8 Continuum Mechanics

8.1 Example: Motion of a String

We have an elastic string lying along the x-axis. Consider when we displace the string in the y direction by $y(x)$. If we slice the string into infinitesimally small segments of mass m, length l, and tension T, balancing the forces gives (using small angle approximation)

$$\begin{aligned} m\ddot{y}_n &= -\frac{T}{l}(y_n - y_{n-1}) + \frac{T}{l}(y_{n+1} - y_n) \\ \ddot{y}_n &= \frac{T}{ml}(y_{n+1} - 2y_n + y_{n-1}) \end{aligned} \tag{1}$$

Recall the limit definition of the second derivative (for intuition on this, refer to timestamp 8:00 of this 3Blue1Brown video):

$$\frac{y_{n+1} - 2y_n + y_{y_{n-1}}}{l^2} \to \frac{\partial^2 y}{\partial x^2} \qquad (\text{as } l \to 0) \tag{2}$$

(see also Newton's "calculus of finite differences").

8.2 The Wave Equation

Taking the limit as $m, l \to 0$ and $\rho = \frac{m}{l}$ fixed, equation (1) becomes

$$\frac{\partial^2 y}{\partial t^2} - \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2} = 0 \tag{Wave Equation}$$

We can look for a solution to this equation two (simple) ways:

1. Fourier method
2. Method of characteristics

8.2.1 Fourier Method

We will solve the wave equation with a Fourier ansatz. Given our domain of $[0, l]$ and assuming the boundary condition $y(0, t) = y(l, t)$, the Fourier series of $y(x)$ is

$$y(x, t) = \sum_{n=-\infty}^{\infty} c_n(t) e^{ik_n x} \qquad k_n = \frac{2\pi n}{l} \tag{3}$$

Plugging this ansatz back into the wave equation,

$$\sum_n \ddot{c_n}(t) e^{ik_n x} + c_s^2 \sum_n c_n(t) k_n^2 e^{ik_n x} = 0 \qquad\qquad (c_s^2 = \frac{T}{\rho})$$

$$\implies \ddot{c_n}(t) + c_s^2 k^2 c_n(t) = 0 \qquad\qquad \text{(by orthogonality of exp)}$$

This is a simple harmonic oscillator (SHO) ODE; substituting $\omega_n^2 = c_s^2 k_n^2$ (this is known as a dispersion relation – we call k_n the wavenumber and ω_n the frequency) and solving the auxiliary equation gives the general solution for a_n,

$$c_n(t) = A_n e^{i\omega_n t} + B_n e^{-i\omega_n t} \tag{4}$$

So our general solution for $y(x,t)$ looks like

$$y(x,t) = \sum_{n=-\infty}^{\infty} A_n e^{i(k_n x + \omega_n t)} + B_n e^{i(k_n x - \omega_n t)} \tag{5}$$

with A_n and B_n determined by initial conditions.

8.2.2 Method of Characteristics

It turns out that the general solution of the wave equation can be expressed another way, known as **D'Alembert's solution**,

$$y(x,t) = \underbrace{f(x - c_s t)}_{\text{right}} + \underbrace{g(x + c_s t)}_{\text{left}}$$

This is most easily derived using the method of *canonical coordinates*. We begin by making the following substitution:

$$\begin{aligned} \xi &= x + ct \\ \eta &= x - ct \end{aligned}$$

Some use of the chain rule gives the following equations,

$$\begin{aligned} y_x &= y_\xi + y_\eta \\ y_t &= c(y_\xi - y_\eta) \\ y_{xx} &= y_{\xi\xi} + 2y_{\xi\eta} + y_{\xi\xi} \\ y_{tt} &= c^2(y_{\xi\xi} - 2y_{\xi\eta} + y_{\eta\eta}) \end{aligned} \tag{6}$$

Substitution into the wave equation produces the conveniently simple

$$y_{\xi\eta} = 0 \tag{7}$$

Integrate both sides by η and ξ to recover d'Alembert's solution.

8.3 Dispersion Relations

Dispersion relations describe the dependence of wave propagation on frequency/wavelength. First introduced by Laplace for water waves, they are essential for understanding the transportation of energy by waves.

They can be given as a relationship between frequency and wavenumber/wavelength or between energy and momentum. Given a relation we can define

- Group velocity, $v_g = \frac{\partial \omega}{\partial k}$, the speed at which the "peak" of a wavepacket propagates.
- Phase velocity, $v_p = \frac{\omega}{k}$, the speed at which a constant phase surface propagates.

If the dispersion relation is linear, a wavepacket will propagate without changing shape. If it is non-linear then the wave will change as it propagates. Note that a dispersion relation is a property of the system through which the wave is traveling, not of the wave itself.

Example 8.3.1 (Water Waves). The naming of water waves is a little misleading, as their classification does not only depend on water depth. A deep water wave is one whose wavelength is more than twice that of depth, $h > \frac{1}{2}\lambda$, whereas a shallow one requires $h < \frac{1}{20}\lambda$. Let's compare their dispersion relations. For deep water waves, $\omega = \sqrt{gk}$. Thus,

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}\frac{\omega}{k} = \frac{1}{2}v_p$$

Since $v_g \neq v_p$, the different ks of the wave envelope move at different frequencies, causing distortion. This produces the Kelvin wake pattern often seen following ducks in a pond.

The dispersion relation for shallow water waves is $\omega = k\sqrt{gh}$. Then $v_g = v_p$! This means the wavepacket never disperses, physically seen by a clean wavefront (no distortion over time).

As a fun aside, back to the duck: theoretically, in shallow enough water, a duck can swim faster than the wavepacket it creates, creating a mach cone and producing a supersonic shock wave!

Additionally, in a system with damping the dispersion relation can become complex. (Will not explore this right now)

8.4 The Klein-Gordon Equation

Let's look at another physical system. This time, we have a row of hanging masses, separated by distance l, hanging from the x-axis a distance r. Each mass is connected by a spring of constant k. We use ϕ_i to measure the angle from vertical of the ith mass. To make things simple, we assume the masses can only move small amounts, so $|\phi_i| << 1$.

The kinetic energy of each mass is

$$T = \frac{1}{2}m(\dot{x}_i)^2 = \frac{1}{2}m(r\dot{\phi}_i)^2 = \frac{1}{2}mr^2\dot{\phi}_i^2$$

and potential energy is

$$V = V_{\text{grav}} + V_{\text{spring}}$$

To find gravitational potential energy we'll use a Taylor expansion to get rid of some higher-order terms and drop constants (the Euler-Lagrange equation isn't affected by them):

$$\begin{aligned}
\Delta h &= r - h_i \\
h_i &= r\cos\phi_i \approx r\left(1 - \frac{\phi_i^2}{2}\right) \\
\implies \Delta h &= \frac{1}{2} r\phi_i^2
\end{aligned}$$

so

$$V_{\text{grav}} = mg\Delta h = \frac{1}{2} mgr\phi_i^2$$

V_{spring} is just given by (using small-angle approximation)

$$V_{\text{spring}} = \frac{1}{2} k(r\sin\phi_{i+1} - r\sin\phi_i)^2 \approx \frac{1}{2} kr^2(\phi_{i+1} - \phi_i)^2$$

The Langrangian of the system is therefore

$$L = \sum_i \frac{1}{2} mr^2\dot{\phi}_i^2 - \frac{1}{2} mgr\phi_i^2 - \frac{1}{2} kr^2(\phi_{i+1} - \phi_i)^2$$

Or more conveniently

$$L = \sum_i \frac{1}{2} l\frac{m}{l} r^2\dot{\phi_i}^2 - \frac{1}{2} l\frac{m}{l} gr\phi_i^2 - \frac{1}{2} kr^2 l^2\left(\frac{\phi_{i+1} - \phi_i}{l}\right)^2 \tag{8}$$

Now think about what we're trying to model with this system. In section 8.1 we used our model to understand where the wave equation comes from, and clearly here we would like to do the same. So rather than think about only these discrete masses, we'd like to construct a continuum. In taking the continuum limit, we now consider the Lagrangian density,

$$\mathcal{L} = \frac{1}{2}\frac{m}{l} r^2\dot{\phi}^2 - \frac{1}{2}\frac{m}{l} gr\phi^2 - \frac{1}{2} kr^2 l\phi'^2$$

We can use the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right] + \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial\mathcal{L}}{\partial\phi'}\right] - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \tag{9}$$

to derive an equation of motion:

$$\frac{m}{l} r^2\ddot{\phi} - kr^2 l\phi'' + \frac{m}{l} gr\phi = 0$$

Let $\mu = \frac{mr^2}{l}$ and $\lambda = kr^2 l$,

$$\mu\ddot{\phi} - \lambda\phi' + \frac{\mu g}{r}\phi = 0$$

$$\ddot{\phi} - \frac{\lambda}{\mu}\phi'' + \omega_0^2\phi = 0 \tag{10}$$

Where $\omega_0^2 = \frac{g}{r}$. In "natural units" (i.e. set all the constants to 1), this is the **Klein-Gordon equation**,

$$\ddot{\phi} - \phi'' + \phi = 0 \tag{11}$$

which has use cases in the field of relativistic quantum field theory.

8.5 The Sine-Gordon Equation