and obtain the following partial F results:

$$F(S_1|C_1, S_2) = 0.01 \quad (P = .90)$$
$$F(S_2|C_1, S_1) = 0.85 \quad (P = .25)$$

Then, S_1 is "less significant" than S_2, controlling for C_1 and the other S variable. Under the strategy of backward elimination then, S_1 should be deleted before the elimination of S_2 is considered. However, to delete both S_1 and S_2 at this point would be incorrect. In fact, when considering the reduced model $Y = \beta_0 + \beta_1 C_1 + \beta_2 S_2 + E$, $F(S_2|C_1)$ may be highly significant. In other words, if S_1 is not significant given S_2 and C_1, and S_2 is not significant given S_1 and C_1, then S_2 is not necessarily unimportant in a reduced model containing S_2 and C_1 but not S_1.

9-4 Multiple-Partial *F* Test

This testing procedure addresses the more general problem of assessing the additional contribution of two or more independent variables over and above that made by other variables already in the model. For the example involving Y = WGT, X_1 = HGT, X_2 = AGE, and X_3 = (AGE)^2, we may be interested in testing whether the AGE variables, taken collectively, significantly improve the prediction of WGT given that HGT is already in the model. In contrast to the partial F test discussed in Section 9-3, the multiple-partial F test concerns the simultaneous addition of two or more variables to a model. Nevertheless, the test procedure is a straightforward extension of the partial F test.

9-4-1 The Null Hypothesis

We wish to test whether the addition of the k variables $X_1^*, X_2^*, \ldots, X_k^*$ significantly improves the prediction of Y once the p variables $X_1, X_2, \ldots, X_p$ are already in the model. The (full) model of interest is thus

$$Y + \beta_0 + \beta_1 X_1 + \beta_p X_p + \beta_1^* X_1^* + \cdots + \beta_k^* X_k^* + E$$

Then, the null hypothesis of interest may be stated as H_0: "$X_1^*, X_2^*, \ldots, X_k^*$ do not significantly add to the prediction of Y given that $X_1, X_2, \ldots, X_p$ are already in the model" or, equivalently, H_0: $\beta_1^* = \beta_2^* = \cdots = \beta_k^* = 0$ in the (full) model.[1]

From the second version of H_0, it follows that the *reduced* model is of the form

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_p X_p + E$$

(i.e., the X_i^* terms are dropped from the full model).

Using the above example, the (full) model is

$$\text{WGT} = \beta_0 + \beta_1 \text{HGT} + \beta_1^* \text{AGE} + \beta_2^* (\text{AGE})^2 + E$$

The null hypothesis here is H_0: $\beta_1^* = \beta_2^* = 0$.

[1] In Chapter 10, an equivalent expression for this null hypothesis will be given in terms of a multiple-partial correlation coefficient.

9-4-2 The Procedure

As with the partial F test, we must compute the extra sum of squares due to the addition of the X_i^* terms to the model. In particular, we have

$$\begin{aligned}
&SS(X_1^*, X_2^*, \ldots, X_k^* | X_1, X_2, \ldots, X_p) \\
&\quad = \text{regression } SS(X_1, X_2, \ldots, X_p, X_1^*, X_2^*, \ldots, X_k^*) - \text{regression } SS(X_1, X_2, \ldots, X_p) \\
&\quad = \text{residual } SS(X_1, X_2, \ldots, X_p) - \text{residual } SS(X_1, X_2, \ldots, X_p, X_1^*, X_2^*, \ldots, X_k^*)
\end{aligned}$$

Using this extra sum of squares, we then obtain the following F statistic:

$$F(X_1^*, X_2^*, \ldots, X_k^* | X_1, X_2, \ldots, X_p) = \frac{SS(X_1^*, X_2^*, \ldots, X_k^* | X_1, X_2, \ldots, X_p)/k}{\text{MS residual } (X_1, X_2, \ldots, X_p, X_1^*, X_2^*, \ldots, X_k^*)} \qquad (9.6)$$

addition reduced

This F statistic has an F distribution with k and $n - p - k - 1$ degrees of freedom under H_0: $\beta_1^* = \beta_2^* = \cdots = \beta_k^* = 0$.

Note that in (9.6) we must divide the extra sum of squares by k, the number of regression coefficients specified to be zero under the null hypothesis of interest. This number k is also the numerator degrees of freedom for the F statistic. The denominator of the F is the mean-square residual for the full model; its degrees of freedom is $n - (p + k + 1)$, which is $n - 1$ minus the number of variables in this model (namely, $p + k$).

An alternative way to write this F statistic is

$$\begin{aligned}
F(X_1^*, X_2^*, \ldots, X_k^* | X_1, X_2, \ldots, X_p) &= \frac{[\text{regression } SS(\text{full}) - \text{regression } SS(\text{reduced})]/k}{\text{MS residual (full)}} \\
&= \frac{[\text{residual } SS(\text{reduced}) - \text{residual } SS(\text{full})]/k}{\text{MS residual (full)}}
\end{aligned}$$

F (2,

Using the information in Table 9-1 involving WGT, HGT, AGE, and $(\text{AGE})^2$, we can test H_0: $\beta_1^* = \beta_2^* = 0$ in the model $\text{WGT} = \beta_0 + \beta_1 \text{HGT} + \beta_1^* \text{AGE} + \beta_2^* (\text{AGE})^2 + E$ as follows:

$$\begin{aligned}
F(\text{AGE}, (\text{AGE})^2 | \text{HGT}) &= \frac{\{\text{regression } SS[\text{HGT, AGE}, (\text{AGE})^2] - \text{regression } SS(\text{HGT})\}/2}{\text{MS residual } [\text{HGT, AGE}, (\text{AGE})^2]} \\
&= \frac{[(588.92 + 103.90 + 0.24) - 588.92]/2}{24.40} \\
&= 2.13
\end{aligned}$$

For $\alpha = .05$, the critical point is

$$F_{k, n-p-k-1, 0.95} = F_{2, 12-1-2-1, 0.95} = 4.46,$$

so that H_0 would not be rejected at $\alpha = .05$.

In the above calculation, note that we used the relationship

$$\begin{aligned}
&\text{regression } SS[\text{HGT, AGE}, (\text{AGE})^2] \\
&\quad = \text{regression } SS(\text{HGT}) + \text{regression } SS(\text{AGE}|\text{HGT}) + \text{regression } SS[(\text{AGE})^2|\text{HGT, AGE})] \\
&\quad = 588.92 + 103.90 + 0.24
\end{aligned}$$

Alternatively, we could form two ANOVA tables (Table 9-2), one for the full and one for the reduced model, and then extract the appropriate regression and/or residual sum-of-

TABLE 9-2 ANOVA tables for WGT regressed on HGT, AGE, and $(AGE)^2$

Full Model

Source	df	SS	MS
Regression (X_1, X_2, X_3)	3	693.06	231.02
Residual	8	195.19	24.40
Total	11	888.25	

Reduced Model

Source	df	SS	MS
Regression (X_1)	1	588.92	588.92
Residual	10	299.33	29.93
Total	11	888.25	

square terms from these tables. More examples of partial F calculations will be given at the end of this chapter.

9-4-3 Comments

As with the partial F test, the multiple-partial F test is very useful for assessing the importance of extraneous variables. In particular, it is often used to test whether a "chunk" (i.e., a group) of variables having some trait in common is important when considered together. An example of a chunk would be a collection of variables all of a certain order (e.g., $(AGE)^2$, HGT $\times$ AGE, and $(HGT)^2$ are all of order 2).

Another example would be a collection of two-way product terms (e.g., X_1X_2, X_1X_3, X_2X_3); this latter group is sometimes referred to as a set of interaction variables (see Chapter 11). It is often of interest to assess the importance of interaction effects collectively before trying to consider individual interaction terms in a model. In fact, the initial use of such a chunk test may reduce the total number of tests to be performed, since variables may be dropped from the model as a group. This, in turn, may help to provide better control of overall Type I error rates, which may be inflated due to multiple testing (Abt, 1981).

9-5 Strategies for Using Partial *F* Tests

In applying the ideas presented in this chapter, the reader will typically use a computer program to carry out the numerical calculations required. Therefore, we will briefly describe the computer output for typical regression programs. In order to understand and use such output, we must discuss two strategies for using partial F tests: *variables-added-in-order tests* and *variables-added-last tests.*

Table 9-3 shows the output from a typical regression computer program[2] for the model

$$WGT = \beta_0 + \beta_1 HGT + \beta_2 AGE + \beta_3 (AGE)^2 + E$$

The results here were computed with *centered* predictors (Section 12-5-2), so (HGT $-$ 52.75), (AGE $-$ 8.833), and $(AGE - 8.833)^2$ were used, with mean HGT = 52.75 and mean AGE = 8.833. Table 9-3 consists of five sections, labeled A through E. Section A provides the overall ANOVA table for the regression model. Note that computer output typically presents numbers with far more significant digits than can be justified. Section B provides a test for significant overall regression, the multiple R^2-value, the mean $(\bar{Y})$ of the dependent variable (WGT), the WGT residual standard deviation or "root-mean-square error" (s), and the coefficient of variation $(100s/\bar{Y})$.

[2] This particular output was produced by the SAS program GLM. We have tried to avoid referring to particular programs, since many good ones exist.