MA3264 Mathematical Modelling

AY2022/23 Semester 1

Basic ODEs and Solutions

1. $M(x) - N(y)y' = 0$

(*Separable*) Separate the variables x and y and rewrite the equation as $\int M(x)\,dx = \int N(y)\,dy$.

2. $y' + P(x)y = Q(x)$

Multiply both sides by an **integrating factor** $\mu(x) = e^{\int P(x)\,dx}$:

$$\mu(x)y' + \mu(x)P(x)y = \mu(x)Q(x)$$
$$\mu(x)y = \int \mu(x)Q(x)\,dx$$

3. $y' + P(x)y = Q(x)y^n$

(*Bernoulli*) Let $z = y^{1-n}$, then $z' = (1-n)y^{-n}y'$. Hence we have:

$$y^{-n}y' + P(x)y^{1-n} = Q(x)$$
$$\frac{z'}{1-n} + P(x)z = Q(x)$$

and use integrating factor.

4. $ay'' + by' + cy = 0$

Consider the **characteristic equation** $ax^2 + bx + c = 0$ with roots λ_1 and λ_2:

- If $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, then $y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$.
- If $\lambda_1 = \lambda_2 \in \mathbb{R}$, then $y = (c_1 + c_2x)e^{\lambda x}$.
- If $\lambda_1 \neq \lambda_2 = \alpha \pm \beta i \in \mathbb{C}$, then $y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$.

5. $ay'' + by' + cy = r(x), r(x) \neq 0$

The goal is to find the **particular solution** y_p:

- If $r(x)$ is a polynomial of order n, guess $y_p(x)$ to be a n-th order polynomial.
- If $r(x)$ is in the form of $g(x)e^{kx}$, let $y_p(x) = u(x)e^{kx}$.
- If $r(x)$ is in the form of $g(x)\cos kx$ or $u(x)\sin kx$, let $z(x) = u(x)e^{ikx}$ and take $Re(z)$ or $Im(z)$).

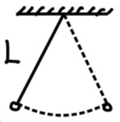
Stability of Solutions

Harmonic Oscillation

$$\underbrace{mL\ddot{\theta}}_{ma} = \underbrace{-mg\sin\theta}_{F}$$

$\theta = 0 \longrightarrow$ stable

$\theta = \pi \longrightarrow$ unstable



Damped Oscillation

$$\underbrace{mL\ddot{\theta}}_{ma} = -mg\sin\theta - \underbrace{sL\dot{\theta}}_{\text{damping force}}$$

$$m\ddot{\theta} + s\dot{\theta} + \frac{mg}{L}\theta = 0$$

Both real: Overdamping
e.g. $\theta = B_1e^{-t} + B_2e^{-2t}$
Dies rapidly to 0.

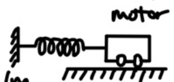
Both complex: Underdamping
e.g. $\theta = e^{-2t}(B_1\cos(3t) + B_2\sin(3t))$
$= Ae^{-2t}\cos(3t - \delta)$
"Quasi-Period"

SHM with amplitude ↓ with time.

Forced Oscillation

$$\underbrace{m\ddot{x}}_{} + \underbrace{kx}_{\text{Hooke's}} = \underbrace{F_0\cos\alpha t}_{\text{Motor}}$$

Frequency $\omega = \sqrt{\frac{k}{m}}$



$$\Downarrow$$

$$x = A\cos(\omega t - \delta) + \frac{F_0/m}{\omega^2 - \alpha^2}\cos(\alpha t)$$

$\frac{\alpha - \omega}{2}$: Beat frequency

$\alpha = \omega$: Resonance $x = \frac{F_0t}{2m\omega}\sin(\omega t)$

Amplitude Response Function:

$$A(\alpha) = \frac{F_0/m}{\sqrt{(\omega^2 - \alpha^2)^2 + \left(\frac{b^2}{m^2}\right)\alpha^2}}$$

When $\alpha^2 = \omega^2 - \frac{b^2}{2m^2}$, max

$$A_{resonance} = \frac{F_0/b\omega}{\sqrt{1 - (b^2/4m^2\omega^2)}}$$

Conservation of Energy

Trick: $\frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) = \dot{x}\frac{d\dot{x}}{dx} = \frac{dx}{dt}\frac{d\dot{x}}{dx} = \ddot{x}$

- Used to draw phase plane diagram ($\dot{x}$ against x).

For SHM we have $m\ddot{x} = -kx$

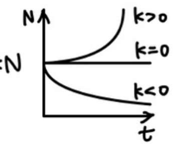
$$m\frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) = -kx$$
$$\frac{1}{2}m\dot{x}^2 = -\frac{1}{2}kx^2 + E$$
$$E = \underbrace{\frac{1}{2}m\dot{x}^2}_{KE} + \underbrace{\frac{1}{2}kx^2}_{PE}$$

Population Models

Malthus' Model

$$\frac{dN}{dt} = (\underbrace{B}_{\text{birth}} - \underbrace{D}_{\text{death}})N = kN$$

$$N(t) = N_0e^{kt}$$

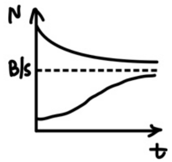


Logistic Model

Assume $D = sN$ (e.g. starvation)

$$\frac{dN}{dt} = BN - sN^2 \quad \text{(Bernoulli)}$$

$$N(t) = \frac{B/s}{1 + e^{-Bt}\left(\frac{B}{N_0 s} - 1\right)}$$

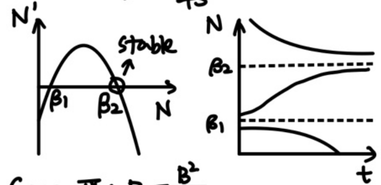


Logistic Model with Harvesting

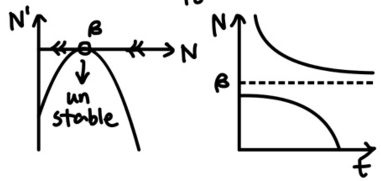
$$\frac{dN}{dt} = (B - sN)N - E$$

Case I: $E > \frac{B^2}{4s}$. Fish dies out.

Case II: $E < \frac{B^2}{4s}$.



Case III: $E = \frac{B^2}{4s}$.



Steady Growth Model

$$\frac{dN}{dt} = (B_0 - \underbrace{\alpha\frac{dN}{dt}}_{\text{birth control}})N - DN \approx \frac{B_0 - D}{\alpha}$$

Model of Ants

rate ↓ when running out of ants

$$\frac{dx}{dt} = (\underbrace{\alpha}_{\text{new ants when no ants on the trail}} + \underbrace{\beta x}_{\text{new ants attracted}})(n - x) - \underbrace{\frac{sx}{r+x}}_{\text{evaporate, ants lose way}}$$

Model of Investment

Profitability $P \equiv \frac{du/dt}{u}$, u → value of company

① Young company. All profits go back.

$$\frac{du}{dt} = Pu$$

② Well-established company ⇒ dividents

$$\frac{du}{dt} = kPu \qquad \frac{dw}{dt} = (1-k)Pu$$

w → investors

$$\Downarrow$$

$$u = Ue^{kPt}$$

$$\begin{cases} k = 0 & w = PUt \\ k \neq 0 & w = \frac{1}{k}(1-k)U[e^{kPt} - 1] \end{cases}$$

Suppose I am in business for a fixed time T, then I pull out and start another business. Given P and T, how should I choose k?

Define $x = kPT$, $y = \frac{w(T)}{U}$,

$$y = (PT - x)\left(\frac{e^x - 1}{x}\right)$$

maximise y. borderline: $PT = 2$

If $PT > 2$, choose $k > 0$ s.t. $\frac{dy}{dx} = 0$

If $PT \leq 2$, choose $k = 0$.

System of 1st Order ODEs

Solve the general system

$$\frac{d}{dt}\begin{pmatrix}x\\y\end{pmatrix}=\underbrace{\begin{pmatrix}a&b\\c&d\end{pmatrix}}_{B}\begin{pmatrix}x\\y\end{pmatrix}\quad \left(\text{i.e.}\begin{cases}\frac{dx}{dt}=ax+by\\ \frac{dy}{dt}=cx+dy\end{cases}\right)$$

$$r=\frac{1}{2}\left[Tr(B)\pm\sqrt{Tr(B)^2-4Det(B)}\right]$$

$$\begin{cases} r_+ \xRightarrow{\text{eigenvector}} \vec{u}_+\\ r_- \qquad\qquad \vec{u}_-\end{cases}$$

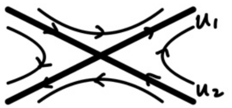
The general solution is

$$\vec{u}(t)=C_+e^{r_+t}\vec{u}_+ + C_-e^{r_-t}\vec{u}_-$$

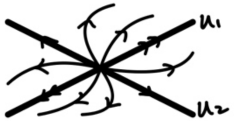
What if we have a non-homogenous equation $\frac{d}{dt}\begin{pmatrix}x\\y\end{pmatrix}=B\begin{pmatrix}x\\y\end{pmatrix}+F$? An obvious particular solution is $\begin{pmatrix}x\\y\end{pmatrix}=-B^{-1}F$.

Phase Plane Classification

Both r_1 and r_2 are real.



$r_1>0,\ r_2<0$
saddle point



$r_1>r_2>0$
Nodal source

$r_1<r_2<0$
Nodal sink

Both r_1 and r_2 are complex.

$Re[r]<0$	$Re[r]>0$	$Re[r]=0$
spiral sink	Spiral source	Centre

- Check the direction from what happen on the x-axis (sign$\left(\frac{dy}{dt}\right)$ when $x>0$ and $y=0$).
- Usually we consider the first quadrant.

Appendix: Common Integrals

Basic

$\int k\,dx = kx + C$
$\int x^n\,dx = \frac{1}{n+1}x^{n+1}+C$
$\int \frac{1}{x}\,dx = \ln|x|+C$
$\int e^x\,dx = e^x + C$

Fractional

$\int \frac{1}{ax+b} = \frac{1}{a}\ln|ax+b|+C$
$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}(\frac{x}{a})+C$
$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}(\frac{x}{a})+C$
$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \sinh^{-1}(\frac{x}{a})+C$
$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}(\frac{x}{a})+C$
$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\tanh^{-1}(\frac{x}{a})+C$
$\int \frac{1}{x\sqrt{1-x^2}}\,dx = -\operatorname{sech}^{-1}x+C$
$\int \frac{1}{|x|\sqrt{1+x^2}}\,dx = -\operatorname{csch}^{-1}x+C$

Logarithmic

$\int \ln x\,dx = x\ln x - x + C$

Trigonometric

$\int \cos x\,dx = \sin x + C$
$\int \sin x\,dx = -\cos x + C$
$\int \tan x\,dx = \ln|\sec x| + C$
$\int \sec x\,dx = \ln|\sec u + \tan u| + C$
$\int \sec^2 x\,dx = \tan x + C$
$\int \sec x\tan x\,dx = \sec x + C$
$\int \csc x\cot x\,dx = -\csc x + C$
$\int \csc^2 x\,dx = -\cot x + C$
$\int \sinh x\,dx = \cosh x + C$
$\int \cosh x\,dx = \sinh x + C$
$\int \operatorname{sech}^2 x\,dx = \tanh x + C$
$\int \operatorname{csch}^2 x\,dx = -\coth x + C$
$\int \operatorname{sech} x\tanh x\,dx = -\operatorname{sech} x + C$
$\int \operatorname{csch} x\coth x\,dx = -\operatorname{csch} x + C$

Appendix: Special Integrals

- Partial fractions
- Integration by parts:
  $\int u\,dv = uv - \int v\,du$
- $\int \sin^n x\cos^m x\,dx$:
  Use trigonometric identities to convert it into $\sin^k x\cos x$ or $\cos^k x\sin x$.

Appendix: Trigonometric Identities

sin, cos: $\sin^2 x+\cos^2 x = 1$
tan: $\tan x = \frac{\sin x}{\cos x}$
sec, csc: $\sec x = \frac{1}{\cos x}$; $\csc x = \frac{1}{\sin x}$;
cot: $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$
$\sec^2 x - \tan^2 x = 1$; $\csc^2 x - \cot^2 x = 1$
$\sin(x+y) = \sin x\cos y + \sin y\cos x$
$\sin 2x = 2\sin x\cos x$
$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$
$\cos(x+y) = \cos x\cos y - \sin x\sin y$
$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = \cos^2 x - 1$
$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$
$\tan(x+y) = \frac{\tan x + \tan y}{1-\tan x\tan y}$
$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$
$\tan\frac{x}{2} = \pm\sqrt{(1-\cos x)(1+\cos x)}$
$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$
$\sin x\sin y = \frac{\cos(x+y)-\cos(x-y)}{2}$
$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$
$\cos x\cos y = \frac{\cos(x-y)+\cos(x+y)}{2}$
$\sin x\cos y = \frac{\sin(x+y)+\sin(x-y)}{2}$
sinh, cosh: $\cosh^2 x - \sinh^2 x = 1$
$\sinh x = \frac{e^x-e^{-x}}{2}$; $\cosh x = \frac{e^x+e^{-x}}{2}$
tanh: $\tanh x = \frac{\sinh x}{\cosh x}$
$\operatorname{sech} x = \frac{1}{\cosh x}$
$\operatorname{csch} x = \frac{1}{\sinh x}$
coth: $\coth x = \frac{1}{\tanh x}$
$\tanh^2 x + \operatorname{sech}^2 x = 1$
$\coth^2 x - \operatorname{csch}^2 x = 1$
$\sinh(x+y) = \sinh x\cosh y + \sinh y\cosh x$
$\sinh 2x = 2\sinh x\cosh x$
$\cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$
$\cosh 2x = \cosh^2 x + \sinh^2 x$
$\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$

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(*Separable*) Separate the variables x and y and rewrite the equation as $\int M(x)\,dx = \int N(y)\,dy$.

2. $y' + P(x)y = Q(x)$

Multiply both sides by an **integrating factor** $\mu(x) = e^{\int P(x)\,dx}$:

$$\mu(x)y' + \mu(x)P(x)y = \mu(x)Q(x)$$

$$\mu(x)y = \int \mu(x)Q(x)\,dx$$

3. $y' + P(x)y = Q(x)y^n$

(*Bernoulli*) Let $z = y^{1-n}$, then $z' = (1-n)y^{-n}y'$. Hence we have:

$$y^{-n}y' + P(x)y^{1-n} = Q(x)$$

$$\frac{z'}{1-n} + P(x)z = Q(x)$$

and use integrating factor

4. $ay'' + by' + cy = 0$

Consider the **characteristic equation** $ax^2 + bx + c = 0$ with roots λ_1 and λ_2:

- If $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, then $y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$.
- If $\lambda_1 = \lambda_2 \in \mathbb{R}$, then $y = (c_1 + c_2x)e^{\lambda x}$.
- If $\lambda_1 \neq \lambda_2 = \alpha \pm \beta i \in \mathbb{C}$, then $y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$.

5. $ay'' + by' + cy = r(x), r(x) \neq 0$

The goal is to find the **particular solution** y_p:

- If $r(x)$ is a polynomial of order n, guess $y_p(x)$ to be a n-th order polynomial.
- If $r(x)$ is in the form of $g(x)e^{kx}$, let $y_p(x) = u(x)e^{kx}$.
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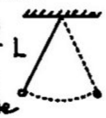
Stability of Solutions

Harmonic Oscillation

$$\underbrace{mL\ddot\theta}_{ma} = \underbrace{-mg\sin\theta}_{F}$$



L

$\ddot{x} = -\omega^2 x$

$\theta = 0 \longrightarrow$ stable

$\theta = \pi \longrightarrow$ unstable

Damped Oscillation

$$\underbrace{mL\ddot\theta}_{ma} = -mg\sin\theta - \underbrace{SL\dot\theta}_{\text{damping force}}$$

$$m\ddot\theta + S\dot\theta + \frac{mg}{L}\theta = 0$$

Both real: Overdamping

e.g. $\theta = B_1e^{-t} + B_2e^{-3t}$

Dies rapidly to 0.

Both complex: Underdamping

e.g. $\theta = e^{-\gamma t}(B_1\cos(\gamma t) + B_2\sin(\gamma t))$

$= Ae^{-\gamma t}\cos(\gamma t - \delta)$

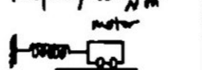
"Quasi-Period"

SHM with amplitude ↓ with time.

Forced Oscillation

Hooke's, Motor, Angular Frequency $\omega = \sqrt{\frac{k}{m}}$

$$m\ddot{x} + kx = F_0\cos\alpha t$$



⇓

$$x = A\cos(\omega t - \delta) + \frac{F_0/m}{\omega^2 - \alpha^2}\cos(\alpha t)$$

$\frac{\alpha - \omega}{2}$: Beat frequency

$\alpha = \omega$: Resonance $x = \frac{F_0 t}{2m\omega}\sin(\omega t)$

Amplitude Response Function:

$$A(\alpha) = \frac{F_0/m}{\sqrt{(\omega^2 - \alpha^2)^2 + \left(\frac{b^2}{m^2}\right)\alpha^2}}$$

When $\alpha^2 = \omega^2 - \frac{b^2}{2m^2}$, max

$$A_{resonance} = \frac{F_0/b\omega}{\sqrt{1 - (b^2/4m^2\omega^2)}}$$

When $A = -\frac{F_0/m}{\omega^2 - \alpha^2}$, $X = A(t)\sin\left[\frac{\alpha + \omega}{2}t\right]$

where $A(t) = \frac{2F_0/m}{\alpha^2 - \omega^2}\sin\left[\frac{\alpha - \omega}{2}t\right]$

Conservation of Energy

Trick $\frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) = \dot{x}\frac{d\dot{x}}{dx} = \frac{dx}{dt}\frac{d\dot{x}}{dx} = \ddot{x}$

- Used to draw phase plane diagram ($\dot{x}$ against x)

For SHM we have $m\ddot{x} = -kx$

$$m\frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) = -kx$$

$$\frac{1}{2}m\dot{x}^2 = -\frac{1}{2}kx^2 + E$$

$$E = \underbrace{\frac{1}{2}m\dot{x}^2}_{KE} + \underbrace{\frac{1}{2}kx^2}_{PE}$$

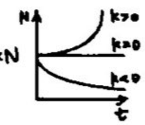
Population Models

Malthus' Model

birth, death

$$\frac{dN}{dt} = (B - D)N = kN$$

$N(t) = N_0e^{kt}$

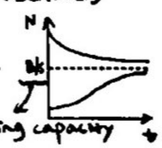


Logistic Model

Assume $D = sN$ (e.g. starvation)

$$\frac{dN}{dt} = BN - sN^2 \quad \text{(Bernoulli)}$$

$$N(t) = \frac{B/S}{1 + e^{-Bt}\left(\frac{B}{N_0S} - 1\right)}$$

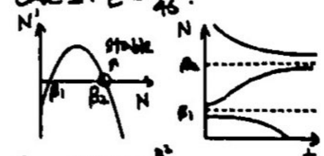


Logistic Model with Harvesting

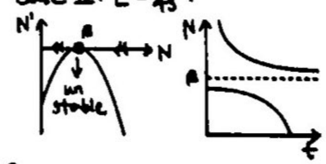
$$\frac{dN}{dt} = (B - sN)N - E$$

Case I: $E > \frac{B^2}{4s}$. Fish dies out.

Case II: $E < \frac{B^2}{4s}$.



Case III: $E = \frac{B^2}{4s}$.



Steady Growth Model

$$\frac{dN}{dt} = (B_0 - \underbrace{\alpha\frac{dN}{dt}}_{\text{birth control}})N - DN \approx \frac{B_0 - D}{\alpha}$$

Model of Ants

$$\frac{dx}{dt} = (\alpha + \beta x)(n - x) - \frac{sx}{r + x}$$

rate ↓ when running out of ants

new ants when no ants on the trail; new ants attracted; desperate ants lose way

Model of Investment

Profitability $P \equiv \frac{du/dt}{u}$, u → value of company

① Young company. All profits go back.

$$\frac{du}{dt} = Pu$$

② Well-established company ⇒ dividends

$\frac{du}{dt} = kPu$ $\quad \frac{dw}{dt} = (1-k)Pu$

w investor

⇓

$u = Ue^{kPt}$

$$\begin{cases} k = 0 & w = PUt \\ k \neq 0 & w = \frac{1}{k}(1-k)U[e^{kPt} - 1] \end{cases}$$

Suppose I am in business for a fixed time T, then I pull out and start another business. Given P and T, how should I choose k?

Define $x = kPT$, $y = \frac{w(T)}{U}$,

$y = (PT - x)\left(\frac{e^x - 1}{x}\right)$

maximise. borderline: $PT = 2$

If $PT > 2$, choose $k > 0$ s.t. $\frac{dy}{dx} = 0$

If $PT \leq 2$, choose $k = 0$.

In real life, friction cannot be ignored.

$$m\ddot{x} + b\dot{x} + kx = F_0\cos(\alpha t)$$

$$x(t) = \frac{\frac{1}{m}F_0\cos(\alpha t - \gamma)}{\sqrt{(\omega^2 - \alpha^2)^2 + \frac{b^2}{m^2}\alpha^2}}$$

Steady oscillation at frequency α

⇒ Amplitude Response Function

System of 1st Order ODEs

Solve the general system

$\frac{d}{dt}\binom{x}{y} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{B}\binom{x}{y}$ (i.e. $\begin{cases} \frac{dx}{dt} = ax+by \\ \frac{dy}{dt} = cx+dy \end{cases}$)

$r = \frac{1}{2}\left[\mathrm{Tr}(B) \pm \sqrt{\mathrm{Tr}(B)^2 - 4\mathrm{Det}(B)}\right]$

$\begin{cases} r_+ \\ r_- \end{cases} \xRightarrow{\text{eigenvector}} \begin{matrix} \vec{u}_+ \\ \vec{u}_- \end{matrix}$

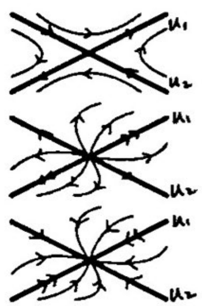
The general solution is

$\vec{u}(t) = C_+ e^{r_+ t}\vec{u}_+ + C_- e^{r_- t}\vec{u}_-$

What if we have a non-homogenous equation $\frac{d}{dt}\binom{x}{y} = B\binom{x}{y} + F$? An obvious particular solution is $\binom{x}{y} = -B^{-1}F$

Phase Plane Classification

Both r_1 and r_2 are real.



$r_1 > 0$, $r_2 < 0$ saddle point

$r_1 > r_2 > 0$ Nodal source

$r_1 < r_2 < 0$ Nodal sink

Both r_1 and r_2 are complex.

Re[r]<0 spiral sink | Re[r]>0 spiral source | Re[r]=0 Centre

- Check the direction from what happen on the x-axis (sign$\left(\frac{dv}{dt}\right)$ when $x > 0$ and $y = 0$)
- Usually we consider the first quadrant

Appendix: Common Integrals

Basic

$\int k\,dx = kx + C$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$

$\int \frac{1}{x}\,dx = \ln|x| + C$

$\int e^x\,dx = e^x + C$

Fractional

$\int \frac{1}{ax+b} = \frac{1}{a}\ln|ax+b| + C$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + C$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}(\frac{x}{a}) + C$

$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \sinh^{-1}(\frac{x}{a}) + C$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}(\frac{x}{a}) + C$

$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\tanh^{-1}(\frac{x}{a}) + C$

$\int \frac{1}{x\sqrt{1-x^2}}\,dx = -\mathrm{sech}^{-1}x + C$

$\int \frac{1}{|x|\sqrt{1+x^2}}\,dx = -\mathrm{csch}^{-1}x + C$

Logarithmic

$\int \ln x\,dx = x\ln x - x + C$

Trigonometric

$\int \cos x\,dx = \sin x + C$

$\int \sin x\,dx = -\cos x + C$

$\int \tan x\,dx = \ln|\sec x| + C$

$\int \sec x\,dx = \ln|\sec u + \tan u| + C$

$\int \sec^2 x\,dx = \tan x + C$

$\int \sec x\tan x\,dx = \sec x + C$

$\int \csc x\cot x\,dx = -\csc x + C$

$\int \csc^2 x\,dx = -\cot x + C$

$\int \sinh x\,dx = \cosh x + C$

$\int \cosh x\,dx = \sinh x + C$

$\int \mathrm{sech}^2 x\,dx = \tanh x + C$

$\int \mathrm{csch}^2 x\,dx = -\coth x + C$

$\int \mathrm{sech}\,x\tanh x\,dx = -\mathrm{sech}\,x + C$

$\int \mathrm{csch}\,x\coth x\,dx = -\mathrm{csch}\,x + C$

$\int \tanh x\,dx = \ln|\cosh x| + C$

$\int \coth x\,dx = \ln|\sinh x| + C$

Appendix: Special Integrals

- Partial fractions
- Integration by parts:
  $\int u\,dv = uv - \int v\,du$
- $\int \sin^n x\cos^m x\,dx$:
  Use trigonometric identities to convert it into $\sin^k x\cos x$ or $\cos^k x\sin x$.

Appendix: Trigonometric Identities

sin, cos: $\sin^2 x + \cos^2 x = 1$

tan: $\tan x = \frac{\sin x}{\cos x}$

sec, csc: $\sec x = \frac{1}{\cos x}$; $\csc x = \frac{1}{\sin x}$;

cot: $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$

$\sec^2 x - \tan^2 x = 1$; $\csc^2 x - \cot^2 x = 1$

$\sin(x+y) = \sin x\cos y + \sin y\cos x$

$\sin 2x = 2\sin x\cos x$

$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$

$\cos(x+y) = \cos x\cos y - \sin x\sin y$

$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = \cos^2 x - 1$

$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$

$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}$

$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$

$\tan\frac{x}{2} = \pm\sqrt{(1-\cos x)(1+\cos x)}$

$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$

$\sin x\sin y = \frac{\cos(x+y)-\cos(x-y)}{2}$

$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$

$\cos x\cos y = \frac{\cos(x-y)+\cos(x+y)}{2}$

$\sin x\cos y = \frac{\sin(x+y)+\sin(x-y)}{2}$

sinh, cosh: $\cosh^2 x - \sinh^2 x = 1$

$\sinh x = \frac{e^x - e^{-x}}{2}$; $\cosh x = \frac{e^x + e^{-x}}{2}$

tanh: $\tanh x = \frac{\sinh x}{\cosh x}$

$\mathrm{sech}\,x = \frac{1}{\cosh x}$

$\mathrm{csch}\,x = \frac{1}{\sinh x}$

coth. $\coth x = \frac{1}{\tanh x}$

$\tanh^2 x + \mathrm{sech}^2 x = 1$

$\coth^2 x - \mathrm{csch}^2 x = 1$

$\sinh(x+y) = \sinh x\cosh y + \sinh y\cosh x$

$\sinh 2x = 2\sinh x\cosh x$

$\cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$

$\int \mathrm{sech}\,x\,dx = \tan^{-1}(\sinh(x)) + C$

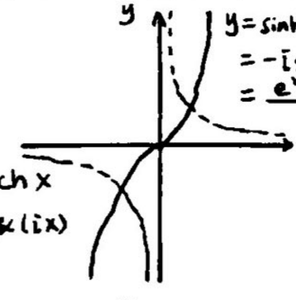
$\int \mathrm{csch}\,x\,dx = \ln|\tanh(\frac{x}{2})| + C$

Romeo and Juliet

$\begin{cases} \frac{dR}{dt} = aJ & R(0) = \alpha \\ \frac{dJ}{dt} = -bR & J(0) = \beta \end{cases}$

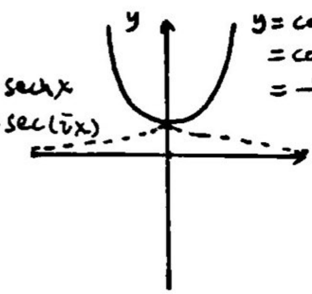
$\begin{cases} R(t) = \alpha\cos(\sqrt{ab}\,t) + \beta\sqrt{\frac{a}{b}}\sin(\sqrt{ab}\,t) \\ J(t) = \beta\cos(\sqrt{ab}\,t) - \alpha\sqrt{\frac{a}{b}}\sin(\sqrt{ab}\,t) \end{cases}$

Hyperbolic Function Graphs



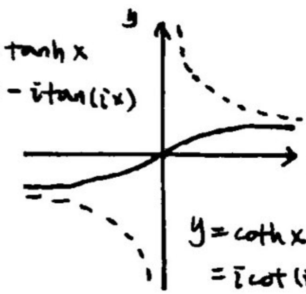
$y = \sinh x = -i\sin(ix) = \frac{e^x - e^{-x}}{2}$

$y = \mathrm{csch}\,x = i\csc(ix)$



$y = \cosh x = \cos(ix) = \frac{e^x + e^{-x}}{2}$

$y = \mathrm{sech}\,x = \sec(ix)$



$y = \tanh x = -i\tan(ix)$

$y = \coth x = i\cot(ix)$