1 The KdV Equation

1.1 Preliminaries

The simplest model of a wave is the one dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

with $u(x,t)$ giving the amplitude of the wave and positive constant c. The wave equation's general solution, expressed in terms of *characteristic variables* $(x \pm ct)$, is

$$u(x,t) = f(x-ct) + g(x+ct) \tag{2}$$

where f, g are determined by given initial conditions. This solution is known as *d'Alembert's solution*. From a glance at the solution it should be clear that it describes two waves, one moving to the left, and one to the right, both at speed c. The linearity of the wave equation allows for the principle of superposition, and as such neither wave interacts with itself nor one another.

> *How does d'Alembert's solution relate to the "Fourier Ansatz" solution to the wave equation?*
>
> Using a trig identity, the general solution to the wave equation that arises from the Fourier Ansatz can be turned into d'Alembert's solution (see here)

Let's make an additional simplification, and consider only a wave which propagates in a single direction (say, $g = 0$. Then $u(x,t) = f(x-ct)$ satisfies the equation

$$u_t + cu_x = 0 \tag{3}$$

In this solution, f does not change shape (justification, which I do not fully understand, is given in the text). Of course, this is therefore the simplest wave; if we lessen our simplifying assumptions, and include, say, dispersion, for which there is a new *dispersive* wave equation:

$$u_t + u_x + u_{xxx} = 0 \tag{4}$$

We can consider a solution known as the *harmonic* wave solution

$$u(x,t) = e^{i(kx-\omega t)} \tag{5}$$

(This is a simplification of the general solution to the standard wave equation – see here)

(5) is a solution to the dispersive wave equation if

$$\omega = k - k^3 \tag{6}$$

This is known as a *dispersion relation* (see ibid.) so we can call k the wavenumber and ω the frequency. Some algebra produces the following equations,

$$kx - \omega t = k(x - (1-k^2)t)$$

$$c = \frac{\omega}{k} = 1 - k^2 \tag{7}$$

This last equation says that the speed at which the wave propagates depends on its wavenumber (this value, known as the phase velocity, describes the velocity of a single component of a wave profile). Thus waves of different wave numbers propagate at different velocities, the defining characteristic of a dispersive wave.

We can also define the *group velocity*,

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = 1 - 3k^2 \tag{8}$$

which describes the velocity of a whole wave packet. This video is a great visualization of the difference between group and phase velocity. For many wave motions it is the case that $c_g \leq c$, though the video above is one such exception.

> Can a wave be described as two components like (5) but with different wavenumbers? That is, can a single wave have two components that move at different velocities?

The book says yes, we *can* represent a wave as a superposition of different harmonic solutions of varying wavenumbers:

$$u(x,t) = \int_{-\infty}^{\infty} A(k) e^{i(kx - \omega(k)t)}\, dk \tag{9}$$

(Note similarity to Fourier transform?) The effect of this superposition is a wave profile which changes as it moves; the differing speeds of the components cause the wave to *disperse*.