CS5461 Algorithmic Mechanism Design

Notes

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1 Games and Nash Equilibria (NE)

Normal-Form Games:

① A set of players $N = \{1, \cdots, n\}$;
② each player has a set of possible actions A_i;
③ action profile $\mathbf{a} \in A_1 \times \cdots \times A_n = A$;
④ utility of player i, $u_i(\mathbf{a})$.

- Best response: $\mathrm{BR}_i(\mathbf{a}_{-i}) = \{b \in A_i : b \in \arg\max u_i(\mathbf{a}_{-i}, b)\}$.

Pure NE: $\forall i \in N\ [a_i \in \mathrm{BR}_i(\mathbf{a}_{-i})]$.

Randomized Actions:

① Each player has a set of possible strategies $\Delta(A_i) = \{\mathbf{p}_i\}$;
② strategy profile $\mathbf{p} \in \Delta(A_i) \times \cdots \times \Delta(A_n)$;
③ utility of player i, $u_i(\mathbf{p})$.

Mixed NE: $\forall \mathbf{q}_i \in \Delta(A_i)\ [u_i(\mathbf{p}) \geq u_i(\mathbf{p}_{-i}, \mathbf{q}_i)]$.

Nash's Theorem: A not necessarily pure NE always exists.

Computing NE in 2×2 Games:

① Compute all NEs in which at least one player plays a pure strategy;
② Compute all NEs in which both players strictly mix;
③ Simplify.

Domination: $\mathbf{p}_i$ dominates $\mathbf{q}_i$ if $\forall \mathbf{p}_{-i}[u_i(\mathbf{p}_{-i}, \mathbf{p}_i) \geq u_i(\mathbf{p}_{-i}, \mathbf{q}_i)]$.

- Strict domination: $u_i(\mathbf{p}_{-i}, \mathbf{p}_i) > u_i(\mathbf{p}_{-i}, \mathbf{q}_i)$.
- Theorem: If action a_i is strictly dominated by some $\mathbf{p}_i$, then a_i is never played with any positive probability in any NE.
  - ▷ Iterated removal of dominated strategies.

2 Auction

Auction Games: $u_i = \begin{cases} v_i - p & \text{if } i \text{ gets the item} \\ 0 & \text{otherwise} \end{cases}$, where

① v_i is i's own valuation of the item;
② p is the price to be paid.

Vickrey Auction: All players submit bids simultaneously in sealed envelops. The highest bidder wins and pays the second highest price.

- Truthfulness: Truthful bidding is a dominant strategy.

Proof. If $v_i \geq b_{n-1}$, if bid higher than b_{n-1} then still pays the second highest price (same utility); if bid lower than b_{n-1} then cannot get the item (0 utility). If $v_i \geq b_{n-1}$, if bid higher than b_{n-1} then will get the item (-ve utility); if bid lower than b_{n-1} then cannot get the item (same utility).

VCG Mechanism: Choose some outcome o^* that maximizes $\sum_i b_i(o^*)$. Each player pays the *externality* it imposes: $\sum_{i \neq j} b_i(o^*_{-j}) - \sum_{i \neq j} b_i(o^*)$.

- Truthfulness:

Proof. Utility of j is

$$v_j - \left(\sum_{i \neq j} b_i(o^*_{-j}) - \sum_{i \neq j} b_i(o^*)\right)$$
$$= v_j + \sum_{i \neq j} b_i(o^*) - \sum_{i \neq j} b_i(o^*_{-j}).$$

The sum of first two terms is maximized when j reports truthfully since VCG selects socially optimal outcomes. The third term does not depend on j.

- Challenges:
  ① Each bidder has 2^m private parameters;
  ② computational issue;
  ③ revenue non-monotonicity: More bidders may lead to less revenue.

3 Facility Location

Facility Location Games: Cost of $i = |f(\mathbf{x}) - x_i|$.

- Minimizing the maximum cost ($f(\mathbf{x}) = \frac{x_1 + x_n}{2}$) is not truthful.
- Median strategy (rounded down if two median) is truthful (strategy proof) and socially optimal (w.r.t. total cost).
- Leftmost/rightmost strategy is truthful.
- $\frac{1}{4}$ leftmost + $\frac{1}{4}$ rightmost + $\frac{1}{2}\frac{x_1 + x_n}{2}$ is truthful.

Theorem: Any deterministic truthful mechanism has a worst-case approximation ratio of at least 2 to the maximum cost; any deterministic truthful mechanism has a worst-case approximation ratio of at least $\frac{3}{2}$ to the maximum cost.

4 Routing

Price of Anarchy: $\mathrm{PoA}(G) = \frac{\mathrm{WorstNE}(G)}{\mathrm{OPT}(G)}$.

Atomic Routing Games:

① $k \in \mathbb{N}$ units of traffic and each unit must be routed as a whole.
② Each edge e has a cost function $c_e : \mathbb{N} \to \mathbb{R}^+$.

- Theorem: A pure NE always exists.

Define a potential function $\Phi(f) = \sum_{e \in E} \sum_{i=1}^{f_e} c_e(i)$. If a player deviates from path P to $\hat{P}$, then

$$\Phi(\hat{f}) - \Phi(f) = \sum_{e \in \hat{P}} c_e(\hat{f}_e) - \sum_{e \in P} c_e(f_e),$$

which is exactly the deviator's individual cost. Thus, the flow that minimizes Φ is an NE.

Non-Atomic Routing Games:

① k units of traffic that can be divided arbitrarily.

- Theorem: A pure NE always exists (change sum to integration).

5 Cooperative Games

Induced Subgraph Games: Players are nodes; value of a coalition is the value of the total edge weights in the subgraph.

Weighted Voting Games: $u(S) = 1$ iff $\sum_{i \in S} w_i \geq q$.

Cooperative Games:

① A set of players $N = \{1, \cdots, n\}$;
② valuation function $v : 2^N \to \mathbb{R}_{\geq 0}$; $v(\emptyset) = 0$;
③ coalition structure CS: A partition of N;
④ $\mathrm{OPT}(\mathcal{G}) \coloneqq \max_{\mathrm{CS}} \sum_{S \in \mathrm{CS}} v(S)$.

- Efficiency: A vector $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$ satisfying $\sum_{i \in N} x_i = v(N)$;
- Individual rationality: $x_i \geq v(i)$ for all $i \in N$;
- Imputation: A vector satisfying EFF + IR.
- Monotone: $S \subseteq T \Rightarrow v(S) \leq v(T)$;
- Simple: Monotone + $v(S) \in \{0, 1\}$ for all S;
- Superadditive: For disjoint S, T, $v(S) + v(T) \leq v(S \cup T)$;
- Convex: $S \subseteq T \Rightarrow m_i(S) \geq m_i(T)$.

The Core: An **imputation** $\mathbf{x}$ is *in the core* if for all $S \subseteq N$, $\sum_{i \in S} x_i \geq v(S)$.

- Veto player: In a simple game, a member of every winning coalition.
- Theorem: The core of a simple game is not empty iff $\mathcal{G}$ has a veto player. The payoff is only distributed among veto players.
- Theorem: The core of an induced subgraph game is not empty iff the graph has no negative cut.
  - ▷ Consider the Shapley value $\phi_i = \sum_{j \in N} \frac{1}{2} w(i, j)$.

The Shapley Value:

- Axioms: EFF, SYM, DUM, LIN;

6 Stable Matching

Stable Matching:

- A pair (s, h) *blocks* M if $h \succ_s M(s)$ and $s \succ_h M^{-1}(h)$.
- A matching M is *stable* if there is no blocking pair.

Gale-Shapley Algorithm:

① Start with all students unassigned;
② While there are unassigned students:
  (a) Each unassigned student proposes to his/her favorite not-yet-proposed-to hospital;
  (b) Each hospital looks at the list of students that proposed to it in this round + whoever is assigned to it now, picks its most preferred student; all others remain unassigned.
③ Return the resulting matching.

- Theorem: The Gale-Shapley algorithm terminates in n^2 iterations with a stable matching.
- Theorem: The Gale-Shapley algorithm (with students proposing) assigns

▷ each student s to the hospital best(s); AND
▷ each hospital h to the student worst(h).

7 Nash Bargaining Solution

Pareto Optimality: (x_1, y_1) *Pareto dominates* (x_2, y_2) if $x_1 \geq x_2$ and $y_1 \geq y_2$. (x_1, y_1) is *Pareto optimal* if it is not Pareto dominated.

Nash Bargaining Solution: If $(v_1, v_2) \in S$ (convex, compact) then (x, y) else (d_1, d_2).

- Axioms:
  - ▷ Efficiency: No outcome (v_1, v_2) Pareto dominates $(f_1(S, \mathbf{d}), f_2(S, \mathbf{d}))$;
  - ▷ Symmetry: $(f_1(S, \mathbf{d}), f_2(S, \mathbf{d})) = (f_2(S^\top, \mathbf{d}^\top), f_1(S^\top, \mathbf{d}^\top))$;
  - ▷ Independence of Irrelevant Alternatives (IIA): If $(f_1(S, \mathbf{d}), f_2(S, \mathbf{d})) \in S'$, then $(f_1(S, \mathbf{d}), f_2(S, \mathbf{d})) = (f_1(S', \mathbf{d}), f_2(S', \mathbf{d}))$;
  - ▷ Invariance under Equivalent Representations (IER): $f_i((\alpha_1, \alpha_2)S + \boldsymbol{\beta}, (\alpha_1, \alpha_2)\mathbf{d} + \boldsymbol{\beta}) = \alpha_i f_i(S, \mathbf{d}) + \beta_i$.
- Nash bargaining solution: $\max(v_1 - d_1)(v_2 - d_2)$ s.t. $(v_1, v_2) \in S$.
  - ▷ Uniquely satisfies EFF, SYM, IIA, IER.

Different fairness:

- Utilitarian: $\max \sum_i u_i(A)$;
- Nash:$\max \prod_i u_i(A)$;
- Egalitarian:$\max \min_i u_i(A)$.

8 Fair Allocation of Indivisible Goods

Allocation:

① Players $N = \{1, \cdots, n\}$;
② goods $G = \{g_1, \cdots, g_m\}$;
③ player i has value $v_i(g)$ for good g;
④ additivity: $v_i(G') = \sum_{g \in G'} v_i(g)$;
⑤ allocation: A partition of goods $A = (A_1, \cdots, A_n)$ where bundle A_i is allocated to player i.

- Proportionality: $v_i(A_i) \geq \frac{1}{n} \cdot v_i(G)$;
- Maximin share (MMS): Each player divides the goods into n bundles so as to maximize the value of the minimum-value bundle.
- Envy-freeness: $v_i(A_i) \geq v_i(A_j)$.
  - ▷ Equiv. to Proportionality when $n = 2$; stronger when 3.
- EF1: $\forall i, j\ \exists g \in A_j$ s.t. $v_i(A_i) \geq v_i(A_j \setminus g)$.
- EFX: $\forall i, j\ \forall g \in A_j\ v_i(A_i) \geq v_i(A_j \setminus g)$.

The Round-Robin Algorithm: Each player takes turn to choose its favorite good.

- Satisfies EF1.

Envy-Cycle Elimination Algorithm: This is for general monotonic valuations: $v_i(S) \leq v_i(T)$ for any $S \subseteq T \subseteq G$.

① Allocate one good at a time in an arbitrary order.
② Maintain an envy graph with the players as its vertices, and a directed edge $i \to j$ if i envies j with respect to the current (partial) allocation.
③ At each step, the next good is allocated to a player with no incoming edge. Any cycle that arises is eliminated by giving j's bundle to i for any edge $i \to j$ in the cycle.

- Satisfies EF1.

Maximum Nash Welfare: $\max_M \prod_i v_i(M_i)$.

- An MNW allocation satisfies EF1. If MNW $= 0$, maximize the number of players with positive utility, then maximize Nash welfare among these players.
- Pareto optimal.

Cut and Choose: When $n = 2$, the first player divides the goods into two bundles that are as equal as possible in her view and the second player chooses.

- Satisfies EFX + MMS.

9 Cake Cutting

Cake Cutting:

① Cake $[0, 1]$;
② agents $N = \{1, \cdots, n\}$;
③ agents' valuations $v_1, \cdots, v_n$;
  - Non-negative;
  - additive: Values of disjoint pieces add up;
  - non-atomic: The value of any single point is 0;
  - normalized: Sum of whole cake is 1.

④ allocation $A = (A_1, \cdots, A_n)$;
  - Connected: If each A_i is a single interval.

- Proportionality: $v_i(A_i) \geq \frac{1}{n}$.
- Envy-freeness: $v_i(A_i) \geq v_i(A_j)$.

Robertson-Webb Model: Allows two types of queries:

- $\text{Eval}_i(x, y)$: Return $v_i(x, y)$.
- $\text{Cut}_i(x, \alpha)$: Return the leftmost y such that $v_i(x, y) = \alpha$, or state that no such point exists.

Cut and Choose: Agent 1 cuts the cake into two pieces according to her opinion; Agent 2 chooses the piece she prefers.

- Satisfies EF and PROP.
- 2 queries.
- Truthful for the chooser, not truthful for the cutter.

Dubins-Spanier Protocol: Repeat: When the piece of cake to the left of the knife is worth $\frac{1}{n}$ to some agent, that agents shout "Stop!" and leaves the procedure with that piece. The last agent gets the remaining piece.

- Satisfies PROP.
- $\mathcal{O}(n^2)$ queries.

Even-Paz Protocol: Each agent marks the point that divides the cake into two halves of equal value, according to his/her own opinion. Let t be mark number $\frac{n}{2}$ from the left. Recurse on $[0, t]$ with the left $\frac{n}{2}$ agents, and on $[t, 1]$ with the right $\frac{n}{2}$ agents. When we are down to one agent, that agent gets the whole cake.

- Satisfies PROP.
- $\mathcal{O}(n \log n)$ queries.

Selfridge–Conway Protocol: 3 agents. Agent 1 divides the cake into 3 equal pieces according to her opinion. If Agent 2 and 3 prefer the same piece, we are done. If they prefer the same piece, Agent 2 is asked to trim the first piece so that it has equal value as the second piece. The trimmed piece is cut further and allocated carefully.

- Satisfies EF.
- 5 cuts and 9 queries.

Envy-Freeness Approximation: Follow Dubin-Spanier. When the piece is worth $\frac{1}{3}$ to someone, she shouts "Stop" and leaves the procedure with that piece. Suppose the knife reaches the right end but some cake is still unallocated:

① If there are still agents left, the remaining cake is given to them arbitrarily;
② if there is no agent left, the remaining cake is given to the last one who shouts.

Truthful Mechanism for Piecewise Uniform Valuations: Two agents. Each agent eats their desired parts at the same speed until they meet. Any part that an agent jumps over goes to the other agent.

9.1 Rent Division

Rent Division:

① Players $N = \{1, \cdots, n\}$;
② rent price r;
③ v_{ij}: player i's value of room j s.t. $\sum_j v_{ij} = r$;
④ room allocation $\sigma : N \to N$;
⑤ rent division $\mathbf{p} = (p_1, \cdots, p_n)$. p_j is j's price.

- Envy-freeness: $v_{i\sigma(i)} - p_i \geq v_{ij} - p_j$.

General Algorithmic Framework:

① Compute a socially optimal allocation (max weighted matching)
② Find an EF price vector (using linear programming).

Equitability under EF Constraint:

$$\min_{\mathbf{p}} \quad D(\mathbf{p}) := \max_{i,j} u_i(\mathbf{p}) - u_j(\mathbf{p})$$
$$\text{s.t.} \quad \mathbf{p} \text{ is envy-free.}$$

Maximin under EF Constraint:

$$\max_{\mathbf{p}} \quad U(\mathbf{p}) := \min_i u_i(\mathbf{p})$$
$$\text{s.t.} \quad \mathbf{p} \text{ is envy-free.}$$

- Theorem: There is a unique maximin price and it is also equitable.
  - ▷ There exist equitable price vectors that are not maximin.

10 Committee Voting

Approval Committee Voting:

- Voters $N = \{1, \cdots, n\}$;
- candidates $|C| = m$;

- voter i approves A_i;
- committee to be chosen $|W| = k$;
- utility $u_i(W) = |A_i \cap W|$.

- Social welfare: Total number of approvals.
- Coverage: Number of voters who approve at least one committee member.

Justified Representation: A group of votes $S \subseteq N$ such that $|S| \geq n/k$ and $|\bigcap_{i \in S} A_i| \geq 1$ is a cohesive group. For any cohesive group of votes S, there exists $i \in S$ such that $|A_i \cap W| \neq \emptyset$.

Extended Justified Representation: A group of votes $S \subseteq N$ such that $|S| \geq t \cdot n/k$ and $|\bigcap_{i \in S} A_i| \geq t$ is a t-cohesive group. For any positive integer t and t-cohesive group of votes S, there exists $i \in S$ such that $|A_i \cap W| \geq t$.

Approval Voting: Maximizing social welfare.

- Does not always satisfy JR.

Chamberlin-Courant: Maximizing coverage.

- Satisfies JR.

GreedyCC: At each step, choose the i that covers as many uncovered voters as possible.

- Satisfies JR.

Thiele Methods: Maximize $W = \sum_{i \in N} (s_1 + \cdots + s_{i_i(W)})$.

- AV: $s_1 = \cdots = 1$;
- CC: $s_1 = 1; s_2 = \cdots = 0$;
- PAV: $s_i = 1/i$.
  - ▷ Satisfies EJR.

Method of Equal Shares:

- Each voter has a budget of k/n.
- Each candidate costs 1; the voters who approve this candidate have to "pool" their money to add this candidate to the committee.
- Start with an empty committee.
- In each round, we want to add a candidate whose approved voters have a total budget of $\geq k$ left.
- If there are several such candidates, choose one such that the maximum amount that any agent has to pay is minimized.
- If no more candidate can be afforded but the committee still has size $< k$, fill in the rest of the committee using some tie-breaking criterion (e.g., by maximizing approval score).

- Satisfies EJR.

11 Tournament

Tournament: $T = (A, \succ)$.

- Outdegree: Number of alternatives dominated by x;
- condorcet winner: Alternatives that dominate all others (outdegree $n - 1$;
- condorcet loser: Alternatives that are dominated by all others (outdegree 0.

- Invariant under isomorphism
  - ▷ For a cycle of 3, every solution must select all 3.
- Condorcet-consistency: If there is a condorcet winner x, then x is uniquely chosen.
- Monotonicity: If x is chosen, then it should still be chosen if it is strengthened against another alternative y.

Copeland Set: Alternatives with highest outdegree.

Top Cycle: Alternatives that can reach every other alternative via a directed path (of any length).

- Equivalent definition: (Unique) smallest nonempty set B of alternatives such that all alternatives in B dominate all alternatives outside B.

Uncovered Set: Alternatives that can reach every other alternative via a directed path of length ≤ 2.

- Covering relation: x covers y if x dominates y and dominates all z dominated by y.
- Equivalent definition: The set of uncovered alternatives.
- UC $\subseteq$ TC.
- CO $\subseteq$ UC.
- BA $\subseteq$ UC.

Banks Set: Alternatives that appear as the maximal (i.e., strongest) element of some transitive subtournament that cannot be extended. A transitive tournament is one such that the alternatives can be ordered as $a_1, \cdots, a_k$ so that a_i dominates a_j for all $i < j$.

Trivial: All alternatives.

Slater Set: Alternatives that are maximal elements in some transitive tournament that can be obtained by inverting as few edges as possible.

Bipartisan set: Alternatives that are chosen with nonzero probability in the (unique) Nash equilibrium of the zero-sum game formed by the tournament matrix.

Markov set: Alternatives that stay most often in the "winner-stays" competition corresponding to the tournament.

Knockout Tournament: An alternative is said to be a knockout winner if it wins a (balanced) knockout tournament under some bracket.

- Strong king: Suppose x beats P and loses to Q. If $|P| \geq \frac{n}{2}$, then x is a knockout winner.

Tournament Fixing Problem: Given a set of alternatives A, the tournament graph and our favorite alternative, is there a bracket for a balanced knockout tournament where our favorite alternative wins?