1 The KdV Equation

1.1 Preliminaries

The simplest model of a wave is the one dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

with $u(x,t)$ giving the amplitude of the wave and positive constant c. The wave equation's general solution, expressed in terms of *characteristic variables* $(x \pm ct)$, is

$$u(x,t) = f(x-ct) + g(x+ct) \tag{2}$$

where f, g are determined by given initial conditions. This solution is known as *d'Alembert's solution*. From a glance at the solution it should be clear that it describes two waves, one moving to the left, and one to the right, both at speed c. The linearity of the wave equation allows for the principle of superposition, and as such neither wave interacts with itself nor one another.

1.1.1 *How does d'Alembert's solution relate to the "Fourier Ansatz" solution to the wave equation?*

Using a trig identity, the general solution to the wave equation that arises from the Fourier Ansatz can be turned into d'Alembert's solution (see here) Let's make an additional simplification, and consider only a wave which propagates in a single direction (say, $g = 0$. Then $u(x,t) = f(x-ct)$ satisfies the equation

$$u_t + cu_x = 0 \tag{3}$$

In this solution, f does not change shape (justification, which I do not fully understand, is given in the text). Of course, this is therefore the simplest wave; if we lessen our simplifying assumptions, and include, say, dispersion, for which there is a new *dispersive* wave equation:

$$u_t + u_x + u_{xxx} = 0 \tag{4}$$