MA3220 Ordinary Differential Equations

AY2022/23 Semester 1 · Midterm Examination Cheatsheet · *Prepared by Tian Xiao @snoidetx*

Differential Equations

Solving a separable ODE

$$y'(t) = P(t)Q(y)$$
$$\int \frac{1}{Q(y)}\,dy = \int P(t)\,dt + C$$

Existence & uniqueness of solutions

• 1st order linear ODE: If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then for any $y_0 \in \mathbb{R}$, there exists a unique solution to the differential equation $y' + p(t)y = g(t)$ for each t in I, with initial condition $y(t_0) = y_0$.

• 1st order non-linear ODE: Consider the equation $y' = f(t, y)$ with initial condition $y(t_0) = y_0$. If f and $\frac{\partial f}{\partial y}$ are both continuous in some rectangle $R = (\alpha, \beta) \times (\gamma, \delta)$ containing the point (t_0, y_0), then in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there exists a unique solution to the IVP.

• 2nd order linear ODE: If the functions p, q, g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique solution to the differential equation $y'' + p(t)y' + q(t)y = g(t)$ for each t in I, with initial condition $y(t_0) = y_0$ and $y'(t_0) = y_0'$.

1st Order ODEs

Terminologies

• Linearity: An ODE is *linear* if it can be written in the form $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1 y = P(x)$.

• Homogeneity: $P(x) = 0$.

• Convexity: If $y''(x) > 0$, then $y(x)$ is *concave*; otherwise, it is *convex*.

• Equilibrium solution: $y'(x) = 0$.

• Exact equation: An ODE $M(x, y) + N(x, y)y' = 0$ is called an *exact ODE* if there exists a function $\psi(x, y)$ such that $\frac{\partial\psi}{\partial x}(x, y) = M(x, y)$ and $\frac{\partial\psi}{\partial y}(x, y) = N(x, y)$.

- If an ODE is exact, $M_y = N_x$.

- If M, N, M_y, N_X are continuous in a simply connected region $D \subset \mathbb{R}^2$, then the equation $M(x, y) + N(x, y)y' = 0$ is an exact equation if and only if $M_y = N_x$.

Solving a 1st order linear ODE

$$y' + P(x)y = Q(x)$$
$$\text{Let } \mu(x) = e^{\int P(x)dx},$$
$$\mu'(x) = \mu(x)P(x);$$
$$\mu(x)y' + \mu'(x)y = \mu(x)Q(x)$$
$$\mu(x)y = \int \mu(x)Q(x)\,dx$$
$$y = \frac{\int \mu(x)Q(x)\,dx}{\mu(x)y} + C$$

Euler's method

1. Partition the interval $[x_0, X]$ into a finite number of mesh points $x_0 < x_1 < \cdots < x_n = X$. If they are uniformly distributed, then the step size $h = \frac{X - x_0}{n}$.

2. For each $i = 1, 2, \cdots, n$, obtain the approximate solution y_i by $y_i = y_{i-1} + y'(i-1)h$.

2nd Order ODEs

Superposition principle

For a linear homogenous equation $L(y) = 0$, if y_1 and y_2 are solutions, then for any constant c_1 and c_2, the linear combination $c_1y_1 + c_2y_2$ is also a solution.

Wronskian and general solution

Let y_1 and y_2 be two solutions of a 2nd order linear homogenous ODE, their *Wronskian* is defined as

$$W[y_1, y_2](t) := \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

Let y_1 and y_2 be two solutions of $y'' + p(t)y' + q(t) = 0$ in an interval I, with p, q continuous in I. Then $y(t) = c_1y_1 + c_2y_2$ is the *general solution* in I if and only if $W[y_1, y_2](t_0) \neq 0$ for some $t_0 \in I$.

Abel's theorem

Let y_1 and y_2 be two solutions of $y'' + p(t)y' + q(t) = 0$ in an interval I, with p, q continuous in I. Then their Wronskian satisfies

$$W[y_1, y_2](t) = ce^{-\int p(t)\,dt}$$

for some constant c. As a result, W is either always 0 or never 0.

Solving a 2nd order linear homogenous ODE

$$ay'' + by' + c = 0$$

Case I: $\Delta > 0$, $x = \lambda_1$ or λ_2.

$$y = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t}$$

Case II: $\Delta < 0$, $x = \alpha \pm \beta i$.

$$y = e^{\alpha t}(c_1 \cos\beta t + \sin\beta t)$$

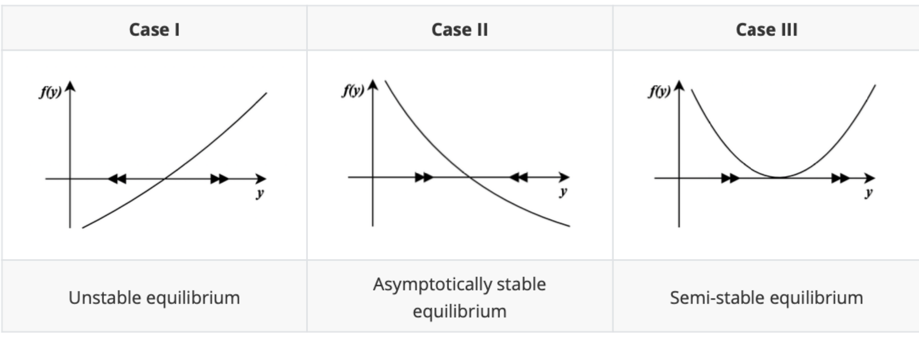
Case III: $\Delta = 0$, $x = \lambda$.

$$y = (c_1 + c_2t)e^{\lambda t}$$

Finding another solution

• Abel's theorem: Plug in the value of $y_1(t)$, $y_1'(t)$ and $ce^{-\int p(t)\,dt}$ and solve for y_2. Set $c = 1$ for convenience.

• Reduction of order: Let $y_2(t) = v(t)y_1(t)$ and plug in to the ODE.



Unstable equilibrium | Asymptotically stable equilibrium | Semi-stable equilibrium

Solving a 2nd order linear non-homogenous ODE

• Making the right guess:

$g(t)$	guess
Ce^{kt}	Ae^{kt}
$C\sin kt$; $C\cos kt$	$A\sin kt + B\cos kt$
degree-n polynomial	degree-n polynomial
sum of different types of terms	sum of their respective guesses
product of different types of terms	product of their respective guesses

- We handle exceptions by multiplying t to our guess.

• Variation of parameters: For the equation $y'' + p(t)y + q(t)y = g(t)$, let the general solution be $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$, where y_1 and y_2 are the solutions to the corresponding homogenous equation. Set $u_1'y_1+u_2'y_2 = 0$, so that $Y' = u_1y_1'+u_2y_2'$ and $Y'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$. Plug this into the ODE and we get:

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = g(t) \end{cases}$$

Solve this simultaneous equation and we get:

$$\begin{cases} u_1(t) = -\int \frac{y_2(t)g(t)}{W[y_1,y_2](t)}\,dt + c_1 \\ u_2(t) = -\int \frac{y_1(t)g(t)}{W[y_1,y_2](t)}\,dt + c_2 \end{cases}$$

• Using power series: A *power series* centered at x_0 is an infinite series of the form $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$. Guess $y = f(x)$ and plug into the ODE. Use *shift of summation index* to get a recurrence relation.

- Taylor series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- Convergence radius: Every power series has a *convergence radius* ρ (can be 0, positive or infinity), such that when $|x-x_0| > \rho$, the series diverges and when $|x-x_0| > \rho$, the series converges absolutely. If $f(x)$ is a polynomial, the power series of the function $\frac{1}{f(x)}$ centered at x_0 has its convergence radius equal to the distance between x_0 and the nearest complex roots of $f(x)$.

- Ratio test for convergence: Consider the expression

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||x-x_0|$$

If the value < 1, the series converges absolutely at x; if the value $= 1$, the test is inconclusive; if the value > 1, the series diverges.

- $f^{(n)}(x_0) = n!a_n$.

- A point t_0 is called an *ordinary point* if both $p(t)$ and $q(t)$ are analytical at t_0; otherwise it is called a *singular point.* If t_0 is an ordinary point, then the ODE has a series solution centered at t_0:

$$y(t) = \sum_{n=0}^{\infty} a_n(t-t_0)^n = a_0y_1(t)+a_1y_2(t)$$

Here y_1 and y_2 form a fundamental set of solutions, and their convergence radius is at least the minimum of the convergence radius of p and q.

Appendix: Integrations

Basic

- $\int k\,dx = kx + C$
- $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$
- $\int \frac{1}{x}\,dx = \ln|x| + C$
- $\int e^x\,dx = e^x + C$

Fractional

- $\int \frac{1}{ax+b} = \frac{1}{a}\ln|ax+b| + C$
- $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \sinh^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\tanh^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{x\sqrt{1-x^2}}\,dx = -^{-1}x + C$
- $\int \frac{1}{|x|\sqrt{1+x^2}}\,dx = -^{-1}x + C$

Logarithmic

- $\int \ln x\,dx = x\ln x - x + C$

Trigonometric

- $\int \cos x\,dx = \sin x + C$
- $\int \sin x\,dx = -\cos x + C$
- $\int \tan x\,dx = \ln|\sec x| + C$
- $\int \sec x\,dx = \ln|\sec u + \tan u| + C$
- $\int \sec^2 x\,dx = \tan x + C$
- $\int \sec x\tan x\,dx = \sec x + C$
- $\int \csc x\cot x\,dx = -\csc x + C$
- $\int \csc^2 x\,dx = -\cot x + C$
- $\int \sinh x\,dx = \cosh x + C$
- $\int \cosh x\,dx = \sinh x + C$
- $\int^2 x\,dx = \tanh x + C$
- $\int^2 x\,dx = -\coth x + C$
- $\int x\tanh x\,dx = -x + C$
- $\int x\coth x\,dx = -x + C$

Appendix: DE

1. $M(x) - N(y)y' = 0$

(*Separable*) Separate the variables x and y and rewrite the equation as $\int M(x)\,dx = \int N(y)\,dy$.

2. $y' + P(x)y = Q(x)$

Multiply both sides by an **integrating factor** $\mu(x) = e^{\int P(x)\,dx}$:

$$\mu(x)y' + \mu(x)P(x)y = \mu(x)Q(x)$$
$$\mu(x)y = \int \mu(x)Q(x)\,dx$$

3. $y' + P(x)y = Q(x)y^n$

(*Bernoulli*) Let $z = y^{1-n}$, then $z' = (1-n)y^{-n}y'$. Hence we have:

$$y^{-n}y' + P(x)y^{1-n} = Q(x)$$
$$\frac{z'}{1-n} + P(x)z = Q(x)$$

and use integrating factor.

4. $ay'' + by' + cy = 0$

Consider the **characteristic equation** $ax^2 + bx + c = 0$ with roots λ_1 and λ_2:

- If $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, then $y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$.
- If $\lambda_1 = \lambda_2 \in \mathbb{R}$, then $y = (c_1 + c_2x)e^{\lambda x}$.
- If $\lambda_1 \neq \lambda_2 = \alpha \pm \beta i \in \mathbb{C}$, then $y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$.

5. $ay'' + by' + cy = r(x), r(x) \neq 0$

The goal is to find the **particular solution** y_p:

- If $r(x)$ is in the form of $g(x)e^{kx}$, let $y_p(x) = u(x)e^{kx}$.
- If $r(x)$ is in the form of $g(x)\cos kx$ or $u(x)\sin kx$, let $z(x) = u(x)e^{ikx}$ and take $Re(z)$ or $Im(z)$).