### **MA3264 Mathematical Modelling**

AY2022/23 Semester 1

### **Basic ODEs and Solutions**

1. 
$$M(x) - N(y)y' = 0$$

(Separable) Separate the variables x and y and rewrite the equation as  $\int M(x) dx = \int N(y) dy$ .

2. 
$$y' + P(x)y = Q(x)$$

Multiply both sides by an integrating factor  $\mu(x) = e^{\int P(x) dx}$ :

$$\mu(x)y' + \mu(x)P(x)y = \mu(x)Q(x)$$
  
$$\mu(x)y = \int \mu(x)Q(x) dx$$

3. 
$$y' + P(x)y = Q(x)y^n$$

(Bernoulli) Let  $z = y^{1-n}$ , then  $z' = (1-n)y^{-n}y'$ . Hence we have:

$$y^{-n}y' + P(x)y^{1-n} = Q(x)$$
  
$$\frac{z'}{1-n} + P(x)z = Q(x)$$

and use integrating factor.

4. 
$$ay'' + by' + cy = 0$$

Consider the **characteristic equation**  $ax^2 + bx + c = 0$  with roots  $\lambda_1$  and  $\lambda_2$ :

- If  $\lambda_1 \neq \lambda_2 \in \mathbb{R}$ , then  $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$ .
- If  $\lambda_1 = \lambda_2 \in \mathbb{R}$ , then  $y = (c_1 + c_2 x)e^{\lambda x}$ .
- If  $\lambda_1 \neq \lambda_2 = \alpha \pm \beta i \in \mathbb{C}$ , then  $y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$ .

5. 
$$ay'' + by' + cy = r(x), r(x) \neq 0$$

The goal is to find the **particular** solution  $y_p$ :

- If r(x) is a polynomial of order n, guess  $y_p(x)$  to be a n-th order polynomial.

- If r(x) is in the form of  $g(x)e^{kx}$ , let  $y_p(x) = u(x)e^{kx}$ .
- If r(x) is in the form of  $g(x) \cos kx$  or  $u(x) \sin kx$ , let  $z(x) = u(x)e^{ikx}$  and take Re(z) or Im(z).

### **Stability of Solutions**

**Harmonic Oscillation** 

$$\frac{mL\ddot{\theta} = -mq\sin\theta}{ma} \xrightarrow{F} L$$

$$\theta = 0 \longrightarrow \text{stable}$$

$$\theta = \pi \longrightarrow \text{unstable}$$

**Damped Oscillation** 

$$\frac{mL\ddot{\theta} = - mg\sin\theta - \underline{SL\dot{\theta}}}{ma}$$

$$\frac{mL\ddot{\theta} = - mg\sin\theta - \underline{SL\dot{\theta}}}{damping}$$

$$\frac{m\ddot{\theta} + \dot{S\dot{\theta}} + \frac{mg}{\dot{\theta}}}{damping} = 0$$

E Both real: Overdamping

e.g. 
$$\theta = B_1 e^{-t} + B_2 e^{-st}$$

Dies rapidly to  $\theta$ .

Both complex: Underdamping

e.g.  $\theta = e^{-st} (B_1 cos(3t) + B_2 sin(3t))$ 
 $= Ae^{-st} cos(3t - S)$ 

\*Bussi-Period"

SHM with amplitude I with time.

Forced Oscillation

Hooke's Motor Frequery 
$$W = \sqrt{\frac{k}{m}}$$
 $mx + kx = F_0 \cos kt$ 
 $X = A\cos(\omega t - \delta) + \frac{F_0 m}{\omega^2 - d^2} \cos (\omega t)$ 
 $\frac{d - \omega}{2}$ : Beat frequency

 $d = \omega$ : Peronance  $x = \frac{F_0 t}{2m\omega} \sin(\omega t)$ 

Amplitude Perponse Function:

$$A(d) = \frac{F_0 m}{\sqrt{(\omega^2 - d^2)^2 + (\frac{b^2}{m^2})^2}}$$

When  $k^2 = \omega^2 - \frac{b^2}{2m^2}$ ,  $max$ 

$$Aresonance = \frac{F_0 b\omega}{\sqrt{|-(k^2/4m^2\omega^2)}}$$

### **Conservation of Energy**

Trick: 
$$\frac{d}{dx} \left( \frac{1}{2} \dot{x}^2 \right) = \dot{x} \frac{d\dot{x}}{dx} = \frac{dx}{dt} \frac{d\dot{x}}{dx} = \ddot{x}$$

- Used to draw phase plane diagram ( $\dot{x}$  against x).

For SHM we have 
$$m\ddot{x} = -kx$$

$$M \frac{d}{dx} (\frac{1}{2} \dot{x}^2) = -kx$$

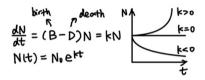
$$\frac{1}{2} m \dot{x}^2 = -\frac{1}{2} k x^2 + E$$

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$$

$$KE PE$$

### **Population Models**

Malthus' Model



Logistic Model

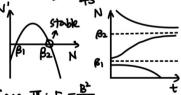
Assume 
$$D = sN$$
 (e.g. starvation)  
 $\frac{dN}{dt} = BN - sN^2$  (Bernoulli)

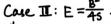


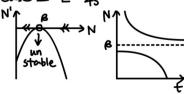
**Logistic Model with Harvesting** 

(age I: 
$$E > \frac{B^2}{45}$$
. Fish dies out.

Case I: E< B2 .







### **Steady Growth Model**

$$\frac{dN}{dt} = \left(B_0 - \alpha \frac{dN}{dt}\right)N - DN \approx \frac{B_0 - D}{\alpha}$$

**Model of Ants** 

rate V when runing out of arts 
$$\frac{dx}{dt} = (x + \beta x)(x - x) - \frac{Sx}{r + x}$$

new auts new auts attracted auts lose way on the trail

**Model of Investment** 

Profitability 
$$P = \frac{du/dt}{u}$$
 value of company

- 1) Young company. All profits go back.

  du = Pu
- Well-established company => dividents  $\frac{du}{dt} = kPu \qquad \frac{dw}{dt} = (1-k)Pu$   $u = Ue^{kPt} \qquad \text{investors}$   $k = 0 \qquad \omega = PUt$   $k \neq 0 \qquad \omega = \frac{1}{L}(1-k)U[e^{kPt}-1]$

Suppose I am in business for a fixed time T, then I pull out and stort another business. Given P and T, how should I choose k?

Define 
$$x = kPT$$
,  $y = \frac{\omega(T)}{U}$ ,  
 $y = (PT - x)(\frac{e^{x} - 1}{x})$   
magazinise borderline:  $PT = 2$ 

If PT>2, choose k=0 s.t.  $\frac{dy}{dx}$  = D If PT \( \text{ } 2, \text{ choose } \text{ } k=0.

### System of 1st Order ODEs

Solve the general system
$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{(i.e. } \begin{cases} \frac{dx}{dt} = ax + by \\ \frac{dy}{dt} = cx + dy \end{cases}$$

$$r = \frac{1}{2} \left[ \text{Tr}(B) \pm \sqrt{\text{Tr}(B)^2} - 4 \text{Det}(B) \right]$$

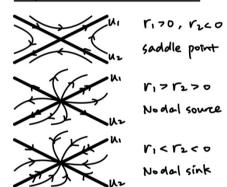
$$r = \frac{1}{2} \left[ \text{Tr}(B) \pm \sqrt{\text{Tr}(B)^2} - 4 \text{Det}(B) \right]$$

The general solution is

What if we have a non-homogenous equation  $\frac{d}{dt} \binom{x}{y} = B \binom{x}{y} + F$ ? An obvious particular solution is  $\binom{x}{y} = -B^{-1}F$ .

#### **Phase Plane Classification**

### Both r, and rz one real.



### Both r, and rz one complex.







Re[r]<0 Re[r]>0 Re[r]=0 Spiral sink Spiral source Centre

- Check the direction from what happen on the x-axis (sign  $\left(\frac{dy}{dt}\right)$  when x > 0 and y = 0).
- Usually we consider the first quadrant.

### **Appendix: Common Integrals**

#### Basic

$$\int k \, dx = kx + C$$

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$$

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

$$\int e^x \, dx = e^x + C$$

#### Fractional

$$\int \frac{1}{ax+b} = \frac{1}{a} \ln|ax+b| + C$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1}(\frac{x}{a}) + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}(\frac{x}{a}) + C$$

$$\int \frac{1}{\sqrt{a^2+x^2}} dx = \sinh^{-1}(\frac{x}{a}) + C$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \cosh^{-1}(\frac{x}{a}) + C$$

$$\int \frac{1}{a^2-x^2} dx = \frac{1}{a} \tanh^{-1}(\frac{x}{a}) + C$$

$$\int \frac{1}{x\sqrt{1-x^2}} dx = - \operatorname{sech}^{-1} x + C$$

$$\int \frac{1}{|x|\sqrt{1+x^2}} dx = - \operatorname{csch}^{-1} x + C$$

### Logarithmic

$$\int \ln x \, dx = x \ln x - x + C$$

### Trigonometric

$$\int \cos x \, dx = \sin x + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \tan x \, dx = \ln|\sec x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

$$\int \csc x \cot x \, dx = -\csc x + C$$

$$\int \csc^2 x \, dx = -\cot x + C$$

$$\int \sinh x \, dx = \cosh x + C$$

$$\int \cosh x \, dx = \sinh x + C$$

$$\int \operatorname{sech}^2 x \, dx = -\coth x + C$$

$$\int \operatorname{sech}^2 x \, dx = -\coth x + C$$

$$\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + C$$

$$\int \operatorname{csch} x \coth x \, dx = -\operatorname{csch} x + C$$

### **Appendix: Special Integrals**

- Partial fractions
- Integration by parts:

$$\int u \, dv = uv - \int v \, du$$

 $-\int \sin^n x \cos^m x \, dx$ :

Use trigonometric identities to convert it into  $\sin^k x \cos x$  or  $\cos^k x \sin x$ .

### **Appendix: Trigonometric Identities**

```
\sin \cos \sin^2 x + \cos^2 x = 1
\tan x = \frac{\sin x}{\cos x}
sec, csc: \sec x = \frac{1}{\cos x}; \csc x = \frac{1}{\sin x}; \cot \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x} \sec^2 x - \tan^2 x = 1; \csc^2 x - \cot^2 x = 1
\sin(x+y) = \sin x \cos y + \sin y \cos x
\sin 2x = 2\sin x \cos x
\sin \frac{x}{2} = \pm \sqrt{\frac{1-\cos x}{2}}
\cos(x+y) = \cos x \cos y - \sin x \sin y
\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = \cos^2 x - 1
\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}
\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}
\tan 2x = \frac{2\tan x}{1-\tan^2 x}
\tan \frac{x}{2} = \pm \sqrt{(1 - \cos x)(1 + \cos x)}
\sin x + \sin y = 2\sin \frac{x+y}{2}\cos \frac{x-y}{2}
\sin x \sin y = \frac{\cos(x+y) - \cos(x-y)}{2}
\cos x + \cos y = 2\cos\frac{\bar{x}+y}{2}\cos\frac{x-y}{2}
\cos x \cos y = \frac{\cos(x-y) + \cos(x+y)}{2}
\sin x \cos y = \frac{\sin(x+y) + \sin(x-y)}{\sin x}
\sinh, \cosh^2 x - \sinh^2 x = 1
\sinh x = \frac{e^x - e^{-x}}{2}; \cosh x = \frac{e^x + e^{-x}}{2}
\tanh x = \frac{\sinh x}{\cosh x}
\operatorname{sech} x = \frac{1}{\cosh x}
\tanh^2 x + \operatorname{sech}^2 x = 1
 \coth^2 x - \operatorname{csch}^2 x = 1
 \sinh(x+y) = \sinh x \cosh y + \sinh y \cosh x
 \sinh 2x = 2 \sinh x \cosh x
\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y
 \cosh 2x = \cosh^2 x + \sinh^2 x
\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}
```

## Just Chrustheet

MA3264 Mathematical Modelling AY2022/23 Senester 1

## Basic ODEs and Solutions

1. M(x) - N(y)y' = 0

(Separable) Separate the variables x and y and rewrite the equation as  $\int M(x) dx = \int N(y) dy$ .

2. y' + P(x)y = Q(x)

Multiply both sides by an integrating factor  $\mu(x) = e^{\int P(x) dx}$ ;

$$\mu(x)y' + \mu(x)P(x)y = \mu(x)Q(x)$$

$$\mu(x)y = \int \mu(x)Q(x) dx$$

3. y' + P(x)y = Q(x)y''

(Bernoulli) Let  $z = y^{1-n}$ , then  $z' = (1-n)y^{-n}y'$ . Hence we have:

$$y^{-n}y' + P(x)y^{1-n} = Q(x)$$
  
 $\frac{z'}{1-n} + P(x)z = Q(x)$ 

and use integrating factor

4. 
$$ay'' + by' + cy = 0$$

Consider the characteristic equation  $ax^2 + bx + c = 0$  with roots  $\lambda_1$  and  $\lambda_2$ :

- If  $\lambda_1 \neq \lambda_2 \in \mathbb{R}$ , then  $y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$ ,
- If  $\lambda_1 = \lambda_2 \in \mathbb{R}$ , then  $y = (c_1 + c_2 x)e^{\lambda_2 x}$ .

If  $\lambda_1 \neq \lambda_2 = \alpha \pm i h \in \mathbb{C}$ , then  $y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$ .

5.  $ay'' + by' + cy - r(x), r(x) \neq 0$ 

The goal is to find the particular solution  $y_p$ :

- If r(x) is a polynomial of order  $n_i$  gives  $y_p(x)$  to be a n-th order polynomial.

- If r(x) is in the form of  $g(x)e^{kx}$ , let  $y_p(x) = u(x)e^{kx}$ .

- If r(x) is in the form of  $g(x)\cos kx$  or  $u(x)\sin kx$ , let  $z(x)=u(x)e^{ikx}$  and take Re(z) or Im(z)).

## Stability of Solutions

Harmonic Oscillation

Damped Oscillation

ml B = -mgcin 8 - 519

m8 + 56 + 198 = 0

Edit real: Overslamping

C.g. B = Biet + B.e. - re

Distribution to 0.

Buth complex: Underdering

C.g. B = e - re (Bicos(149) + Bishlots)

A R - re cos (14 - 6)

Busi-Peted

When K'=W'- Elbert Traductory

6=C : Pertonence K= 1500 (1918)

Anglitude Petponse Function:

A (6) = \frac{F\_1}{\sqrt{100}} \text{sin(wb)}

A (6) = \frac{F\_1}{\sqrt{100}} \text{sin(wb)}

A (6) = \frac{F\_1}{\sqrt{100}} \text{sin(wb)}

For the sin(wb) = \frac{F\_1}{200} \text{sin(wb)}

When K'=W'-\frac{F\_1}{200} \text{sin(wb)}

Arymore 11-(8/40/20)

When Az - Frim , X= A(+) sin (27/2) + 3

where A(+) = 25-1/20 + 3

where A(+) = 25-1/20 + 3

Conservation of Energy First  $\frac{d}{dt}(\frac{1}{2}x^2) = x\frac{dx}{dt} = \frac{dt}{dt}\frac{dt}{dt} = x$ Used to draw phase plane diagram (x against x)

For Street one have mile - but m Age (4/4) 2 - tot be a but be a b

### Population Models

Malthus' Model

Logistic Model
Assume D= SN (e.g. starveston)

LN = BN - SN\* (Barranta)

Logistic Model with Harvesting

A = (B-5N)N-E

CAL I: E> A : BN ARL out

in real life. triation (pand be ignoral)

MX + bx + kx = Fo (ostat)

X(t) = AFo (os(at-Y)

X(t) = A(w-A) + AFA

Standy oscillation at prespecty of as Amplitude Prespond Faculting

Prepared by Tass Jims

Steady Growth Model

A = (B\_n - ot A) N - DN = B - D

A = (B\_n - ot A) N - DN = B - D

A = ot A = ot A = ot A)

Model of A ats

of the x)(n-x) - Sx

At = (4+6x)(n-x) - Sx

nevert new outs structed of y

on the trail

Nodel of Investment

Profeshing Party and the company

(a) Yang company. As parties go bank.

Aff = Pu

(b) Welf-cat-based company addusters.

| kee waput | kto watuku[eff-1] Super I am is beginning for a fined the [],
then I pail and and strart and the beginning.

Given P and T. I have should I down b?

Define X= kPT. y = \(\frac{\pi\_{\infty}}{\pi\_{\infty}}\))

y = (PT-x)(\(\frac{\pi\_{\infty}}{\pi\_{\infty}}\))

prostruct. Invariant pT = 2.

HPT>2. Chank kine (1:\frac{\pi\_{\infty}}{\pi\_{\infty}}\)

HPT>2. Chank kine (1:\frac{\pi\_{\infty}}{\pi\_{\infty}}\)

HPT>3. Chank kine (1:\frac{\pi\_{\infty}}{\pi\_{\infty}}\)

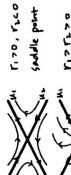
## System of 1" Order ODEs

12th - C.e "t 12+ C.e" 12. r= = [ T-(B) ± JT-(B) - + (Det(B) Solve the general system The general solution is Fr. Santor Fr.

What if we have a non-homogenous equation  $\frac{d}{dt} {x \choose x} = B {x \choose x} + F^{*}$  An obvious particular solution is  $\binom{x}{y} = -B^{-1}F$ 

## Phase Plane Classification

Both r, and rs are real.



No del source 112520

アートアトゥ

Nodel sink

# Both r, and ra are complex.

### 0

9

Re[r]so Pe[r]so Pa[r]=0 9

 Check the direction from what happen on the xspiral sak spiral burge centure (sign  $\left(\frac{dy}{dt}\right)$  when x > 0 and y = 0)

- Usually we consider the first quadrant

# Appendix: Common Integrals

 $x^n dx = \frac{1}{n+1}x^{n+1} + C$  $\frac{1}{z} dx = \ln|x| + C$  $e^x dx = e^x + C$  $\int k dx = kx + C$ 

### Fractional

 $\frac{1}{|x|\sqrt{1+x^2}} dx = -\operatorname{csch}^{-1} x + C$  $r\sqrt{\frac{1}{1-x^2}} dx = -\operatorname{sech}^{-1} x + C$  $\frac{1}{\sqrt{x^2 - a^2}} dx = \cosh^{-1}(\frac{x}{a}) + C$  $a_{2} - \frac{1}{x^{2}} d\varepsilon = \frac{1}{a} \tanh^{-1}(\frac{\pi}{a}) + C$  $\frac{1}{\sqrt{a^2+x^2}} dx = \sinh^{-1}(\frac{x}{a}) + C$  $\frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}(\frac{x}{a}) + C$  $\sqrt{a^2 - x^2} dx = \sin^{-1}(\frac{x}{a}) + C$  $\frac{1}{ax+b} = \frac{1}{a} \ln|ax+b| + C$ 

### ogarithmic

 $\ln x \, dx = x \ln x - x + C$ 

### **Prigonometric**

 $\sec x \, dx = \ln|\sec u + \tan u| + C$  $\operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + C$  $\operatorname{csch} x \operatorname{coth} x \, dx = -\operatorname{csch} x + C$  $\csc x \cot x \, dx = -\csc x + C$  $\operatorname{csch}^2 x \, dx = -\coth x + C$  $\sec x \tan x \, dx = \sec x + C$  $\tan x \, dx = \ln|\sec x| + C$  $\operatorname{sech}^2 x \, dx = \tanh x + C$  $\csc^2 x \, dx = -\cot x + C$  $\sinh x \, dx = \cosh x + C$  $\cosh x dx = \sinh x + C$  $\sin x \, dx = -\cos x + C$  $\sec^2 x \, dx = \tan x + C$  $\cos x \, dx = \sin x + C$ 

Scoths dx = in |sinhs i + C [tonkx dx = 1= | cosh x (+ C

」は)=はいるに対も)できるいまし) R(t)=xco(あも)+Aにあも) 4 = - PR J(0) = 4 第=c) Rio)=以 Romes and Juliet

(メンルドーニ 2 Cx - C y-sinh X Hyperbolic Function Graphs 一になには カラ Cach X

= (0)(1)x) = = 3= Cosh x = Sec(TX) 3=500

= [ wat (Ex) SHLOTEK = - itan((x) S = tent x

## centr2r - centra + sluba tanh 2z = 141anh3

JGGLKXAX= In tank(長) 1+C

# Appendix: Special Integrals

Prepared by Tian Xiao

- $\int u \, dv = uv \int v \, du$ - Integration by parts: - Partial fractions
- f sin" z cos" z dz:

Use trigonometric identities to convert it into sink x cos x or cosk x sin x.

# Appendix: Trigonometric Identities

con 2x = cos2 x - sin2 x = 1 - 2 sin2 x = cos2 x - 1 tan; tan  $x = \frac{\sin x}{\cos x}$ ;  $\cos x = \frac{1}{\sin x}$ ;  $\cos x = \frac{1}{\sin x}$ ;  $\cot x = \frac{\cos x}{\cos x} = \frac{\cos x}{\sin x}$ ;  $\cot x = \frac{\cos x}{\cos x} = \frac{\sin x}{\sin x}$ ;  $\sec^2 x - \tan^2 x = 1$ ;  $\cot^2 x - \cot^2 x = 1$  $\cos(x+y) = \cos x \cos y - \sin x \sin y$  $\sin(x+y) = \sin x \cos y + \sin y \cos x$ with x = = ==== ; cosh z = = ++==  $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$   $\cos x \cos y = \frac{\cos(x-y)+\cos(x+y)}{2}$  $\sinh_1 \cosh^2 x - \sinh^2 x = 1$  $\tan \frac{x}{2} = \pm \sqrt{(1 - \cos x)(1 + \cos x)}$  $\sin x + \sin y = 2\sin \frac{x+k}{2} \cos \frac{x-1}{2}$  $\sin x \sin y = \frac{\cos(x+y) - \cos(x-y)}{2}$  $\sin x \cos y = \frac{\sin(x+y) + \sin(x-y)}{2}$ sin, cos:  $\sin^2 x + \cos^2 x = 1$  $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$  $\sin 2x = 2 \sin x \cos x$  $\sin \frac{\pi}{2} = \pm \sqrt{\frac{1-\cos \pi}{2}}$ 008 \$ = # \\ 1+con =  $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$ 

tanh: tanh r = tinh s  $\coth. \ \coth x = \frac{1}{(anh^2 x + soch^2 x = 1)}$ 

 $\cot h^2 x - \operatorname{cach}^2 y = 1$ 

 $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$  $\sinh(x+y) = \sinh x \cosh y + \sinh y \coth x$ sinh 2r = 2 sinh z cosh 3

Sector x dx = tan (sinbix)+C