MA3252 Linear Programming

Final Examination Helpsheet

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1 Linear Programming Problem

$$
\begin{array}{lll|lll}
\text{(P)} \min\limits_{\mathbf{x}\in\mathbb{R}^n} & \mathbf{c}^\top\mathbf{x} & & \text{(D)} \max\limits_{\mathbf{p}\in\mathbb{R}^m} & \mathbf{p}^\top\mathbf{b} & \\
\text{s.t.} & \mathbf{a}_i^\top\mathbf{x} \geq b_i & \text{for } i \in M_+; & \text{s.t.} & \mathbf{p}_i \geq 0 & \text{for } i \in M_+; \\
& \mathbf{a}_i^\top\mathbf{x} \leq b_i & \text{for } i \in M_-; & & \mathbf{p}_i \leq 0 & \text{for } i \in M_-; \\
& \mathbf{a}_i^\top\mathbf{x} = b_i & \text{for } i \in M_0; & & \mathbf{p}_i \text{ free} & \text{for } i \in M_0; \\
& x_j \geq 0 & \text{for } j \in N_+; & & \mathbf{p}^\top\mathbf{A}_j \leq c_j & \text{for } j \in N_+; \\
& x_j \leq 0 & \text{for } j \in N_-; & & \mathbf{p}^\top\mathbf{A}_j \geq c_j & \text{for } j \in N_-; \\
& x_j \in \mathbb{R} & \text{for } j \in N_{\mathbb{R}}. & & \mathbf{p}^\top\mathbf{A}_j = c_j & \text{for } j \in N_{\mathbb{R}},
\end{array}
$$

where $\mathbf{a}_i = (a_{i,1}, a_{i,1}, \cdots, a_{i,n})^\top \in \mathbb{R}^n, b_i \in \mathbb{R}$.

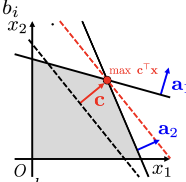
- The feasible region $P \in \mathbb{R}^n$ is a *polyhedron*.
- An LP problem may have
  - ▷ one unique solution; OR
  - ▷ one finite optimal cost with multiple optimal solutions; OR
  - ▷ unbounded optimal cost with no optimal solution; OR
  - ▷ empty feasible set, where optimal cost equals $+\infty$.
- Each variable/constraint in (P) gives a constraint/variable in D.

Graphical Representation: In $\mathbb{R}^n$, $\{\mathbf{x} \mid \mathbf{a}^\top\mathbf{x} = b\}$ is a hyperplane with normal vector $\mathbf{a}$.

- Vector $\mathbf{c}$ corresponds to the direction of increasing $\mathbf{c}^\top\mathbf{x}$.

Standard Form: Minimization + equality + non-negative.

- Maximization objective: $\max \mathbf{c}^\top\mathbf{x} \Rightarrow \min -\mathbf{c}^\top\mathbf{x}$.
- Inequality constraints: $\mathbf{a}_i^\top\mathbf{x} \leq / \geq b_i \Rightarrow \begin{cases}\mathbf{a}_i^\top\mathbf{x} \pm s_i = b_i \\ s_i \geq 0\end{cases}$
  - ▷ s_i is *slack* variable.
- Non-positive variables: $x_i \leq 0 \Rightarrow x_i^- \geq 0$.
- Free variables: $x_i \Rightarrow (x_i^+ - x_i^-); x_i^+, x_i^- \geq 0$.



Convex Sets and Convex Functions:

- Convex set: $\forall \mathbf{x}, \mathbf{y} \in S\, \forall\lambda \in [0,1]\, [\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in S]$.
- Convex combination: $\mathbf{x} = \sum\limits_{i=1}^{k} \lambda_i \mathbf{x}^i$, where $\lambda_i \in [0,1]$ s.t. $\sum\limits_{i=1}^{k} \lambda_i = 1$.
  - ▷ Any convex combination of two optimal solutions is also an optimal solution.
- Convex hull: Set of convex combinations.
- Convex function: $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n\, \forall\lambda \in [0,1]\, [f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1-\lambda)f(\mathbf{y})]$.
  - ▷ f is *concave* if $-f$ is convex.
  - ▷ *Affine function* $d + \mathbf{c}^\top\mathbf{x}$ is both convex and concave.
  - ▷ **Thm 1.5.1.** If $f_1, f_2, \cdots, f_m : \mathbb{R}^n \to \mathbb{R}$ are convex, then $f(\mathbf{x}) = \max\{f_1(\mathbf{x}), f_2(\mathbf{x}), \cdots, f_m(\mathbf{x})\}$ is also convex.
    - ∗ **Cor 1.5.2.** $\max\limits_{i=1,2,\cdots,m} \{d_i + \mathbf{c}_i^\top\mathbf{x}\}$ is convex.

Example. Reformulate as LP problem:

- $\max\min(x_1, x_2) \Rightarrow \max t$ s.t. $t \leq x_1; t \leq x_2$.
- $|x_1 - x_2| \leq 2 \Rightarrow x_1 - x_2 \leq 2; x_1 - x_2 \geq -2$.
- $\min |x| \Rightarrow \min\max(x, -x) \Rightarrow \min t$ s.t. $t \geq x; t \geq -x$.

Polyhedra and Extreme Points:

- Polyhedron: $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$.
  - ▷ A polyhedron is a finite intersection of half-spaces.
  - ▷ A polyhedron has finite number of vertices/BFS.
- 3 definitions of corner points: Consider a convex set $P \subseteq \mathbb{R}^n$,
  - ▷ Extreme point: A point $\mathbf{x}^* \in P$ is an *extreme point* if whenever points $\mathbf{y}, \mathbf{z} \in P$ and scalar $\lambda \in (0,1)$ are such that $\mathbf{x}^* = \lambda\mathbf{y} + (1-\lambda)\mathbf{z}$, we have $\mathbf{y} = \mathbf{z} = \mathbf{x}^*$.
  - ▷ Vertex: A point $\mathbf{x}^* \in P$ is a *vertex* if there is a $\mathbf{c} \in \mathbb{R}^n$ such that $\mathbf{c}^\top\mathbf{x}^* > \mathbf{c}^\top\mathbf{y}$ for all $\mathbf{y} \in P \setminus \{\mathbf{x}^*\}$.
  - ▷ Basic feasible solution (BFS): $\mathbf{x}^*$ is a *BFS* of a polyhedron if n **linearly independent** constraints are active at $\mathbf{x}^*$ and $\mathbf{x}^* \in P$.
    - ∗ Basic solution: A point where n linearly independent constraints are active but not necessarily in P.
  - ▷ **Thm 2.1.5.** In a non-empty polyhedron, an extreme point, a vertex and a BFS are equivalent.
- Degenerate: A basic solution (not necessarily feasible) is *degenerate* if more than n contraints are active at $\mathbf{x}^*$.

Basic Feasible Solutions for Standard Polyhedra:

$$\{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\},$$

where $\mathbf{A} \in \mathbf{R}^{m\times n}, m < n$ contains m linearly independent rows.

- Basic solution for standard polyhedra: $\mathbf{x}^*$ is a *basic solution* iff
  - ▷ the equality constraints $\mathbf{A}\mathbf{x}^* = \mathbf{b}$ hold; AND
  - ▷ $x_i^* = 0$ for $n - m$ indices; AND
  - ▷ these n binding constraints are linearly independent.
- **Thm 2.2.1.** A vector $\mathbf{x}^* \in \mathbb{R}^n$ is a basic solution of the standard form LP iff
  - ▷ $\mathbf{A}\mathbf{x}^* = \mathbf{b}$; AND
  - ▷ There exists $B = \{B(1), B(2), \cdots, B(m)\} \subset \{1, 2, \cdots, n\}$ such that
    - ∗ the columns of $\mathbf{A}_B = \left(\mathbf{A}_{B(1)}, \mathbf{A}_{B(2)}, \cdots, \mathbf{A}_{B(m)}\right)$ are linearly independent; AND
    - ∗ $x_i^* = 0$ for $i \in N = \{1, 2, \cdots, n\} \setminus B$.

  If in addition, $\mathbf{x}_B^* \geq \mathbf{0}$, then $\mathbf{x}^*$ is a BFS.
  - ▷ $\mathbf{x}_B^* = \mathbf{A}_B^{-1}\mathbf{b}$.
  - ▷ A degenerate basic solution $\mathbf{x}^*$ has more than $n - m$ zero components.
  - ▷ If $n = m + 1$, then there are at most two BFSs.
- Adjacent BFS: Extreme points connected by an edge on the boundary.
  - ▷ The corresponding bases share all but one basic column.
  - ▷ There are common $n - 1$ linearly independent constraints that are active at both of them.

Optimal Solutions at Extreme Points:

- A polyhedron $P \subseteq \mathbb{R}^n$ *contains a line* if $\exists \mathbf{x}^* \in P\, \exists \mathbf{d} \neq \mathbf{0} \in \mathbb{R}^n\, \forall\lambda \in \mathbb{R}\, [\mathbf{x}^* + \lambda\mathbf{d} \in P]$. A polyhedron containing an infinite line does not contain an extreme point.
- **Thm 2.3.1.** Let $\mathbf{A} \subseteq \mathbb{R}^{m\times n}, m \geq n$. Suppose $P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}\} \neq \emptyset$. The following are equivalent:
  - ▷ P does not contain a line;
  - ▷ P has a BFS;
  - ▷ P has n linearly independent constraints.
  - ▷ Implication: Every non-empty bounded polyhedron and every non-empty standard form polyhedron has at least one BFS.
- **Thm 2.3.3.** If an LP has a BFS and an optimal solution, then there exists an optimal solution that is a BFS.
  - ▷ Hence, it suffices to check BFS.

1.1 The Simplex Method

Feasible Direction and Reduced Cost:

- Feasible direction: For a polyhedron P and a point $\mathbf{x} \in P$, a vector d is a *feasible direction* if $\mathbf{x} + \theta\mathbf{d} \in P$ for some $\theta > 0$.
  - ▷ For standard polyhedra, $\mathbf{A}\mathbf{d} = \mathbf{0}$.
- **Clm †.** Let $\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N)$ with $\mathbf{x}_B \geq 0, \mathbf{x}_N = 0$ be a BFS. A direction d moving from $\mathbf{x}$ to an adjacent BFS is of the form $\mathbf{d}^j = \left(\mathbf{d}_B^j, \mathbf{d}_N^j\right)$ for some $j \in N$, where
  - ▷ $\mathbf{d}_N^j = \mathbf{e}_j$ where $e_{j,j} = 1$ and $e_{j,i} = 0$ for $i \in N \setminus \{j\}$; AND
  - ▷ $\mathbf{d}_B^j = -\mathbf{A}_B^{-1}\mathbf{A}_j$.
- Reduced cost: Let $\mathbf{x}$ be a basic solution. Let $\mathbf{c} = (\mathbf{c}_B, \mathbf{c}_N)$. For each $j \in \{1, 2, \cdots, n\}$, the *reduced cost* $\bar{c}_j$ of variable x_j is defined by
$$\bar{c}_j = c_j - \mathbf{c}_B^\top\mathbf{A}_B^{-1}\mathbf{A}_j.$$
  - ▷ For $j \in B$, $\bar{c}_j = 0$.
  - ▷ If $\bar{c}_j \geq 0$ for all $j \in N$, then current BFS is the unique optimal solution.
  - ▷ A direction $\mathbf{d}^j$ is an *improving direction* if $\bar{c}_j < 0$.
  - ▷ Change in cost in any direction $\mathbf{d}$:
$$\mathbf{c}^\top\mathbf{d} = \mathbf{c}_B^\top\mathbf{d}_B + \mathbf{c}_N^\top\mathbf{d}_N = -\mathbf{c}_B^\top\mathbf{A}_B^{-1}\mathbf{A}_N\mathbf{d}_N + \mathbf{c}_N^\top\mathbf{d}_N.$$
- **Clm.** Let $\mathbf{x}$ be a BFS with basis B. Any feasible direction at $\mathbf{x}$ can be represented as
$$\sum_{j\in N} \lambda_j \mathbf{d}^j \text{ for } \lambda_j \geq 0.$$
- Degenerate: A BFS is *degenerate* if some element of $\mathbf{x}_B$ is zero. A BFS is *non-degenerate* if $\mathbf{x}_B = \mathbf{A}_B^{-1}\mathbf{b} > \mathbf{0}$.
- **Thm 3.1.6.** (Optimality conditions) Consider a BFS $\mathbf{x}$ associated with basis matrix $\mathbf{A}_B$, and let $\bar{\mathbf{c}}$ be corresponding vector of reduced costs.
  - ▷ If $\bar{\mathbf{c}} \geq \mathbf{0}$, then $\mathbf{x}$ is optimal.
  - ▷ If $\mathbf{x}$ is optimal and non-degenerate, then $\bar{\mathbf{c}} \geq \mathbf{0}$.

Special Cases:

- Some $x_{B(k)} = 0$ at optimum $\Rightarrow$ degenerate solution.
- Some nonbasic $\bar{c}_j = 0$ at optimum:
  - ▷ $\mathbf{u} \leq \mathbf{0} \Rightarrow$ unbounded optimum set;
  - ▷ Otherwise $\Rightarrow$ alternate optimum.
  - ▷ $\mathbf{u} \leq \mathbf{0}$ and $\bar{c}_j < 0 \Rightarrow$ unbounded problem.
  - ▷ Some $y_i > 0$ at optimum for auxiliary problem $\Rightarrow$ infeasible.

Simplex Method:

1. Start with basis B and its basic columns $\mathbf{A}_B$ and BFS $\mathbf{x}$.
   - ▷ Check that $\mathbf{x}$ is indeed a BFS.
2. Compute reduced costs $\bar{c}_j = c_j - \mathbf{c}_B^\top\mathbf{A}_B^{-1}\mathbf{A}_j$ for all $j \in N$.
   - ▷ If $\bar{c}_j \geq 0$ for all $j \in N$, then current BFS is optimal. END.
   - ▷ Otherwise, choose some j for which $\bar{c}_j < 0$.
3. Compute $\mathbf{d}_B^j = -\mathbf{A}_B^{-1}\mathbf{A}_j$ (see **Clm †.**).
   - ▷ If $\mathbf{d}_B^j \geq \mathbf{0}$, then problem is unbounded. END.
   - ▷ Otherwise, let $\theta^* = \min\left\{\frac{x_i}{-d_i^j} \,\middle|\, i \in B, d_i < 0\right\}$.
4. Let $l \in B$ be such that $\theta^* = \frac{x_l}{-d_l^j}$. The corresponding x_l is the leaving variable.
5. Form a new basis $\bar{B} = (B \setminus \{l\}) \cup \{j\}$.
6. The other basic variables are $x_i + \theta^* d_i^j$ for $i \neq l$.
7. The entering variable x_j assumes $\theta^* = \frac{x_l}{-d_l^j}$. Go to Step 1.

Big-M Method:

1. Multiply constraints by -1 to make $\mathbf{b} \geq \mathbf{0}$ as needed.
2. Add artificial variables $y_1, y_2, \cdots, y_m$ to constraints without positive slack.
3. Apply simplex method on LP with cost $\min \mathbf{c}^\top\mathbf{x} + M \sum\limits_{y=1}^{m} y_i$, where $M \gg 0$ is treated as some algebraic variable.

Tableau Method:

① Start from basis B and its basic columns $\mathbf{A}_B$ (preferably $\mathbf{I}$, and the corresponding BFS $\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N)$ (check)).

Basic	$x_j, j \in N$	$x_{B(1)}$	$x_{B(2)}$	Solution
$\mathbf{c}$	c_j	$c_{B(1)}$	$c_{B(2)}$	
$\bar{\mathbf{c}}$	$c_j - \mathbf{c}_B^\top \mathbf{A}_B^{-1}\mathbf{A}_j$	0	0	Obj: $-\mathbf{c}_B^\top \mathbf{x}_B$
$B(1)$	$-d_1^j = \left(\mathbf{A}_B^{-1}\mathbf{A}_j\right)_1$	1	0	$x_{B(1)}$
$B(2)$	$-d_2^j = \left(\mathbf{A}_B^{-1}\mathbf{A}_j\right)_2$	0	1	$x_{B(2)}$

② Choose some j such that $x_j < 0$. At that column, for all $-d_i^j > 0, i \in B$, calculate $\frac{x_i}{-d_i^j}$ and pick the smallest one i^* (0 is also considered).

③ i^* leaves and j enters. Normalize the row where this happens such that the cell $(x_j, x_j) = 1$.

④ Perform row operations to all rows including $\bar{\mathbf{c}}$ such that the column of x_j is all 0 but one 1.

⑤ If all $\bar{\mathbf{c}} \geq \mathbf{0}$, END; else, return to ② again.

Two-Phase Method:

Phase I: Find BFS using auxiliary LP.

① Multiply constraints by -1 to make $\mathbf{b} \geq \mathbf{0}$ as needed.

② Add artificial variables $y_1, y_2, \cdots, y_m$ to constraints without positive slack.

③ Apply simplex method on auxiliary LP with cost min $\sum_{y=1}^{m} y_i$.

④ If the optimal cost in auxiliary LP is:

- ▷ zero: A BFS to original LP is found.
- ▷ positive: Original LP is infeasible. END.

Phase II: Solve original LP.

① Take BFS found in Phase I to start Phase II.

② Use cost coefficients of original LP to compute reduced costs.

③ Apply simplex method to original LP.

- ▷ Either finds an optimum, or detects unboundedness.

1.2 The Dual Simplex Method

- **Thm. 4.1.5.** The dual of the dual is the primal.
- **Weak Duality Thm.** If $\mathbf{x}$ is feasible in (P) and $\mathbf{p}$ is feasible in (D), then $\mathbf{p}^\top\mathbf{b} \leq \mathbf{c}^\top\mathbf{x}$ and thus $\sup_{\mathbf{p}\text{ feasible}} \mathbf{p}^\top\mathbf{b} \leq \inf_{\mathbf{x}\text{ feasible}} \mathbf{c}^\top\mathbf{x}$.
  - ▷ **Col.** If feasible and $\mathbf{p}^\top\mathbf{b} = \mathbf{c}^\top\mathbf{x}$, then $\mathbf{x}$ and $\mathbf{p}$ optimal.
  - ▷ **Col.** Unboundedness in one implies infeasibility in another.
    - ∗ (P) and (D) can be both infeasible.
- **Strong Duality Thm.** If an LP has an optimum, so does its dual, and both optimal objective values are equal.
  - ▷ An optimal solution to (D) is $\mathbf{p}^\top = \mathbf{c}_B^\top\mathbf{A}_B^{-1}$, where B is an optimal basis for (P).
  - ▷ If there is a basis B_0 s.t. $\mathbf{A}_{B_0} = \mathbf{I}$, then an optimal solution to (D) is $\mathbf{p}^\top = \mathbf{c}_{B_0}^\top - \bar{\mathbf{c}}_{B_0}^\top$.
- **Complementary Slackness Thm.** If $\mathbf{x}$ is feasible in (P) and $\mathbf{p}$ is feasible in (D), then both are optimal if and only if

$$p_i\left(\mathbf{a}_i^\top\mathbf{x} - b_i\right) = 0 \text{ for all } i;$$

$$\left(c_j - \mathbf{p}^\top\mathbf{A}_j\right)x_j = 0 \text{ for all } j.$$

  - ▷ **Prop.** If $\mathbf{x}$ is feasible, then $\mathbf{x}$ is optimal iff $\exists\mathbf{p}$ CS.

Dual Simplex Method: Nonnegative $\mathbf{c}$ and only $\leq$ constraints.

① Start from basis B and its basic columns $\mathbf{A}_B$ (preferably $\mathbf{I}$, and the corresponding BFS $\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N)$ (check)).

Basic	$x_j, j \in N$	$x_{B(1)}$	$x_{B(2)}$	Solution
$\bar{\mathbf{c}}$	c_j	0	0	Obj: 0
$B(1)$	A_{1j}	1	0	b_1
$B(2)$	A_{2j}	0	1	b_2

② Choose some i such that $b_i < 0$. At that row, for all columns j that are negative (neg), calculate $\frac{\bar{c}_j}{|\text{neg}|}$ and pick the smallest one j^*.

③ i leaves and j^* enters. Normalize the row where this happens such that the cell $(x_{j^*}, x_{j^*}) = 1$.

④ Perform row operations to all rows including $\bar{\mathbf{c}}$ such that the column of x_j is all 0 but one 1.

⑤ If all $b \geq 0$, END; else, return to ② again.

1.3 Sensitivity Analysis

- Feasibility: $\mathbf{A}_B^{-1}\mathbf{b} \geq \mathbf{0}$.
- Optimality: $\mathbf{c}^\top - \mathbf{c}_B^\top\mathbf{A}_B^{-1}\mathbf{A} \geq \mathbf{0}$.

Change in b: $b_i = b_i + \delta$.

- Feasibility is checked by $\mathbf{x}_B^* + \delta(\mathbf{A}_B^{-1}\mathbf{e}_i) \geq \mathbf{0}$; optimality not affected.
- If not feasible, use dual simplex method.
- Dual p_i is the marginal cost of b_i. When b_i changes δ, the optimal cost changes by δp_i.

Change in c: $c_j = c_j + \delta_j$.

- Optimality: If x_j nonbasic $\bar{c}_j \leftarrow \bar{c}_j + \delta_j$; else for all $i \in N$, $\bar{c}_i \leftarrow \bar{c}_i - \delta_j\mathbf{e}_j^\top\mathbf{A}_B^{-1}\mathbf{A}_i$. Feasibility not affected.
- If x_j nonbasic and not optimal, use primal simplex method.

Change in Nonbasic Column of A: $a_{ij} = a_{ij} + \delta$.

- Optimality: Only $\bar{c}_j \leftarrow \bar{c}_j - \delta p_i$. Feasibility not affected.
- If not optimal, use primal simplex method.

Add a New Variable: Add c_{n+1} and $\mathbf{A}_{n+1}$.

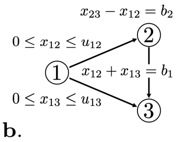
- Check optimality at $(\mathbf{x}^*, 0)$.
- If not optimal, continue primal simplex method by adding a new column $\begin{bmatrix}\bar{c}_{n+1}\\ \mathbf{A}_B^{-1}\mathbf{A}_{n+1}\end{bmatrix}$ to the final tableau.

Add a New Constraint: Add $\mathbf{a}_{m+1}^\top\mathbf{x} \leq b_{m+1}$.

- Check if the original solution is feasible.
- If not feasible, add new constraint to the bottom of the final tableau. Use row operations to make $(\mathbf{x}_B, x_{n+1})$ a basic solution. Use dual simplex method to solve new problem.

2 Network Flow Problem

$$\begin{aligned}\min_{\mathbf{x}\in\mathbb{R}^n} \quad & \mathbf{c}^\top\mathbf{x}\\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \text{ at all vertices;}\\ & \mathbf{0} \leq \mathbf{x} \leq \mathbf{u} \text{ at all edges.}\end{aligned}$$



- Flow-outs − Flow-ins = Supply $\mathbf{b}$.
- Network has feasible flow $\Rightarrow \sum b_i = 0$.
- Formulation of minimum cost flow problem.

Shortest Path Problem: Find the shortest path from s to t.

$$\text{(P)} \min_{\mathbf{x}\in\mathbb{R}^n} \ \mathbf{c}^\top\mathbf{x} \quad \text{s.t.} \ \mathbf{A}\mathbf{x} = \mathbf{b}; \ \mathbf{x} \geq \mathbf{0}. \qquad \text{(D)} \max_{\mathbf{p}\in\mathbb{R}^m} \ \mathbf{p}^\top\mathbf{b} \quad \text{s.t.} \ \mathbf{A}^\top\mathbf{p} \leq \mathbf{c}; \ \mathbf{p} \text{ free.} \qquad \max_{\mathbf{p}\in\mathbb{R}^m} \ p_s - p_t \quad \text{s.t.} \ p_i - p_j \leq c_{ij}, \ \forall(i,j) \in E.$$

- $b_s = 1; b_t = -1; b_{-s-t} = 0$.
- $\mathbf{x} \in \{0,1\}^n$ is equivalent as $\mathbf{x} \geq \mathbf{0}$ if no negative cycle.

Maximum Flow Problem: Find the maximum flow from s to t.

$$\text{(P)} \max_{\mathbf{x}\in\mathbb{R}^n} \ v \quad \text{s.t.} \ \mathbf{A}\mathbf{x} = \mathbf{d}v; \ \mathbf{x} \leq \mathbf{u}; \ \mathbf{x} \geq \mathbf{0}. \qquad \text{(D)} \min_{\mathbf{z}\in\mathbb{R}^m} \ \mathbf{u}^\top\mathbf{z} \quad \text{s.t.} \ \mathbf{d}^\top\mathbf{y} = 1; \ \mathbf{z} \geq \mathbf{A}^\top\mathbf{y}; \ \mathbf{z} \geq \mathbf{0}. \qquad \min_{\mathbf{z}\in\mathbb{R}^m} \ \sum u_{ij}z_{ij} \quad \text{s.t.} \ y_i - y_j \leq z_{ij} \ \& \ z_{ij} \geq 0 \ \forall(i,j) \in E; \ y_s - y_t = 1.$$

- $d_s = 1; d_t = -1; d_{-s-t} = 0$.
- The dual is the minimum cut capacity problem.
- **Thm.** The maximum flow is equal to the capacity of the min cut.

2.1 The Network Simplex Method

Feasible Tree Solution and Reduced Cost:

- Truncated matrix: $\tilde{\mathbf{A}}\mathbf{x} = \tilde{\mathbf{b}}$ by removing any row from $\mathbf{A}$.
- Tree solution: ① $\tilde{\mathbf{A}}\mathbf{x} = \tilde{\mathbf{b}}$; ② A spanning tree.
- Feasible tree solution: Tree solution $\mathbf{x}$ with $\mathbf{x} \geq \mathbf{0}$.
- **Thm. 7.1.1.** The columns corresponding to $n-1$ arcs form a basis of $\tilde{\mathbf{A}}$ iff these arcs form a spanning tree.
- Dual vector: Given basis B, $\mathbf{p}^\top = \mathbf{c}_B^\top\tilde{\mathbf{A}}_B^{-1}$.
- Reduced cost: $\bar{\mathbf{c}}^\top = \mathbf{c}^\top - \mathbf{p}^\top\tilde{\mathbf{A}}$.
  - ▷ Let $p_n = 0$ for the truncated node n. Then $\bar{c}_{ij} = c_{ij} - (p_i - p_j)$ for all $(i,j) \in E$.

① Start with a spanning tree T, feasible tree solution $\mathbf{x}$.

② Compute dual vector $\mathbf{p}$ and $\bar{c}_{ij} = c_{ij} - p_i + p_j$ for all arcs $(i,j) \notin T$.

- ▷ If $\bar{c}_{ij} \geq 0$ for all $(i,j) \in E$, then current $\mathbf{x}$ optimal. END.
- ▷ Otherwise, choose some (i,j) for which $\bar{c}_{ij} < 0$.

③ Follow the flow update scheme:

- ▷ Enter (i,j) gives a unique cycle. Identify the cycle.
- ▷ Orientate the cycle s.t. (i,j) is a forward arc.
- ▷ Let C_f and C_b be sets of forward and backward arcs in cycle.
- ▷ If $C_b \neq \emptyset$, set $\theta^* = \min_{(k,l)\in C_b} x_{kl}$, attained by arc (p,q).
- ▷ If $C_b = \emptyset$, then $\theta^* = \infty$, so objective is $-\infty$.
- ▷ Update $\mathbf{x}$ in cycle: if in C_f add θ^*; if in C_b minus θ^*.

④ Form a new tree $T = (T \setminus \{p,q\}) \cup \{(i,j)\}$ and go to Step ②.

Two-Phase Method:

Phase I: Find initial BFS.

① For any $i \in V \setminus \{n\}$, if $b_i \geq 0/b_i < 0$ and $(i,n)/(n,i) \notin E$, create an artificial arc $(i,n)/(n,i)$.

② Initial basis $B = \{(i,n) \text{ if } b_i \geq 0 \text{ or } (n,i) \text{ if } b_i < 0 \mid i \in V \setminus \{n\}\}$.

③ Initial flow $x_{in} = b_i$ when $b_i \geq 0$ and $x_{ni} = -b_i$ when $b_i < 0$.

④ Solve this using the Simplex method.

Phase II: Solve original LP.

Integrality:

- **Thm. 7.3.1.** Consider an uncapacitated network flow problem where underlying graph is connected. Then
  - ① For every basis matrix $\tilde{\mathbf{A}}_B$, $\tilde{\mathbf{A}}_B^{-1}$ has integer entries.
  - ② If $\mathbf{b}$ is integral, then every primal basic solution $\mathbf{x}$ is integral.
  - ③ If $\mathbf{c}$ is integral, then every dual basic solution $\mathbf{p}$ is integral.
  - ▷ **Col.** Consider an uncapacitated network flow problem and assume that the optimal cost is finite, then
    - ① If $\mathbf{b}$ is integral, then there is an integral optimal flow vector.
    - ② If $\mathbf{c}$ is integral, then there is an integral optimal solution to the dual problem.