Math 202a Lecture Notes
Lecture 1

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28 August, 2025

1 Conceptual Introduction

A main theme of this course will be taking some classical space and through some expansion considering it a core to a larger modern space. A good example of this is the rational $\to$ real numbers relationship.

We have the rationals, which we can relate to a sieve; there are clear holes in it, but the rationals themselves are still countable and dense. We use a procedure for "completion" that will expand the rationals into the "solid line" of the reals.

Just a reminder,

Definition 1 (Rationals). A **rational number** q is represented as

$$q = \frac{m}{n}, \quad m \in \mathbb{Z}, n \in \mathbb{N}^+$$

This set forms a metric space with the following distance function,

$$d(q_1, q_2) = \frac{|m_1 n_2 - m_2 n_1|}{n_1 n_2}$$

2 Equivalence

Our goal now is to express the irrationals as a limit of a sequence of rational numbers. First we'll define some stuff that will help with that.

Definition 2 (Equivalence Relation). An **equivalence relation** is a set and operation $(S, \sim)$ that satisfies the following 3 axioms. We can think of $\sim \subseteq S \times S$. For $x, y, z \in S$,

1. (reflexive) $x \sim x$
2. (symmetric) $x \sim y \implies y \sim x$
3. (transitive) $x \sim y \wedge y \sim z \implies x \sim z$

We can partition S into equivalence classes, where the equivalence class for some $s \in S$ is the set $\{x \in S : x \sim s\}$.

3 Convergence

Consider a sequence in $\mathbb{Q}$, $(x_n : n \in \mathbb{N})$. We can define convergence of this sequence to some point x by saying that $\lim_n |x_n - x| = 0$, but what if our limit lands outside of $\mathbb{Q}$? To keep things internal, we will instead use the idea of a Cauchy sequence.

Definition 3 (Cauchy Sequence)**.** A sequence is **Cauchy** if $\forall \epsilon > 0$, $\exists n_0$ s.t. $\forall n, m \geq n_0$,

$$d(x_n, x_m) < \epsilon$$

i.e. the points in the sequence eventually get very close together. This fixes our problem of defining irrational numbers as the limit of a sequence of rationals; is it true that

$$0.999\ldots = 1.000\ldots$$

using our notion of equivalence relation, we can say that

$$(x_n) \sim (y_n) \iff |x_n - y_n| \to 0$$

as well as express the distance between two irrational numbers,

$$d(x, y) = \lim_n |x_n - y_n|$$

This completes our definition of an irrational number internally (within $\mathbb{Q}$).

4 Size of Sets

Refer to section 2.2 of course notes

We'll now consider intervals with an open left and closed right. Define the set of all finite disjoint unions of these to be $\mathcal{I}$. Formally,

$$\mathcal{I} = \bigcup_{i=1}^{n} (a_i, b_i]$$

where $n \in \mathbb{N}^+$ and $a_1 < b_1 < a_2 < \ldots < a_n < b_n$. Also permit $a_i, b_i \in \{-\infty, \infty\}$.

$\mathcal{I}$ is closed under complement, pairwise intersection, and finite intersection (shown by induction).

We define $l(I) = \sum_{i=1}^{n}(b_i - a_i) \in [0, \infty) \cup \{\infty\}$. Now we want to define the notion of distance.

We'll do this with symmetric difference:

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$

and then define $d(I_1, I_2) = l(I_1 \triangle I_2)$, for $I_1, I_2 \in \mathcal{I}$. It can be shown that d is a metric on $\mathcal{I}$.

Now consider the closure of $\mathcal{I}$, $\overline{\mathcal{I}}$. Our aim is to make this a metric space. Let $I, J \in \overline{\mathcal{I}}$. We'll define distance like this:

$$d(I, J) = \lim_n d(I_n, J_n)$$

From this, we have a notion of measure:

$$l(I) = \lim_n l(I_n)$$

5 Integration

Section 2.3 of course notes

From calculus, we understand the notion of integration as "area under the curve" $= \int_a^b f$. From analysis (Math 104) we know that we the idea of Riemann integration centers around a set of partitions P and infimum and supremum integrals

$$I^{\downarrow}(f,P) = \sum_{i=1}^{n} \inf\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1})$$

and similarly for $I^{\uparrow}$.

From this, we define integrability:

Definition 4 (Riemann Integrable)**.** $f\colon [a,b] \to \mathbb{R}$ is **Riemann integrable** if $\exists I \in \mathbb{R}$ s.t. $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $\forall P$ with $\text{mesh}(P) < \delta$,

$$|I^{\uparrow}(f,P) - I| < \epsilon$$

Where $\text{mesh}(P)$ is the length of the longest interval in P.

The integral-defined function

$$d(f,g) = \int |f-g|$$

is a metric on the space of continuous functions called the L^1 norm. We can define pointwise convergence in L^1.

Now consider

$$f_n \to \mathbb{1}_{[1/2,1]}$$

where $f_n\colon [0,1] \to [0,1]$ are pointwise decreasing. We can have the problem where each f_n is Riemann integrable, but

$$f = \lim_n f_n$$

is not. Thus, we wish to broaden our integration to Lebesgue theory.

An important example to highlight is the integral of the indicator function. What is

$$\int_0^1 \mathbb{1}_{\mathbb{Q}}$$

where $\mathbb{1}_{\mathbb{Q}} = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{else} \end{cases}$

Riemann integration says this integral doesn't exist ($\mathbb{Q}$ is dense in $\mathbb{R}$). But intuitively we think it should really be 0.