9 Continuum Mechanics

9.1 Example: Motion of a String

We have an elastic string lying along the x-axis. Consider when we displace the string in the y direction by $y(x)$. If we slice the string into infinitesimally small segments of mass m, length l, and tension T, balancing the forces gives (using small angle approximation)

$$\begin{aligned} m\ddot{y}_n &= -\frac{T}{l}(y_n - y_{n-1}) + \frac{T}{l}(y_{n+1} - y_n) \\ \ddot{y}_n &= \frac{T}{ml}(y_{n+1} - 2y_n + y_{n-1}) \end{aligned} \tag{1}$$

Recall the limit definition of the second derivative (for intuition on this, refer to timestamp 8:00 of this 3Blue1Brown video):

$$\frac{y_{n+1} - 2y_n + y_{y_{n-1}}}{l^2} \to \frac{\partial^2 y}{\partial x^2} \qquad (\text{as } l \to 0) \tag{2}$$

(see also Newton's "calculus of finite differences").

9.2 The Wave Equation

Taking the limit as $m, l \to 0$ and $\rho = \frac{m}{l}$ fixed, equation (1) becomes

$$\frac{\partial^2 y}{\partial t^2} - \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2} = 0 \tag{Wave Equation}$$

We can look for a solution to this equation two (simple) ways:

1. Fourier method
2. Method of characteristics

9.2.1 Fourier Method

We will solve the wave equation with a Fourier ansatz. Given our domain of $[0, l]$ and assuming the boundary condition $y(0,t) = y(l,t)$, the Fourier series of $y(x)$ is

$$y(x,t) = \sum_{n=-\infty}^{\infty} c_n(t) e^{ik_n x} \qquad k_n = \frac{2\pi n}{l} \tag{3}$$

Plugging this ansatz back into the wave equation,

$$\sum_n \ddot{c_n}(t) e^{ik_n x} + c_s^2 \sum_n c_n(t) k_n^2 e^{ik_n x} = 0 \qquad (c_s^2 = \frac{T}{\rho})$$

$$\implies \ddot{c_n}(t) + c_s^2 k^2 c_n(t) = 0 \qquad\qquad \text{(by orthogonality of exp)}$$

This is a simple harmonic oscillator (SHO) ODE; substituting $\omega_n^2 = c_s^2 k_n^2$ (this is known as a dispersion relation – we call k_n the wavenumber and ω_n the frequency) and solving the auxiliary equation gives the general solution for a_n,

$$c_n(t) = A_n e^{i\omega_n t} + B_n e^{-i\omega_n t} \tag{4}$$

So our general solution for $y(x,t)$ looks like

$$y(x,t) = \sum_{n=-\infty}^{\infty} A_n e^{i(k_n x + \omega_n t)} + B_n e^{i(k_n x - \omega_n t)} \tag{5}$$

with A_n and B_n determined by initial conditions.

9.2.2 Method of Characteristics

It turns out that the general solution of the wave equation can be expressed another way, known as **D'Alembert's solution**,

$$y(x,t) = \underbrace{f(x - c_s t)}_{\text{right}} + \underbrace{g(x + c_s t)}_{\text{left}}$$

This is most easily derived using the method of *canonical coordinates*. We begin by making the following substitution:

$$\begin{aligned} \xi &= x + ct \\ \eta &= x - ct \end{aligned}$$

Some use of the chain rule gives the following equations,

$$\begin{aligned} y_x &= y_\xi + y_\eta \\ y_t &= c(y_\xi - y_\eta) \\ y_{xx} &= y_{\xi\xi} + 2y_{\xi\eta} + y_{\xi\xi} \\ y_{tt} &= c^2(y_{\xi\xi} - 2y_{\xi\eta} + y_{\eta\eta}) \end{aligned} \tag{6}$$

Substitution into the wave equation produces the conveniently simple

$$y_{\xi\eta} = 0 \tag{7}$$

Integrate both sides by η and ξ to recover d'Alembert's solution.

9.3 Dispersion Relations

Dispersion relations describe the dependence of wave propagation on frequency/wavelength. First introduced by Laplace for water waves, they are essential for understanding the transportation of energy by waves.

They can be given as a relationship between frequency and wavenumber/wavelength or between energy and momentum. Given a relation we can define

- Group velocity, $v_g = \frac{\partial \omega}{\partial k}$, the speed at which the "peak" of a wavepacket propagates.
- Phase velocity, $v_p = \frac{\omega}{k}$, the speed at which a constant phase surface propagates.

If the dispersion relation is linear, a wavepacket will propagate without changing shape. If it is non-linear then the wave will change as it propagates. Note that a dispersion relation is a property of the system through which the wave is traveling, not of the wave itself.

Example 9.3.1 (Deep Water Waves). For deep water waves, $\omega = \sqrt{gk}$. Thus,

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}\frac{\omega}{k} = \frac{1}{2}v_p$$

Since $v_g \neq v_p$, the different ks of the wave envelope move at different frequencies, causing distortion. This produces the Kelvin wake pattern often seen following ducks in a pond!

Additionally, in a system with damping the dispersion relation can become complex. (Will not explore this right now)