MOBIUS TRANSFORMATIONS AND THEIR INVARIANT CURVES

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Abstract. This project investigates the notion that Loxodromic Möbius transformations are so called because each “keeps invariant a loxodrome.” The goals of the project are to determine the invariant curves kept invariant by each type of Möbius transformation, and develop a method to explicitly identify all the invariant spiral curves of any arbitrary chosen Möbius transformation. To both of these ends, methods from Complex Analysis were used. The Loxodrome, or Rhumb line, on a sphere, is a curve that keeps constant bearing, and it has been used historically to navigate on a globe by keeping the direction headed with constant angle with lines of latitude and longitude. Because of this history it was useful to study Halley’s Proof, from 1690, which showed that the stereographic projections of the loxodromic curves in a plane are logarithmic spirals (which has applications for cartography, but for the purposes of this project allowed the loxodromic curves to be analyzed in the plane rather than always in three dimensions). This topic is often little attended to in the mathematical literature, despite its various applications such as to navigation and understanding weather patterns, and despite its rich history.

1. Introduction

In the Fall of 2015, I began to study Hyperbolic Geometry with Dr. Melkana Brakalova. In order to familiarize myself with the material, I started to read Anderson’s text on the subject. Early on, in a chapter introducing Möbius transformations one line in particular stood out: ”the reason [loxodromic Möbius transformations] are called loxodromic is that each one keeps invariant a loxodrome.” [2] This assertion of fact is where this thesis began. What is a loxodrome? How can one determine which curves are kept invariant by any particular transformation? The section ended directly after this statement, leaving these questions (and more) unanswered.

This paper will first briefly discuss the historical background of the issue, and define significant terms. Then a proof will be provided for a method to determine the invariant curve of a transformation given the curve of its standardized form. Next, invariant curves of each type of transformation will be explored and classified, with examples provided of each. Following, a historical proof by Halley with regards to stereographic projection of loxodromic spirals will be called upon. An area in which this paper requires further study, the choice of conjugating map, will be addressed. Finally, some applications for this material will be mentioned.

1.1. Historical Background.

This topic has historically been of interest for navigational purposes. While sailing along a course determined by a great circle is the shortest possible path, it requires constantly recalculating the angle to travel. So

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courses would be set according to Rhumb Lines, or Loxodromes, which are lines of constant bearing (or angle to latitudes). Thus it was of great importance for navigators to calculate and express these rhumb lines, ideally on a flat plane to be used as a navigational map. [1]

2. Definition of Terms

2.1. **Hyperbolic Geometry.** A very brief mention of one of the key differences between Hyperbolic and Euclidean Geometry is that Euclidean Geometry maintains Euclid's Parallel Postulate: that there is only one line l parallel to a line l_0 through a given point. This postulate is not used in Hyperbolic Geometry, where there are infinite lines (that is, circles) that do not intersect l_0 and also go through one given point. This is because in Hyperbolic Geometry there are no straight lines, only circles and arcs of circles between points.

2.2. **Complex Plane.** The Complex Plane is the set of all numbers $z = x + yi$ so that x and y are real numbers, and i is defined as $\sqrt{-1}$. (x is referred to as the real component and yi as the imaginary component of z). Here, the extended complex plane is considered (that is, all of the points of the complex plane in addition to the point at infinity), taken as the Riemann Sphere, that is, the unit sphere (of radius 1) around the origin in the Complex Plane, with the point $(0, 0, 1)$ taken to be the point at infinity. The plane is lifted onto the sphere by the process of Stereographic Projection.

2.3. **Stereographic Projection.** Stereographic projection is a process that connects points on the plane to ones on the sphere, $x^2 + y^2 + z^2 = 1$, which has the point at infinity as its North Pole, the point $N = (0, 0, 1)$. That is, using a function $\xi : \mathbb{S}^1 - \{N\} \to \mathbb{C}$, a Euclidean line is connected between a point P on the sphere (x_P, y_P, z_P) and the North pole, and where the line intersects with the plane (where $z = 0$) is the output of the function[2]. Explicitly, define ξ to be:

$$\xi(P) = \begin{cases} x = \frac{x_P}{1-z_P} \\ y = \frac{y_P}{1-z_P} \\ z = 0 \end{cases} \tag{1}$$

In the same way, the inverse function ξ^{-1}, which takes the point $\xi(P) = (x_\xi, y_\xi, z_\xi = 0)$ on the plane to a point on the sphere can be expressed:

$$\xi^{-1}(\xi(P)) = P = \begin{cases} x = \frac{2x_\xi}{x_\xi^2+y_\xi^2+1} \\ y = \frac{2y_\xi}{x_\xi^2+y_\xi^2+1} \\ z = \frac{x_\xi^2+y_\xi^2-1}{x_\xi^2+y_\xi^2+1} \end{cases} \tag{2}$$

2.4. **Möbius Transformations.** A Möbius transformation, or a fractional linear transformation, is a function from the Riemann sphere (the extended Complex Plane) to the Riemann sphere of the form:

$$m(z) = \frac{az + b}{cz + d} \tag{3}$$

where a, b, c, and d are complex numbers and $ad - bc \neq 0$. (Also note that $x/0$ is assigned to be the point at infinity). [2]

2.5. **Standard and Normal forms.** A transformation in standard form is $n(z)$ so that $n(z) = q(m(q^{-1}(z)))$, where $q(z)$ is a conjugating function that takes the fixed points to 0 and ∞.

A transformation that has been normalized has a matrix representation whose determinant is equal to 1. To calculate this normal form, consider the transformation $m(z) = \frac{az+b}{cz+d}$ as a matrix, that is $m(z) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The determinant of this matrix is $ad - bc$. To find the constant by which this matrix should be multiplied by to arrive at a determinant of 1, consider the matrix $m_\alpha(z) = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$, with determinant $\alpha^2(ad - bc)$. Set $\alpha^2 = \frac{1}{(ad-bc)}$ and thus $\alpha = \sqrt{\frac{1}{(ad-bc)}}$. Then $m_n(z) = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$ is the matrix of the transformation $m(z)$ in normalized form, with determinant 1. [2]

2.6. **Fixed Points.** A fixed point is a point that the particular transformation takes it into itself. That is, when the transformation is applied to the fixed point, that same point is the output.

2.7. **Loxodromic Curve.** A loxodromic curve is a curve on the sphere that maintains the same angle with all lines of longitude and latitude (which are all at right angles to each other). When this curve is used in particular for navigational purposes, it is known as a Rhumb Line. [1]

2.8. **Invariant curve.** Here, an invariant curve is defined to be a curve for which the Möbius transformation takes the points on that curve, as a set, into that same set; the curve as a set is taken into itself by the transformation. There will be infinitely many of these for each transformation, determined by the choice of initial points of the curve; but this paper endeavors to show that the set of invariant curves can be determined and classified as a set for each transformation.

More precisely, the invariant curve is defined $z = C(t)$ for $t \in (a, b)$ for some open set (a, b) where C is continuous and for every $t^* \in (a, b)$ there is a $t^{**} \in (a, b)$ so that $m(C(t^*)) = C(t^{**})$. That is, every point on the curve is taken by the transformation to some point on the curve.

2.9. **Classifying by Trace.** The types of transformations discussed in this paper can be classified by trace. The trace of a transformation is defined as $trace(m) = a + d$; but to eliminate the ambiguity of signs, following Anderson's text, this paper will consider the function from the group of Möbius Transformations to the Complex Plane defined as $\tau(m) = (a + d)^2$, for the normalized transformation $m(z) = \frac{az+b}{cz+d}$. Then, the transformation m is:

1. parabolic if and only if $\tau(m) = 4$
2. elliptic if and only if $\tau(m)$ is real and $0 \le \tau(m) < 4$
3. loxodromic if and only if either the imaginary component of $\tau(m)$ is nonzero, or $t(m)$ is real and $\tau(m) < 0$ or $\tau(m) > 4$ [2]

There are other ways of classifying the types of transformations, which nevertheless yield nearly similar results, such as in Katok classifying using the function $Tr(m) = |a + d|$, whose classifications fall in the same categories, but with cutoffs of $Tr(m) < 2, = 2, > 2$. [5]

2.10. **Pencil of Circles.** A pencil of circles is a set of circles aligned on an axis. A hyperbolic pencil of circles are a set of orthogonal, non-intersecting circles that recedes into (are centered around) two point circles, which are the points of intersection of the elliptic pencil of circles, which are orthogonal circles that intersect only at those two points [8].

3. Finding the invariant curve of $m(z)$ given that of $n(z)$

This paper is concerned with finding the invariant curves of any particular, non-standardized transformation. However, it is certainly simpler to find the invariant curves once the transformation has been standardized. Thus, it is important to have a way to conjugate the invariant curve of $n(z)$ itself, not simply the transformation; the following proposition permits such an endeavor.

Given $m(z)$, a Möbius transformation, and let $q(z)$ be a transformation so that $q(m(q^{-1}(z))) = n(z)$ is the standard form. Assume $n(z)$, the standard transformation, keeps invariant some curve, C_n. Then $C_m = q^{-1}(C_n)$ is the curve kept invariant under m.

Proof. Define $C_m = q^{-1}(C_n)$ where C_n is the invariant curve of the transformation in standard form, $n(z) = q(m(q^{-1}))$.

Part 1: Prove that $m(C_m) \subset C_m$ (m takes points in C_m to points in C_m; for some $z_0 \in C_m$, there must exist a $z \in C_m$ so that $z = m(x_0)$)

Treating C_m as a set, choose $z_1 \in C_m$. Then $m(z_1) = q^{-1}(n(q(z_1)))$. Now, by definition $q(z_1) \in C_n$. And since C_n is the invariant curve of $n(z)$, $n(q(z_1)) \in C_n$. Then by definition of C_m, $q^{-1}(n(q(z_1))) \in C_m$. Thus for all $z_1 \in C_m$, $m(z_1) \in C_m$, therefore as a set, $m(C_m) \subset C_m$

Part 2: Prove that $C_m \subset m(C_m)$ (every point in C_m has a preimage in C_m; if $z_0 \in C_m$ and $z_0 = m(z_1)$ for some z_1, then $z_1 \in C_m$)

Let $z_0 = m(z_1) \in C_m$. By definition, $m(z_1) = q^{-1}(n(q(z_1))) \in C_m$. Then since by definition $q(C_m) = C_n$, $n(q(z_1)) \in C_n$, and since C_n is the invariant curve of $n(z)$, and by the following Lemma, $q(z_1) \in C_m$.

Lemma. Prove that $n^{-1}(C_n) \subset C_n$ (for $z \in C_n$, $n^{-1}(z) \in C_n$)

Parabolic Case: The transformation in standard form can be represented $n(z) = z + b$ (where b is some real number). Let z_0 be some complex number. Then $n_p(z_0) = z_0 + b, n(n(z_0)) = z_0 + 2b, n(n(n(z_0))) = z_0 + 3b, ..., z_0 + nb$; the invariant curve is described by the set of points

$$C_{n_p} := \{z_0 + tb; t \in R\}$$

Then, $n(z) = z + b$ and $n^{-1}(z) = z - b$. Choose some $\alpha \in C_n$. Then $n(\alpha) \in C_n$. And choose some $\beta \in C_n$. Then $n^{-1}(\beta) = \beta - b$, and where $t = -1$, $n^{-1}(\beta) \in C_n$.

Elliptic Case: The transformation in standard form can be represented $n_e(z) = e^{2i\theta}z$. Let $z_0 \in \mathbb{C}$. Then when applying n_e, the following can be observed:

$$n_e(z_0) = e^{2i\theta}z_0, n_e(n_e(z_0)) = e^{2(2i\theta)}z_0, n_e(n_e(n_e(z_0))) = e^{3(2i\theta)}z_0, ..., e^{n(2i\theta)}z_0$$

So the invariant curve can be described

$$C_n = \{e^{t2i\theta}z_0; \text{Given complex } z, \text{ real parameters } \theta \text{ and } t\ \}$$

Now, $n(z) = e^{2i\theta}z$, and $n^{-1}(z) = e^{-2i\theta}z$. Choose some $\alpha \in C_n$. Then $n(\alpha) \in C_n$. And choose some $\beta \in C_n$. Then $n^{-1}(\beta) = e^{-2i\theta}\beta$, and where $t = \frac{2\pi}{\theta} - 1$, $e^{2i(\frac{2\pi}{\theta}-1)\theta} = e^{2i(2\pi-\theta)} = e^{-2i\theta}z = n^{-1}(z)$, so $n^{-1}(\beta) \in C_n$.

Loxodromic Case: The transformation in standard form can be represented $n_l(z) = \rho^2 e^{2i\theta} z$. Let $z_0 \in \mathbb{C}$. Then when applying n_l, the following can be observed:

$$n_l(z_0) = \rho^2 e^{2i\theta} z_0, n_l(n_l(z_0)) = \rho^{2(2)} e^{2(2i\theta)} z_0, n_l(n_l(n_l(z_0))) = \rho^{3(2)} e^{3(2i\theta)} z_0, ..., \rho^{n(2)} e^{n(2i\theta)} z_0$$

So the invariant curve can be described

$$C_n = \{\rho^{2t} e^{t2i\theta} z_0; \text{Given complex } z,\ \rho, \text{ real parameters } \theta \text{ and } t\ \}$$

Now, $n(z) = \rho^2 e^{2i\theta} z$, and $n^{-1}(z) = \rho^{-2} e^{-2i\theta} z$. Choose some $\alpha \in C_n$. Then $n(\alpha) \in C_n$. And choose some $\beta \in C_n$. Then $n^{-1}(\beta) = \rho^{-2} e^{-2i\theta} \beta$, and where $t = \frac{2\pi}{\theta} - 1$, since $e^{2i(\frac{2\pi}{\theta}-1)\theta} = e^{2i(2\pi-\theta)} = e^{-2i\theta} z = n^{-1}(z)$, so $n^{-1}(\beta) \in C_n$.

□

4. Invariant Curves of Parabolic Transformations

4.1. **Parabolic Transformations.** Parabolic Möbius transformations are those with one fixed point, and can be written in standard form as $n(z) = z + b$ (where b and z are complex numbers). To take a parabolic transformation, $m(z)$ to its standard form, use the conjugating map $q : (z_1, y, m(y)) \to (\infty, 0, 1)$, where z_1 is the fixed point and y is a point in the complex plane other than z_1. The invariant curves of the parabolic transformations in standard form will all be straight lines. If one continually applies the transformation, one will simply add the same number, obtaining the series of points $\{z_0 + tb\}$ where z_0 is the initial point and t is a complex parameter. This is not a curve but a sequence of points separated by some parameter t determined by the number of times the transformation is applied. But as it appears to be a straight line, it suggests the form of the set of invariant curves.

Choosing an initial point z_0, the line through this line with the slope determined by the b factor that is added by the transformation can be determined, through to its y-intercept, a point $i\eta$. Thus the set of all such lines between every point z in the plane and some y-intercept can be categorized, and when this set is combined with the set of all vertical lines (which have no y-intercept and are thus "missed" by the first set) the resulting union of sets is the set of all invariant curves of the given parabolic transformation. More precisely, $C_n = \{z; z = i\eta + bt, b \text{ a complex number}, t \in (-\infty, \infty)\} \cup \{z; z = \eta + bit \text{with } b \text{and } t \text{as before}\}$. That is, the set of curves kept invariant by a parabolic transformation is the set of all straight lines with slope determined by b, the translation factor of the transformation.

This is supported geometrically by the fact that the translation contains no dilation or rotation on z, simply a translation, which means that the transformation moves the initial point along the line determined by b.

Now, since the inverse of the conjugating map, q^{-1}, is defined to take the point at infinity to the fixed point, and the points 0 and 1 to an arbitrary pair of points y, $m(y)$, it seems intuitive that the invariant curve will be the line going through the fixed point (and recall that in Hyperbolic Geometry straight lines are circles). The curves kept invariant by the original transformation keep invariant a parabolic pencil of circles which share a common point of the fixed points, with radius determined by the imaginary component of the initial point, while the standard form keeps invariant the horizontal lines. [7]

What follows is a demonstration of the method used to determine the invariant curve of a parabolic transformation.

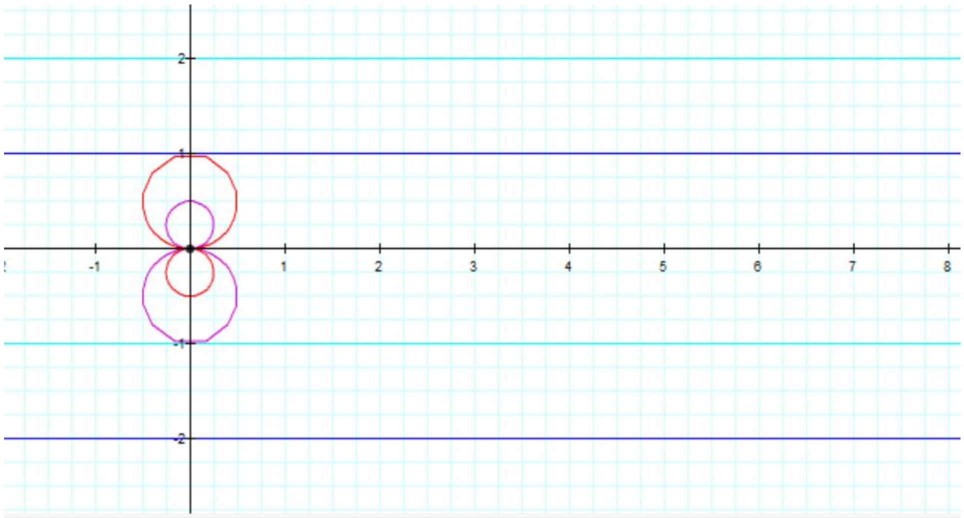

FIGURE 1. Parabolic example C_n (blue) beside C_m (red).

4.2. **Parabolic Example.**

4.2.1. $m(z) = \frac{z}{z+1}$.

1. Determine the fixed points: $z = \frac{z}{z+1} \to z = 0$

2. Standardize: Define $q : (0, 1, 1/2) \to (\infty, 0, 1)$: $q(z) = \frac{1-z}{z}$ with $q^{-1}(z) = \frac{1}{z+1}$. Now, to find the standard form, take $n(z) = q(m(q^{-1}(z)))$. Substituting, one finds that $n(z) = z + 1$.

3. Find the invariant curves of $n(z)$: In the complex plane, this standardized transformation involves a translation by 1 on the Real axis; so it can be represented by the curves, with initial point $z_0 = \alpha + i\beta$, $C_n := \{z = t + i\beta\}$.

4. Find the invariant curves of $m(z)$ using the conjugating map: By the above proof, $C_m = q^{-1}(C_n)$; so, $C_m = \frac{1}{t+i\beta+1}$.

5. Graph the results: Figure 1 shows several of the invariant curves of C_n (in blue) alongside those of C_m (in red). The choices of z_0 were such that the imaginary components were $\beta = \{-2, -1, 1, 2\}$, so that the equations which generated these curves are:

$$C_n : z = t - \beta i$$
$$C_m : z = \frac{1}{t + 1 + \beta i}$$

5. Invariant Curves of Elliptic Transformations

5.1. **Elliptic Transformations.** Elliptic Transformations have two fixed points, and can be written in standard form as $n(z) = az$ for some multiplier a so that $|a| = 1$; or, in polar coordinates, as $n(z) = e^{i\theta}z$. Furthermore, if $m(z)$ is a transformation that has been normalized, it can be classified using the trace (as defined in Anderson) in the following way: elliptic iff $\tau(m)$ is real and $0 \leq \tau(m) < 4$. Now, for an elliptic transformation in standard form, every invariant curve will be a circle of radius r, (depending on the initial point, $z_0 = re^{i\theta_0}$) centered around the origin. That is, the set of invariant curves for every elliptic $n(z)$ can be defined:

$C_n = \{re^{it}\}$; for some real parameter t, and given some initial point $z = re^{i\theta_0}$. This is evident from simply examining the standard form, which can be expressed $n(z) = e^{i\theta} z$. The coefficient in front of z is a complex number with radius 1; that is, it only takes a rotation, without any translation or dilation. Thus, geometrically, any initial point $re^{i\theta}$ will simply be rotated around a circle of radius r. Particularly, consider an initial point $z_0 = re^{i\theta_0} \in \mathbb{C}$. Then taking $n(z_0) = z_1 = re^{i(\theta_0+2\theta)}$, $n(z_1) = z_2 = re^{i(\theta_0+4\theta)}$, ..., $n(z_k) = re^{i(\theta_0+(2k+2))}$ shows that the transformation simply rotates the initial point to another point around the circle of radius r centered at the origin.

Two elliptic or loxodromic transformations have equivalent standard form if and only if they have equivalent trace.

Proof. Let m_1, m_2 be two elliptic or loxodromic transformations, $m(z) = \frac{\alpha_i z - \beta_i}{\gamma_i z - \delta_i}$, $i = 1, 2$, in normal form (that is, with $\alpha_i\delta_i - \beta_i\gamma_i = 1$). Note that in standard form, since both types can be written $n(z) = \rho e^{i\theta} z$, both types will be in fractional form; $n(z) = \frac{a_i z}{b_i}$.

=> Presume $\tau(m_1) = \tau(m_2)$.

$$
\begin{aligned}
&= \tau(q \circ m_1 \circ q^{-1}) = \tau(q \circ m_2 \circ q^{-1}) && \text{(Since } \tau \text{ remains unchanged under conjugation [2])}\\
&= \tau(n_1) = \tau(n_2)\\
&(a_1 + d_1)^2 = (a_2 + d_2)^2 && \text{(definition of trace)}\\
&a_1^2 + 2a_1d_1 + d_1^2 = a_2^2 + 2a_2d_2 + d_2^2\\
&a_1^2 + 2 + d_1^2 = a_2^2 + 2 + d_2^2 && \text{(} n_1,\ n_2 \text{ normal elliptic or loxodromic implies } a_1d_1 = 1 = \\
&a_1^2 + d_1^2 = a_2^2 + d_2^2\\
&\frac{a_1}{d_1} + \frac{d_1}{a_1} = \frac{a_2}{d_2} + \frac{d_2}{a_2} && \text{(since } a_id_i = 1,\ a_i = \frac{1}{d_i} \text{ and } d_i = \frac{1}{a_i} \text{ for } i = 1, 2)\\
&\frac{a_1^2 + d_1^2}{a_1d_1} = \frac{a_2^2 + d_2^2}{a_2d_2}\\
&a_2d_2a_1^2 + a_2d_2d_1^2 = a_1d_1a_2^2 + a_1d_1d_2^2\\
&a_1d_2(a_2a_1) + a_2d_1(d_2d_1) = a_2d_1(a_1a_2) + a_1d_2(d_1d_2)\\
&a_1d_2(a_2a_1 - d_1d_2) = a_2d_1(a_1a_2 - d_1d_2)\\
&\frac{a_1d_2}{a_2d_1} = \frac{a_2a_1 - d_1d_2}{a_1a_2 - d_1d_2}\\
&\frac{a_1d_2}{a_2d_1} = 1\\
&a_1d_2 = a_2d_1\\
&\frac{a_1}{d_1} = \frac{a_2}{d_2}
\end{aligned}
$$

thus the standard forms are equivalant.

<= Presume $n_1 = n_2$.

$\tau(n_1) = \tau(n_2)$

$\tau(q_1^{-1}(n_1(q_1))) = \tau(q_2^{-1}(n_2(q_2)))$ (Since τ remains unchanged under conjugation)

$\tau(m_1) = \tau(m_2)$

thus the traces are equivalant. □

In summary, since the standard elliptic and loxodromic forms can be written $n(z) = \rho e^{i\theta}$, ($\rho = 1$ in the elliptic case) where the transformation is defined as $n(z) = \frac{az+b}{cz+d}$, $b = 0$ and $c = 0$, the transformation can be written $n(z) = \frac{az}{d}$ for some $a, d \in \mathbb{C}$. Thus the proof above shows that if and only if two normalized elliptic or loxodromic transformations have equivalent traces, then the transformations have equivalent standard forms.

The reason that this fact could not be discussed in the parabolic case is that for all normalized parabolic transformations, the above a and d terms will be equal to one, and so the trace will be four. However, for a normalized parabolic transformation, the c term will be equal to zero, so that the fact that the determinant of the transformation's coefficient matrix is one implies nothing about the value of b, and so the traces being equal, even for a normalized transformation, as it only concerns a and d, implies nothing about the equivalence of the transformations.

Now, turn towards finding a method to find the invariant curve of any, non-standard elliptic transformation. It can be conjugated to the standard form according to the conjugating transformation $q : (z_1, z_2) -> (0, \infty)$, defined $q = \frac{z-z_1}{z-z_2}$ (implying $q^{-1} = \frac{z_2 z - z_1}{z-1}$). Using the previously proven proposition that the invariant curve of the original transformation can be determined by the relation $C_m = q^{-1}(C_n)$, then the invariant curves of the non-standard transformations will be of the form $C_m = \frac{z_2(re^{it})-z_1}{(re^{it})}$, where t is a parameter and r is the distance of the initial point from the origin.

As can be observed algebraically, as r goes to zero, the curve will approach z_1, the first fixed point, and as r gets large (goes to infinity), the curve will approach z_2, the second fixed point; so as a whole, the curves are the hyperbolic pencil of circles with point circles at the fixed points along the axis determined by the line through the fixed points [7]. That is, the invariant forms make up a set of circles that are orthogonal to each other and do not intersect each other; and have point circles at the fixed points of the transformation, lying on either side of the straight line between the fixed points [8].

So, if after normalizing the transformations, the traces are equivalent, then the standard forms are equivalent. And given a standardized elliptic transformation, given the conjugacy map q^{-1}, it is possible to determine the invariant curve of the original, non-standard transformation. What follows are two examples of (normal) elliptic transformations, and their invariant curves.

5.2. **Elliptic Examples.**

5.2.1. $m_1(z) = \frac{z-1}{z}$.

1. Determine the fixed points:

$$z = \frac{z-1}{z} \to z_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i, z_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

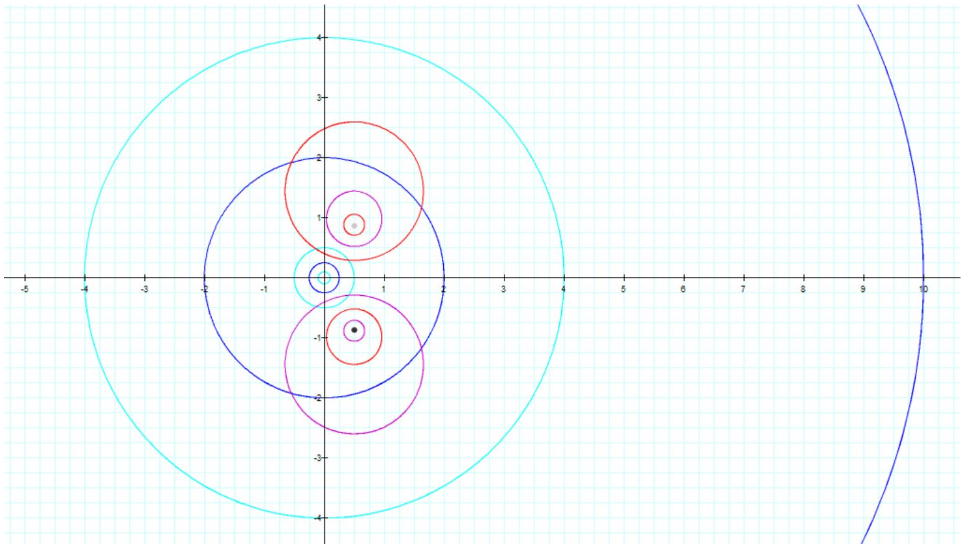

FIGURE 2. First elliptic example C_n (blue) beside C_m (red).

2. Standardize: Given the fixed points, define $q : (z_1, z_2) \to (0, \infty)$: $q(z) = \frac{z-z_1}{z-z_2}$, and so $q^{-1} = \frac{z_2 z - z_1}{z-1}$. To find the standard form, take $n(z) = q(m(q^{-1}(z)))$. Substituting, one finds that $n(z) = \frac{-1-\sqrt{3}i}{2}z$. Converting into exponential form, this is $n(z) = e^{i\frac{4\pi}{3}}z$. It has radius of 1.

3. Find the invariant curves of $n(z)$: Choose an initial point $z_0 = re^{i\theta}$. Continually applying the standard transformation to this initial point; $z_0 \to n(z_0) = z_1 \to n(z_1) = z_2 \to ...$; concludes that the invariant curves can be defined: $C_n := \{re^{i(4\pi/3+t\theta)}, \text{for some initial } z_0 = re^{i\theta}\}$. And this, on the complex plane, is all of the circles centered around the origin, with radius r determined by the distance between the origin and the chosen initial point.

4. Find the invariant curves of $m(z)$ using the conjugating map: define $Cm = q^{-1}(C_m)$. Reparameterize C_n as $C_n = re^{it}$, as it is equivalently the circle centered at the origin of radius r. Define $C_m := \{z = \frac{z_2(re^{e^i t} - z_1)}{(re^{it})-1}\}$, which is the pencil of circles whose point circles are the fixed points, and which is along the axis determined by the real parts of the fixed points. This makes sense because as $|r| \to \infty$, $C_m \to z_2$, and as $|r| \to 0$, $C_m \to z_1$.

5. Graph the results: Figure 2 shows several of the invariant curves of C_n (in blue) alongside those of C_m (in red), along with the two fixed points (in black and grey). The choices of z_0 were: $z_0 = re^{i\theta}$ for $r = \{-10, -4, -2, 1/2, 1/4, 1/10\}$, so that the equations which generated these curves are, for each r:

$$C_n := re^{it}$$

$$C_m := \frac{(\frac{1}{2} - \frac{\sqrt{3}}{2}i)re^{it} - (\frac{1}{2} + \frac{\sqrt{3}}{2}i)}{re^{it} - 1}$$

The curves for the C_n are clearly the circles centered at the origin, and the ones for C_n are the pencil of circles whose point circles are the fixed points, as discussed above.

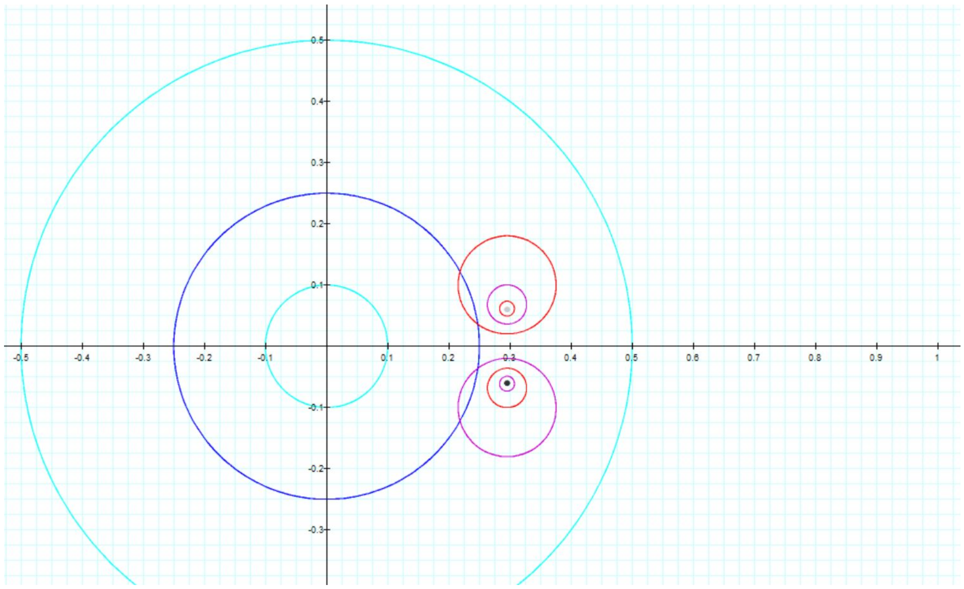

FIGURE 3. Second elliptic example C_n (blue) beside C_m (red).

5.2.2. $m_2(z) = \frac{4z-1}{11z-2.5}$.

1. Determine the fixed points:

$$z = \frac{4z-1}{11z-2.5} \to z_1 = \frac{6.5+\sqrt{1.75}i}{22}, z_2 = \frac{6.5-\sqrt{1.75}i}{22}$$

2. Standardize: Define $q:(z_1,z_2)\to(0,\infty)$: $q(z) = \frac{z-z_1}{z-z_2}$ with $q^{-1}(z) = \frac{z_2 z - z_1}{z-1}$. Now, to find the standard form, take $n(z) = q(m(q^{-1}(z)))$. Substituting in, one finds that $n(z) = \frac{1-3\sqrt{7}i}{8}z$. Converting into exponential form, $n(z) = e^{i(-.46\pi)}z$ and has radius of 1 as in the previous example.

3. Find the invariant curves of $n(z)$: As in the previous example, since the only impact of this transformation is in the exponential component, it will keep invariant the circles centered at the origin; given an initial $z_0 = re^{i\theta}$, it will keep invariant: $C_n := \{re^{it}\}$ for some parameter t.

4. Find the invariant curves of $m(z)$ using the conjugating map: As above, define $Cm = q^{-1}(C_m)$. Reparameterize C_n and consider $n(z) = re^{it}$, as it is the circles centered at the origin. Define $C_m := \{z = \frac{z_2(re^{e^{i}t}-z_1)}{(re^{it})-1}\}$, which is the pencil of circles whose point circles are the fixed points, and which is along the axis determined by the real parts of the fixed points.

5. Graph the results: Figure 3 shows some of the invariant curves of C_n (in blue) alongside those of C_m (in red), along with the two fixed points (in black and grey). The choices of z_0 were, as before: $z_0 = re^{i\theta}$ for $r = \{-10, -4, -2, 1/2, 1/4, 1/10\}$, so that the equations which generated these curves are, for each r:

$$C_n := re^{it}$$

$$C_m := \frac{(\frac{6.5-\sqrt{1.75}i}{22})(re^{it}) - (\frac{6.5+\sqrt{1.75}i}{22})}{re^{it}-1}$$

As above, the curves for the C_n are clearly the circles centered at the origin, and the ones for C_m are the pencil of circles whose point circles are the fixed points, as discussed above. Note that because of the scaling of this image, the curves for C_n where $r = -2, -4, -10$ do not appear in this image, as they are the circles centered at the origin with radius $2, 4, 10$ respectively.

6. Invariant Curves of Loxodromic Transformations

6.1. **Loxodromioc Transformations.** Loxodromic transformations have two fixed points, and can be written in standard form as $n(z) = az$ for some multiplier a so that $|a| \neq 1$; or, as $n(z) = \rho^2 e^{2i\theta} z$, where $\rho \neq \pm 1$ [2]. Geometrically, this transformation involves both a rotation ($e^{i\theta}$) as well as a dilation (ρ^2), and thus will most likely keep invariant some kind of logarithmic spiral, as it keeps constant the degree of rotation and dilation at every point.

It is important to note that this category is broken up with more specificity in Schwerdtfeger's book, where the categories "hyperbolic," "improper hyperbolic," and "loxodromic" are used. Hyperbolic refers to a transformation that only involves a dilation (as in the first example below); improper hyperbolic refers to a transformation that is a dilation as well, but with a coefficient that is a negative or negative inverse of a proper hyperbolic transformation. Meanwhile a loxodromic transformation is one where the coefficients are not all real numbers, and involves a rotation and dilation. It is noted that the hyperbolic keeps invariant an elliptic pencil of circles through the two fixed points; the improper hyperbolic is an involution of one of these, and the loxodromic does not keep invariant any pencil of circles. [7]

This type of transformation is of particular interest for its historical use to map loxodromic spirals on the sphere to logarithmic ones on a plane, as is discussed in the historical background. In this section, several examples of loxodromic transformations and their invariant curves in the complex plane will be provided, as well as a discussion of Halley's proof that the stereographic projection of loxodromic spirals on the sphere onto the plane are logarithmic spirals.

In Anderson, it is stated outright that "the reason [loxodromic Möbius transformations] are called loxodromic is that each one keeps invariant a loxodrome." [2] And, as has been proven by Halley, a loxodromic spiral can be stereographically projected to a logarithmic spiral in the plane.[4] This is somewhat intuitive, as mentioned above, but the steps of his proof will be discussed below, after the following illustrative examples.

6.2. **Loxodromic Examples.**

6.2.1. $m_1(z) = 7z + 6$.

1. Determine the fixed points:

$$z = 7z + 6 \to z_1 = -1, z_2 = \infty$$

2. Standardize: Take $n(z) = q(m(q^{-1}(z)))$, with q the conjugating map defined as above. Substituting in and simplifying, one gets that $n(z) = 7z$.

3. Find the invariant curves of $n(z)$: Take some initial $z_0 = re^{i\theta}$. Repeatedly applying the transformation yields the set of invariant curves of $C_n := \{7^t(re^{i\theta})\}$ for some real parameter $t \in (-\infty, \infty)$. This is the set of all of the half straight lines from the point 0 to the point at ∞.

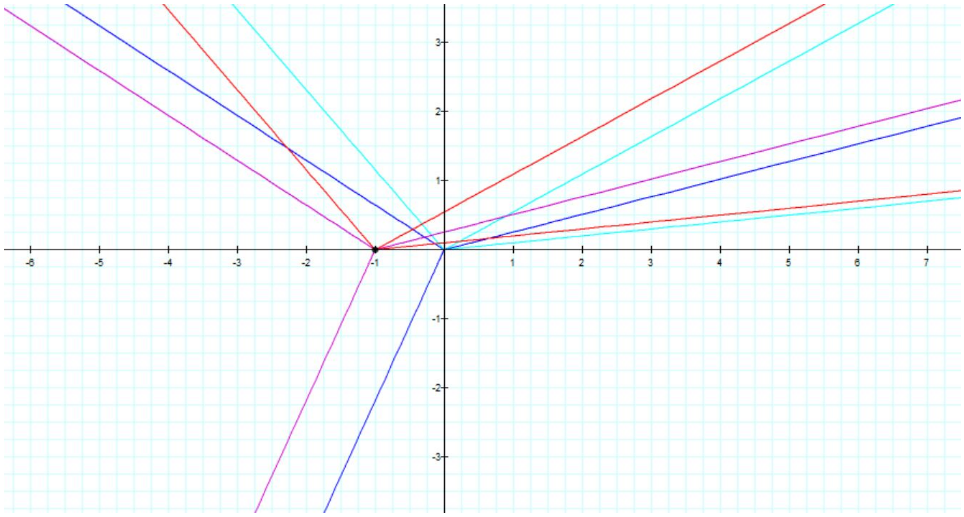

FIGURE 4. First loxodromic example C_n (blue) beside C_m (red).

4. Find the invariant curves of $m(z)$ using the conjugating map: As proven above, the set of invariant curves for the non-standard transformation can be determined $C_m = q^{-1}(C_n)$; thus, $C_m := \{7^t(ne^{i\theta} - 1)\}$. This is the set of all lines through the fixed points, $z_1 = -1$ and $z_2 = \infty$

5. Graph the results: Figure 4 shows several of the invariant curves of C_n (in blue) alongside that of C_m (in red), along with the two fixed points (in black and grey). The choices of z_0 were: $z_0 = re^{i\theta}$ for $r = \{1\}$, and $\theta = -10, -4, -2, 1/2, 1/4, 1/10$, so that the equations which generated these curves are, for each r:

$$C_n := 7^t re^{i\theta}$$

$$C_m := 7^t re^{i\theta} - 1$$

6.2.2. $m_2(z) = \frac{(3i+3)z-8i+2}{(4i-1)z-5i+5}$.

1. Determine the fixed points:

$$z = \frac{(3i+3)z - 8i + 2}{(4i-1)z - 5i + 5} \to z_1 = 1 + i, z_2 = 1 - i$$

2. Standardize: Take $n(z) = q(m(q^{-1}(z)))$, with q the conjugating map defined as above. Substituting in and simplifying, one gets that $n(z) = 4iz$ (in exponential form, this is $n(z) = 4e^{\pi/2i}z$).

3. Find the invariant curves of $n(z)$: Take some initial $z_0 = re^{i\theta}$. Repeatedly applying the transformation yields the invariant curve of $C_n := \{4^t re^{i(t\pi/2+\theta)}\}$ for some real parameter $t \in (-\infty, \infty)$, and some initial point $z_0 = re^{i\theta}$. This is a set of logarithmic spirals from the point 0 to the point at ∞.

4. Find the invariant curve of $m(z)$ using the conjugating map: As proven above, the invariant curve for the non-standard transformation can be determined $C_m = q^{-1}(C_n)$; thus, $C_m := \{\frac{(4^t re^{i(t\pi/2+\theta)})z_2 - z_1}{4^t re^{i(t\pi/2+\theta)} - 1}\}$. This is the set of all logarithmic spirals through the fixed points.

5. Graph the results: Figure 5 shows the invariant curve of C_n (in blue) alongside that of C_m (in red), along with the two fixed points (in black and grey). The choices

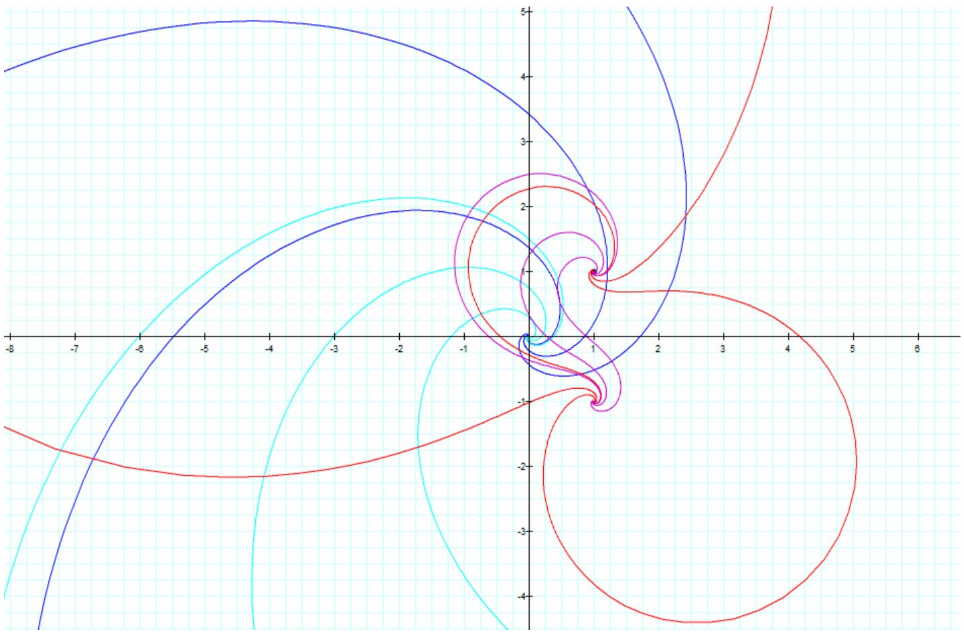

FIGURE 5. Second loxodromic example C_n (blue) beside C_m (red).

of z_0 were, of the form $z_0 = re^{i\theta}$: $z_0^a = 2e^{2i}, z_0^b = 2e^{10i}, z_0^c = 5e^{2i}, z_0^d = 5e^{10i}, z_0^e = 10e^{2i}, z_0^f = 10e^{10i}$, so that the equations which generated these curves are, for each r:

$$C_n := 4^t re^{it\pi/2+\theta i}$$

$$C_m := \frac{(4^t re^{it\pi/2+\theta i})(1-i)-1-i}{4^t re^{it\pi/2+\theta i}-1}$$

6.3. **Proof Loxodromic Transformations Keep Invariant a Loxodrome.** When written in standard form, $n(z) = \rho^2 e^{2i\theta} z$, it is clear geometrically that the loxodromic transformation involves both a rotation and a dilation, but no translation. Thus it intuitively makes sense that the curve kept invariant would be some type of spiral, which maintains the same degree of rotation and dilation at each step, and maintains an angle of intersection with a line from the origin at every point. Such a spiral is called logarithmic and what follows is an outline of the historical proof that it is just such a curve that is maintained by a loxodrome.

7. Edmund Halley and the Stereographic Projection of Loxodromic Spirals

7.1. **Background to Halley's Proof.** As has been discussed above, historically understanding how to set a "course following a constant bearing" was of paramount importance. [3] It was desirable to set a course that kept the same angle with every meridian, as opposed to following the path of a great circle. For this purpose, mathematicians and cartographers created the notion of a "loxodromic chart," or a table of values that would allow for a conformal map between the sphere and a flat plane (to create a map). One of the most famous conformal map is the Mercator Projection, which follows a "modified cylindrical projection" - "all parallels of latitude are straight lines parallel and equal to the equator." [3] In his paper "An Easie Demonstration of the Analogy of the Logarithmick Tangents to the Meridian Line or Sum of the Secants: With Various Methods for Computing the Same to the Utmost Exactness," Edmund Halley is concerned with the spacing of the parallels of latitude in the Mercator Projection, as well as demonstrating methods to determine the distances between them. Ultimately he proves that "the meridian line is a scale of logarithmick tangents of the half complements of the latitudes." [4]

The importance of Halley's proof to historical methods of navigation (as well as its influence on today's navigational pursuits) is significant. Additionally, it is of use analytically to any one who studies processes that occur on a globe, and whose courses or impacts can be analyzed directionally with regard to the lines of latitude and longitude on the globe.

In the context of this paper, Halley's work is significant because it proves that stereographic projection is conformal, or angle-preserving. He demonstrates in his third Lemmma that the stereographic projection of a loxodromic spiral from the sphere onto the plane is the logarithmic, or proportional spiral.

This makes sense of the fact that the premise of this project is that a loxodromic transformation keeps invariant a loxodrome, and a since by Halley a loxodrome on the sphere is a logarithmic spiral in the plane, our conclusions about invariant curves in previous sections make sense. In other words, as stated by the textbook, every loxodromic transformation keeps invariant a loxodrome. Halley proved that a loxodrome on the sphere is equivalent to a logarithmic spiral in the plane. This supports the results obtained in previous sections, where it is demonstrated that the loxodromic transformations keep invariant logarithmic spirals in the extended complex plane.

7.1.1. *Outline/Interpretation of Halley's Proof.* In his proof, Halley makes use of four main lemmas towards achieving his goal of proving that the stereographic projection of a loxodromic spiral (or Rhumb Line) from the unit sphere onto the plane is a logarithmic spiral.

Lemma 1. The first Lemma proves that in the stereographic projection, the distances from the pole to points on the loxodromic spiral are the tangents of half of the distances; or, half the complements of the latitudes.

Lemma 2. The second Lemma proves that the spherical angle is the same as the angle projected in the plane; that is, stereographic projection is conformal.

Lemma 3. The third Lemma proves that the angle between the loxodromic spiral and the meridians is preserved in the plane under stereographic projection; and so, since the loxodromic curve makes equivalent angles with every meridian, the curve on the plane makes equal angles with the polar radii (the lines outward from the origin), which is the definition of a logarithmic spiral.

Lemma 4 The fourth Lemma considers the proportional (logarithmic) spiral, and proves that the angle of every arc BD along the spiral is an exponent of the ratio of the distance from point B to the origin, and from point C (a point on the line between point D and the origin) to the origin.

Conclusion Thus he proves that since angles are preserved under stereographic projection, and as the angles of the loxodromic spiral are constant with every line of longitude, the angles of the projected curve will also be constant with every line from the origin, and thus the curve in the plane is the logarithmic spiral.

8. Choice of Conjugating Map

Throughout the above sections, the transformation q which conjugates the original to the standard form has been defined as taking the fixed point or points to 0 and ∞, respectively. This paper has used specifically the transformation $q = \frac{z-z_1}{z-z_2}$ in the elliptic and parabolic cases, and a more case-by-case approach for the parabolic case. It is believed that the choice of q plays no significant role in terms of standardizing the transformation [2]. However, it remains to be seen how this choice of conjugating transformation will effect the invariant curves. Here, the inverse $q^{-1} = \frac{z_2 z - z_1}{z-1}$ of that particular transformation has been used to take the invariant curves of the standardized transformations to the invariant curves of the original transformations. However it remains to be seen what role the choice of q plays into that process; for a different q^{-1} (that still takes the points $(0, \infty)$ to the fixed points), will a different set of invariant curves be determined for the original transformation? If so, is it possible to classify all of the curves of a given transformation independent from the choice of q? Or given the current method of relying on the invariant curves of the standard transformation to determine those of the original transformation, is the choice of conjugating transformation integral to the determination of the original transformation's invariant curve, and thus indispensable?

9. Applications

While this topic may seem esoteric, it builds upon knowledge that was developed for very practical navigational purposes. And it remains useful for practical purposes to this day. It is still important to develop and refine methods to take curves on a sphere to curves on a plane in a way that preserves angles for navigation as well as other studies such as weather patterns and other fields of earth science.

Clearly the process of stereographic projection directly takes points on a sphere to points on the plane, and is thus important for the construction of various representational models for the earth (such as maps). More broadly, the Möbius transformation is a process that maps a point on the plane to a point on the plane, after a transformation which maps them to the sphere. In other words, this process is itself a study in the relationship between spherical and planar points, and understanding these transformations can aid in the understanding of the relationship between the sphere and the plane, as well as points on each.

This paper has explained, justified, and given examples of methods for classifying and determining invariant curves of transformations. It has briefly discussed the history of loxodromic spirals, as well as Halley's historical proof on that subject. Though this paper is ultimately a small piece of research on an abstract topic, it is of interest because it bridges a logical gap left by a major textbook on Hyperbolic Geometry, and because it is a part of a long history of important mathematics.

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