MA3264 Mathematical Modelling

AY2022/23 Semester 1 · *Prepared by Tian Xiao*

Basic ODEs

Separable Equation

$$M(x) - N(y)y' = 0 \Rightarrow \int M(x)dx = \int N(y)dy.$$

Integrating Factor

$$y' + P(x)y = Q(x)$$
$$\text{Let } \mu(x) = e^{\int P(x)dx}:$$
$$\mu(x)y' + \mu'(x)y = \mu(x)Q(x)$$
$$y = \frac{\int \mu(x)Q(x)dx}{\mu(x)}$$

Bernoulli Equation

$$y' + P(x)y = Q(x)y^n$$
$$\text{Let } z = y^{1-n}, \text{ then } z' = (1-n)y^{-n}y':$$
$$y^{-n}y' + P(x)y^{1-n} = Q(x)$$
$$\frac{z'}{1-n} + P(x)z = Q(x)$$

2nd Order Equation

$ay'' + by' + cy = r(x) \Leftrightarrow ax^2 + bx + c = 0.$

$$\begin{cases} y = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x} & \lambda_1 \neq \lambda_2 \in \mathbb{R} \\ y = (C_1 + C_2x)e^{\lambda x} & \lambda_1 = \lambda_2 \in \mathbb{R} \\ y = e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x) & \lambda = \alpha \pm \beta i \in \mathbb{C} \end{cases}$$

To find the particular solution:

- If $r \in \mathcal{P}^n$, $y_p(x) \leftarrow p^n$.
- If $r(x) = g(x)e^{kx}$, $y_p(x) \leftarrow u(x)e^{kx}$.
- If $r(x) = g(x)\cos kx$ or $g(x)\sin kx$, let $z(x) \leftarrow u(x)e^{ikx}$ and take $Re(z)$ or $Im(z)$.

Population Models

Malthus Model

$\frac{dN}{dt} = (B-D)N \Leftrightarrow N(t) = N_0e^{(B-D)t}.$

Logistic Model

$\frac{dN}{dt} = BN - sN^2 \Leftrightarrow N(t) = \frac{B/S}{1+e^{-Bt}(\frac{B}{N_0 s}-1)}.$

B/S is a stable equilibrium point.

System of Linear ODEs

General Linear ODE System

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{B}\begin{bmatrix} x \\ y \end{bmatrix} \Leftrightarrow \begin{cases} \frac{dx}{dt} = ax + by \\ \frac{dy}{dt} = cx + dy \end{cases}.$$

Eigenvalues: $r = \frac{1}{2}\left[\mathrm{Tr}(B) \pm \sqrt{\mathrm{Tr}(B)^2 - 4\mathrm{Det}(B)}\right].$

Solutions: $\boldsymbol{u}(t) = C_+e^{r_+t}\boldsymbol{u}_+ + C_-e^{r_-t}\boldsymbol{u}_-.$

Nonhomogenous System

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = B\begin{bmatrix} x \\ y \end{bmatrix} + F \Rightarrow -B^{-1}F.$$

Phase Plane Classification

Both r_1 and r_2 are real:

$r_1 > 0$, $r_2 < 0$ saddle point

$r_1 > r_2 > 0$ Nodal source

$r_1 < r_2 < 0$ Nodal sink

Both r_1 and r_2 are complex:

Re[r]<0 Spiral sink | Re[r]>0 Spiral source | Re[r]=0 Centre

System of Non-Linear ODEs

Linearisation

$$\begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{cases} \text{ with equilibrium point } (a,b).$$

By Taylor expansion,

$$\begin{cases} \frac{dx}{dt} \approx f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ \frac{dy}{dt} \approx g_x(a,b)(x-a) + g_y(a,b)(y-b) \end{cases}.$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{bmatrix}}_{J(a,b)}\begin{bmatrix} x-a \\ y-b \end{bmatrix}.$$

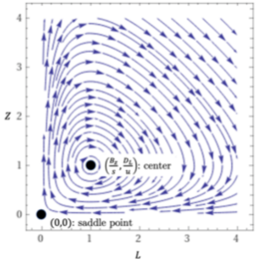
Lotka-Volterra Model

$$\begin{cases} \frac{dL}{dt} = uZL - D_LL \\ \frac{dZ}{dt} = B_zZ - sLZ \end{cases} \quad L\text{: lion; } Z\text{: zebra.}$$

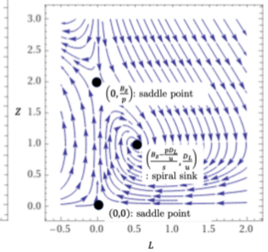
Every trajectory in the contour is periodic:

$$\left(\frac{B_z}{L} - s\right)\frac{dL}{dt} + \left(\frac{D_L}{Z} - u\right)\frac{dZ}{dt} = 0$$
$$\underbrace{B_z\ln L - sL + D_L\ln Z - uZ}_{} = C$$
$$F(L,Z) \text{ has only 1 maximum} \Rightarrow \text{closed.}$$

$$J(L,Z) = \begin{bmatrix} uZ - D_L & uL \\ -sZ & B_z - sL \end{bmatrix}.$$



LV



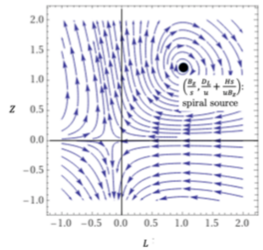
Logistic LV

Logistic LV Model

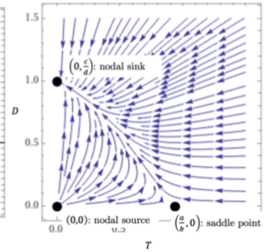
$$\begin{cases} \frac{dL}{dt} = uZL - D_LL \\ \frac{dZ}{dt} = (B_zZ - pZ^2) - sLZ \end{cases}.$$

Lion Hunting

$$\begin{cases} \frac{dL}{dt} = uZL - D_LL - H \\ \frac{dZ}{dt} = B_zZ - sLZ \end{cases}$$



LV with Hunting



Competing Species

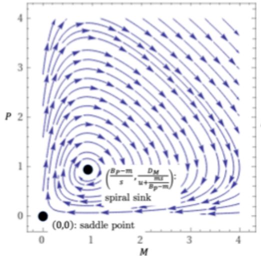
Competing Species

$$\begin{cases} \frac{dT}{dt} = (a - kD)T - bT^2 & T\text{: thylacine;} \\ \frac{dD}{dt} = (c - \sigma T)D - dD^2 & D\text{: dingo.} \end{cases}$$

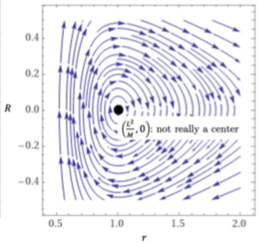
[Principle of Competitive Exclusion]: If two species are too similar, one will wipe out another.

Where Is Everybody?

$$\begin{cases} \frac{dM}{dt} = uPM - D_MM + mP & M\text{: mutant probe;} \\ \frac{dP}{dt} = B_PP - sMP - mP & P\text{: normal probe.} \end{cases}$$



Mutant Probes



Planetary Orbits

Non-Linear 2nd Order ODEs

Consider the Earth moving around the Sun:

$$\ddot{r} = -\frac{M}{r^2} + \frac{L^2}{r^3} \Rightarrow \begin{cases} \dot{r} = R \\ \dot{R} = -\frac{M}{r^2} + \frac{L^2}{r^3} \end{cases}.$$

The equilibrium point is almost a center.

Partial Differential Equations

A PDE is an equation containing an unknown function u of 2 or more independent variables $x, y, \cdots$ and its partial derivatives with respect to them.

Separation of Variables

PDE: $u_x = f(x)g(y)u_y$. Suppose $u = X(x)Y(y)$, then $X'(x)Y(y) = f(x)g(y)X(x)Y'(y) = k$.

$$\begin{cases} X'(x) = kf(x)X(x) \\ Y'(y) = \frac{k}{g(y)}Y(y) \end{cases}$$

Wave Equation

$$c^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}, \text{ where } \quad \begin{matrix} y(t,0) = 0 & y(t,\pi) = 0 \\ y(0,x) = f(x) & \frac{\delta y}{\delta t}(0,x) = 0 \end{matrix}.$$

We need 4 pieces of information for a solution.

d'Alembert's Solution

$y(t,x) = \frac{1}{2}[f(x+ct) + f(x-ct)]$.

Separation of Variables

Let $y(t,x) = u(x)v(t)$, then $\begin{cases} u'' + \lambda u = 0 \\ v'' + \lambda c^2 v = 0 \end{cases}$.

From $\begin{cases} y(t,0) = u(0)v(t) = 0 \\ y(t,\pi) = u(\pi)v(t) = 0 \end{cases}$,
for u to cut the x-axis twice, we have $\lambda > 0$.
Let $\lambda = n^2$ and $u = C\cos(nx) + D\sin(nx)$.
Since $u(0) = 0$, $C = 0$ and $u = D\sin(nx)$.
Since $u(\pi) = 0$, $n \in \mathbb{Z}$.
Similarly, $v(t) = A\cos(nct)$.
Therefore, $y = b_n \sin(nx)\cos(nct)$ and only $y(0,x) = f(x)$ is not satisfied yet.

Fourier Series

Any odd function $f(x)$ of period 2π on $[0,\pi]$ can be expressed as $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$, where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx$. So the complete solution is: $y(t,x) = \sum_{n=1}^{\infty} b_n \sin(nx)\cos(nct)$.

If we want $[0,L]$ instead of $[0,\pi]$, the Fourier formulae becomes $g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ and $b_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$. The complete solution is: $y(t,x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$.

Tsunami (Korteweg-de Vries)

$\partial_t \eta + \sqrt{gh}\partial_x \eta + \frac{3}{2}\sqrt{\frac{g}{h}}\eta\partial_x\eta + \frac{1}{6}h^2\sqrt{gh}\partial_x^3\eta = 0$,
where η denotes the elevation above sea level.

Suppose $\eta = E(x-ct)$, then we can simplify to
$-2AE' + 6BEE' + 2CE''' = 0$.
$-2AE + 3BE^2 + 2CE'' = 0$ (integrate).
$-AE^2 + BE^3 + C(E')^2 = K$ (integrate w.r.t. E).

Heat Equation

$u_t = c^2 u_{xx}$, where $\begin{matrix} u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{matrix}$.

Solution: $u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} e^{-\frac{\pi^2 n^2 c^2}{L^2}t}$.

Heat Equation Variant

$u_t = c^2 u_{xx}$, where $\begin{matrix} u(0,t) = 0 & u(L,t) = T \\ u(x,0) = f(x) & \end{matrix}$.

Let $u^*(x,t) = u(x,t) - \frac{Tx}{L}$.

Consider the Fourier series of $f(x) - \frac{Tx}{L}$, then $u^*(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\frac{\pi^2 n^2 c^2}{L^2}t}$, where $B_n = b_n - \frac{2}{L}\int_0^L \frac{Tx}{L}\sin\frac{n\pi x}{L}dx = b_n + \frac{2T}{\pi n}(-1)^n$.

Fisher's Equation

$u_t = \alpha u_{xx} + \beta u(1-u) \Rightarrow$ heat + rumour.
We seek a solution of the form $u(x,t) = U(x-ct) \equiv U(s)$, moving to the right at constant speed c, starting at $x = 0$. As $x \to \infty$, $s \to \infty$; but as $t \to \infty$, $s \to -\infty$. Now $u_x x = U''$ and $u_t = -cU'$. We can reduce Fisher's Equation to

$$\alpha U'' + cU' + \beta U - \beta U^2 = 0.$$

$$\begin{cases} U' = V \\ V' = -\frac{c}{\alpha}V - \frac{\beta}{\alpha}U + \frac{\beta}{\alpha}U^2 \end{cases}$$

The system has two equilibrium points $(0,0)$ and $(1,0)$. $(0,0)$ is a spiral sink if $c < 2\sqrt{\alpha\beta}$, which is rejected since U cannot be negative; hence $c \geq 2\sqrt{\alpha\beta}$, where $(0,0)$ is a nodal sink.

Diffusion of Lions (Laplace)

When $u_t = c^2(u_{xx} + u_{yy})$ and everything has settled down to a steady state ($u_t = 0$), we have:
$u_{xx} + u_{yy} = 0$, $u(x,0) = u(0,y) = u(\pi,y) = 0$.
Suppose $0 \leq x, y \leq \pi$. Take 4 boundary conditions:

$$u(x,0) = 0; \quad u(0,y) = 0$$
$$u(\pi,y) = 0; \quad u(x,\pi) = f(x)$$

$f(x)$ describes the density of lions along the border that has the river. Let $u(x,y) = X(x)Y(y)$, we have $X''Y + XY'' = 0 \Rightarrow -\frac{X''}{X} = \frac{Y''}{Y} = \lambda$ and $X(0) = X(\pi) = 0$. Let $\lambda = n^2$, then $X(x) = \sin(nx), Y(y) = c_n\sinh(ny), c_n \in \mathbb{R}$. Hence, $u(x,y) = \sum_{n=1}^{\infty} c_n \sin(nx)\sinh(ny)$. Putting $u(x,\pi) = f(x)$, we have $f(x) = \sum_{n=1}^{\infty} c_n \sin(nx)\sinh(n\pi)$. $c_n\sinh(n\pi) = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx$ is the Fourier series of the odd extension of $f(x)$. Since $n \geq 1$, $\sinh(n\pi) \neq 0$. For example, if $f(x) = \sin(x) + 0.2\sin(4x)$, then $u(x,y) = c_1\sin(x)\sinh(y) + c_2\sin(4x)\sinh(4y)$, where $c_1 = \frac{1}{\sinh(\pi)}$ and $c_2 = \frac{0.2}{\sinh(4\pi)}$.

Appendix

Trigonometric Identities

- sin, cos: $\sin^2 x + \cos^2 x = 1$
- tan: $\tan x = \frac{\sin x}{\cos x}$
- sec, csc: $\sec x = \frac{1}{\cos x}$; $\csc x = \frac{1}{\sin x}$;
- cot: $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$
- $\sec^2 x - \tan^2 x = 1$; $\csc^2 x - \cot^2 x = 1$
- $\sin(x+y) = \sin x\cos y + \sin y\cos x$
- $\sin 2x = 2\sin x\cos x$
- $\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$
- $\cos(x+y) = \cos x\cos y - \sin x\sin y$
- $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = \cos^2 x - 1$
- $\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}$
- $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$
- $\tan\frac{x}{2} = \pm\sqrt{(1-\cos x)(1+\cos x)}$
- $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$
- $\sin x\sin y = \frac{\cos(x+y) - \cos(x-y)}{2}$
- $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$
- $\cos x\cos y = \frac{\cos(x-y) + \cos(x+y)}{2}$
- $\sin x\cos y = \frac{\sin(x+y) + \sin(x-y)}{2}$
- sinh, cosh: $\cosh^2 x - \sinh^2 x = 1$
  $\sinh x = \frac{e^x - e^{-x}}{2}$; $\cosh x = \frac{e^x + e^{-x}}{2}$
- tanh: $\tanh x = \frac{\sinh x}{\cosh x}$
- $\operatorname{sech} x = \frac{1}{\cosh x}$
- $\operatorname{csch} x = \frac{1}{\sinh x}$
- coth: $\coth x = \frac{1}{\tanh x}$
- $\tanh^2 x + \operatorname{sech}^2 x = 1$
- $\coth^2 x - \operatorname{csch}^2 x = 1$
- $\sinh(x+y) = \sinh x\cosh y + \sinh y\cosh x$
- $\sinh 2x = 2\sinh x\cosh x$
- $\cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$
- $\cosh 2x = \cosh^2 x + \sinh^2 x$
- $\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$

Integrals

Basic

- $\int k\,dx = kx + C$
- $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$
- $\int \frac{1}{x}\,dx = \ln|x| + C$
- $\int e^x\,dx = e^x + C$

Fractional

- $\int \frac{1}{ax+b} = \frac{1}{a}\ln|ax+b| + C$
- $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \sinh^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\tanh^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{x\sqrt{1-x^2}}\,dx = -\operatorname{sech}^{-1} x + C$
- $\int \frac{1}{|x|\sqrt{1+x^2}}\,dx = -\operatorname{csch}^{-1} x + C$

Logarithmic

- $\int \ln x\,dx = x\ln x - x + C$

Trigonometric

- $\int \cos x\,dx = \sin x + C$
- $\int \sin x\,dx = -\cos x + C$
- $\int \tan x\,dx = \ln|\sec x| + C$
- $\int \sec x\,dx = \ln|\sec u + \tan u| + C$
- $\int \sec^2 x\,dx = \tan x + C$
- $\int \sec x\tan x\,dx = \sec x + C$
- $\int \csc x\cot x\,dx = -\csc x + C$
- $\int \csc^2 x\,dx = -\cot x + C$
- $\int \sinh x\,dx = \cosh x + C$
- $\int \cosh x\,dx = \sinh x + C$
- $\int \operatorname{sech}^2 x\,dx = \tanh x + C$
- $\int \operatorname{csch}^2 x\,dx = -\coth x + C$
- $\int \operatorname{sech} x\tanh x\,dx = -\operatorname{sech} x + C$
- $\int \operatorname{csch} x\coth x\,dx = -\operatorname{csch} x + C$

Special Integrals

- Partial fractions
- Integration by parts:
  $\int u\,dv = uv - \int v\,du$
- $\int \sin^n x\cos^m x\,dx$:
  Use trigonometric identities to convert it into $\sin^k x\cos x$ or $\cos^k x\sin x$.