PC1101 Frontiers of Physics

AY2022/23 Semester 1 · Prepared by Tian Xiao

Elementary Physics

Fundamental Units of Measurement

Base Quantity	SI Unit	Derived Quantity	SI Unit
Length, l	Metre, m	Volume, V	m^3
Mass, m	Kilogram, kg	Density, ρ	$
Time, t	Second, s	Velocity, v	$
Temperature, T	Kelvin, k	Force, F	N
Electrical current, I	Ampere, A	Acceleration, a	$m\ s^{-2}$

Kinematics

Kinematics is the study of motions with respect to time t within the framework of 3-dimensional space. In a basic type of motion, where a body travels in a straight line with initial velocity u and constant or uniform acceleration a, we have:

- Final velocity, $v = u + at$ ①.
- Displacement, $s = \frac{1}{2}(u+v)t = ut + \frac{1}{2}at^2$ ②. Displacement is represented by the area under the velocity-time curve.
- From the two equations above, we have $v^2 = u^2 + 2as$ ③. Equations ①, ② and ③ are called ***Equations of Motion***.

A commonly seen motion with uniform acceleration is falling under gravity. Suppose a body falls a distance h in the first second, then the total distance s it falls after t seconds can be represented by

$$s = t^2 h$$

This equation is called the ***Law of Falling Bodies under the Influence of Gravity***. By differentiating this equation with regards to t, we get the velocity of the falling body at time t

$$v_t = \frac{ds}{dt} = 2ht$$

Dynamics

Dynamics is the study of motions with the consideration of forces. The relationship between forces and motions can be summarised by the three Newton's Laws as follows:

- [***Newton's First Law***] Every body remains stationary or moves with uniform velocity unless it is made to change this state by external forces.

- [***Newton's Second Law***] If a force acts on a body and produces a certain acceleration, then the force is proportional to the product of mass of the body and the acceleration. The acceleration takes place in the direction of the forces. Mathematically,

$$\sum F = ma$$

- [***Newton's Third Law***] If 2 bodies A and B are in contact, A will exert a force on B and B will exert an equal but opposite force on A.

Momentum

Momentum of a body p is defined as the product of its mass and velocity, i.e $p = mv$. When the velocity of a body changes, there is a change to its momentum and this quantity is called the **impulse** of the force on the body. We have:

$$\Delta p = mv - mu = F \cdot t$$

- [***Principle of Conservation of Momentum***] The total momentum within a closed system remains constant. For example, in the collision of two bodies A and B with masses m_A and m_B and velocities u_A and u_B, their final velocities after collision, v_A and v_B, should satisfy $m_A u_A + m_B u_B = m_a v_A + m_B v_B$.

Work and Energy

When a force acts to a body and causes it to move, the force is doing work on the body. Formally, if a force F moves the body through a distance s in the direction of the force, the work done W by the force satisfies $W = F \cdot s$.

The energy of a body due to its motion is called **kinetic energy**, and is defined by the function $KE = \frac{1}{2}mv^2$. If the velocity of a body changes due to a force exerting on it, then the work done by the force $W = \Delta KE$.

A body can also have energy when it is raised through a height h. This type of energy is called **gravitational potential energy** and satisfies $GPE = mgh$. Similar to kinetic energy, when the height of body changes due to a force exerting on it, then the work done by the force $W = \Delta mgh$.

- [***Principle of Conservation of Energy***] The total energy in a closed system is constant. Specifically, if we do not consider energy loss through fraction, heat, etc., then $KE + GPE$ remains constant.

Gravitational potential energy is a result of the attraction between any two masses. This force of attraction is generalised through the ***Universal Law of Gravitation***, which states that

1. Every mass attracts every other mass.
2. Attraction F_g is directly proportional to the product of their masses M_1 and M_2.
3. Attraction F_g is inversely proportional to the square of the distance r between their centers.

Mathematically,

$$F_g = G\frac{M_1 M_2}{r^2}$$

To represent the work done by the force of attraction, we define the **Newtonian Gravitational Potential** as

$$V(r)=G\frac{M_1M_2}{r}$$

Since **force is equal to the negative derivative of potential energy**, the relationship between F_g and $V(r)$ is

$$F_g=-\frac{d}{dr}V(r)$$

Another similar inverse square law is about the attraction between two charges. When we bring a positive and a negative charge together, they will atract; when we bring two positive or two negative charges together, they will repel. The force is represented by

$$F_e=k\frac{e_1e_2}{d^2}=-\frac{d}{dr}EPE$$

Another example of the relationship between force and potential energy is the elastic spring. According to Hooke's Law, the force needed to extend an elastic spring through a distance x is $F=-kx$, where k is some constant measuring the elasticity of the spring. Since force is the negative derivative of potential energy, the elastic potential energy here is

$$U=\int_{-x}^{x}-(-kx)\,dx=\frac{1}{2}kx^2$$

Physics Quantities

In physics, there are two type of quantities, **scalar** and **vector**. A scalar is a quantity that is described by one number, whereas a vector is a quantity that is described by more than one number.

Elementary Mathematics

Limits

- [***L'Hôpital's Rule***] Consider two functions f and g that are both differentiable. If at a point x^* we have $\lim\limits_{x\to x^*}f(x)=0$ and $\lim\limits_{x\to x^*}g(x)=0$, or $\lim\limits_{x\to x^*}|f(x)|=\infty$ and $\lim\limits_{x\to x^*}|g(x)|=\infty$, then $\lim\limits_{x\to x^*}\frac{f(x)}{g(x)}=\lim\limits_{x\to x^*}\frac{f'(x)}{g'(x)}$.

Differentiation

Function	x^n	$\sin x$	$\cos x$	e^x	$\ln x$	$c\,f(x)$
Derivative	nx^{n-1}	$\cos x$	$-\sin x$	e^x	$\frac{1}{x}$	$c\,f'(x)$

There are several differentiation rules:

- [***Product Rule***] $\frac{d}{dx}(uv)=u'v+uv'$.
- [***Chain Rule***] $\frac{d}{dx}(f(u))=\frac{d}{du}(f(u))\times\frac{du}{dx}$.
- Total differential: Suppose we have a function $f(x_1,x_2,\cdots,x_n)$. Then its total derivative with regards to t is equal to

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial x_1} dx_1\right) + \left(\frac{\partial f}{\partial x_2} dx_2\right) + \cdots + \left(\frac{\partial f}{\partial x_n} dx_n\right)$$

Integration

Function	$x^n, n \neq -1$	$\frac{1}{x}$	$\sin x$	$\cos x$	$(ax+b)^n, n \neq -1$	$\frac{f'(x)}{f(x)}$	$\sin(ax+b)$	$\cos(ax+b)$
Integral	$\frac{x^{n+1}}{n+1}$	$\ln\|x\|$	$-\cos x$	$\sin x$	$\frac{(ax+b)^{n+1}}{a(n+1)}$	$\ln\|f(x)\|$	$-\frac{1}{a}\cos(ax+b)$	$\frac{1}{a}\cos(ax+b)$

There are some special integration techniques:

- [***Integration by Parts***] $\int u\, dv = uv - \int v\, du$.

Complex Numbers

The introduction of complex numbers allow us to express the identity $i = \sqrt{-1}$. A complex number can be represented on an Argand diagram with a real axis and an imaginary axis. Each complex number z has the following properties:

1. Magnitude, $|z| = \sqrt{Re(z)^2 + Im(z)^2} = \sqrt{z^* z}$.
2. Direction, $\theta = \tan^{-1} \frac{Im(z)}{Re(z)}$.
3. Conjugate, $z^* = Re(z) - Im(z)i$.

In engineering, a complex number can also be represented with its magnitude r and direction θ as

$$r\angle\theta$$

- [***Euler's Equation***] $e^{i\pi} + 1 = 0$.
- [***De Moivre's Formula***] $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$. From this formula, we have $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ and $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$.
- Hyperpolic functions: $\cosh x = \frac{e^x+e^{-x}}{2}$ and $\sinh x = \frac{e^x-e^{-x}}{2}$.
- Derivatives and second derivatives of a complex number z: $\frac{dz}{d\theta} = iz$; $\frac{d^2z}{d\theta^2} = -z$.

Here we have a very important group of quantities, exponentials. They are commonly seen in nature such as

- [***Boltzmann's Distribution Law***] Consider a gas in equilibrium at temperature T. The probability of finding a given molecule in a given phase space cell of fixed energy E is $P = Ae^{-\frac{E}{kT}}$.
- The radioactive decay phenomenon can be described as $\frac{dN}{dt} = -\lambda N$. Equivalently, $N = Ce^{-\lambda t}$.
- In **Poisson Distribution**, $P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}$.
- Exponentials also occur in the famous **Schrödinger Equation**.

Waves

In physics, waves can be modelled as sine and cosine curves. A moving wave satisfies the following equations:

$$v = f\lambda$$
$$f = \frac{1}{T}$$

where v is the velocity of the wave, f is the frequency, λ is the wavelength and T is the period.

A wave with equation $y = a\sin\omega t$ satisfies $\omega = 2\pi f = \frac{2\pi}{T}$. Its accelerate is equal to $a = -\omega^2 x$ by differentiating its velocity.

Relativity

Galilean and Newtonian Relativity

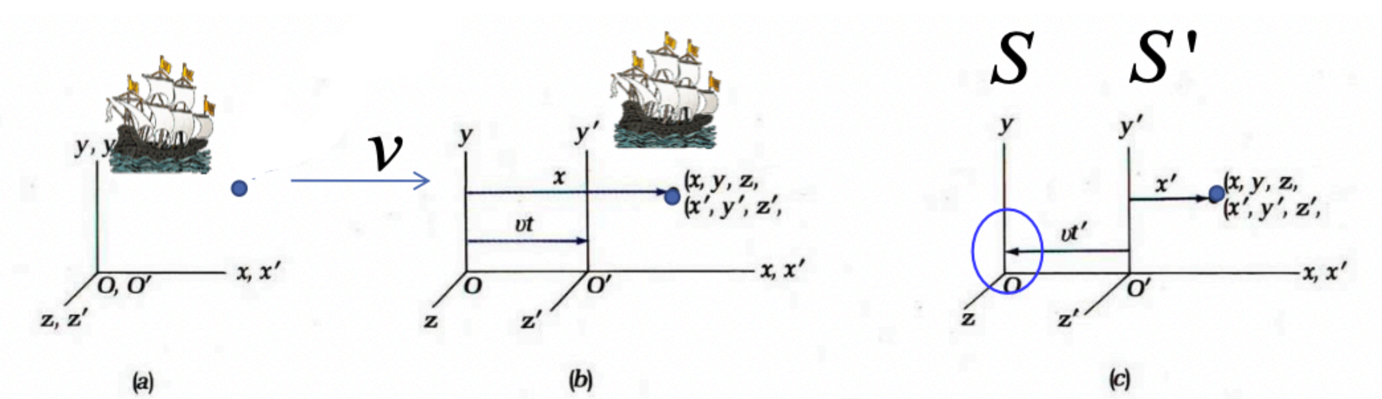
It was believed in ancient times, that force is needed to sustain motion. Galileo was the first one to realise that this reasoning is wrong. He raised the ***Law of Inertia***a, which states that a body on which forces have ceased to act keeps moving with the same speed and in the same direction as it had at the instant when these forces ceased. The reluctance to change is called **inertia**. Newton restated the same idea in his Newton's First Law, with respect to the **absolute space** which in its own nature without regard to anything external remains always similar and immovable, and **absolute time** which of itself and from its own nature flows equably without regard to anything external.

This leads us to the ***Principle of Relativity***, which states that ① the laws of mechanics in a frame of reference moving rectilinearly and uniformly through absolute space are exactly the same as in another frame which is at rest in absolute space. These frames are called **inertial frame of reference**, where no forces are acting on it such that it has a constant velocity. Principle of Relativity in its second form states that consider O and O' to be observers with O at rest (absolute space) and O' moving with a uniform velocity and there is an object in the absolute space which is acted on by a force. Then O would describe it by saying $F = ma$ and O' would say $F' = ma'$. Principle of Relativity in its third form states that ③ the laws of mechanics remain the same for observers in inertial frames that are in uniform motion with respect to each other.

Galilean Relativity

Say observer O is in the S frame and observer O' is in the S' frame. Both S and S' are inertial frames and S' is moving at a velocity v with respect to S. Consider an object at (x, y, z) at time t observed by O. Under O''s observation, the position of the object would be

$$\begin{cases} x' = x - vt \\ y' = y \\ z' = z \\ t' = t \qquad\qquad \text{assuming clocks behave in the same way in } S \text{ and } S' \end{cases}$$



The consequences of these transformations, called **Galilean Relations**, are as follows:

- [***Law of Addition of Velocities***] $u' = u - v$. Hence $a' = \frac{du'}{dt} = \frac{du}{dt} = a$, meaning that $F = ma$ still holds. Newton's Second Law obeys Galilean Relations.

Before Einstein's Relativity, people consider relativity under Galilean Transformations. It was believed that even under transformations, some physics quantities should be **invariant** (it does not change) and physic principles should be **covariant** (the formula does not change). It turns out that Maxwell's Equations get changed under Galilean transformations, so it is not covariant. This means

1. The principle of relativity does not apply to electromagnetic waves. OR
2. Maxwell's Equations are wrong. OR
3. Galilean Transformations are wrong.

Before Einstein's Relativity, people also believed that when there is no air, light transmits in some medium called **ether** in the absolute frame. The Michelson-Morley Experiment proves that there is no either.

Einstein's Special Relativity

Einstein believed that the Laws of Physics should apply in the same fashion everywhere, so he tried to find a new transformation that may preserve the Principle of Relativity and the Speed of Light, c. His axioms are:

1. The Laws of Physics are covariant in all inertial frames.
2. [***Principle of Constancy of Speed of Light***] The Speed of Light in free space has the same value c in all intertial systems.

Einstein believed that the transformation between the two frames must be linear, in order to ensure that Newton's Laws hold in all inertial frames. Hence, we can write the coordinates of an event in one frame as linear combinations of the coordinates in the other frame.

Einsteinian Transformation

Say observer O is in the S frame and observer O' is in the S' frame. Both S and S' are inertial frames and S' is moving at a velocity v with respect to S. Consider an object at (x, y, z) at time t observed by O. Under O''s observation, the position of the object would be

$$\begin{cases} x' = (x - vt)\gamma \\ y' = y \\ z' = z \\ t' = (t - \frac{vx}{c^2})\gamma \end{cases}$$

The term $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} > 1$ is called the **Lorentz Factor** and this transformation is also called Lorentz Transformation. In matrix form, we have

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\frac{v}{c^2} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

We notice that $t' \neq t$, meaning that moving clocks run differently as compared with stationary clocks. It also depends on its location. We also notice that the Lorentz Factor ensures that $v < c$.

This transformation preserves both Newton's Laws and Maxwell's Equations, hence satisfies Einstein's axioms.

Reversely, if observer O' observes that an object is at (x', y', z', t'), then the attributes observed by observe O would be the ***Inverse Lorentz Transformation***:

$$\begin{cases} x = (x' + vt)\gamma \\ y = y' \\ z = z' \\ t = (t' + \frac{vx}{c^2})\gamma \end{cases}$$

If we draw the space-time diagram of S' frame, the x'-t' axes will be skewed because Lorentz Transformations are not orthogonal.

This transformation has several consequences different from Galilean Relations:

1. **Velocity Addition**. $u = \frac{u'+v}{1+\frac{u'v}{c^2}}$. This implies that the Speed of Light is always constant.
2. **Simultaneity**. Consider two events at (x_1, t_1) and (x_2, t_2). In the S' frame, we have

$$t'_2 - t'_1 = \frac{v}{c^2}(x_1 - x_2)\gamma$$

   which means that simultaneous events in one frame are not simultaneous in moving frames.
3. **Lorentz Contraction**. $L_0 = L\gamma$ or $L = \frac{L_0}{\gamma}$. Lengths observed by observers in different frames are not the same. Specifically, L_0 is called the **proper length** measured by the person at rest with the object.
4. **Time Dilation**. $T = T_0\gamma$. Time ticks more slowly in moving frames.
5. **Momentum**. $p = \gamma m_0 v$.
6. **Mass**. $m = \gamma m_0$.
7. **Kinetic Energy**. $KE = mc^2 - m_0c^2 = (\gamma - 1)m_0c^2$. Here $E = mc^2$ represents the energy or the mass of a body.
8. **Total Energy**. Total energy equals the sum of rest enerfy and momentum of any moving particle: $E^2 = (m_0c^2)^2 + (pc)^2$.

Einstein's General Relativity

In mathematical physics, **Minkowski Space** is a combination of three-dimsensional Euclidean space and time into a four-dimensional manifold where the space-time interval between any two events is independent of the inertical frame of reference in which they are recorded. In all frames of reference, (x, y, z, t) satisfies $-c^2t^2 + x^2 + y^2 + z^2 = 0$. According to Einstein's summation convention,

$$ds^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}dx_\mu dx_\nu = g_{\mu\nu}dx_\mu dx_\nu$$

In the last term, $g_{\mu\nu}$ is to a matrix. It has some special cases:

1. If motion of any body not travelling at the speeds close to c and in the vicinity of low intensity gravitational field, this is a generalisation of the 3-dimensional distance relation to 4 dimensions.
2. [***Einstein's Field Equation***] $R_{ij} - \frac{1}{2}Rg_{ij} = \frac{8\pi GT_{ij}}{c^4}$. Here T_{ij} contains all forms of energy and momentum. Hence in an empty space time equation where no matter and energy is present, we have $R_{ij} = 0$.

There is no force concept in Einstein's General Relativity. Instead, he thinks space determines how mass should move and mass determines how space should curve.

Einstein made a statement about the equality of intertial mass and gravitational mass:

$$\text{Inertial mass} \times \text{Acceleration} = \text{Intensity of gravitational field} \times \text{Gravitational mass}$$

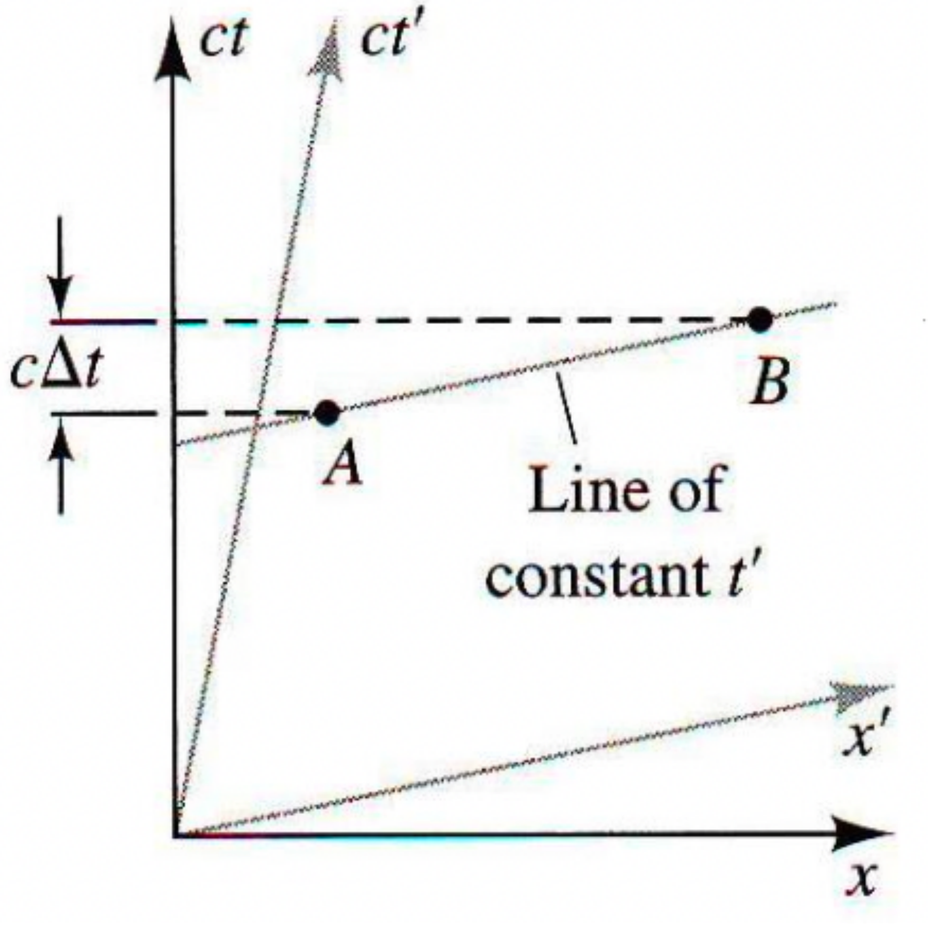
This suggests that only when inertial mass and gravitational mass are equal, the acceleration is independent of the nature of the body. As a result, he stated the ***Equivalence Principle*** as all objects fall at the same rate, assuming negligible air resistance. Considering a spaceship far away from gravitational influence in uniform acceleration with respect to distant stars. When an object is thrown horizontally, an observer outside the spaceship observes a straight line path of the object, but the observer inside the spaceship observes a curved parabola path. Einstein argues that the same holds true for a beam of light.

Equivalence Principle has the following consequences:

1. Gravitation causes time to slow down.
2. Gravitation causes space to be non-Euclean but rather Riemannian.
3. When an object moves, the surrounding warp of space and time moves to readjust to a new position. This readjustments produce ripples in the overall geometry of space-time, called **gravitational waves**.

Space-Time Diagrams

The x'-t' diagram is skewed because $\frac{\Delta x}{\Delta t} = \frac{\Delta x'}{\Delta t'} = c$. This also demonstrates why there is time dilation and length contraction:



Therefore, time dilation is always accompanied by length contraction.

In a skewed space-time diagram, the axes are not orthogonal to each other. Suppose

$$x = x(x_1, x_2, x_3); y = y(x_1, x_2, x_3); z = z(x_1, x_2, x_3)$$

Then in the new coordinates, by total differentiation we have

$$dr^2 = dx^2 + dy^2 + dz^2 = \sum_{\mu=1}^{3} \sum_{\nu=1}^{3} g_{\mu\nu} dx_{\mu} dx_{\nu}$$

Intuitively, in a right angle triangle, $c^2 = a^2 + b^2$, but in other triangles, $c^2 = a^2 + b^2 - 2ab\cos\theta$.

Black Holes

In the 1700's, John Michell and Pierre Laplace both asked the possibility of an astronomical object whose mass and radius satisfy the criteria $R = \frac{2Gm}{c^2}$, then even light cannot escape. Under Newtonian Relativity, the velocity required to escape from a planet is equal to $v = \sqrt{\frac{2Gm}{R}}$.

Consider B on the surface of a very massive star. B sends out signals every second to A, but the star starts to collapse. As the star collapses, B experiences an increasing force of gravity. Gravitational red shift comes into effect because time slows down. Eventually, the dying star will approach a boundary, called **Schwarzschild Radius** where A will have to wait infinitely long for the next signal from B. Therefore, when B crosses the sphere of Schwarzschild radius $R = R_s = \frac{2Gm}{c^2}$, no signals from B will ever reach A. R_s is called the event horizon, because no events taking place within the event horizon will ever be observed by an outside observer like A. No mass can escape from a black hole not even light.