Exploring a Type-Theoretic Environment for Python

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Set Theory vs. Type Theory

- Russell's paradox, 1901: Let $R = \{x | x \notin x\}$. $R \in R \implies R \notin R$ and $R \notin R \implies R \in R$, contradiction
- Need to put mathematics on a strong logical/axiomatic foundation
- Competing candidates: set theory & type theory
- The expressions of Type Theory are **terms**; all terms have a **type**
- The expression "x has type T" is written $x : T$
- $0 : \mathbb{N}$ means 0 has type "natural number"
- Every object in Type Theory has a type: there are types for functions, types for proofs, types for types themselves, etc.
- **Theorems are types, proofs are terms**
- Set Theory is built on top of propositional and predicate logic, and elements can belong to multiple sets; In Type Theory, propositional and predicate logic are encoded as types, and terms can only belong to one type
- **It is easier for a machine to check type-theoretic proofs**

Inductive & Record Types

- An **Inductive Type** is a type equipped with rules (called **constructors**) that explain how the terms of a type are built
- Defining $\mathbb{N}$ as the inductive type `nat` in Coq:

```
Inductive nat :  Set := 0 :  nat, S : nat -> nat
```

- Using the constructors `0` and `S`, which have type `nat` and `nat -> nat`, we construct terms of type `nat`
- Represent $0, 1, 2, ... \in \mathbb{N}$ by `0, S(0), S(S(0)),...  :  nat`
- A **Record Type** is a type composed of fields of different types
- Defining $\mathbb{Q}$ as the record type `rat` in Coq:

```
Record rat :  { num :  nat, denom :  nat, sign :  bool
 denom_cond :  bottom_neq_0, irred_cond :  irreducible}
```

- Can represent a wide variety of mathematical structures as inductive/record types
- Easier to prove statements about inductive/record types because terms of these types are **constructive** (we can build them), and their types are made explicit

Syntactic Rules

- Formally, a **proof** is a sequence of applications of **syntactic rules**
- Γ is a set of formulas, φ, ψ, and θ are formulas, and $\vdash$ means "proves"
- $\frac{A}{B}$ means if A is True, then deduce B
- Examples:

$\Gamma \vdash \varphi$ if $\varphi \in \Gamma$ (Assume)
$\Gamma \vdash t = t$ for all terms t (Reflexivity)

$$\frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi}(\wedge EL) \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}(\wedge ER) \qquad \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi}(\wedge I)$$

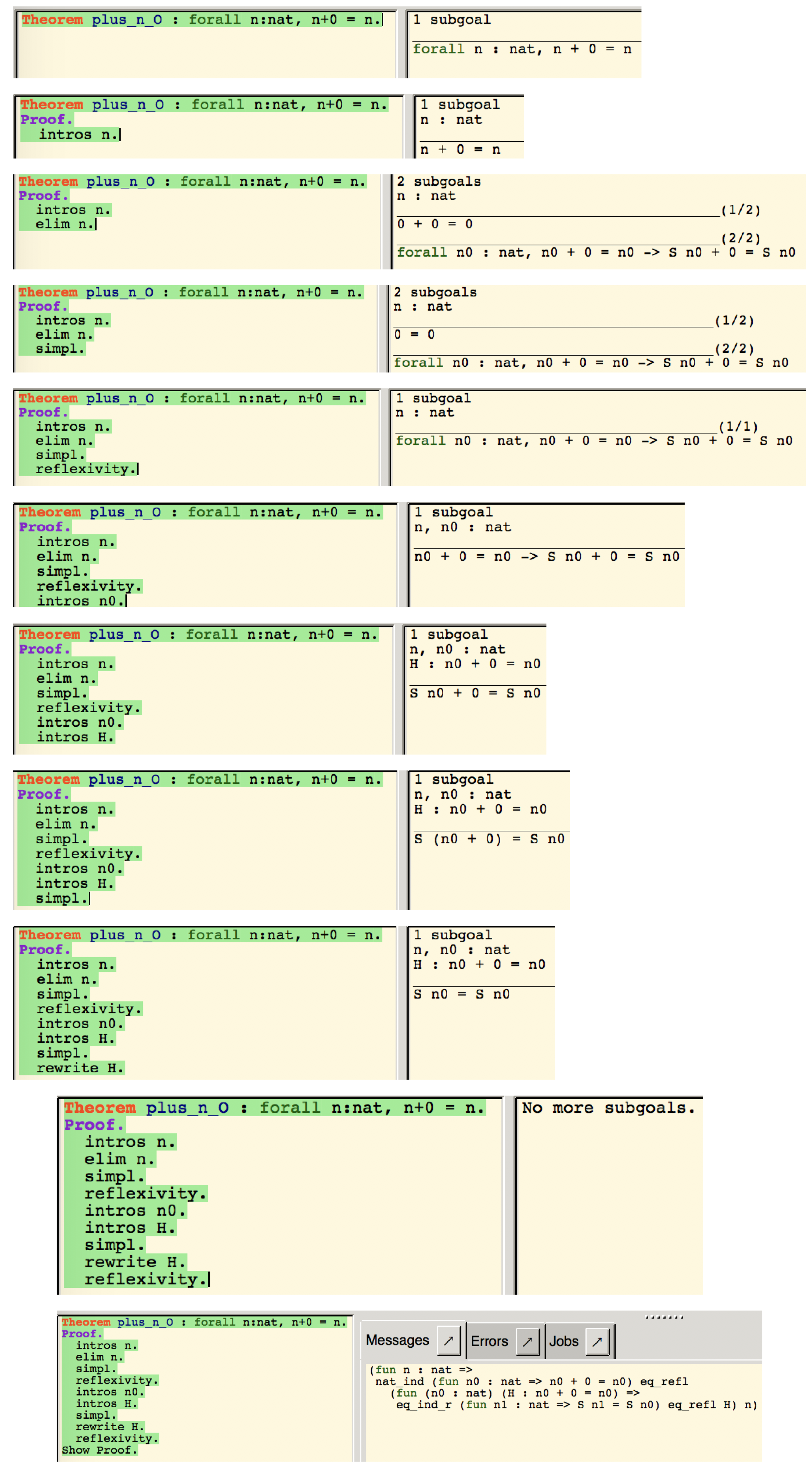
$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}(\vee IL) \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}(\vee IR)$$

$$\frac{\Gamma \vdash \varphi \to \psi}{\Gamma \cup \{\varphi\} \vdash \psi}(\to E) \qquad \frac{\Gamma \cup \{\varphi\} \vdash \psi}{\Gamma \vdash \varphi \to \psi}(\to I)$$

$$\frac{\Gamma \cup \{\varphi\} \vdash \theta \quad \Gamma \cup \{\psi\} \vdash \theta}{\Gamma \cup \{\varphi \wedge \psi\} \vdash \theta}(\wedge PC) \qquad \frac{\Gamma \cup \{\psi\} \vdash \varphi \quad \Gamma \cup \{\neg\psi\} \vdash \varphi}{\Gamma \vdash \varphi}(\neg PC)$$

Interactive Theorem Proving in Coq

- Theorems are types, proofs are terms
- Interactive Theorem Provers are **Type-Checkers**: build a proof term, and check that its type matches your desired theorem
- Build proof terms by applying syntactic rules, using backwards-chaining logic
- Example: prove "`forall n :  nat, n + 0 = n`" in Coq:



Interactive Theorem Proving in Python?

- **Motivation:** Nvidia, our project's sponsor, wants a type-theoretic environment in Python, because Python is ubiquitous these days, and because Nvidia wants machine learning and "machine reasoning" on the same platform (Python)
- **Main Challenge #1:** A Coq expert told us it would take a team of experts > 3 years to write a robust interactive theorem prover in Python
- **Main Challenge #2:** Successful interactive theorem provers (Coq, Isabelle) are written in functional programming languages (ML, OCAML), whereas Python allows functional programming, imperative programming, and object-oriented programming
- We think an **object-oriented** layout is best for a type-theoretic library in Python
- Classes Type and `Term`
- `Term` has subclasses `Variable, Constant, Application, Abstraction, OrderedPair, RecordTerm, ...`
- `Type` has subclasses `Inductive, Record, Implication, Conjunction, Disjunction, ...`
- Using Nat (i.e. $\mathbb{N}$) in Python:

```
nat = Inductive("nat")
O = Const("O")
S = Const("S")
InductiveTypeIntro(O, nat)
InductiveTypeIntro(S, Implication(nat, nat))

one = Application(S, O)
two = Application(S, Application(S, O))
print(type_check(two))
>>> nat
```

- Implement syntactic rules as methods of the `Thm` class:

0. $A \to B \vdash A \to B$ by assume $A \to B$
1. $A \vdash A$ by assume A
2. $A, A \to B \vdash B$ by implication elimination on 0,1
3. $A \vdash (A \to B) \to B$ by implication introduction on 2
4. $\vdash A \to (A \to B) \to B$ by implication introduction on 3

```
th0 = Thm.assume(Term.mk_implies(A, B))
th1 = Thm.assume(A)
th2 = Thm.implies_elim(th0, th1)
th3 = Thm.implies_intr(Term.mk_implies(A, B), th2)
th4 = Thm.implies_intr(A, th3)
print(printer.print_thm(thy, th4, unicode=True))

⊢ A ⟶ (A ⟶ B) ⟶ B
```

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