Math 142 Lecture Notes
Lecture 2

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1 Announcements

- Midterm 1: Monday, October 6
- Midterm 2: Friday, November 14
- Homework logistics: Each lecture Professor Rieffel will assign reading/exercises, they will be due on Fridays at 11:59pm. No late homework will be accepted, but the lowest 4 scores will be dropped.
- **Reading Assignment**: Read §12 and page 119
- **Assigned HW Problems**: Exercises 2 & 3 on page 83

2 Generalizing Continuity

Recall from Lecture 1: Given metric spaces $(X, d^X), (Y, d^Y)$ and $f\colon X \to Y$,

Definition 1 (Continuity at a point)**.** f is continuous at x_0 if for any open $O \subseteq Y$ with $f(x_0) \in O$ there is an open set $U \subseteq X$ containing x_0 such that $U \subseteq f^{-1}(O)$.

We can broaden this definition to continuity of a function:

Definition 2 (Continuity)**.** f is continuous if it is continuous at every $x \in X$. Formally, for every $x \in X$ and $O \subseteq Y$ with $f(x) \in O$, there is an open $U \subseteq X$ s.t. $U \subseteq f^{-1}(O)$.

In general terms we can say that f is continuous if for any open O in Y the preimage of O is open in X. These definitions are equivalent because any point $x \in f^{-1}(O)$ corresponds to a point $x_0 \in O$, and by definition of continuity there is an open set around x.

3 Topology

Consider a collection $\mathcal{T}^X$ of open sets in (X, d^X). It has the following properties:

1. $X, \emptyset \in \mathcal{T}^X$
2. Any arbitrary (possibly infinite) union of a collection of open sets is open.
   If x is in the union then it is part of one of the open sets, therefore it must have an open ball

around it which is also in the union.

3. Any **finite** intersection of open sets is open.
   If x is in the finite intersection of open sets then it must be in all open sets. Therefore for the jth open set there is a $B(x, r_j)$ in the open set. Let $r = \min\{r_0 \dots r_n\}$, then $B(x, r) \subseteq \bigcap_{i=1}^{n} O_i$.

In a kind of roundabout way, we can abstract these properties to a structure known as a topology.

Definition 3 (Topology)**.** Given a set X, a **topology** $\mathcal{T}$ on X is a collection of subsets of X that satisfies the properties above. Rewritten, those are

1. $X, \emptyset \in \mathcal{T}$
2. Any arbitrary union of elements in $\mathcal{T}$ is in $\mathcal{T}$
3. Any finite intersection of elements in $\mathcal{T}$ is in $\mathcal{T}$

Note that there is no metric involved in this definition.

4 Comparing Topologies

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X, we say that $\mathcal{T}_1$ is **stronger/finer/bigger** than $\mathcal{T}_2$ if $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

Conversely, we say $\mathcal{T}_1$ is **weaker/coarser/smaller** than $\mathcal{T}_2$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$.

There is a biggest topology on X, namely the topology consisting of all subsets of X, the power set of X. This topology corresponds to a metric, where $d(x_1, x_2) = 1$ if $x_1 \neq x_2$ and 0 otherwise. This is called the **discrete** topology (use balls of radius 1/2 to show openness).

There is also a smallest topology, consisting of $\{X, \emptyset\}$, called the **trivial** or **indiscrete** topology. Given that $|X| \geq 2$, this does not come from a metric.