1.
Integral Calculus on $\mathbb{R}^2$

Review of Notions

(1.1) Recall the following things about $\mathbb{R}^2$.

Recall 1.1. **Norm**
We define the ***norm*** $|\cdot| \to \mathbb{R}$ on $\mathbb{R}^2$ by

$$|(x,y)| = \sqrt{x^2+y^2}$$

for all $(x,y) \in \mathbb{R}^2$.

Note that $|z-\zeta|$ for any $z,\zeta \in \mathbb{R}^2$ gives the distance between z,ζ.

Recall 1.2. **Open Disc**
Given $z_0 \in \mathbb{R}^2, r > 0$, we define the ***open disc*** of radius r centered at z_0, denoted as $D(z_0;r)$, to be

$$D(z_0;r) = \left\{z \in \mathbb{R}^2 : |z-z_0| < r\right\}.$$

Recall 1.3. **Topology** on a Set
Let X be a set. We say $\mathcal{T} \subseteq \mathcal{P}(X)$ is a ***topology*** if

(a) $\varnothing, X \in \mathcal{T}$;

(b) for any $\{U_i\}_{i\in I} \subseteq \mathcal{T}$ where I is an index set, $\bigcup_{i\in I} U_i \in \mathcal{T}$; and

(c) for any $\{U_k\}_{k=1}^n \subseteq \mathcal{T}$, $\bigcap_{k=1}^n U_k \in \mathcal{T}$.

In particular, we are going to work with the *standard topology* on $\mathbb{R}^2$ which is constructed as follows.

Recall 1.4. **Interior, Boundary** Point
Let $A \subseteq \mathbb{R}^2$ and let $z \in \mathbb{R}^2$.

(a) We say z is an ***interior*** point of A if there exists $r > 0$ such that $D(z;r) \subseteq A$.

(b) We say z is a ***boundary*** point of A if, for every $r > 0$, $D(z;r) \cap A \neq \varnothing$ and $D(z;r) \cap A^C \neq \varnothing$.

Observe that if $z \in \mathbb{R}^2$ is an interior point of $A \subseteq \mathbb{R}^2$, then $z \in A$.

Recall 1.5. **Interior, Boundary, Closure** of a Set
Let $A \subseteq \mathbb{R}^2$.

(a) We define the ***interior*** of A, denoted as A°, by

$$A^\circ = \left\{z \in \mathbb{R}^2 : z \text{ is an interior point of } A\right\}.$$

(b) We define the ***boundary*** of A, denoted as ∂A, by

$$\partial A = \left\{z \in \mathbb{R}^2 : z \text{ is a boundary point of } A\right\}.$$

(c) We define the ***closure*** of A, denoted as $\overline{A}$, by

$$\overline{A} = A^\circ \cup \partial A.$$

Recall 1.6. **Open, Closed** Set

Let $A \subseteq \mathbb{R}^2$.

(a) We say A is ***open*** if $A = A^\circ$.

(b) We say A is ***closed*** if $A = \overline{A}$.

Recall 1.7. **Bounded, Compact** Set

Let $A \subseteq \mathbb{R}^2$.

(a) We say A is ***bounded*** if there exists $r > 0$ such that $A \subseteq D(0;r)$.

(b) We say A is ***compact*** if A is closed and bounded.[a]

[a] Of course, we are using the Heine-Borel theorem here.

Recall 1.8. **Connected** Set

Let $A \subseteq \mathbb{R}^2$.

(a) We say A is ***disconnected*** if there exist open $U_1, U_2 \subseteq \mathbb{R}^2$ such that $U_1 \cup U_2 \supseteq A$ and $U_1 \cap U_2 = \varnothing$.

(b) We say A is ***connected*** if A is not disconnected.

Def'n 1.9. **Domain**

We say $A \subseteq \mathbb{R}^2$ is a ***domain*** if A is open and connected.

Def'n 1.10. **Path**

Let $A \subseteq \mathbb{R}^2$ be nonempty and let $z_1, z_2 \in A$. A ***path*** in A from z_1 to z_2 is a continuous function $\gamma : [a,b] \to A$ such that $\gamma(a) = z_1, \gamma(b) = z_2$, where $[a,b] \subseteq \mathbb{R}$ is a closed interval.

Def'n 1.11. **Path-connected** Set

Let $A \subseteq \mathbb{R}^2$. We say A is ***path-connected*** if, for every $z_1, z_2 \in A$, there exists a path in A from z_1 to z_2.

Proposition 1.1. *Let $A \subseteq \mathbb{R}^2$ be open. Then A is connected if and only if A is path-connected.*

Parameterized Curves

Def'n 1.12. **Smooth Parameterization** of a Set

Let $\Gamma \subseteq \mathbb{R}^2$ and let $[a,b] \subseteq \mathbb{R}$. We say $\gamma : [a,b] \to \Gamma$ is a ***smooth parameterization*** of Γ if

(a) $\operatorname{image}(\gamma) = \Gamma$;

(b) $\gamma'(t) \neq 0$ for all $t \in [a,b]$; and

(c) if $\gamma(a) = \gamma(b)$, then $\gamma'(a) = \gamma'(b)$.

(1.2) Recall that the *velocity vector* $\gamma'(t)$ is tangent to Γ at the point $\alpha(t)$ whenever $\alpha'(t) \neq 0$. So (b) of Def'n 1.12 ensures that we can specify a tangent direction to Γ at every point. Moreover, in the case when Γ is closed (i.e. when $\gamma(a) = \gamma(b)$), (c) of Def'n 1.12 ensures that the tangent direction is well-defined.

Def'n 1.13. **Closed, Simple** Parameterization

Let $\gamma : [a,b] \to \Gamma$ be a parameterization of $\Gamma \subseteq \mathbb{R}^2$.

(a) We say γ is ***closed*** if $\gamma(a) = \gamma(b)$.

(b) We say γ is simple if γ is injective on $[a,b), (a,b]$.

Def'n 1.14. **Piecewise Smooth** Parameterization

Let $\gamma : [a,b] \to \Gamma$ be a parameterization of $\Gamma \subseteq \mathbb{R}^2$. If there exists finite number of subsets $\Gamma_1, \ldots, \Gamma_n \subseteq \Gamma$ with smooth parameterizations $\gamma_1 : [a_1, a_2] \to \Gamma_1, \ldots, \gamma_n : [a_n, a_{n+1}] \to \Gamma_n$ such that

(a) $a = a_1 < \cdots < a_n = b$;

(b) $\gamma_{i-1}(a_{i-1}) = \gamma_i(a_i)$ for all $i \in \{1, \ldots, n\}$; and

(c) $\Gamma = \bigcup_{i=1}^n \Gamma_i$;

then we say γ is a ***piecewise smooth*** parameterization of Γ.

In connection to Def'n 1.14, we say the ordered pair (Γ, γ) a *piecewise smooth curve*.

Def'n 1.15. **Jordan** Curve

A ***Jordan*** curve is a simple, closed piecewise smooth curve.

Theorem 1.2. Jordan Curve Theorem

If Γ is a Jordan curve, its complement $\mathbb{R}^2 \setminus \Gamma$ consists of two disjoint domains, one bounded (the "inside") and one not (the "outside"), each domain having the curve Γ as its boundary. If a point inside Γ is joined by a path to a point outside Γ, then the path must cross Γ.

Proof. Proof of Theorem 1.2 is out of scope of this note. ∎

Def'n 1.16. **Jordan** Domain

A Jordan domain is a bounded domain $A \subseteq \mathbb{R}^2$ whose boundary is the union of a finite number of positively oriented Jordan curves, in the sense that when one walks along a given curve in the direction specified by its parametrisation, then A always lies to the left.

Def'n 1.17. **Equivalence**

Let $E : [c,d] \to [a,b]$ be a smooth bijection with $E'(s) > 0$ for all $s \in [c,d]$. Then we call E an ***equivalence***.

(1.3)

(a) By the inverse function theorem, equivalences have smooth inverses.

(b) Let $\alpha : [a,b] \to \Gamma$ be a smooth parameterization of a curve $\Gamma \subseteq \mathbb{R}^2$. Given an equivalence $E : [c,d] \to [a,b]$, we may define $\beta : [c,d] \to \Gamma$ by $\beta = \alpha \circ E$. Then β is also a smooth parameterization of Γ that traverses the curve in the smae direction as α, since E is increasing. In this sense, we may call α, β to be equivalent.

Def'n 1.18. **Equivalent** Parameterizations

Let $\alpha : [a,b] \to \Gamma, \beta : [c,d] \to \Gamma$ be smooth parameterizations of a curve $\Gamma \subseteq \mathbb{R}^2$. We say α, β are ***equivalent*** if there exists an equivalence $E : [c,d] \to [a,b]$ such that $\beta = \alpha \circ E$.

(EX 1.4) Arclength Parameterization

Let $\alpha : [a,b] \to \Gamma$ be a smooth parameterization of $\Gamma \subseteq \mathbb{R}^2$. Then for any $s \in [a,b]$, we have

$$\int_a^s |\alpha'(t)| = \text{length of the curve from } \alpha(a) \text{ to } \alpha(s).$$

Note that, if $|\alpha'(t)| = 1$ for all $t \in [a,b]$, then

$$s - a = \int_a^s |\alpha'(t)| = \text{length of the curve from } \alpha(a) \text{ to } \alpha(s). \qquad [1.1]$$

If [1.1] holds, then we say α an *arclength parameterization*.

Proposition 1.3. Finding Equivalent Arclength Parameterization

Let $\alpha : [a,b] \to \Gamma$ be a smooth parameterization of $\Gamma \in \mathbb{R}^2$ of length $L > 0$. Then, there exists another parameterization $\sigma : [0,L] \to \Gamma$ that is equivalent to α and is such that

$$|\sigma'(s)| = 1$$

for all $s \in [0,L]$.

Proof. Let $F : [a,b] \to [0,L]$ be defined by

$$F(\tau) = \int_a^\tau |\alpha'(t)| \, \mathrm{dt}$$

for all $\tau \in [a,b]$.

(a) By the fundamental theorem of calculus, F is differentiable with

$$F'(\tau) = |\alpha'(\tau)| \qquad [1.2]$$

for all $\tau \in [a,b]$.

(b) Notice that $F(a) = 0, F(b) = L$. Since (a) guarantees that F is strictly increasing and continuous, F is bijective. It is also clear from [1.2] that F is smooth, so F is an equivalence. As a result, its inverse $E = F^{-1} : [0,L] \to [a,b]$ is an equivalence.

(c) Define $\sigma : [0,L] \to \Gamma$ by

$$\sigma(s) = \alpha(E(s))$$

for all $s \in [0,L]$. Then by the chain rule and inverse function theorem,

$$\sigma'(s) = (\alpha(E(s)))' = \alpha'(E(s))E'(s) = |F'(E(s))| \frac{1}{F'(E(s))} = 1$$

for all $s \in [0,L]$, as desired.

∎

Def'n 1.19. **Unit Tangent Vector, Outward Normal Vector**
Given an acrlength parameterization $\sigma : [0,L] \to \Gamma$ of $\Gamma \subseteq \mathbb{R}^2$, we define

(a) the ***unit tangent vector*** to Γ at the point $\sigma(s)$ to be

$$\sigma'(s) = \left(\sigma_1'(s), \sigma_2'(s)\right),$$

the derivative of σ at s; and

(b) the ***outward normal vector*** to Γ at $\sigma(s)$, say $N(s)$, to be

$$N(s) = \left(\sigma_2'(s), -\sigma_1'(s)\right).$$

Integral Calculus on $\mathbb{R}^2$

(1.5) Line Integral

The setting is that we desire to integrate a continuous vector-valued function $f : \mathbb{R}^2 \to \mathbb{R}^2$ along a curve, defined by

$$(x,y) \mapsto (p(x,y), q(x,y)),$$

where $p, q : \mathbb{R}^2 \to \mathbb{R}$ are some real-valued functions.

Def'n 1.20. **Line Integral** along a Smooth Curve
Consider the setting of (1.5), and suppose that we have a smooth parameterization $\alpha : [a,b] \to \Gamma$ of $\Gamma \subseteq A$, where $A \subseteq \mathbb{R}^2$ is a domain. When $p, q : A \to \mathbb{R}$ are continuous, we define the ***line integral*** of $(p,q) : A \to \mathbb{R}^2$ along Γ, denoted as $\int_\Gamma p \, \mathrm{d}x + q \, \mathrm{d}y$ by

$$\int_\Gamma p \, \mathrm{d}x + q \, \mathrm{d}y = \int_a^b p(\alpha(t)) \alpha_1'(t) \, \mathrm{d}t + \int_a^b q(\alpha(t)) \alpha_2'(t) \, \mathrm{d}t.$$

Proposition 1.4. Uniqueness of Line Integral up to Equivalence

Let $\Gamma \subseteq \mathbb{R}^2$ and let $\alpha : [a,b] \to \Gamma, \beta : [c,d] \to \Gamma$ be equivalent parameterizations, where $E : [c,d] \to [a,b]$ is the equivalence such that $\beta = \alpha \circ E$. Let $A \subseteq \mathbb{R}^2$ be a domain and let $p, q : A \to \mathbb{R}$ be continuous. Then

$$\int_a^b p(\alpha(t)) \alpha_1'(t) \, \mathrm{d}t = \int_c^d p(\beta(s)) \beta_1'(s) \, \mathrm{d}s$$

and

$$\int_a^b q(\alpha(t)) \alpha_2'(t) \, \mathrm{d}t = \int_c^d q(\beta(s)) \beta_2'(s) \, \mathrm{d}s$$

We can surely extend our notion of line integrals to piecewise-smooth curves.

Def'n 1.21. **Line Integral** along a Piecewise-smooth Curve
Let (Γ, α) be a piecewise-smooth parameterized curve in a domain $A \subseteq \mathbb{R}^2$ given by the smooth parameterizations $\alpha_i : [a_{i-1}, a_i] \to \Gamma_i$, where $i \in \{1, \ldots, n\}$ is the running index. For any continuous functions $p, q : A \to \mathbb{R}$, we define the ***line integral*** of $(p,q) : A \to \mathbb{R}^2$ along Γ, denoted as $\int_\Gamma p \, \mathrm{d}x + q \, \mathrm{d}y$ as

$$\int_\Gamma p \, \mathrm{d}x + q \, \mathrm{d}y = \sum_{i=1}^n \int_{\Gamma_i} p \, \mathrm{d}x + q \, \mathrm{d}y.$$

Proposition 1.5. Properties of Line Integrals

Let (Γ, α) be a piecewise smooth curve in a domain $A \subseteq \mathbb{R}^2$ and let $p, q, p_1, q_1, p_2, q_2 : A \to \mathbb{R}$ be continuous. Let $a \in R$. Then

(a) *linearity:* $\int_\Gamma ap_1 + p_2 \,\mathrm{d}x + aq_1 + q_2 \,\mathrm{d}y = a \int_\Gamma p_1 \,\mathrm{d}x + q_1 \,\mathrm{d}y + \int_\Gamma p_2 \,\mathrm{d}x + q_2 \,\mathrm{d}y$*; and*

(b) *reversing the orientation:* $\int_{-\Gamma} p \,\mathrm{d}x + q \,\mathrm{d}y = -\int_\Gamma p \,\mathrm{d}x + q \,\mathrm{d}y$.

Proof. Exercise. ∎

Def'n 1.22. **Exact** Differential

We say $p \,\mathrm{d}x + q \,\mathrm{d}y$, where $p, q : A \to \mathbb{R}$ are real-valued continuous functions on a domain $A \subseteq \mathbb{R}^2$, is ***exact*** if there exists a C^1-function $u : A \to \mathbb{R}$ such that $p = \partial_x u, q = \partial_y u$.

Theorem 1.6. Independence of Path

Let Γ be a piecewise-smooth curve from z_1 to z_2, lying inside a domain $A \subseteq \mathbb{R}^2$ and let $u : A \to \mathbb{R}$ be C^1. Then

$$\int_\Gamma 1 \,\mathrm{d}u = u(z_2) - u(z_1).$$

In paticular, when Γ is a loop so that $z_1 = z_2$,

$$\oint_\Gamma 1 \,\mathrm{d}u = 0.$$

Def'n 1.23. **Double Integral** of a Continuous Function

Let $A \subseteq \mathbb{R}^2$ and let $f : A \to \mathbb{R}$ be continuous. Then we know that f is integrable on A so that

$$\iint_A f(x,y) \,\mathrm{d}x \,\mathrm{d}y$$

exists. We call $\iint_A f$ the ***double integral*** of f over A.

Proposition 1.7. Properties of Double Integrals

Let $f, g : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^2$ is bounded, and let $c \in \mathbb{R}$.

(a) *linearity:* $\iint_A (cf + g) = c \iint_A f + \iint_A g$.

(b) *If $f \geq 0$ on A, then $\iint_A f \geq 0$. Similarly, if $f > 0$ on A, then $\iint_A f > 0$.*

Theorem 1.8. Bump Principle

Let $A \subseteq \mathbb{R}^2$ be a bounded domain and let $q : A \to \mathbb{R}$ be continuous. If q is nonnegative on A, then q is identically zero on A if and only if

$$\iint_A q = 0. \qquad [1.3]$$

Proof. Observe that the forward direction is clear. To prove the reverse direction, assume [1.3] and, for the sake of contradiction, also assume that $q(x_0, y_0) > 0$ for some $(x_0, y_0) \in A$. Since A is open and q is continuous, there is a neighborhood B of (x_0, y_0) contained in A such that $q(x,y) > 0$ for all $(x,y) \in B$. But this means that

$$\iint_A q = \iint_{A \setminus B} q + \iint_B q \geq \iint_B q > 0,$$

which is a contradiction. ∎

Def'n 1.24. ***k*-connected** Jordan Domain

Let $A \subseteq \mathbb{R}^2$ be a Jordan domain. We say A is ***k-connected***, where $k \in \mathbb{N}$, if ∂A consists of k distinct Jordan curves.

(1.6) *k*-connected Jordan Domains

It is equivalent to say that a Jordan domain $A \subseteq \mathbb{R}^2$ is k-connected if $\mathbb{R}^2 \setminus A$ consists of k disjoint conected components. In case when $k = 1$, we often say that A is ***simply-connected***.

Theorem 1.9. Green's Theorem

Let $A \subseteq \mathbb{R}^2$ be a k-connected Jordan domain and let $p, q : A^+ \to \mathbb{R}$ be C^1-functions defined on a domain A^+ containing $\overline{A}$. Then,

$$\oint_{\partial A} p \,\mathrm{d}x + q \,\mathrm{d}y = \iint_A (q_x - p_y) \,\mathrm{d}x \,\mathrm{d}y.$$

Corollary 1.9.1. Poincare Lemma

Let $A \subseteq \mathbb{R}^2$ be a simply connected domain and let $p, q : A \to \mathbb{R}$ be C^1. Then $p \,\mathrm{d}x + q \,\mathrm{d}y$ is exact if and only if $p_y = q_x$.

(1.7)

Recall that, given an arclength parameterization $\sigma : [0, L] \to \Gamma$ for some $\Gamma \subseteq \mathbb{R}^2$, we define the unit outward normal vector to Γ at $\sigma(s)$, denoted as $N(s)$, is defined as

$$N(s) = \left(\sigma_2'(s), -\sigma_1'(s)\right)$$

for all $s \in [0, L]$. We often write $N(z)$ to denote

$$N(z) = N(s),$$

where $s \in [0, L]$ is such that $\sigma(s) = z$.

Notation 1.25. **Outward Normal Derivative**

Let $s_0 \in [0, L]$ and let $z_0 = \sigma(s_0)$. We write $\partial_n u(z_0)$ to denote

$$\partial_n u(z_0) = \nabla u(z_0) N(z_0),$$

the ***outward normal derivative***.

Note that when u is differentiable at z_0, then $\partial_n u(z_0)$ is the directional derivative of u in the direction $N(z_0)$. An important consequence of Green's theorem is the inside outside theorem.

Theorem 1.10. Inside Outside Theorem

Let $A \subseteq \mathbb{R}^2$ be a k-connected Jordan domain and let $u : A^+ \to \mathbb{R}$ be C^2, where $A^+ \subseteq \mathbb{R}^2$ is a domain containing $\overline{A}$. Then

$$\oint_{\partial A} u_n \,\mathrm{d}s = \iint_A \Delta u \,\mathrm{d}x \,\mathrm{d}y.$$

Note that Δu denotes the Laplacian of u, $u_{xx} + u_{yy}$.

Proof of Theorem 1.10. Let $\sigma : [0, L] \to \partial A$ be an arclength parameterization of ∂A oriented positively, where L is the arclength of A. Then observe that

$$\begin{aligned} u_n(\sigma(s)) &= \langle \nabla u(\sigma(s)), N(\sigma(s)) \rangle \\ &= \left\langle (u_x(\sigma(s)), u_y(\sigma(s))), \left(\sigma_2'(s), -\sigma_1'(s)\right) \right\rangle \\ &= -u_y(\sigma(s))\sigma_1'(s) + u_x(\sigma(s))\sigma_2'(s) \end{aligned} \tag{1.4}$$

for all $s \in [0, L]$, and we also know that

$$\sigma_1'(s)\,\mathrm{ds} = \mathrm{dx}, \sigma_2'(s)\,\mathrm{ds} = \mathrm{dy} \tag{1.5}$$

on ∂A. So

$$\begin{aligned} \oint_{\partial A} u_n\,\mathrm{ds} &= \oint_{\partial A} -u_y\,\mathrm{dx} + u_x\,\mathrm{dy} && \text{by [1.4], [1.5]} \\ &= \iint_A u_{xx} + u_{yy}\,\mathrm{dx}\,\mathrm{dy}, && \text{by Green's theorem} \end{aligned}$$

since u_x, u_y are C^1 on a domain $A^+ \supseteq \overline{A}$ and A is a k-connected Jordan domain. ∎

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2.
Harmonic Functions

Harmonic Functions

Def'n 2.1. **Harmonic** Function

Let $A \subseteq \mathbb{R}^2$ be a domain and let $u : A \to \mathbb{R}$ be C^2. We say u is ***harmonic*** if $\Delta u = 0$.

(2.1) Harmonic Functions

In other words, harmonic functions are solutions to the *Laplace equation*:

$$u_{xx} + u_{yy} = 0.$$

Proposition 2.1. Characterization of Harmonic Functions I

Let $A \subseteq \mathbb{R}^2$ be a domain and let $u : A \to \mathbb{R}$ be C^2. Then u is harmonic if and only if

$$\oint_\Gamma u_n \, \mathrm{d}s = 0 \tag{2.1}$$

for every Jordan curve $\Gamma \subseteq A$ whose interior lies in A.

Proof. Let $\Omega \subseteq \mathbb{R}^2$ be the interior of Γ. Then by assumption, $\Gamma \subseteq A$.

- ($\Longrightarrow$) Assume that u is harmonic. Then

$$\begin{aligned} \oint_\Gamma u_n \, \mathrm{ds} &= \iint \Delta u \, \mathrm{dx \, dy} && \text{by the inside outside theorem} \\ &= \iint 0 \, \mathrm{dx \, dy} && \text{since } u \text{ is harmonic} \\ &= 0. \end{aligned}$$

- ($\Longleftarrow$) Assume that [2.1] holds for every Jordan curve $\Gamma \subseteq A$ whose interior lies in A. Then given any $z_0 \in A$, we can find a neighborhood $D(z_0; 2r_0)$ of z_0 contained in A since A is open. This means $\overline{D(z_0; r_0)} \subseteq A$, so

$$\begin{aligned} \iint_{D(z_0;r_0)} \Delta u \, \mathrm{dx \, dy} &= \oint_{\partial D(z_0;r_0)} u_n \, \mathrm{ds} && \text{by the inside outside theorem} \\ &= 0. && \text{since we assumed [2.1]} \end{aligned}$$

  Hence by the bump primciple (which we can invoke for Δu because u is C^2 so Δu is continuous), $\Delta u = 0$ on $D(z_0; r_0)$, in particular at z_0. Since we assumed z_0 to be an arbitrary point of A, it follows that $\Delta u = 0$ on A, as desired. ∎

Mean Value Property

(2.2) Mean Value Property

Recall that we have the mean value theorem for real-valued univariate functions. The theorem states that, given a function $f : [a,b] \to \mathbb{R}$ continuous on $[a,b]$ and differentiable on (a,b), where $[a,b] \subseteq \mathbb{R}$ is a closed interval, there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Notice that, by the mean value theorem and the fundamental theorem of calculus, when we let $g = f' : [a,b] \to \mathbb{R}$,

$$g(c) = \frac{1}{b-a} \int_a^b g(t)\, \mathrm{d}t, \tag{2.2}$$

for some $c \in (a,b)$. This means g attains its mean (or average) at c. We shall see that an analogue of [2.2] is satisfied by harmonic functions.

Theorem 2.2. Circumferencial Mean Value Theorem

Let $u : A \to \mathbb{R}$ be harmonic, where $A \subseteq \mathbb{R}^2$ is a domain, and suppose that $\overline{D} = \overline{D(z_0;R)} \subseteq A$ for some $z_0 = (x_0, y_0) \in A, R > 0$. Then

$$u(z_0) = \frac{1}{2\pi R} \oint_C u(z)\, \mathrm{d}z = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + R\cos(\theta), y_0 + R\sin(\theta))\, \mathrm{d}\theta. \tag{2.3}$$

Proof. First note that

$$\begin{aligned} \oint_C u(z)\, \mathrm{d}z &= \int_0^{2\pi} u(x_0 + R\cos(\theta), y_0 + R\sin(\theta)) \left|(-R\sin(\theta), R\cos(\theta))\right| \mathrm{d}\theta \\ &= R\int_0^{2\pi} u(x_0 + R\cos(\theta), y_0 + R\sin(\theta))\, \mathrm{d}\theta \end{aligned}$$

which verifies the second equality of [2.3]. So it suffices to prove that

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + R\cos(\theta), y_0 + R\sin(\theta))\, \mathrm{d}\theta \tag{2.4}$$

is true. Let $r \in (0,R]$ and consider $\Gamma = \partial D(z_0;r)$. We know that

$$\frac{\partial}{\partial n}\left(u\left(x_0 + r\cos\left(\frac{s}{r}\right)\right), y_0 + r\sin\left(\frac{s}{r}\right)\right) = \frac{\partial}{\partial r} u(x_0 + r\cos(\theta), y_0 + r\sin(\theta)),$$

when $s = r\theta$, so

$$\begin{aligned} 0 &= \iint_{D(z_0;r)} \Delta u\, \mathrm{d}x\, \mathrm{d}y && \text{since } u \text{ is harmonic} \\ &= \oint_\Gamma \frac{\partial u}{\partial n}\, \mathrm{d}s && \text{by the inside outside theorem} \\ &= \int_0^{2\pi} \frac{\partial}{\partial r} u(x_0 + r\cos(\theta), y_0 + r\sin(\theta))\, r\, \mathrm{d}\theta \\ &= r\frac{\mathrm{d}}{\mathrm{dr}} \int_0^{2\pi} u(x_0 + r\cos(\theta), y_0 + r\sin(\theta))\, \mathrm{d}\theta, && \text{by Leibniz integral rule} \end{aligned}$$

and in particular

$$\frac{\mathrm{d}}{\mathrm{dr}} \int_0^{2\pi} u(x_0 + r\cos(\theta), y_0 + r\sin(\theta)) = 0,$$

which means $\int_0^{2\pi} u(x_0 + r\cos(\theta), y_0 + r\sin(\theta))$ is independent of r. Moreover, observe that by using the mean value theorem with respect to θ, there exists $\theta_r \in [0, 2\pi]$ such that

$$u(x_0 + r\cos(\theta_r), y_0 + r\sin(\theta_r)) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos(\theta), y_0 + r\sin(\theta))\, \mathrm{d}\theta. \tag{2.5}$$

Thus,

$$\begin{aligned}
u(z_0) &= \lim_{r\downarrow 0} u(x_0 + r\cos(\theta_r), y_0 + r\sin(\theta_r)) && \text{by continuity of } u \text{ at } z_0 \\
&= \lim_{r\downarrow 0} \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + r\cos(\theta), y_0 + r\sin(\theta))\,\mathrm{d}\theta && \text{by [2.5]} \\
&= \lim_{r\downarrow 0} \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + R\cos(\theta), y_0 + R\sin(\theta))\,\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + R\cos(\theta), y_0 + R\sin(\theta))\,\mathrm{d}\theta,
\end{aligned}$$

as desired. ∎

In fact, the converse of Theorem 2.2 is also true, giving us a new characterization of harmonic functions.

Def'n 2.2. **Circumferential Mean Value Property**
We say a C^2-function $u : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^2$ is a domain, has the ***circumferential mean value property*** if for every $z_0 \in \Omega$ and every ball $D = D(z_0; R)$ such that $\overline{D} \subseteq A$, we have

$$u(z_0) = \frac{1}{2\pi R}\int_{\partial D} u(z)\,\mathrm{d}s\,.$$

Proposition 2.3. Circumferential Mean Value Property Characterizes Harmonic Functions

Let $A \subseteq \mathbb{R}^2$ be a domain and let $u : A \to \mathbb{R}$ be C^2. Then u is harmonic if and only if u has the circumferential mean value property.

Proof. The forward direction is provided by Theorem 2.2. Conversely, suppose that u has a circumferential mean value property. Then given any $z_0 \in A$, we can find a neighborhood $D = D(z_0; r_0) \in A$ such that $\overline{D} \subseteq A$. Thus

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{dr}} u(z_0) \\
&= \frac{\mathrm{d}}{\mathrm{dr}} \oint_{\partial D} u(z)\,\mathrm{dz} && \text{by the circumferential mean value property of } u \\
&= \iint_D \Delta u(x,y)\,\mathrm{dx}\,\mathrm{dy}, && \text{see Theorem 2.2}
\end{aligned}$$

so by the bump principle, $\Delta u(x,y) = 0$ for all $(x,y) \in D$, in particular $\Delta u(z_0) = 0$. ∎

It turns out that every C^2 function that has the circumferential mean value property is C^∞. In other words, all harmonic functions are smooth. Although we are not going to prove this now, we shall see this result later when we begin to discuss holomorphic functions.

Theorem 2.4. Solid Mean Value Theorem

Let $u : A \to \mathbb{R}$ be harmonic, where $A \subseteq \mathbb{R}^2$ is a domain, and let $\overline{D} = \overline{D(z_0; r_0)} \subseteq A$, where $z_0 \in A, r_0 > 0$. Then

$$u(z_0) = \frac{1}{\pi R^2}\iint_D u\,\mathrm{d}x\,\mathrm{d}y\,.$$

Proof. By the circumferential mean value theorem,

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + R\cos(\theta), y_0 + R\sin(\theta))\,\mathrm{d}\theta$$

for all $r \in (0, R]$. Then

$$\begin{aligned} u(z_0)\frac{R^2}{2} &= \int_0^R u(z_0)\, r \,\mathrm{dr} \\ &= \frac{1}{2\pi}\int_0^R \int_0^{2\pi} u(x_0 + R\cos(\theta), y_0 + R\sin(\theta))\, r \,\mathrm{d}\theta \,\mathrm{dr} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \int_0^R u(x_0 + R\cos(\theta), y_0 + R\sin(\theta))\, r \,\mathrm{dr}\,\mathrm{d}\theta\,. && \text{by the Fubini-Tonelli theorem} \end{aligned}$$

Thus

$$\begin{aligned} u(z_0) &= \frac{1}{\pi R^2}\int_0^{2\pi} \int_0^R u(x_0 + R\cos(\theta), y_0 + R\sin(\theta))\, r \,\mathrm{dr}\,\mathrm{d}\theta \\ &= \frac{1}{\pi R^2}\iint_D u(x, y)\,\mathrm{dx}\,\mathrm{dy}, && \begin{aligned} \text{let } x &= x_0 + R\cos(\theta) \\ y &= y_0 + R\cos(\theta) \end{aligned} \end{aligned}$$

as desired. ∎

Maximum Principle and Liouville's Theorem

Theorem 2.5. Strong Maximum Principle

Let $u : A \to \mathbb{R}$ be harmonic, where $A \subseteq \mathbb{R}^2$ is a domain, and suppose that u attains a maximum or minimum in Ω. Then u is constant.

Proof. Suppose that u has a maximum at $z_0 \in \Omega$ and write $c = u(z_0)$. Then

$$u(z) \leq c \tag{2.6}$$

for all $z \in A$. Since u is continuous, the preimage $B = u^{-1}(\{c\})$ is closed, and clearly $z_0 \in B$ so $B \neq \varnothing$. So it suffices to show that B is also open, since the only nonempty clopen subset of A is A itself. ∎

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3.
Complex Functions

Limits and Continuity

(3.1) Real and Imaginary Parts of a Complex Function

Given a complex function $f : A \to \mathbb{C}$ for some $A \subseteq \mathbb{C}$, we can define the *real* and *imaginary parts* $u, v : A \to \mathbb{R}$ of f as follows:

$$u(x,y) = \mathrm{Re}(f(x+iy)), v(x,y) = \mathrm{Im}(f(x+iy))$$

for all $(x,y) \in A$. Note that u, v are precisely the component functions of f, when we consider $A \subseteq \mathbb{R}^2$.

Def'n 3.1. **Limit** of a Complex Function

Let $f : A \to \mathbb{C}$ be a complex function for some $A \subseteq \mathbb{C}$ and let $z_0 \in A$. If

$$\forall \varepsilon > 0 \exists \delta > 0 \forall z \in A \left[0 < |z - z_0| < \delta \implies |f(z) - \zeta_0| < \varepsilon\right]$$

for some $\zeta_0 \in \mathbb{C}$, then we say the ***limit*** of f as z approaches z_0 exists and is equal to ζ_0. Symbolically, we write

$$\lim_{z \to z_0} f(z) = \zeta_0.$$

The following are some useful properties of limits. Since they follow directly from multivariate calculus, we shall omit their proof. Fix $A \subseteq \mathbb{C}$ throughout this section.

Proposition 3.1. Limit of Real and Imaginary Parts

Let $f : A \to \mathbb{C}$ and let $u, v : A \to \mathbb{R}$ be the real and imaginary parts of f. Then given any $z_0 = x_0 + iy_0 \in A$ ($x_0, y_0 \in \mathbb{R}$),

$$\lim_{z \to z_0} f(z) = \xi_0 + i\eta_0$$

for some $\xi_0, \eta_0 \in \mathbb{R}$ if and only if

$$\lim_{(x,y) \to (x_0,y_0)} u(x,y) = \xi_0, \quad \lim_{(x,y) \to (x_0,y_0)} v(x,y) = \eta_0.$$

Corollary 3.1.1. Uniqueness of Limit

If $\lim_{z \to z_0} f(z)$ exists, then it is unique.

Proposition 3.2. Arithmatic of Limits

Let $f, g : A \to \mathbb{C}$, with $A \subseteq \mathbb{C}$, and let $z_0 \in A$. Suppose that $\lim_{z \to z_0} f(z) = f_0, \lim_{z \to z_0} g(z) = g_0$.

(a) $\lim_{z \to z_0} (f + g)(z) = f_0 + g_0$.

(b) $\lim_{z \to z_0} (fg)(z) = f_0 g_0$.

(c) $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f_0}{g_0}$, *provided that* $g_0 \neq 0$.

Notation 3.2. **Limit** at ∞

Let $f : A \to \mathbb{C}$.

(a) If $\lim_{z \to z_0} |f(z)| = \infty$ for some $z_0 \in A$, then we write $\lim_{z \to z_0} f(z) = \infty$.

(b) We write $\lim_{z \to \infty} f(z)$ to denote $\lim_{z \to 0} f\left(\frac{1}{z}\right)$, provided that the latter limit exists.

Def'n 3.3. **Continuous** Complex Function
Let $f : A \to \mathbb{C}$ and let $z_0 \in A$. We say f is ***continuous*** at z_0 if $f(z_0) = \lim_{z \to z_0} f(z)$.

(3.2) Continuous Complex Functions

We show the following results without proof, because they follow immediately from multivariate calculus.

Proposition 3.3. Continuity of Real and Imaginary Parts

Let $f : A \to \mathbb{C}$ and let $u, v : A \to \mathbb{R}$ be the real and imaginary parts of f. Then f is continuous at $z_0 = x_0 + iy_0 \in A$ if and only if u, v are continuous at (x_0, y_0).

Proposition 3.4. Arithmetic of Continuous Functions

Let $f, g : A \to \mathbb{C}$ be continuous at $z_0 \in A$.

(a) $f + g$ is continuous at z_0.

(b) fg is continuous at z_0.

(c) $\frac{f}{g}$ is continuous at z_0, provided that $g(z_0) \neq 0$.

Proposition 3.5. Composition of Continuous Functions

Let $f : A \to \mathbb{C}, g : B \to \mathbb{C}$ be continuous, where $B \subseteq \mathbb{C}$ is such that $g(B) \subseteq A$. Then $f \circ g : B \to \mathbb{C}$ is continuous.

Proposition 3.6.

Let $f : A \to \mathbb{C}$ be continuous at $z_0 \in A^\circ$ with $f(z_0) \neq 0$. Then there is a neighborhood $D \subseteq A$ of z_0 such that $f(z_0) \neq 0$ for all $z \in D$.

Theorem 3.7. Extreme Value Theorem (EVT)

Let $f : A \to \mathbb{C}$ be continuous. If A is compact, then $|f|$ attains its extremum on A.

Differentiation

(3.3) Fix $A \subseteq \mathbb{C}$ throughout.

Def'n 3.4. **Derivative** of a Complex Function
Let $f : A \to \mathbb{C}$ and let $z_0 \in A^\circ$. If the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{3.1}$$

exists, then we say f is ***differentiable*** at z_0, and define the ***derivative*** of f at z_0, denoted as $f'(z_0)$, to be the limit in [3.1]:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{3.2}$$

Def'n 3.5.

Analytic, Entire Function

Let $f : A \to \mathbb{C}$ and let $z_0 \in A^\circ$. We say f is ***analytic*** at z_0 if f is differentiable at some neighborhood of z_0. When $A = \mathbb{C}$ and f is differentiable everywhere, we say f is ***anlaytic***.

Theorem 3.8.
Differentiation Rules

Let $f, g : A \to \mathbb{C}$ be differentiable at some $z_0 \in A$.

(a) *linearity:* $(cf+g)'(z_0) = cf'(z_0) + g'(z_0)$ *given any* $c \in \mathbb{C}$.

(b) *product rule:* $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$.

(c) *quotient rule:* $\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}$ *if* $g(z_0) \neq 0$.

Theorem 3.9.
Chain Rule

Let $f : A \to \mathbb{C}, g : B \to \mathbb{C}$ for some $B \subseteq \mathbb{C}$ and let $z_0 \in B$ with $g(B) \subseteq A$. If g is differentiable at z_0 and f is differentiable at $g(z_0)$, then $f \circ g : B \to \mathbb{C}$ is differentiable at z_0 and

$$(f \circ g)'(z_0) = f'(g(z_0))\, g'(z_0).$$

Proposition 3.10.
Differentiability Implies Continuity

Let $f : A \to \mathbb{C}$ be differentiable at $z_0 \in A$. Then f is continuous at z_0.

Cauchy-Riemann Equations

(3.4)
Cauchy-Riemann Equations

The motivation for the upcoming discussion begins when one desires to find out the necessary conditions for a function to be differentiable. Fix $A \subseteq \mathbb{C}, f : A \to \mathbb{C}$ and suppose that f is differentiable at $z_0 \in A$. This means that the Newton quotient

$$\frac{f(z) - f(z_0)}{z - z_0} \tag{3.3}$$

converges to $f'(z_0)$ along any path leading z to z_0. In particular, we can conisder the following two paths.

(a) *horizontal path*: Suppose that we use a path of the form $z = x + iy_0$. Then [3.3] becomes

$$\frac{f(z) - f(z_0)}{x - x_0} = \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + i\frac{v(x, y_0) - v(x_0, y_0)}{x - x_0}, \tag{3.4}$$

where $u, v : A \to \mathbb{C}$ are the real and imaginar parts of f, respectively. Therfore, by [3.3], [3.4], Proposition 3.1,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

if and only if

$$\lim_{x \to x_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} = \operatorname{Re}(f'(z_0)), \lim_{x \to x_0} \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0} = \operatorname{Im}(f(z_0)). \tag{3.5}$$

But note that the limits in [3.5] are precisely what defines $u_x(x_0, y_0), v_x(x_0, y_0)$, respectively, where we assumed $f'(z_0)$ exists. Thus $u_x(x_0, y_0), v_x(x_0, y_0)$ exist and

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0). \tag{3.6}$$

(b) *vertical path*: By a similar process where we use a path of the form $z = x_0 + iy$, we obtain

$$f'(z_0) = v_y(x_0, y_0) - iu_y(x_0, y_0). \tag{3.7}$$

Combining [3.6], [3.7] gives the following system of differential equations:

Cauchy-Riemann Equations (CRE)

$$\begin{cases} u_x &= v_y \\ u_y &= -v_x \end{cases}. \tag{3.8}$$

Proposition 3.11. Necessary Conditions for Differentiability

Let $f : A \to \mathbb{C}$ be differentiable at $z_0 \in A$. Then the real and imaginary parts of f satisfy the Cauchy-Riemann equations at z_0.

(3.5) Sufficient Conditions for Differentiability

Now that Proposition 1.11 provides necessary conditions for differentiability, it is natural to ask which condition do we have to add in order to obtain sufficient conditions. First note that the conditions listed in Proposition 1.11 is not sufficient. For instance, $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

is such that its real and imaginary parts have partials and satisfy the Cauchy-Riemann equations at 0 but not differentiable at 0.

Proposition 3.12. Sufficient Conditions for Differentiability

Let $f : A \to \mathbb{C}$ and let $z_0 = x_0 + iy_0 \in A^\circ$. If there is a neighborhood $U \subseteq A$ of z_0 such that the real and imaginary parts $u, v : A \to \mathbb{R}$ have partials on U, C^1 and satisfies the Cauchy-Riemann equations at z_0, then f is differentiable at z_0. In particular,

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0). \tag{3.9}$$

Proof. We desire to show that

$$\lim_{\Delta z} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists. First note that

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{u(z_0 + \Delta z) - u(z_0) + i(v(z_0 + \Delta z) - v(z_0))}{\Delta z}. \tag{3.10}$$

Moreover, since u, v are C^1 at (x_0, y_0), we have linear approximations

$$\begin{aligned} L_u(x, y) &= u(x_0, y_0) + \langle \nabla u(x, y), \Delta z \rangle \\ L_v(x, y) &= v(x_0, y_0) + \langle \nabla v(x, y), \Delta z \rangle \end{aligned}$$

for all $(x, y) \in U$, with a property that

$$\lim_{\Delta z \to 0} \frac{u(x_0, y_0) - L_u(x, y)}{\|\Delta z\|} = \lim_{\Delta z \to 0} \frac{v(x_0, y_0) - L_v(x, y)}{\|\Delta z\|} = 0. \tag{3.11}$$

For convenience, define $R_u, R_v : U \to \mathbb{R}$ by

$$\begin{aligned} R_u(x, y) &= u(x_0, y_0) - L_u(x, y) \\ R_v(x, y) &= v(x_0, y_0) - L_v(x, y) \end{aligned}, \tag{3.12}$$

then

$$\lim_{\Delta z\to 0}\frac{R_u(x,y)}{\|\Delta z\|}=\lim_{\Delta z\to 0}\frac{R_v(x,y)}{\|\Delta z\|}\overset{[3.11]}{\underset{[3.12]}{=}}0. \quad [3.13]$$

This exactly means that

$$\lim_{\Delta z\to 0}\frac{R_u(x,y)}{\Delta z}=\lim_{\Delta z\to 0}\frac{R_v(x,y)}{\Delta z}\overset{[3.13]}{=}0. \quad [3.14]$$

But note that

$$u(z_0+\Delta z)-u(z_0)+i(v(z_0+\Delta z)-v(z_0))=(u_x(z_0)+iv_x(z_0))\Delta z+R_u(z_0+\Delta z)+iR_v(z_0+\Delta z) \quad [3.15]$$

after some rearrangement using the Cauchy-Riemann equations. Therefore,

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\overset{[3.10]}{\underset{[3.15]}{=}}u_x(z_0)+iv_x(z_0)+\frac{R_u(z_0+\Delta z)}{\Delta z}+i\frac{R_v(z_0+\Delta z)}{\Delta z}, \quad [3.16]$$

where

$$\lim_{\Delta z\to 0}u_x(z_0)+iv_x(z_0)+\frac{R_u(z_0+\Delta z)}{\Delta z}+i\frac{R_v(z_0+\Delta z)}{\Delta z}\overset{[3.14]}{=}u_x(z_0)+iv_x(z_0). \quad [3.17]$$

Hence

$$\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\overset{[3.16]}{\underset{[3.17]}{=}}u_x(z_0)+iv_x(z_0), \quad [3.18]$$

so f is differentiable at z_0 with

$$f'(z_0)\overset{[3.18]}{=}u_x(z_0)+iv_x(z_0). \quad [3.19]$$

Thus by using the Cauchy-Riemann equations,

$$f'(z_0)=u_x(z_0)+iv_x(z_0)=v_y(0)-iu_y(z_0),$$

as desired. ∎

(3.6) Analytic Functions

Observe that, given any open $U\subseteq A$, we have

$$f \text{ is differentiable on } U \iff f \text{ is analytic on } U.$$

Moreover, we showed in (3.4) that

$$f \text{ is analytic on } U \implies \text{the real and imaginary parts of } f \text{ satisfy the CRE on } U.$$

The last implication allows us to derive several important properties of analytic functions.

Proposition 3.13. *Let $f,g:A\to\mathbb{C}$ be analytic.*

(a) f is constant on A if and only if $f'=0$ on A.

(b) If $f'=g'$ on A, then $f=g+c$ for some $c\in\mathbb{C}$ on A.

(c) Let $u,v:A\to\mathbb{R}$ be the real and imaginary parts of f, respectively. Then u is constant on A if and only if v is constant on A.

(d) If $|f|$ is constant on A, then f is constant on A.

Proof.

(a) The reverse direction is clear. By the analyticity of f, we have

$$f(x+iy) = u_x(x,y) + iv_x(x,y) = v_y(x,y) - u_y(x,y).$$

This means

$$\forall x+iy \in A\,[f(x+iy)=0] \iff \forall (x,y) \in A\,[\nabla u(x,y) = \nabla v(x,y) = 0].$$

So by the connectedness of A, u, v are constant on A. Thus $f = u+iv$ is also constant on A. ◁

(b) This follows immediately from (a). ◁

(c) Assume that u is constant on A. Then $\nabla u = 0$ on A, so by the CRE, $\nabla v = 0$ on A. The reverse direction can be proven in a similar manner. ◁

(d) Let $u, v : \Omega \to \mathbb{R}$ be the real and imaginary parts of f, respectively. Then observe that

$$\begin{aligned}
|f| \text{ is constant on } \Omega &\iff \sqrt{u^2+v^2} \text{ is constant on } \Omega \\
&\iff u^2+v^2 \text{ is constant on } \Omega \\
&\iff \nabla\left(u^2+v^2\right) = 0 \text{ on } \Omega \\
&\iff 2u\nabla u + 2v\nabla v = 0 \text{ on } \Omega \\
&\iff u\nabla u + v\nabla v = 0 \text{ on } \Omega \\
&\iff uu_x + vv_x = 0 \text{ and } uu_y + vv_y = 0 \text{ on } \Omega. \qquad [3.20]
\end{aligned}$$

But f is analytic, so u, v satisfy the Cauchy-Riemann equations (see [3.8]), which means

$$\begin{aligned}
|f| \text{ is constant on } \Omega &\overset{[3.20]}{\iff} uu_x + vv_x = 0 \text{ and } uu_y + vv_y = 0 \text{ on } \Omega \\
&\iff uu_x + vu_x = 0 \text{ and } uu_y - vu_y = 0 \text{ on } \Omega \\
&\iff (u+v)u_x = 0 \text{ and } (u-v)u_y = 0 \text{ on } \Omega. \qquad [3.21]
\end{aligned}$$

[3.21] means we have four cases:

(i) $u+v=0$ and $u-v=0$. This means $2u = (u+v)+(u-v) = 0$, so $u=0$, which also means $v=0$. Therefore, $f = u+iv = 0$, a constant function.

(ii) $u+v=0$ and $u_y=0$. By taking the gradient of $u+v$, we obtain

$$u_x + v_x = u_y + v_y = 0. \qquad [3.22]$$

Combining [3.22] with the Cauchy-Riemann equations (see [3.8]), we obtain that

$$u_x - u_y = u_y + u_x. \qquad [3.23]$$

But we assumed that $u_y = 0$, so [3.23] implies $u_x = 0$ as well. It follows by the Cauchy-Riemann equations that $v_x = v_y = 0$, so u, v are constant on Ω by the connectedness of Ω. Thus $f = u+iv$ is also constant on Ω.

(iii) This case can be proven similarly to (ii).

(iv) $u_x = 0$ and $u_y = 0$. Then it immediately follows from the Cauchy-Riemann equations that $v_x = v_y = 0$, so u, v are constant on Ω by the connectedness of Ω. Thus $f = u+iv$ is also constant on Ω. ∎

Elementary Complex Functions

(3.7) Elementary Complex Functions

We begin to discuss about elementary complex functions other than polynomials: exponential, trigonometric, logarithmic, hyperbolic, and power functions.

(3.8) Exponential Function

Consider the function $z : \mathbb{R} \to \mathbb{C}$ by

$$z(\theta) = \cos(\theta) + i\sin(\theta)$$

for all $\theta \in \mathbb{R}$. Then given any $\theta_1, \theta_2 \in \mathbb{R}$,

$$\begin{aligned} z(\theta_1)\, z(\theta_2) &= (\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) \\ &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) && \text{by trigonometric identities} \\ &= z(\theta_1 + \theta_2) \end{aligned}$$

and, since $z \neq 0$ on $\mathbb{R}$,

$$\frac{1}{z(\theta_1)} = \frac{1}{\cos(\theta_1) + i\sin(\theta_1)} = \cos(-\theta_1) + i\sin(-\theta_1) = z(-\theta_1), \tag{3.24}$$

suggesting that z behaves like an exponential, motivating the following definition.

Notation 3.6. **Euler's Formula**
For every $\theta \in \mathbb{R}$, we write

$$e^{i\theta} = \cos(\theta) + i\sin(\theta). \tag{3.25}$$

Observe that $e^{i\theta}$ share many properties with the real exponential function:

- $e^{i0} = 1$.
- $\left(e^{i\theta}\right)^k = e^{ik\theta}$ for all $k \in \mathbb{R}$.

We also have the following formula: given any $n \in \mathbb{Z}$,

de Moivre's Formula

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta). \tag{3.26}$$

Def'n 3.7. **Exponential Function**
We define the (complex) ***exponential function*** $e^z : \mathbb{C} \to \mathbb{C}$ by

$$e^z = e^{\mathrm{Re}(z)} e^{\mathrm{Im}(z)}$$

for all $z \in \mathbb{C}$.

Note that, by Euler's formula,

$$e^z = e^{\mathrm{Re}(z)} (\cos(\mathrm{Im}(z)) + i\sin(z))$$

for all $z \in \mathbb{C}$. Observe that

(a) e^z is entire with $(e^z)' = e^z$;

(b) $\arg(e^z) = \mathrm{Im}(z) + 2k\pi$, $k \in \mathbb{Z}$; and

(c) $e^{\mathbb{C}} = \mathbb{C} \setminus \{0\}$.

Periodicity is a property that the complex exponential function differ from the real one.

4. Complex Integration

Complex-valued Functions of a Real Variable

(4.1)
Motivation

We desire to find a meaningful way of integrating complex functions. Suppose that $f:[a,b]\to\mathbb{C}$ is a complex-valued function of one *real* variable so that

$$f(t)=u(t)+iv(t)$$

for all $t\in[a,b]$ with $u,v:[a,b]\to\mathbb{R}$, then the simplist way to achieve our goal is the following: define $\int_a^b f(t)\text{ dt}$ by

$$\int_a^b f(t)\text{ dt}=\int_a^b u(t)\text{ dt}+i\int_a^b v(t)\text{ dt},$$

provided that the integrals in the RHS exist. We start by deriving a few more properties of f by considering f as a function of a complex variable (this is possible since $\mathbb{R}$ is (isomorphic to) a proper subset of $\mathbb{C}$). For convenience, fix $t_0\in[a,b]$. Note that, we have

$$\lim_{t\to t_0} f(t)=u_0+iv_0$$

for some $u_0,v_0\in\mathbb{R}$ if and only if

$$\begin{array}{l}\lim_{t\to t_0} u(t)=u_0\\ \lim_{t\to t_0} v(t)=v_0\end{array}.$$

In other words

$$\lim_{t\to t_0} f(t)=\lim_{t\to t_0} u(t)+i\lim_{t\to t_0} v(t) \qquad [4.1]$$

proivded that the limits exist. A direct consequence of [4.1] is that

$$f\text{ is continuous at }t_0\iff u,v\text{ are continuous at }t_0.$$

Moreover, the derivative $f'(t_0)$ of f at t_0 is given by

$$f'(t_0)=\lim_{t\to t_0}\frac{f(t)-f(t_0)}{t-t_0}$$

provided that the limit exists. Note that this is the case if and only if

$$u'(t_0)=\lim_{t\to t_0}\frac{u(t)-u(t_0)}{t-t_0},v'(t_0)=\lim_{t\to t_0}\frac{v(t)-v(t_0)}{t-t_0}$$

both exist, in which case

$$f'(t_0)=u'(t_0)+iv'(t_0).$$

(4.2)
Antiderivative of f

We define the ***antiderivative*** of f to be a differentiable function $F:[a,b]\to\mathbb{C}$ such that $F'=f$ on $[a,b]$. Note that, if $F=U+iV$ for some $U,V:[a,b]\to\mathbb{R}$, then U,V are antiderivatives of u,v, respectively. As seen before, we also define the ***definite integral*** of f on $[a,b]$ by

$$\int_a^b f(t)\text{ dt}=\int_a^b u(t)\text{ dt}+i\int_a^b v(t)\text{ dt}$$

provided that the integrals in the RHS exists. The following properties follow immediately.

Proposition 4.1.

(a) If $c \in \mathbb{C}$ and $f, g : [a,b] \to \mathbb{C}$ are integrable, then

$$\int_a^b cf(t) + g(t)\,\mathrm{d}t = c\int_a^b f(t)\,\mathrm{d}t + \int_a^b g(t)\,\mathrm{d}t.$$

(b) $\int_b^a f(t)\,\mathrm{d}t = -\int_a^b f(t)\,\mathrm{d}t$.

(c) If f is continuous on $[a,b]$, then $\int_a^b f(t)\,\mathrm{d}t$ exists.

(d) If F is an antiderivative of f on $[a,b]$, then $\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a)$.

(4.3) Parameterized Curves in $\mathbb{C}$

Let Γ be a smooth curve in $\mathbb{R}^2$, parameterized by some $\alpha : [a,b] \to \mathbb{R}^2$. By identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the usualy way, we can think Γ as a parameterized curve in $\mathbb{C}$. Indeed, let $z : [a,b] \to \mathbb{C}$ be defined by

$$z(t) = \alpha_1(t) + i\alpha_2(t)$$

for all $t \in [a,b]$, where α_1, α_2 are the component functions of α. Then z is a complex function of a real variable with real part α_1 and imaginary part α_2. This means, in particular,

$$z'(t) = \alpha_1'(t) + i\alpha_2'(t)$$

for all $t \in [a,b]$, since α_1, α_2 are C^1. Furthermore, since $\alpha'(t) \neq 0$ for all $t \in [a,b]$, we have that $z'(t) \neq 0$ for all $t \in [a,b]$. We also have that $|z'(t)| = |\alpha'(t)|$ for all $t \in [a,b]$, so

$$\int_a^b \left|z'(t)\right|\,\mathrm{d}t = \int_a^b \left|\alpha'(t)\right|\,\mathrm{d}t,$$

which is the arclength of Γ. This motivates the following definition.

Smooth Parameterization

Def'n 4.1. Let $\Gamma \subseteq \mathbb{C}$ and let $[a,b] \subseteq \mathbb{R}$. A smooth complex function $z : [a,b] \to \Gamma$ with real and complex parts $x, y : [a,b] \to \mathbb{R}$, respectively, is called a ***smooth parameterization*** of Γ if the following hold.

(a) $z([a,b]) = \Gamma$.

(b) $z'(t) = x'(t) + iy'(t) \neq 0$ for all $t \in [a,b]$.

(c) If $z(a) = z(b)$, then $z'(a) = z'(b)$.

One notes that Def'n 4.1 is equivalent to Def'n 1.12. That is, we can think of smooth parameterized curves in $\mathbb{R}^2$ and $\mathbb{C}$ interchangeably. Also note that piecewise smooth and simple parameterized curves in $\mathbb{C}$ are as defined as in $\mathbb{R}^2$.

Contour Integrals

Contour Integral of a Complex Function

Def'n 4.2. Let $f : \Omega \to \mathbb{C}$ be a continuous on a domain $\Omega \subseteq \mathbb{C}$ and let $z : [a,b] \to \Gamma \subseteq \Omega$ be a parameterization of Γ, a smooth curve in Ω. We define the ***contour integral*** of f on the curve Γ, denoted as $\int_\Gamma f(z)\,\mathrm{d}z$, to be

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_a^b f(z(t))\,z'(t)\,\mathrm{d}t.$$

Moreover, if Γ is piecewise smooth with smooth pieces $\Gamma_1, \ldots, \Gamma_m$, we define

$$\int_\Gamma f(z)\,\mathrm{d}z = \sum_{i=1}^m \int_{\Gamma_i} f(z)\,\mathrm{d}z.$$

Proposition 4.2. Properties of Contour Integrals

Let Γ be a piecewise smooth parameterized curve in a domain $\Omega \subseteq \mathbb{C}$ and let $f, g : \Omega \to \mathbb{C}$ be continuous.

(a) linearity: For all $\alpha \in \mathbb{C}$,

$$\int_\Gamma \alpha f(z) + g(z)\,\mathrm{d}z = \alpha \int_\Gamma f(z)\,\mathrm{d}z + \int_\Gamma g(z)\,\mathrm{d}z.$$

(b) reversal of orientation:

(4.4) Antiderivatives

Recall that the fundamental theorem of calculus states that if a real univariate function $f : [a,b] \to \mathbb{R}$ has an antiderivative $F : [a,b] \to \mathbb{R}$, then

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a).$$

In particular, $\int_a^b f(t)\,\mathrm{d}t$ only depends on the endpoints of $[a,b]$. We desire to find if an analogue of the fundamental theorem of calculus holds for complex functions.

Def'n 4.3. **Antiderivative** of a Complex Function
Let $f : \Omega \to \mathbb{C}$ be continuous, where $\Omega \subseteq \mathbb{C}$ is a domain. A function $F : \Omega \to \mathbb{C}$ is called an ***antiderivative*** of f if $F' = f$ on Ω.

It follows immediately from Def'n 4.3 that, if an antiderivative of f exists, then it is unique up to addition of a constant, since it is analytic.

Proposition 4.3. Independence of Path

Let $f : \Omega \to \mathbb{C}$, where $\Omega \subseteq \mathbb{C}$ is a domain, and let $z_1, z_2 \in \Omega$. Suppose that f has an antiderivative $F : \Omega \to \mathbb{C}$ on Ω. Then for every piecewise smooth Γ lying in Ω with initial point z_1 and terminal point z_2,

$$\int_\Gamma f(z)\,\mathrm{d}z = F(z_2) - F(z_1).$$

Proof. Withous loss of generality, assume that Γ is smooth. Then

$$\begin{aligned}\int_\Gamma f(z)\,\mathrm{d}z &= \int_a^b f(z(t))\,z'(t)\,\mathrm{d}t = \int_a^b F'(z(t))\,z'(t)\,\mathrm{d}t = \int_a^b (F \circ z)'(t)\,\mathrm{d}t \\ &\overset{\text{FTC}}{=} F(z(b)) - F(z(a)) = F(z_2) - F(z_1).\end{aligned}$$

∎

Corollary 4.3.1.

Consider the setting of Proposition 4.3. If Γ is closed, then

$$\int_\Gamma f(z)\,\mathrm{d}z = 0.$$

Proposition 4.3 guarantees that the contour integrals of f are independent of path, provided that f has an antiderivative. Proposition 4.4 shows that the converse is also true.

Proposition 4.4.

Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f : \Omega \to \mathbb{C}$ be continuous. Then the following are equivalent.

(a) f has an antiderivative on Ω.

(b) $\int_\Gamma f(z)\,\mathrm{d}z$ is independent of path for any piecewise smooth curve Γ in Ω.

(c) $\int_\Gamma f(z)\,\mathrm{d}z = 0$ for every closed curve Γ in Ω.

Cauchy's Theorems

(4.5)
Cauchy Integral Theorem

Let $\Omega \subseteq \mathbb{C}$ be a domain throughout.

Proposition 4.5.

Let $\Gamma \subseteq \mathbb{C}$ be a piecewise smooth curve and let $f : \Gamma \to \mathbb{C}$ be continuous. Suppose that $|f|$ is bounded on Γ, say $|f| \le M$ for some $M \ge 0$. Then

$$\left|\int_\Gamma f(z)\,\mathrm{d}z\right| \le \int_\Gamma |f(z)|\,\mathrm{d}z \le ML,$$

where L is the arclength of Γ and $|\mathrm{d}z|$ denotes $|z'(t)|\,\mathrm{d}t$ for any parameterization z of Γ.

There are several versions of the Cauchy integral theorem. We start with a more restrictive version. Suppose that $f : \Omega \to \mathbb{C}$ is analytic on and let $u, v : \Omega \to \mathbb{R}$. By the analyticity, the Cauchy-Riemann equation applies:

$$f'(z) = u_x(z) + iv_x(z) = v_y(z) - iu_y(z).$$

Hence we see that u, v are C^1 if and only if f is C^1.

Theorem 4.6.
Cauchy Integral Theorem

Let $f : \Omega \to \mathbb{C}$ be analytic and C^1. Then for every closed Jordan curve Γ in Ω whose interior is contained in Ω,

$$\oint_\Gamma f(z)\,\mathrm{d}z = 0.$$

(4.6)
Cauchy-Goursat Theorem

It turns out that we can relax the condition that f' is continuous in the statement of Cauchy integral theorem. This result is known as the Cauchy-Goursat theorem, which is central in the theory of complex integration.

Theorem 4.7.
Cauchy-Goursat Theorem

Let $f : \Omega \to \mathbb{C}$ be analytic. Then for any closed Jordan curve Γ in Ω whose interior is contained in Ω,

$$\oint_\Gamma f(z)\,\mathrm{d}z = 0.$$

Theorem 4.8.
Deformation Principle

Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f : \Omega \to \mathbb{C}$ be analytic. Let $\Gamma_1, \Gamma_2 \subseteq \Omega$ be similarly oriented piecewise smooth Jordan curves inside Ω such that Γ_2 lies in the interior of Γ_1 and f is analytic at all points between the curves. Then

$$\oint_{\Gamma_1} f(z)\,\mathrm{d}z = \oint_{\Gamma_2} f(z)\,\mathrm{d}z.$$

Corollary 4.8.1. Cauchy Integral Theorem for Jordan Domains

Let $\Omega \subseteq \mathbb{C}$ be a k-connected Jordan domain and let $f : \Omega + \to \mathbb{C}$ be analytic where Ω^+ is a domain containing $\overline{\Omega}$. Then

$$\int_{\partial\Omega} f(z)\,\mathrm{d}z = 0.$$

Corollary 4.8.2. Cauchy Integral Theorem for Simply Connected Domains

Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and let $f : \Omega \to \mathbb{C}$ be analytic. Then

$$\int_{\Gamma} f(z)\,\mathrm{d}z = 0$$

for every closed curve Γ in Ω.

Corollary 4.8.3.

Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain and let $f : \Omega \to \mathbb{C}$ be analytic. Then

(a) f has an antiderivative throughout Ω;

(b) $\int_\Gamma f(z)\,\mathrm{d}z$ is path independent for any contour Γ in Ω; and

(c) $\oint_\Gamma f(z)\,\mathrm{d}z = 0$ for any loop Γ in Ω.

Theorem 4.9. Cauchy's Integral Formula

Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f : \Omega \to \mathbb{C}$ be analytic. Let Γ be a positively oriented Jordan curve inside Ω whose interior is contained in Ω. If z_0 is any point interior to Γ, then

$$f(z_0) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{z - z_0}\,\mathrm{d}z. \tag{4.2}$$

Theorem 4.10. Analtyic Functions Are Smooth

Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f : \Omega \to \mathbb{C}$ be analytic. Then derivatives of f of all order exist in Ω. Moreover, for each $z_0 \in \Omega$, we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z. \tag{4.3}$$

Corollary 4.10.1.

Let $\Omega \subseteq \mathbb{C}$ be a domain and let $f : \Omega \to \mathbb{C}$ be analytic. Let $u, v : \Omega \to \mathbb{R}$ be the real and imaginary parts of f, respectively. Then u, v are C^∞.

5.
Complex Series

Complex Sequences

(5.1) Sequences

We begin by stating the complex analog of several definitions and basic theorems for real sequences.

Def'n 5.1. **Bounded** Sequence

We say a complex sequence $(z_n)_{n=1}^{\infty}$ is ***bounded*** if there exists $M > 0$ such that $|z_n| < M$ for all $n \in \mathbb{N}$.

Def'n 5.2. **Convergent, Divergent** Sequence

We say a sequence $(z_n)_{n=1}^{\infty}$ is

- ***convergent*** if there exists $z \in \mathbb{C}$ such that

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N \left[|z_n - z| < \varepsilon\right], \qquad [5.1]$$

  and we say $(z_n)_{n=1}^{\infty}$ ***converges*** to z; and
- ***divergent*** otherwise.

The following result is immediate from multivariate calculus:

Proposition 5.1. *Let $(z_n)_{n=1}^{\infty}$ be a complex sequence and let $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}$ be the real and imaginary parts of $(z_n)_{n=1}^{\infty}$. Then $(z_n)_{n=1}^{\infty}$ converges to $z \in \mathbb{C}$ if and only if $(x_n)_{n=1}^{\infty}$ converges to $x \in \mathbb{R}$, $(y_n)_{n=1}^{\infty}$ converges to $y \in \mathbb{R}$, and $z = x + iy$.*

Complex Series

Def'n 5.3. **Convergent** Series

We say a series $\sum_{n=1}^{\infty} z_n$, where $(z_n)_{n=1}^{\infty}$ is a complex sequence, is ***convergent*** if the sequence $(S_N)_{N=1}^{\infty}$ of partial sums,

$$S_N = \sum_{n=1}^{N} z_n,$$

is convergent. We write $\sum_{n=1}^{\infty} z_n = S$ and say $\sum_{n=1}^{\infty} z_n$ ***converges*** to S if $(S_N)_N^{\infty}$ converges to S. Otherwise, we say $\sum_{n=1}^{\infty} z_n$ is ***divergent***.

(5.2) Complex Series

We have the following results.

Proposition 5.2. *Let $\sum_{n=1}^{\infty} z_n$ be a complex series and let $x_n, y_n \in \mathbb{R}$ be such that $z_n = x_n + iy_n$ for all $n \in \mathbb{N}$.*

(a) $\sum_{n=1}^{\infty} z_n = x + iy$ $(x, y \in \mathbb{R})$ if and only if $\sum_{n=1}^{\infty} x_n = x$ and $\sum_{n=1}^{\infty} y_n = y$.

(b) divergence test: If $\lim_{n\to\infty} z_n \neq 0$, then $\sum_{n=1}^{\infty} z_n$ diverges.

(c) absolute convergence test: If $\sum_{n=1}^{\infty} |z_n|$ is convergent, then $\sum_{n=1}^{\infty} z_n$ is convergent.

Proof.

(a) This is clear from multivariate calculus. ◁

(b) Observe that

$$\begin{aligned}\lim_{n\to\infty} z_n \neq 0 &\iff \lim_{n\to\infty} x_n \neq 0 \text{ or } \lim_{n\to\infty} y_n \neq 0 \\ &\implies (x_n)_{n=1}^{\infty} \text{ diverges or } (y_n)_{n=1}^{\infty} \text{ diverges} \\ &\iff (z_n)_{n=1}^{\infty} \text{ diverges.}\end{aligned}$$

◁

(c) Assume that $\sum_{n=1}^{\infty} |z_n|$ converges. Since

$$|z_n| = \sqrt{x_n^2 + y_n^2} \geq \max(|x_n|, |y_n|)$$

for all $n \in \mathbb{N}$, by the monotone convergence theorem, $\sum_{n=1}^{\infty} |x_n|, \sum_{n=1}^{\infty} |y_n|$ converge. So by the absolute convergence test for reals, $\sum_{n=1}^{\infty} x_n, \sum_{n=1}^{\infty} y_n$ converge. Thus $\sum_{n=1}^{\infty} z_n$ converges. ∎

Proposition 5.3.
Geometric Series Test

Let $z \in \mathbb{C}$. Then

$$\sum_{n=0}^{\infty} b^n$$

converges if and only if $|b| < 1$.

Proof.

- ($\implies$) Assume that $\sum_{n=0}^{\infty} b^n$ converges. Then $\lim_{n\to\infty} b^n = 0$ by the divergence test, which means $|b| < 1$. ◁
- ($\impliedby$) Assume that $|b| < 1$. Then by the geometric test for reals, $\sum_{n=1}^{\infty} |b^n| = \sum_{n=1}^{\infty} |b|^n$ converges. Thus by the absolute convergence test, $\sum_{n=1}^{\infty} b^n$ converges. ∎

Proposition 5.4.
Ratio Test

Let $\sum_{n=1}^{\infty} z_n$ be a complex series and suppose that $\left(\left|\frac{z_{n+1}}{z_n}\right|\right)_{n=1}^{\infty}$ is convergent and let

$$L = \lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right|.$$

We have the following:

- *if $L < 1$, then $\sum_{n=1}^{\infty} z_n$ converges; and*
- *if $L > 1$, then $\sum_{n=1}^{\infty} z_n$ diverges.*

Note that the ratio test is inconclusive when $L = 1$.

Proof of Proposition 5.5.

(a) If $L > 1$, then there exists $N \in \mathbb{N}$ such that $|z_{n+1}| > |z_n|$ for all $n > N$, which means $\lim_{n\to\infty} |z_n| \neq 0$. This means $\lim_{n\to\infty} z_n \neq 0$, so by the divergence test $\sum_{n=1}^{\infty} z_n$ diverges. ◁

(b) If $L < 1$, then $\sum_{n=1}^{\infty} |z_n|$ converges by the real ratio test, so by the absolute convergence test $\sum_{n=1}^{\infty} z_n$ converges. ∎

Proposition 5.5. Comparison Test

Let $\sum_{n=1}^{\infty} z_n$ be a complex series and suppose that there exists $n_0 \in \mathbb{N}$ and a real sequence $(r_n)_{n=1}^{\infty}$ such that

$$|z_n| \leq r_n$$

for all $n \geq n_0$. If $\sum_{n=1}^{\infty} r_n$ converges, then $\sum_{n=1}^{\infty} z_n$ converges.

Note that the comparison test for complex series is more restrictive than the real version.

Proof of Proposition 5.6. Let $(z_n)_{n=1}^{\infty}$ be a complex sequence and suppose that there exists a real sequence $(r_n)_{n=1}^{\infty}$ such that

$$|z_n| < r_n$$

for all $n \in \mathbb{N}$ and that $\sum_{n=1}^{\infty} r_n$ converges. Then by the monotone convergence theorem, $\sum_{n=1}^{\infty} |z_n|$ converges. Thus by the absolute convergence test, $\sum_{n=1}^{\infty} z_n$ converges. ∎

Theorem 5.6. Cauchy Convergence Criterion

Let $\sum_{n=1}^{\infty} z_n$ be a complex series. Then the series is convergent if and only if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N \forall p \in \mathbb{N} \left[\left| \sum_{i=1}^{p} z_{n+i} \right| < \varepsilon \right]. \qquad [5.2]$$

Proof. Let $(Z_N)_{N=1}^{\infty}$ be the sequence of partial sums: for all $N \in \mathbb{N}$,

$$Z_N = \sum_{n=1}^{N} z_n.$$

Then observe that

$$\begin{aligned}
\sum_{n=1}^{\infty} z_n \text{ converges} &\iff (Z_N)_{N=1}^{\infty} \text{ converges} \\
&\iff (Z_N)_{N=1}^{\infty} \text{ is Cauchy} \\
&\iff \forall \varepsilon > 0 \exists N_0 \in \mathbb{N} \forall N, M > N_0 \left[|Z_N - Z_M| < \varepsilon \right] \\
&\iff \forall \varepsilon > 0 \exists N_0 \in \mathbb{N} \forall N, M > N_0 \left[\left| \sum_{n=\min(N,M)}^{\max(N,M)} z_n \right| < \varepsilon \right] \\
&\iff \forall \varepsilon > 0 \exists N_0 \in \mathbb{N} \forall n > N_0 \forall p \in \mathbb{N} \left[\left| \sum_{i=1}^{p} z_{n+i} \right| < \varepsilon \right].
\end{aligned}$$

∎

Power Series

Def'n 5.4. **Power Series** about a Point

Let $z_0 \in \mathbb{C}$. A ***power series*** about z_0 is a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for some complex sequence $(z_n)_{n=0}^{\infty}$. z_0 is called the ***center*** of $\sum_{n=0}^{\infty} a_n (z - z_0)^n$.

Proposition 5.7.

Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series and suppose that $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists (on the extended number line). Let $L \in [0, \infty]$.

(a) If $L = \infty$, then the power series only converges at z_0.

(b) If $L = 0$, then the power series converges for all $z \in \mathbb{C}$.

(c) Otherwise, the power series converges for all $z \in D\left(z_0; \frac{1}{L}\right)$ and diverges for all $\mathbb{C} \setminus \overline{D}\left(z_0; \frac{1}{L}\right)$.

(5.3) $R = \frac{1}{L}$ of Proposition 5.7 is called the ***radius*** of convergence and $D\left(z_0; \frac{1}{L}\right)$ is called the ***disc*** of convergence.

Proof of Proposition 5.7.

(a) Clearly the sequence converges at z_0. When $z \neq z_0$, we have $\frac{a_{n+1}(z-z_0)^{n+1}}{a_n(z-z_0)^n} = \frac{a_{n+1}}{a_n}(z - z_0)$, so

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(z-z_0)^{n+1}}{a_n(z-z_0)^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n}(z-z_0) \right| = L|z-z_0| = \infty.$$

Thus by the ratio test $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ diverges. ◁

(b) Let $z \in \mathbb{C}$. Then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(z-z_0)^{n+1}}{a_n(z-z_0)^n} \right| = L|z-z_0| = 0,$$

so by the ration test $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges. ◁

(c) Let $z \in D\left(z_0; \frac{1}{L}\right)$. Then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(z-z_0)^{n+1}}{a_n(z-z_0)^n} \right| = L|z-z_0| < L\frac{1}{L} = 1,$$

so by the ratio test $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges. On the other hand, if $z \in \mathbb{C} \setminus \overline{D}\left(z_0; \frac{1}{L}\right)$, then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(z-z_0)^{n+1}}{a_n(z-z_0)^n} \right| = L|z-z_0| > L\frac{1}{L} = 1,$$

so $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ diverges by the ratio test. ∎

Def'n 5.5. **Convergence** of Sequence of Functions

Let $(f_n)_{n=0}^{\infty}$ be a sequence of functions $f_1, \ldots : \Omega \to \mathbb{C}$ for some domain $\Omega \subseteq \mathbb{C}$. We say $(f_n)_{n=0}^{\infty}$ ***converges*** on Ω if $\lim_{n\to\infty} f_n(z)$ exists for all $z \in \Omega$. In case $(f_n)_{n=0}^{\infty}$ is convergent, we denote $f : \Omega \to \mathbb{C}$ defined by

$$f(z) = \lim_{n\to\infty} f_n(z)$$

for all $z \in \Omega$ as $f = \lim_{n\to\infty} f_n$.

Def'n 5.6. **Uniform Convergence** of Sequence of Functions

Let $(f_n)_{n=0}^{\infty}$ be a sequence of functions $f_0, \ldots : \Omega \to \mathbb{C}$ for some domain $\Omega \subseteq \mathbb{C}$. We say $(f_n)_{n=0}^{\infty}$ ***converges uniformly*** to a function $f : S \to \mathbb{C}$ on $S \subseteq \Omega$ if

$$\forall \varepsilon > 0 \exists N_\varepsilon \in \mathbb{N} \forall n \geq N_\varepsilon \forall z \in S \left[|f_n(z) - f(z)| < \varepsilon \right].$$

In this case, f is called the ***uniform limit*** of $(f_n)_{n=0}^{\infty}$.

Proposition 5.8. *Let $(f_n)_{n=0}^{\infty}$ be a sequence of complex functions defined on a domain $\Omega \subseteq \mathbb{C}$.*

(a) If $(f_n)_{n=0}^{\infty}$ converges uniformly and each f_n is continuous, then the uniform limit of $(f_n)_{n=0}^{\infty}$ is continuous.

(b) If $\Gamma \subseteq \mathbb{C}$ is a curve such that $(f_n)_{n=0}^{\infty}$ converges uniformly on Γ and each f_n is continuous on Γ, then

$$\lim_{n\to\infty} \int_\Gamma f_n(z)\,\mathrm{d}z = \int_\Gamma \lim_{n\to\infty} f_n(z)\,\mathrm{d}z.$$

Proof.

(a) Let $f : \Omega \to \mathbb{C}$ be the uniform limit of $(f_n)_{n=0}^{\infty}$ on Ω, l;et $z_0 \in \Omega$, and let $\varepsilon > 0$. Then it suffices to show that

$$\exists \delta > 0 \forall z \in D(z_0;\delta) \cap \Omega \left[|f(z_0) - f(z)| < \varepsilon\right]. \tag{5.3}$$

By the uniform convergence of $(f_n)_{n=0}^{\infty}$, there exists $m \in \mathbb{N}$ such that

$$|f_m(z) - f(z)| < \frac{\varepsilon}{3}$$

for all $z \in \Omega$. Moreover, by the continuity of f_m, there exists $\delta > 0$ such that

$$|f_m(z) - f_m(z_0)| < \frac{\varepsilon}{3}$$

for all $z \in \Omega \cap D(z_0;\delta)$. This means

$$|f(z_0) - f(z)| \le |f(z_0) - f_m(z_0)| + |f_m(z_0) - f_m(z)| + |f_m(z) - f(z)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

for all $z \in \Omega \cap D(z_0;\delta)$, so [5.3] is established. ◁

(b) We may assume that Γ is not singleton. It suffices to show that

$$\lim_{n\to\infty} \left| \int_\Gamma f_n(z) - f(z)\,\mathrm{d}z \right| = 0, \tag{5.4}$$

where $f : \Gamma \to \mathbb{C}$ is the uniform limit of $(f_n)_{n=0}^{\infty}$ on Γ. Let $L > 0$ be the arclength of Γ. Then by the unifom convergence of $(f_n)_{n=0}^{\infty}$,

$$\forall \varepsilon > 0 \exists N_\varepsilon \in \mathbb{N} \forall n \ge N_\varepsilon \forall z \in \Gamma \left[|f_n(z) - f(z)| < \frac{\varepsilon}{L}\right]. \tag{5.5}$$

SO by the ML-inequality,

$$\left| \int_\Gamma f_m(z) - f(z)\,\mathrm{d}z \right| < \frac{\varepsilon}{L} L = \varepsilon$$

for all $n \ge N_\varepsilon$. Thus [5.4] is established. ∎

Theorem 5.9. *Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a series convergent on $D(z_0;R) \subseteq \mathbb{C}$ ($z_0 \in \mathbb{C}, R > 0$) and let $f : D(z_0;R) \to \mathbb{C}$ be defined by*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in D(z_0;R)$. Then for each $r < R$, the sequence $(f_N)_{N=0}^{\infty}$ defined by

$$f_N = \sum_{n=0}^{N} a_n (z - z_0)^n$$

for all $N \in \mathbb{N}, z \in \overline{D}(z_0; r)$ converges uniformly to $f|_{\overline{D}(z_0;r)}$.

Proof. We first establish two inequalities that we are going to utilize in proving the result.

- Let $r \in (0, R)$. Then for every $n \in \mathbb{N} \cup \{0\}, p \in \mathbb{N}, z \in \overline{D}(z_0; r)$,

$$\left|\sum_{j=1}^{p} a_{n+j} (z - z_0)^{n+j}\right| < M \sum_{j=1}^{p} \rho^{n+j} \tag{5.6}$$

  for some $\rho \in (0, 1), M > 0$.

  Proof. Let $r \in (0, R)$. Then we may choose $\zeta \in D(z_0; R) \setminus \overline{D}(z_0; r)$. Since ζ is a point in the disc of convergence, the series $\sum_{n=0}^{\infty} a_n (\zeta - z_0)^n$ is convergent. So by the divergence test,

$$\lim_{n\to\infty} a_n (\zeta - z_0)^n = 0,$$

  which means the sequence $(a_n (\zeta - z_0)^n)_{n=0}^{\infty}$ is bounded. So fix $M > 0$ such that

$$\forall n \in \mathbb{N} \cup \{0\} \left[|a_n (\zeta - z_0)^n| < M\right].$$

  Now, fix $z \in \overline{D}(z_0; r)$, with respect to which we are going to show [5.6]. Since $|z - z_0| \leq r < |\zeta - z_0|$,

$$|a_n (z - z_0)^n| = \underbrace{|a_n (\zeta - z_0)^n|}_{<M} \left|\frac{(z - z_0)^n}{(\zeta - z_0)^n}\right| < M\rho^n$$

  for all $n \in \mathbb{N} \cup \{0\}$, where we are taking $\rho = \frac{r}{|\zeta - z_0|}$. This ρ is positive since both r, $|\zeta - z_0|$ are positive and $\rho < 1$ since $r < |\zeta - z_0|$. Then by simply summing over indices, we obtain [5.6]. ◁

- The series

$$\sum_{n=0}^{\infty} \rho^n$$

  is convergent by the geometric series test. Therefore, by the Cauchy convergence criterion,

$$\forall \varepsilon > 0 \exists N_\varepsilon \in \mathbb{N} \forall n \geq N_\varepsilon \forall p \in \mathbb{N} \left[\left|\sum_{j=1}^{p} \rho^{n+j}\right| < \varepsilon\right]. \tag{5.7}$$

Now, suppose that $\varepsilon > 0$ is given. It suffices to show that there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\forall N \geq N_\varepsilon \forall z \in \overline{D}(z_0; r) \left[|f_N(z) - f(z)| < \varepsilon\right].$$

By [5.7], we may choose $N_\varepsilon \in \mathbb{N}$ such that

$$\forall N \geq N_\varepsilon \forall p \in \mathbb{N} \left[\left|\sum_{j=1}^{p} \rho^{N+j}\right| < \frac{\varepsilon}{M}\right],$$

since M is a positive constant. Then we have

$$\forall N \geq N_\varepsilon \forall p \in \mathbb{N} \forall z \in \overline{D}(z_0; r) \left[\left|\sum_{j=1}^{p} a_{N+j} (z - z_0)^{N+j}\right| < \varepsilon\right]$$

by [5.6]. So by the Cauchy convergence criterion,

$$\forall N \geq N_\varepsilon \forall z \in \overline{D}(z_0; r) \left[|f_N(z) - f(z)| < \varepsilon\right],$$

as desired. ∎

Corollary 5.9.1.

Consider the setting of Theorem 5.9. f is continuous.

Theorem 5.10. *Let $(f_n)_{n=0}^{\infty}$ be a sequence of analytic $f_0, \ldots : \Omega \to \mathbb{C}$, where $\Omega \subseteq \mathbb{C}$ is a domain. If $(f_n)_{n=0}^{\infty}$ converges uniformly to $f : \Omega \to \mathbb{C}$, then f is analytic and $(f_n')_{n=0}^{\infty}$ converges uniformly to f' on Ω.*

Corollary 5.10.1. *Every power series defines an analytic function on its disc of convergence.*

Proposition 5.11. *Let $(f_n)_{n=0}^{\infty}$ be a sequence of analytic $f_0, \ldots : \Omega \to \mathbb{C}$ for some domain $\Omega \subseteq \mathbb{C}$ and suppose that the sequence $(S_N)_{N=0}^{\infty}$ of partial sums defined by*

$$S_N = \sum_{n=0}^{N} f_n$$

for all $N \in \mathbb{N} \cup \{0\}$ converges uniformly to $f : S \subseteq \Omega$ on some subset $S \subseteq \mathbb{C}$. Then

(a) term-by-term differentiation: $f' = \sum_{n=0}^{\infty} f_n'$; and

(b) term-by-term integration: $\int_\Gamma f(z)\,\mathrm{d}z = \sum_{n=0}^{\infty} \int_\Gamma f_n(z)\,\mathrm{d}z$ for any curve $\Gamma \subseteq S$.

Theorem 5.12. *Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series with the disc of convergence $D(z_0; r)$ and let $f : D(z_0; r) \to \mathbb{C}$ by*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in \mathbb{C}$.

(a) For all $z \in D(z_0; r)$, $k \geq 0$,

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n (z - z_0)^{n-k}.$$

In particular,

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

for all $n \geq 0$.

(b) For any curve $\Gamma \subseteq D(z_0; r)$, we have

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_{n=0}^{\infty} a_n \int_\Gamma (z - z_0)^n\,\mathrm{d}z.$$

Theorem 5.13.
Taylor's Theorem

Let $z_0 \in \mathbb{C}, r > 0$ and let $f : D(z_0; r) \to \mathbb{C}$ be analytic. Then f has a power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

for all $n \geq 0$.

Def'n 5.7. **Taylor Series, Taylor Polynomial** about a Point

Let $z_0 \in \mathbb{C}, r > 0$ and let $f : D(z_0; r) \to \mathbb{C}$ be analytic.

(a) The ***Taylor series*** of f about z_0 is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (z - z_0)^n .$$

(b) The ***Taylor polynomial*** of degree $N \in \mathbb{N} \cup \{0\}$ about z_0 is the partial sum

$$\sum_{n=0}^{N} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n .$$

(c) The ***remainder*** of the Taylor polynomial of degree $N \in \mathbb{N} \cup \{0\}$ is

$$\frac{1}{2\pi i} \oint_{\partial D(z_0; r')} \frac{f(s)}{(s - (z - z_0))} \left(\frac{z}{s}\right)^{N+1} \mathrm{ds},$$

where $0 < r' < r$.

(EX 5.4) We have the following Taylor series.

(a) $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ for all $z \in D(0; 1)$.

(b) $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ for all $z \in \mathbb{C}$.

(c) $\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{2n!}$ for all $z \in \mathbb{C}$.

(d) $\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$ for all $z \in \mathbb{C}$.

(e) $\mathrm{Log}(1 - z) = \sum_{n=0}^{\infty} -\frac{z^{n+1}}{n+1}$ for all $z \in D(0; 1)$.

Identity Theorem

Proposition 5.14. Zeroes of Nonconstant Analytic Functions Are Isolated

Let $f : \Omega \to \mathbb{C}$ be analytic, where $\Omega \subseteq \mathbb{C}$ is a domain, and suppose that there exists $z_0 \in \Omega$ such that $f(z_0) = 0$. Then there exists a disc $D \subseteq \Omega$ centered at z_0 such that one of the following happens:

(a) $f(z) = 0$ for all $z \in D$.

(b) There exist $n_0 \in \mathbb{N}$ and nowhere vanishing analytic $g : D \to \mathbb{C}$ such that

$$f(z) = (z - z_0)^{n_0} g(z)$$

for all $z \in D$.

Proof. Clearly (a), (b) exclude each other. To show that one of them must be true, suppose that (a) is false. Since f is analytic at z_0, it has a Taylor series expansion about z_0 given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

on some disc $D(z_0;R)$, with

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

for all $n \in \mathbb{N} \cup \{0\}$. Note that $a_0 = f(z_0) = 0$, so

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n.$$

Let n_0 be the smallest index such that $a_{n_0} \neq 0$. Then

$$f(z) = \sum_{n=n_0}^{\infty} a_n (z - z_0)^n.$$

Clearly the series

$$\sum_{k=0}^{\infty} a_{n_0+k} (z - z_0)^k$$

is convergent, so we may define $g : D(z_0;R) \to \mathbb{C}$ by

$$g(z) = \sum_{k=0}^{\infty} a_{n_0+k} (z - z_0)^k$$

for all $z \in D(z_0;R)$. This g is defined by a convergent power series, so is convergent. Moreover, $g(z_0) = a_{n_0} \neq 0$, so by the continuity of g, we may choose $r \in (0,R]$ such that g does not vanish on $D(z_0;r)$. ∎

Proposition 5.15. *Let $f : \Omega \to \mathbb{C}$ be analytic, where $\Omega \subseteq \mathbb{C}$ is a domain, and let $(z_j)_{j=1}^{\infty}$ be a sequence of distinct points in Ω such that $(z_j)_{j=1}^{\infty}$ converges in Ω and $f(z_j) = 0$ for all $j \in \mathbb{N}$. Then $f = 0$.*

Theorem 5.16. Identity Theorem

Let $f, g : \Omega \to \mathbb{C}$ be analytic, where $\Omega \subseteq \mathbb{C}$ is a domain. If $f|_D = g|_D$ for some $D \subseteq \Omega$ that has a limit point, then $f = g$.

Isolated Singularities and Laurant Series

(5.5) We begin to look at isolated singularities of analytic functions.

Def'n 5.8. **Isolated Singularity** of an Analytic Function

Let Ω be a domain and let $z_0 \in \Omega$. If f is a function analytic on $\Omega \setminus \{z_0\}$, then we say z_0 is an ***isolated singularity*** of f.

There are three types of isolated singularities.

Def'n 5.9. **Removable Singularity, Pole, Essential Singularity** of an Analytic Function

Let f be an analytic function and suppose that $z_0 \in \mathbb{C}$ is an isolated singularity of f.

(a) If $\lim_{z \to z_0} f(z)$ exists, say $w_0 \in \mathbb{C}$, then we call z_0 a ***removable singularity***.

(b) If $\lim_{z \to z_0} f(z) = \infty$ does not exist but $\lim_{z \to z_0} |f(z)| = \infty$, then we call z_0 a ***pole***.

(c) Otherwise, we call z_0 an ***essential singularity***.

(5.6) Removable Singularities

We begin by considering removable singularities. In particular, we shall see that removable singularities are characterized by the fact that the function is always bounded in some punctured disc around the removable singularity.

Theorem 5.17. Riemann's Criterion for Removable Singularities

Let f be analytic on $\Omega \setminus \{z_0\}$, where $\Omega \subseteq \mathbb{C}$ is a domain and $z_0 \in \Omega$. Then the isolated singularity z_0 of f is removable if and only if $|f|$ is bounded in some punctured disc about z_0. In this case, $\tilde{f} : \Omega \to \mathbb{C}$ by

$$\tilde{f}(z) = \begin{cases} f(z) & \text{if } z \neq z_0 \\ \lim_{z \to z_0} f(z) & \text{otherwise} \end{cases}$$

is analytic.

Laurent Series

Theorem 5.18. Laurent's Theorem

Let f be analytic on a punctured disc $D = D(z_0; R) \setminus D(z_0; r)$ with $0 < r < R$, and let $\Gamma \subseteq D$ be a positively oriented simple closed curve whose interior contains z_0. Then f can be expressed as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

for all $z \in D$, where

$$c_n = \frac{1}{2\pi i} \oint_\Gamma \frac{f(s)}{(s - z_0)^{n+1}} \, \mathrm{d}s$$

for all $n \in \mathbb{Z}$.

(5.7) Laurent Series

We call

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

with

$$c_n = \frac{1}{2\pi i} \oint_\Gamma \frac{f(s)}{(s - z_0)^{n+1}} \, \mathrm{d}s$$

for all $n \in \mathbb{Z}$ the ***Laurent series*** representation of f centered at z_0.