded. For, we know that [1.1] has a feasible solution, say $\mathbf{x} \in \mathbb{F}^n$. Moreover, [1.3] shows that, given any $k \in \mathbb{F}$,

$$\begin{bmatrix} A \\ B \end{bmatrix} (\mathbf{x} + k\mathbf{w}) = \begin{bmatrix} A \\ B \end{bmatrix} \mathbf{x} + \underbrace{\begin{bmatrix} A \\ B \end{bmatrix} k\mathbf{w}}_{\leq k\mathbf{0} = \mathbf{0}} \overset{[1.3]}{\leq} \begin{bmatrix} A \\ B \end{bmatrix} \mathbf{x} \overset{[1.1]}{\leq} \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix}$$

so $\mathbf{x} + k\mathbf{w}$ is also a feasible solution for [1.1]. Finally, note that $\mathbf{w} \neq 0$ as $-a_i \cdot \mathbf{w} > 0$. This means, given any $r > 0$, we can choose a scalar $k \in \mathbb{F}$ so that $A(\mathbf{x} + k\mathbf{w}) \leq \mathbf{b}, B(\mathbf{x} + k\mathbf{w}) \leq \mathbf{c}$, and $-\mathbf{a}_i^T (\mathbf{x} + k\mathbf{w}) > r$, since $-\mathbf{a}_i^T \mathbf{w} > 0$ from [1.3]. Therefore

$$-b_i < -\mathbf{a}_i^T \mathbf{z}_i,$$

for some $\mathbf{z}_i = \mathbf{x} + k\mathbf{w}$, which precisely satisfies the properties that we want. ◁

(b) Assume that [1.2] has a feasible solution. Note that [1.1] also has a feasible solution, as we have assumed that $A\mathbf{x} \leq b, B\mathbf{x} \leq c$. So we can apply strong duality to conclude that there is $m \in \mathbb{F}$ such that

$$m = \max\left\{ -\mathbf{a}_i^T \mathbf{x} : \begin{bmatrix} A \\ B \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix} \right\} = \min\left\{ \mathbf{u}^T \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix} : \mathbf{u} \geq \mathbf{0}, \mathbf{u}^T \begin{bmatrix} A \\ B \end{bmatrix} = -\mathbf{a}_i \right\}. \qquad [1.51]$$

But we know that

$$\mathbf{u}^T \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix} > -b_i$$

for every feasible solution $\mathbf{u}$ to [1.2], because

$$\mathbf{y} \geq 0, \mathbf{z} \geq 0, \mathbf{y}^T A + \mathbf{z}^T B = -\mathbf{a}_i^T, \mathbf{y}^T \mathbf{b} + \mathbf{z}^T \mathbf{c} \leq -b_i$$

has no solution. This means

$$-b_i < m. \tag{1.52}$$

So by choosing $\mathbf{z}_i \in \mathbb{F}^n$ to be an optimal solution for [1.1], we obtain that

$$\mathbf{a}_i^T \mathbf{z}_i \overset{[1.4]}{\underset{}{\overset{[1.5]}{<}}} b_i.$$ ∎

Proof of Theorem 1.15.

- ($\Longrightarrow$) Assume that we have $\mathbf{x} \in \mathbb{F}^n$ such that $A\mathbf{x} < \mathbf{b}, B\mathbf{x} \le \mathbf{c}$. Then given any $\mathbf{y}, z \in \mathbb{F}^m$ with $\mathbf{y}, \mathbf{z} \ge \mathbf{0}$ and $\mathbf{y}^T A + \mathbf{z}^T B = \mathbf{0}$, we have

$$\mathbf{y}^T\mathbf{b} + \mathbf{z}^T\mathbf{c} \ge \underbrace{\mathbf{y}^T}_{\ge \mathbf{0}^T} \underbrace{A\mathbf{x}}_{>\mathbf{b}} + \underbrace{\mathbf{z}^T}_{\ge \mathbf{0}^T} \underbrace{B\mathbf{x}}_{\ge \mathbf{c}} = \underbrace{\left(\mathbf{y}^T A + \mathbf{z}^T B\right)}_{=\mathbf{0}^T} \mathbf{x} = 0.$$

When $\mathbf{y} > \mathbf{0}$, we have

$$\mathbf{y}^T\mathbf{b} + \mathbf{z}^T\mathbf{c} > \underbrace{\mathbf{y}^T}_{> \mathbf{0}^T} \underbrace{A\mathbf{x}}_{>\mathbf{b}} + \underbrace{\mathbf{z}^T}_{\ge \mathbf{0}^T} \underbrace{B\mathbf{x}}_{\ge \mathbf{c}} = \underbrace{\left(\mathbf{y}^T A + \mathbf{z}^T B\right)}_{=\mathbf{0}^T} \mathbf{x} = 0.$$ ◁

- ($\Longleftarrow$) Assume that (a), (b) are true. Then by the Farkas' lemma, there exists a solution to $\begin{bmatrix} A \\ B \end{bmatrix} \mathbf{x} \le \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix}$. Moreover, (b) implies that, for each $i \in \{1, \ldots, m\}$, there is no solution to

$$\mathbf{y}, \mathbf{z} \ge \mathbf{0}, \mathbf{y}^T A + \mathbf{z}^T B = -\mathbf{a}_i, \mathbf{y}^T\mathbf{b} + \mathbf{z}^T\mathbf{c} \le -b_i.$$

By Lemma 1.15.1, this is equivalent to saying that, for each $i \in \{1, \ldots, m\}$, there exists $\mathbf{z}_i \in \mathbb{F}^n$ such that $A\mathbf{z}_i \le \mathbf{b}, B\mathbf{z}_i \le \mathbf{c}, \mathbf{a}_i^T \mathbf{z}_i < \mathbf{b}$. Note that, by taking

$$\mathbf{x}_0 = \frac{1}{m} \sum\nolimits_{i=1}^m \mathbf{z}_i,$$

$\mathbf{x}_0$ satisfies $A\mathbf{x}_0 < \mathbf{b}, B\mathbf{x}_0 \le \mathbf{c}$. ∎

Structure of Polyhedra

(1.13) Throughout, let $A \in M_{m \times n}(\mathbb{F})$ and let $\mathbf{b} \in \mathbb{F}^m$ be such that the system $A\mathbf{x} \le \mathbf{b}$ is feasible and let $P = \{\mathbf{x} \in \mathbb{F}^n : A\mathbf{x} \le \mathbf{b}\}$ be the corresponding polyhedron.

Def'n 1.21. **Characteristic Cone** of a Polyhedra

We define the ***characteristic cone*** of P, denoted as $\operatorname{char.cone}(P)$, to be the polyhedral cone

$$\operatorname{char.cone}(P) = \{\mathbf{y} \in \mathbb{F}^n : \forall \mathbf{x} \in P\,[\mathbf{x} + \mathbf{y} \in P]\}.$$

Proposition 1.16. Properties of Characteristic Cone

(a) Given any $\mathbf{y} \in \mathbb{F}^n$, $\mathbf{y} \in \operatorname{char.cone}(P)$ if and only if there is $\mathbf{x} \in P$ such that $\mathbf{x} + \lambda \mathbf{y} \in P$ for all $\lambda \geq 0$.

(b) $\operatorname{char.cone}(P) = \{\mathbf{y} \in \mathbb{R}^n : A\mathbf{y} \geq \mathbf{0}\}$.

(c) P is bounded if and only if $\operatorname{char.cone}(P) = \varnothing$.

(d) If $P = Q + C$ for some polytope Q and polyhedral cone C, then $C = \operatorname{char.cone}(P)$.

Def'n 1.22. **Affinely Independent** Vectors

We say $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{F}^n$ are ***affinely independent*** if

$$\sum_{i=1}^{m} \lambda_i \mathbf{x}_i = \mathbf{0}, \sum_{i=1}^{m} \lambda_i = 0$$

has the unique solution $\lambda_1 = \cdots = \lambda_m = 0$.

Def'n 1.23. **Dimension** of a Set

Let $K \subseteq \mathbb{R}^n$. The ***dimension*** of K, denoted as $\dim(K)$, is the maximal number of affinely independent elements of K.

Proposition 1.17.

Let $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{F}^n$. If $\mathbf{x}_1, \ldots, \mathbf{x}_m$ are affinely independent, then given any $\mathbf{w} \in \mathbb{F}^n$, $\mathbf{x}_1 - \mathbf{w}, \ldots, \mathbf{x}_m - \mathbf{w}$ are affinely independent.

Proof. We may assume that $\mathbf{w} \neq \mathbf{0}$. Suppose that

$$\sum_{i=1}^{m} \lambda_i (\mathbf{x}_i - \mathbf{w}) = \mathbf{0}, \sum_{i=1}^{m} \lambda_i = 0$$

for some $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$. Then we have

$$\mathbf{0} = \sum_{i=1}^{m} \lambda_i (\mathbf{x}_i - \mathbf{w}) = \sum_{i=1}^{m} \lambda_i \mathbf{x}_i + \underbrace{\sum_{i=1}^{m} \lambda_i}_{=0} \mathbf{w} = \sum_{i=1}^{m} \lambda_i \mathbf{x}_i,$$

so by the affine independence of $\mathbf{x}_1, \ldots, \mathbf{x}_m$, $\lambda_1 = \cdots = \lambda_m = 0$. ∎

(1.14) Proposition 1.17 shows that the affine independence is *invariant under translations*. Also note that, given $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{F}^n$,

$$\mathbf{x}_1, \ldots, \mathbf{x}_m \text{ are affinely independent} \iff \mathbf{x}_2 - \mathbf{x}_1, \ldots, \mathbf{x}_n - \mathbf{x}_1 \text{ are affinely independent.}$$

Def'n 1.24. **Affine Hull** of a Set

Let $X \subseteq \mathbb{R}^n$. The ***affine hull*** of X, denoted as $\operatorname{aff}(X)$, is the set of every affine combination of finitely many elements in X:

$$\operatorname{aff}(X) = \left\{ \sum_{i=1}^{m} \lambda_i \mathbf{x}_i : m \in \mathbb{N}, \mathbf{x}_1, \ldots, \mathbf{x}_m \in X, \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

Def'n 1.25. **Implicit Equality**
An inequality $\mathbf{a}_i^T\mathbf{x} \le b_i$ of $A\mathbf{x} \le \mathbf{b}$ is called an ***implicit equality*** if it holds with an equality for all $\mathbf{x} \in P$. We write $A^=, b^=$ to denote the matrix and vector corresponding to the implicit equalities, respectively, and A^+, b^+ to denote the matrix and vector with the remaining rows, respectively.

(1.15) Note that there is $\mathbf{x} \in P$ such that $A^=\mathbf{x} = \mathbf{b}$ and $A^+\mathbf{x} < \mathbf{b}^+$. Also note that

$$\dim(P) = n - \operatorname{rank}(A^=). \qquad [1.53]$$

Proposition 1.18. *We have*

$$\operatorname{aff}(P) = \{\mathbf{x} \in \mathbb{F}^n : A^=\mathbf{x} = \mathbf{b}^=\} = \{\mathbf{x} \in \mathbb{F}^n : A^=\mathbf{x} \le \mathbf{b}^=\}.$$

Proof. We verify three things.

(a) $\operatorname{aff}(P) \subseteq \{\mathbf{x} \in \mathbb{F}^n : A^=\mathbf{x} = \mathbf{b}^=\}$.

Proof. Suppose $\mathbf{x} = \sum_{i=1}^m \lambda_i \mathbf{x}_i$ for some $\mathbf{x}_1, \ldots, \mathbf{x}_m \in P$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$ such that $\sum_{i=1}^m \lambda_i = 1$. Then

$$A^=\mathbf{x} = A^= \sum_{i=1}^m \lambda_i \mathbf{x}_i = \sum_{i=1}^m \lambda_i \underbrace{A^=\mathbf{x}_i}_{=\mathbf{b}} = \underbrace{\sum_{i=1}^m \lambda_i}_{=1} \mathbf{b} = \mathbf{b},$$

where each $A^=\mathbf{x}_i = 1$ is because $P \subseteq \{\mathbf{x} \in \mathbb{F}^n : A^=\mathbf{x} = \mathbf{b}^=\}$ by definition. ◁

(b) $\{\mathbf{x} \in \mathbb{F}^n : A^=\mathbf{x} = \mathbf{b}^=\} \subseteq \{\mathbf{x} \in \mathbb{F}^n : A^=\mathbf{x} \le \mathbf{b}^=\}$

Proof. This is trivial. ◁

(c) $\{\mathbf{x} \in \mathbb{F}^n : A^=\mathbf{x} \le \mathbf{b}^=\} \subseteq \operatorname{aff}(P)$.

Proof. Let $\mathbf{x} \in \mathbb{F}^n$ be such that $A^=\mathbf{x} \le \mathbf{b}^+$ and let $\mathbf{z} \in P$ be such that $A^=\mathbf{z} = \mathbf{b}^=, A^+\mathbf{z} < \mathbf{b}^+$. We have two cases.

(i) If $\mathbf{x} = \mathbf{z}$, then $\mathbf{x} = \mathbf{z} \in P \subseteq \operatorname{aff}(P)$.

(ii) If $\mathbf{x} \ne \mathbf{z}$, then consider the line seqment $L \subseteq \mathbb{F}^n$ connecting $\mathbf{x}, \mathbf{z}$ (i.e. $L = \operatorname{conv}\{\mathbf{x}, \mathbf{z}\}$). Then any $\mathbf{y} \in L$ is of the form

$$\mathbf{y} = \lambda\mathbf{x} + (1-\lambda)\mathbf{z}$$

for some $\lambda \in [0,1]$. So by choosing a small nonzero λ, we find $\mathbf{y}$ distinct from $\mathbf{z}$ such that $\mathbf{y} \in P$. It follows that $L \subseteq \operatorname{aff}\{\mathbf{y}, \mathbf{z}\} \subseteq (P)$. ◁

Combining (a), (b), (c) gives the desired result. ∎

Def'n 1.26. **Full-dimension**
We say P has ***full-dimension*** if $\dim(P) = n$.

By [1.53], P has full-dimension if and only if A does not have any implicit equality.

Def'n 1.27. **Vaded** Inequality for a Polyhedron

Let $\mathbf{c} \in \mathbb{F}^n$ and let $\delta \in \mathbb{F}$. We say the inequality $\mathbf{c}^T\mathbf{x} \leq \delta$ is ***vaded*** if it is satisfied for every $\mathbf{x} \in P$.

Def'n 1.28. **Supporting Hyperplane** of a Polyhedron

Let $\mathbf{c} \in \mathbb{F}^n$ and let $\delta \in \mathbb{F}$. We say the hyperplane $\left\{\mathbf{x} \in \mathbb{F}^n : \mathbf{c}^T\mathbf{x} = \delta\right\}$ is a ***supporting hyperplane*** of P if

$$\mathbf{c} \neq \mathbf{0}, \delta = \max\left\{\mathbf{c}^T\mathbf{x} : A\mathbf{x} \leq \mathbf{b}\right\}. \qquad [1.54]$$

[1.54] means the hyperplane $\left\{\mathbf{x} \in \mathbb{F}^n : \mathbf{c}^T\mathbf{x} = \delta\right\}$ intersects with P and the inequality $\mathbf{c}^T\mathbf{x} \leq \delta$ is vaded.

Def'n 1.29. **Face** of a Polyhedron

Let $F \subseteq P$. We say F is a ***face*** of P if $F = P$ or F is the intersection of P and a supporting hyperplane of P.

Proposition 1.19. Characterization of Faces

Let $F \subseteq P$. Then F is a face of P if and only if $F \neq \varnothing$ and $F = \{\mathbf{x} \in P : A'\mathbf{x} = \mathbf{b}'\}$ for some subsystem $A'\mathbf{x} \leq \mathbf{b}'$ of $A\mathbf{x} \leq \mathbf{b}$.[a]

[a] A subsystem $A'\mathbf{x} \leq \mathbf{b}'$ of $A\mathbf{x} \leq \mathbf{b}$ is a system obtained by deleting some rows of $A\mathbf{x} \leq \mathbf{b}$.

Lemma 1.19.1.

Let $F \subseteq P$. Then F is a face of P if and only if there exist $\mathbf{c} \in \mathbb{F}^n, \delta \in \mathbb{F}$ such that

$$\delta = \max_{\mathbf{x} \in P} \mathbf{c}^T\mathbf{x}$$

and that $F = P \cap \left\{\mathbf{x} \in \mathbb{F}^n : \mathbf{c}^T\mathbf{x} = \delta\right\}$.

Proof of Proposition 1.19.

- ($\Longrightarrow$) Assume that F is a face of P (so $F \neq \varnothing$), and let $\mathbf{c} \in \mathbb{F}^n, \delta \in \mathbb{F}$ be such that $\delta = \max_{\mathbf{x} \in P} \mathbf{c}^T\mathbf{x}$ by Lemma 1.19.1. This means the LP problem

$$\begin{aligned} &\text{maximize } \mathbf{c}^T\mathbf{x} \\ &\text{subject to } A\mathbf{x} \leq \mathbf{b} \end{aligned} \qquad [1.55]$$

has an optimal solution, so its dual

$$\begin{aligned} &\text{minimize } \mathbf{y}^T\mathbf{b} \\ &\text{subject to } \mathbf{y}^T A = \mathbf{c} \\ &\qquad\qquad \mathbf{y} \geq \mathbf{0} \end{aligned} \qquad [1.56]$$

is feasible, in particular with

$$\delta = \min\left\{\mathbf{y}^T\mathbf{b} : \mathbf{y}^T A = \mathbf{c}, \mathbf{y} \geq \mathbf{0}\right\}.$$

Let $\mathbf{y}_0 \in \mathbb{F}^n$ be an optimal solution for [1.56], and let

$$I = \{i \in \{1, \ldots, m\} : y_i > 0,\}$$

where $\mathbf{y}_0 = (y_1, \ldots, y_n)$. Then

$$\begin{aligned} \mathbf{x} \in F &\iff \mathbf{x} \text{ is an optimal solution for [1.55]} \\ &\iff \mathbf{x} \in P \text{ and } A'\mathbf{x} = \mathbf{b}', \end{aligned}$$

where the second if and only if holds by the complementary slackness. Thus $F = \{\mathbf{x} \in P : A'\mathbf{x} = \mathbf{b}\}$.

- ($\Longleftarrow$) Suppose
$$F = \left\{\mathbf{x} \in P : A'\mathbf{x} = \mathbf{b}'\right\}$$
for some subsystem $A'\mathbf{x} \leq \mathbf{b}'$ of $A\mathbf{x} \leq \mathbf{b}$. Then by letting $\mathbf{c}$ be the sum of the rows of A', $\mathbf{x}_0 \in P$ is an optimal solution to [1.55] if and only if $\mathbf{x}_0 \in P$ and $A'\mathbf{x}_0 = \mathbf{b}'$. So
$$F = \left\{\mathbf{x} \in P : \mathbf{c}^T\mathbf{x} = \delta\right\},$$
where $\delta = \max_{\mathbf{x} \in P} \mathbf{c}^T\mathbf{x}$. ∎

Corollary 1.19.2.

(a) P has finitely many faces.

(b) Each face of P is a polyhedron.

(c) If $F \subseteq P$ is a face of P and $E \subseteq F$, then E is a face of P if and only if E is a face of F.

Def'n 1.30. **Facet** of a Polyhedron
A ***facet*** of P is a maximal proper face of P.

Def'n 1.31. **Redundant** Inequality of a System
An inequality (or equality) in a system is called ***redundant*** if it is implied by the otehr constraints in the system.

Proposition 1.20.

Suppose no inequality in $A^+\mathbf{x} \leq \mathbf{b}^+$ is redundant in $A\mathbf{x} \leq \mathbf{b}$. Then there is a one-to-one correspondence between the facets of P and the inequalities in $A^+\mathbf{x} \leq \mathbf{b}^+$, given by
$$F = \left\{\mathbf{x} \in P : \mathbf{a}_i^T\mathbf{x} = \mathbf{b}_i\right\}$$
for all $i \in \{1, \ldots, m\}$ such that $\mathbf{a}_i^T\mathbf{x} \leq b_i$ is an inequality from $A^+\mathbf{x} \leq \mathbf{b}^+$.

Proposition 1.21.

Let $F \subseteq P$. If F is a facet of P, then
$$\dim(F) = \dim(P) - 1. \qquad [1.57]$$

Proposition 1.22.

If P is full-dimensional and $A\mathbf{x} \leq \mathbf{b}$ is irredundant (i.e. no inequalities in $A\mathbf{x} \leq \mathbf{b}$ is redundant), then $A\mathbf{x} \leq \mathbf{b}$ is the unique minimal representation of P, up to multiplication of inequalities by positive scalars.

Def'n 1.32. **Minimal** Face of a Polyhedron
Let $F \subseteq P$ be a face of P. We say F is ***minimal*** if no face of P is contained in F.

Proposition 1.23. Characterizations of Minimal Faces

Let $F \subseteq P$ be a face of P. Then the following are equivalent.

(a) F is minimal.

(b) F is an affine subspace.

(c) $F = \{\mathbf{x} \in \mathbb{F}^n : A'\mathbf{x} = \mathbf{b}'\}$ for some subsystem $A'\mathbf{x} \le \mathbf{b}'$ of $A\mathbf{x} \le \mathbf{b}$.

(1.16) Suppose that

$$F = \left\{\mathbf{x} \in \mathbb{F}^n : A'\mathbf{x} = \mathbf{b}'\right\}$$

is a minimal face of P, where $A'\mathbf{x} \le \mathbf{b}'$ is a subsystem of $A\mathbf{x} \le \mathbf{b}$. Then note that $\operatorname{rank}(A') = \operatorname{rank}(A)$. This means all minimal faces of P have the same dimension.

Vertex of a Polyhedron

Def'n 1.33. Let $V \subseteq P$ be a face of P. If V is a singleton, then we say V is a ***vertex*** of P.

Pointed Polyhedron

Def'n 1.34. P is called ***pointed*** if it there exists a face of P that is a vertex of P.

Proposition 1.24.

(a) If P is bounded, then P is pointed.

(b) If P is pointed, then every minimal face is a vertex.

Edge, Ray of a Polyhedron

Def'n 1.35. Let $E \subseteq P$ be a face of P.

(a) If $\dim(E) = 1$, then E is called an ***edge*** of P.

(b) If E is a half-line, then E is called a ***ray*** of P.

Adjacent (Neighboring) Vertices

Def'n 1.36. Let $U, V \subseteq P$ be vertices of P. We say U, V are ***adjacent*** (or ***neighboring***) if there exists an edge $E \subseteq P$ of P such that $U, V \subseteq E$.

(1.17) Consider a cone

$$C = \{\mathbf{x} \in \mathbb{F}^n : A\mathbf{x} \le \mathbf{0}\}.$$

Then the only minimal face of C is the subspace

$$\underbrace{\{\mathbf{x} \in \mathbb{F}^n : A\mathbf{x} = \mathbf{0}\}}_{=Z}.$$

Let $t = \dim(Z)$ and let $G_1, \ldots, G_s \subseteq P$ be the faces of P of dimension $t+1$.

Extreme Ray

Def'n 1.37. Consider the setting of (1.17). We say $G_1, \ldots, G_s$ are ***extreme*** rays if C is pointed.

For each $i \in \{1, \ldots, s\}$, let $\mathbf{y}_i \in G_i \setminus Z$. Also choose $\mathbf{z}_1, \ldots, \mathbf{z}_m \in Z$ such that

$$W = \operatorname{cone}\{\mathbf{z}_1, \ldots, \mathbf{z}_m\}.$$

Proposition 1.25. *Consider the setting of (1.17). Then*

$$C = \operatorname{cone}\{\mathbf{y}_1, \dots, \mathbf{y}_s, \mathbf{z}_1, \dots, \mathbf{z}_m\}.$$

Corollary 1.25.1. *If C is pointed, then C is generated by nonzero representations from each of its extreme rays.*

Linear Programming Complexity

(1.18) Consider the following question.

LP Feasibility Question

$$\text{Given } A \in M_{m\times n}(\mathbb{Q}), \mathbf{b} \in \mathbb{Q}^m, \text{ does } A\mathbf{x} \leq \mathbf{b} \text{ have a solution?} \qquad [1.58]$$

A certificate for the answer of [1.58] can be found as follows.

(a) If the answer for [1.58] is positive, then a certificate is simply $\mathbf{x}_0 \in \mathbb{Q}^n$ with $A\mathbf{x}_0 \leq \mathbf{b}$.

(b) Otherwise, find $\mathbf{y} \in \mathbb{Q}^m$ such that $\mathbf{y}^T A = \mathbf{0}, \mathbf{y} \geq \mathbf{0}, \mathbf{y}^T\mathbf{b} = -1$.[1] This guarantees the infeasibility of $A\mathbf{x} \leq \mathbf{b}$ by the Farkas' lemma.

We desire to show that the above approach has a good characterization (i.e. NP, co-NP characterization) by showing that the sizes of some $\mathbf{x}, \mathbf{y}$ can be bounded polynomially.

Proposition 1.26. *If the system $A\mathbf{x} \leq \mathbf{b}$ has a solution, it has one of size polynomially bounded by the sizes of $A, \mathbf{b}$.*

Proof. Let $F = \{\mathbf{x} \in \mathbb{Q}^n : A'\mathbf{x} = \mathbf{b}'\}$ be a minimal face of $P = \{\mathbf{x} \in \mathbb{Q}^n : A\mathbf{x} \leq \mathbf{b}\}$, where $A'\mathbf{x} \leq \mathbf{b}'$ is a subsystem of $A\mathbf{x} \leq \mathbf{b}$. Then by Lemma 1.2.2, F has a solution of size polynomially bounded by the sizes of $A', \mathbf{b}'$, hence by the sizes of $A, \mathbf{b}$. ∎

Showing that some $\mathbf{y} \in \mathbb{Q}^m$ satisfying (b) has a polynomially bounded size can be done in a similar manner.

Facet Complexity, Vertex Complexity

Def'n 1.38. Let $P \subseteq \mathbb{Q}^n$ be a polyhedron. We define the ***facet complexity*** of P to be the smallest $\varphi \in \mathbb{Q}$ such that

- $\varphi \geq n$; and
- there exist $A \in M_{m\times n}(\mathbb{Q}), \mathbf{b} \in \mathbb{Q}^m$ such that $A\mathbf{x} \leq \mathbf{b}$ defines P and that each inequality in $A\mathbf{x} \leq \mathbf{b}$ has size at most φ.

We define the ***vertex complexity*** of P to be the smallest $\nu \in \mathbb{Q}$ such that

- $\nu \geq n$; and
- there exist $\mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{y}_1, \dots, \mathbf{y}_l \in \mathbb{Q}^n$ of sizes at most ν such that

$$P = \operatorname{conv}\{\mathbf{x}_i\}_{i=1}^k + \operatorname{cone}\{\mathbf{y}_j\}_{j=1}^l.$$

[1] Or we can use any negative constant instead of -1.

Proposition 1.27. *Let $P \subseteq \mathbb{Q}^n$ be of facet complexity $\varphi \in \mathbb{Q}$ and vertex complexity $\nu \in \mathbb{Q}$. Then*

(a) $\nu \leq 4n^2\varphi$; *and*

(b) $\varphi \leq 4n^2\nu$.

2.
Integer Programming

Integer Programming

(2.1) Given $A \in M_{m\times n}(\mathbb{Q}), \mathbf{b} \in \mathbb{Q}^m, \mathbf{c} \in \mathbb{Q}^n$, an ***integer programming*** (***IP***) problem is a problem of the form

IP Problem

$$\begin{aligned} &\text{maximize } \mathbf{c}^T\mathbf{x} \\ &\text{subject to } A\mathbf{x} \leq \mathbf{b} \quad . \\ &\mathbf{x} \in \mathbb{Z}^n \end{aligned} \tag{2.1}$$

In other words, an IP problem is equivalent to an LP problem with an additional constraint that the solution must be integral. The feasibility problem

Feasibility Problem of IP

$$\text{Does } A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \in \mathbb{Z}^n \text{ have a solution?} \tag{2.2}$$

is an NP-hard problem.

(2.2) Techniques for IP depend heavily on the theory of polyhedra.

Def'n 2.1. **Integer Hull** of a Polyhedron

Let $P \subseteq \mathbb{R}^n$ be a polyhedron. The set

$$P_I = \operatorname{conv}(P \cap \mathbb{Z}^n)$$

is called the ***integer hull*** of P.

Note the following.

(a) If P is bounded, then $P \cap \mathbb{Z}^n$ is finite, so P_I is a polyhedron.

(b) If $C \subseteq \mathbb{R}^n$ is a rational cone, then $C = C_I$. Indeed, take $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{Q}^n$ be such that $C = \operatorname{cone}\{\mathbf{v}_i\}_{i=1}^m$. For each $i \in \{1, \ldots, m\}$, let

$$\lambda_i = \text{a positive common multiple of the denominators of the entries of } \mathbf{v}_i,$$

then $\lambda_1\mathbf{v}_1, \ldots, \lambda_m\mathbf{v}_m$ are integer vectors which generate C.

Theorem 2.1. *If $P \subseteq \mathbb{R}^n$ is a rational polyhedron, then P_I is also a polyhedron.*

Proof. By the decomposition theorem for polyhedra, write $P = Q + C$ for some polytope $Q \subseteq \mathbb{R}^n$ and cone $C \subseteq \mathbb{R}^n$. By (b) of (2.2), let $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{Z}^n$ be such that $C = \operatorname{cone}\{\mathbf{v}_i\}_{i=1}^m$. Define

$$B = \left\{\sum_{i=1}^m \lambda_i\mathbf{v}_i : \lambda_i \in [0,1]\right\}.$$

Observe that the following claim implies the result, since $(Q+B)_I$ is a polytope by the fact that $Q+B$ is bounded (and (a) of (2.2)).

- We claim that $P_I = (Q+B)_I + C$.

  Proof. Let $\mathbf{p} \in P \cap \mathbb{Z}^n$. Then $\mathbf{p} = \mathbf{q} + \mathbf{c}$ for some $\mathbf{q} \in Q, \mathbf{c} \in C$. It follows that $\mathbf{c} = \mathbf{b} + \mathbf{d}$ for some $\mathbf{b} \in B, \mathbf{d} \in C \cap \mathbb{Z}^n$ (i.e. $\mathbf{d}$ is the integer part of $\mathbf{c}$ and $\mathbf{b}$ is the vector of remaining decimal places). We also have

  $$\mathbf{p} = \mathbf{q} + \mathbf{b} + \mathbf{d},$$

  which means $\mathbf{q} + \mathbf{b} \in \mathbb{Z}^n$. Therefore, $\mathbf{p} \in (Q+B)_I + \mathbf{c}$, so $P_I \subseteq (Q+B)_I + C$. On the other hand,

  $$(Q+B)_i + C \subseteq P_I + C = P_I + C_I \subseteq (P+C)_I = P_I.$$ ∎

(2.3) Theorem 2.1 allows us to use LP techniques in the solution of IP problems. But how can we get information about P_I given a polyhedron $P \subseteq \mathbb{R}^n$? A special case is when $P = P_I$ (such P is called to be ***integral***), for which we have the folloing equivalent statements.

$$\begin{aligned} P \text{ is integral} &\iff \text{each minimal face of } P \text{ contains integral vectors} \\ &\iff \text{given } \mathbf{c} \in \mathbb{Q}^n, \max_{\mathbf{x} \in P} \left(\mathbf{c}^T \mathbf{x}\right) \text{ has an integral optimal solution if bounded.} \end{aligned}$$

In fact, if P is integral, then [2.1] can be solved in a polynomial time.

(2.4) If $\mathbf{w}^T \mathbf{x} \leq \delta$ is valid for P and $\mathbf{w} \in \mathbb{Z}^n$, then $\mathbf{w}^T \mathbf{x} \leq \lfloor \delta \rfloor$ is valid for P_I.

Def'n 2.2. **Chvatal-Gomory Cut** for a Polyhedron
Consider (2.4). The inequality $\mathbf{w}^T \mathbf{x} \leq \lfloor \delta \rfloor$ is called a ***Chvatal-Gomory cut*** for P.

Def'n 2.3. **Chvatal Closure** of a Polyhedron
Consider (2.4). The ***Chvatal closure*** of P is

$$P' = \{\mathbf{x} \in P : \mathbf{x} \text{ satisfies all Chvatal-Gomory cut for } P\}.$$

We have inclusions $P_I \subseteq P' \subseteq P$. In other words, P' is a better approximation to P_I than P. Let us examine P' for some

$$P = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}\} \subseteq \mathbb{R}^n,$$

where $A \in M_{m\times n}(\mathbb{Q}), \mathbf{b} \in \mathbb{Q}^m$. By scaling, we may assume that $A, \mathbf{b}$ are integral. Suppose that we are given an integral $\mathbf{w} \in \mathbb{Z}^n$ such that $\mathbf{w}^T \mathbf{x} \leq \delta$ for some $\delta \in \mathbb{R}$ is valid for P. Then by the affine Farkas lemma, there exists nonnegative $\mathbf{y} \in \mathbb{Q}^n$ such that $\mathbf{y}^T A = \mathbf{w}, \mathbf{y}^T \mathbf{b} \leq \delta$. Then the Chvatal-Gomory cut $\mathbf{w}^T \mathbf{x} \leq \lfloor \delta \rfloor$ is implied by $\left(\mathbf{y}^T A\right) \mathbf{x} \leq \left\lfloor \mathbf{y}^T \mathbf{b} \right\rfloor$. Note that by the Caratheodory's theorem, we may assume that $\mathbf{y}$ has at most n positive components.