Quiz 1 Cheatsheet

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Fourier Approximation and Analysis

- Piecewise continuous: $\exists\, a = x_0 < x_1 < \cdots < x_n = b$ s.t. $f(a^+), f(b^-), f(x_i^-), f(x_i^+)$ exists.
- Piecewise smooth: Both f' and f are piecewise continuous on $[a, b]$.
- Fourier Sine series: $f(x) = \sum_{k=1}^{\infty} b_k \sin kx$, $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$ for all n.
- Fourier Cosine series: $f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx$, $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\, dx$

  $a_k = \frac{2}{\pi}\int_0^{\pi} f(x)\cos kx\, dx$
- Fourier Series: $f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx$

  $[-\pi, \pi]$

  $$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$
- If f has period L, then

  $$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos\frac{2k\pi x}{L} + \sum_{k=1}^{\infty} b_k \sin\frac{2k\pi x}{L}$$

  $$\frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\,dx \qquad \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\cos\frac{2\pi n x}{L}\,dx \qquad \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\sin\frac{2\pi n x}{L}\,dx$$
- Inner product $\forall f, g, h \in V$

  $$\begin{cases} \langle f, g\rangle = \langle g, f\rangle & \text{(symmetric)} \\ \langle f, g + ch\rangle = \langle f, g\rangle + c\langle f, h\rangle & \text{(bilinear)} \\ \langle f + ch, g\rangle = \langle f, g\rangle + c\langle h, g\rangle & \\ \langle f, f\rangle \geq 0 \text{ with eq only when } f = 0 & \text{(+ve def)} \end{cases}$$
- Orthogonal: $\forall f, g \in \mathcal{F}$, $f \neq g$ $[\langle f, g\rangle = 0]$.
- Orthonormal: $\forall f \in \mathcal{F}$, $[\langle f, f\rangle = 1]$. Norm: $\|x\| = \sqrt{\langle x, x\rangle}$
- Cauchy Schwartz Inequality: $|\langle x, y\rangle| \leq \|x\|\|y\|$
- Parallelogram rule: $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$
- Pythagorean law: $\|\sum_{j=1}^{k} v_j\|^2 = \sum_{j=1}^{k}\|v_j\|^2$ if $\langle v_j, v_l\rangle = 0$ whenever $j \neq l$
- Hilbert space: complete inner product space
- An orthogonal family of vectors is linearly independent.
- Every Hilbert space has a maximal (complete) orthonormal set.
- An orthonormal set $\{e_1, e_2, \ldots\}$ is said to be an orthonormal basis of V if $\forall f \in V$, $\exists!\, \{c_k\} \subset \mathbb{R}$ s.t. $f = \sum_{k=1}^{\infty} c_k e_k$. In particular, it is a complete orthonormal set.
- Bessel's inequality: Let F be a family of orthonormal vectors in V.

  Then $\sum_{v_\alpha \in F} |\langle v, v_\alpha\rangle|^2 \leq \|v\|^2$, $\forall v \in V$.
- Parseval's identity: If F is complete (hence an orthonormal basis),

  then $\sum_{v_\alpha \in F} |\langle v, v_\alpha\rangle|^2 = \|v\|^2 \quad \forall v \in V$

Proposition 3.8. $\{v_1, \cdots, v_n\}$ be an orthonormal family of vectors.

$$\inf\{\|u-v\| : v \in \operatorname{span}\{v_1, \cdots, v_n\} = M\} = \|\underbrace{\sum_{j=1}^{n} \langle u, v_j\rangle v_j}_{= P_M u \text{ orthogonal projection}} - u\|$$

(P) $\|P_M u - P_M v\| \le \|u-v\| \Rightarrow P_M$ is continuous.

If $u \in M$, then $P_M(u) = u$.

Least Squares Approximation

Homework 3.9. $\{V_j\}$ orthonormal basis, H Hilbert space. $\langle x, y\rangle = \sum_{j=1}^{\infty} \langle x, v_j\rangle\langle y, v_j\rangle$

$$\|A\alpha^* - b\| = \min_{\alpha \in \mathbb{R}^m} \|A\alpha - b\| \Rightarrow A^T A \alpha^* = A^T b$$

$$\Downarrow$$

$$P_M b = A\alpha^* = \sum_{i=1}^{m} \alpha_i^* a_i$$

$$\langle P_M b, a_j\rangle = \langle b, a_j\rangle$$

$$\begin{bmatrix} \langle a_1, a_1\rangle & \cdots & \langle a_m, a_1\rangle \\ \vdots & & \\ \langle a_1, a_m\rangle & \cdots & \langle a_m, a_m\rangle \end{bmatrix} \begin{pmatrix} \alpha_1^* \\ \vdots \\ \alpha_m^* \end{pmatrix} = \begin{pmatrix} \langle b, a_1\rangle \\ \vdots \\ \langle b, a_m\rangle \end{pmatrix}$$

Gram determinant: $\|b - P_M b\|^2 = \dfrac{G(b, a_1, \cdots, a_m)}{G(a_1, \cdots, a_m)}$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \qquad \cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x} \qquad \sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$

$$\tan\frac{x}{2} = \pm\sqrt{(1-\cos x)(1+\cos x)} \qquad \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\cos x \cos y = \frac{\cos(x-y) + \cos(x+y)}{2} \qquad \sin x \cos y = \frac{\sin(x+y) + \sin(x-y)}{2}$$

$$\sin x \sin y = \frac{\cos(x+y) - \cos(x-y)}{2}$$