Math 202a Lecture Notes
Lecture 1

Walter Cheng

28 August, 2025

1 Conceptual Introduction

A main theme of this course will be taking some classical space and through some expansion considering it a core to a larger modern space. A good example of this is the rational $\to$ real numbers relationship.

We have the rationals, which we can relate to a sieve; there are clear holes in it, but the rationals themselves are still countable and dense. We use a procedure for "completion" that will expand the rationals into the "solid line" of the reals.

Just a reminder,
Definition 1 (Rationals)**.** A **rational number** q is represented as

$$q = \frac{m}{n}, \quad m \in \mathbb{Z}, n \in \mathbb{N}^{+}$$

This set forms a metric space with the following distance function,

$$d(q_1, q_2) = \frac{|m_1 n_2 - m_2 n_1|}{n_1 n_2}$$

2 Equivalence

Our goal now is to express the irrationals as a limit of a sequence of rational numbers. First we'll define some stuff that will help with that.

Definition 2 (Equivalence Relation)**.** An **equivalence relation** is a set and operation $(S, \sim)$ that satisfies the following 3 axioms. We can think of $\sim \subseteq S \times S$. For $x, y, z \in S$,

1. (reflexive) $x \sim x$
2. (symmetric) $x \sim y \implies y \sim x$
3. (transitive) $x \sim y \cap y \sim z \implies x \sim z$

We can partition S into equivalence classes, where the equivalence class for some $s \in S$ is the set $\{x \in S : x \sim s\}$.

3 Convergence

Consider a sequence in $\mathbb{Q}$, $(x_n : n \in \mathbb{N})$. We can define convergence of this sequence to some point x by saying that $\lim_n |x_n - x| = 0$, but what if our limit lands outside of $\mathbb{Q}$? To keep things internal, we will instead use the idea of a Cauchy sequence.

Definition 3 (Cauchy Sequence)**.** A sequence is **Cauchy** if $\forall \epsilon > 0$, $\exists n_0$ s.t. $\forall n, m \geq n_0$,

$$d(x_n, x_m) < \epsilon$$

i.e. the points in the sequence eventually get very close together. This fixes our problem of defining irrational numbers as the limit of a sequence of rationals; is it true that

$$0.999\ldots = 1.000\ldots$$

using our notion of equivalence relation, we can say that

$$(x_n) \sim (y_n) \iff |x_n - y_n| \to 0$$

as well as express the distance between two irrational numbers,

$$d(x, y) = \lim_n |x_n - y_n|$$

This completes our definition of an irrational number internally (within $\mathbb{Q}$).

4 Size of Sets

We'll now consider intervals with an open left and closed right. Define the set of all finite disjoint unions of these to be $\mathcal{I}$. Formally,

$$\mathcal{I} = \bigcup_{i=1}^{n} (a_i, b_i]$$

where $n \in \mathbb{N}^+$ and $a_1 < b_1 < a_2 < \ldots < a_n < b_n$. Also permit $a_i, b_i \in \{-\infty, \infty\}$.

$\mathcal{I}$ is closed under complement, pairwise intersection, and finite intersection (shown by induction).

We define $l(I) = \sum_{i=1}^{n} (b_i - a_i) \in [0, \infty) \cup \{\infty\}$. Now we want to define the notion of distance.