

Kelly Betting

Card game demo

- Split up into pairs
- each pair grab a deck of cards
- each person starts with, say, \$100
- let person A in the pair be the dealer and person B be the bettor.

Person B bets as much as he wants of his bankroll on

whether the next card will be Red or Black at Fair Odds,

person A flips a card. Record the new bankroll after the card is flipped.

Play until 35 cards have been flipped.

- Then reverse the roles A and B and replay
- The person who makes the most money wins!

We want to bet on Basketball games over the course of the entire NBA season and beyond.

Initial Bankroll $B = \$100$.

P_i = true prob. that team A_i wins (assume known) ^{vs. team B_i} in game i

Decimal odds $\alpha_{A_i} = \# \text{ dollars returned for a } \$1 \text{ bet on team } A_i;$
 $\alpha_{B_i} = 1 + \text{profit from } \$1 \text{ bet on team } A_i;$

Q How should we bet?

* Someone's going to say: maximize EV

* EV of a \$1 bet on team A : ignoring subscript i ,

$$\begin{aligned}EV_A &= E(\text{profit}_A) = \left(\Pr(A \text{ wins}) \right) \cdot \left(\begin{array}{c} \text{profit} \\ \text{if } A \text{ wins} \end{array} \right) + \left(\Pr(A \text{ loses}) \right) \cdot \left(\begin{array}{c} \text{profit} \\ \text{if } A \text{ loses} \end{array} \right) \\&= P(\alpha_A - 1) + (1-P)(-1) \\&= P\alpha_A - 1\end{aligned}$$

$$E(\text{profit}_B) = (1-P)\alpha_B - 1$$

* Let's say $EV_A < 0$. How much will you bet?

→ they'll say zero. Fair enough.
Come back to this

* Let's say $EV_A > 0$. How much will you bet?

→ they'll say make a bet!

* Even if $EV_A > 0$, if you bet your entire bankroll on A and lose, then you're out of money :)

How to account for this?

→ What about, over N bets, just split up your money evenly betting B/N on team A each time?

→ Make money on average but can we make more money?
What are we missing?

* Not taking advantage of the sequential nature of the bets.
If I make money on the first bet, I can use that profit to bet more on the second bet! compounding!

* How do we actually achieve compounding?

→ bet a fraction $f \in (0, 1)$ of your bankroll

Bet size $B \cdot f_i$ on team A_i in game i

Make $(\alpha_{A_i} - 1) B f_i$ w.p. p

Lose make $-B f_i$ w.p. $1-p$

Profit $B \cdot f_i (\alpha_{A_i} X_i - 1)$

where $X_i = \begin{cases} 1 & \text{if } A_i \text{ wins in game } i \\ 0 & \text{if } A_i \text{ loses in game } i \end{cases}$

* Is this what we want to maximize though?
Profit? What do we actually want to have at the end?
A high bankroll!

$$\text{Bankroll } B + B f_i(\alpha_{A_i} x_i - 1) = B [1 + f_i(\alpha_{A_i} x_i - 1)]$$

↓

After first bet, we have $B [1 + f_i(\alpha_{A_i} x_i - 1)]$

Bet size $B [1 + f_i(\alpha_{A_i} x_i - 1)] \cdot f_2$ on team A_2 in game 2

Profit $B [1 + f_i(\alpha_{A_i} x_i - 1)] \cdot f_2 (\alpha_{A_2} x_2 - 1)$ after game 2

Bankroll $B [1 + f_i(\alpha_{A_i} x_i - 1)] [1 + f_2 (\alpha_{A_2} x_2 - 1)]$ by same logic

↓

After N games,

$$\text{Bankroll} = B \prod_{i=1}^N [1 + f_i (\alpha_{A_i} x_i - 1)]$$

B = initial bankroll (say, \$100)

α_{A_i} = decimal odds for betting on team A in game i (known)

$x_i = 1$ if team A wins in game i , else 0 (random variable)

f_i = fraction of bankroll on game i (want to find)

* Want to Maximize Bankroll
 but, bankroll is a Random Variable
 → Maximize Expected Bankroll (a number)

$$\begin{aligned} & \underset{f}{\operatorname{argmax}} \mathbb{E} \text{Bankroll} \\ = & \underset{f}{\operatorname{argmax}} \mathbb{E} \left(\prod_{i=1}^N \left[1 + f_i (\alpha_i X_i - 1) \right] \right) \end{aligned}$$

↓
 the random variables
 X_1, \dots, X_N

* Good luck doing this!!
 Too hard due to the product.

So, we're stuck because of the product.

How to get rid of a product?

→ log

Kelly's brilliant idea:

TRY $\underset{f}{\operatorname{argmax}} \mathbb{E} \log \left(B \prod_{i=1}^N \left[1 + f_i (\alpha_{A_i} X_i - 1) \right] \right)$

Shannon-McMillan-Breiman 1950s.

the f that maximizes the log bankroll
has more money asymptotically as N goes to ∞
than any other allocation f !

$$= \underset{f}{\operatorname{argmax}} \mathbb{E} \sum_{i=1}^n \log (1 + f_i (\alpha_{A_i} X_i - 1))$$

$$= \sum \mathbb{E} \log$$

$$= \underset{f}{\operatorname{argmin}} \sum_{i=1}^n \log (1 + f_i (\alpha_{A_i} - 1)) \cdot P_i + \log (1 - f_i) \cdot (1 - P_i)$$

Same minimization for each i , due to the \sum

$$\underset{f}{\operatorname{argmin}} \quad \log(1+f(\alpha-1))P + \log(1-f)(1-P)$$

* How to solve this? Calculus!

$$\frac{d}{df} \left[\log(1+f(\alpha-1))P + \log(1-f)(1-P) \right]$$

$$= \frac{\alpha-1}{1+f(\alpha-1)} \cdot P + \frac{-1}{1-f} \cdot (1-P) = 0$$

$$\Rightarrow \frac{1-P}{1-f} = \frac{P(\alpha-1)}{1+f(\alpha-1)}$$

$$\Rightarrow (1-P)(1+f(\alpha-1)) = P(\alpha-1)(1-f)$$

$$\Rightarrow f(1-P)(\alpha-1) + (1-P) = -fp(\alpha-1) + p(\alpha-1)$$

$$\Rightarrow f(-P(\alpha-1) + P(\alpha-1)) = P(\alpha-1) - (1-P)$$

$$\Rightarrow f(\alpha-1) = P \cancel{\alpha-1} - \cancel{P-1} + P$$

$$\Rightarrow f = \frac{P\alpha-1}{\alpha-1}$$

$$f = \max(0, \frac{P\alpha-1}{\alpha-1})$$

Kelly
Fraction

Exs

- If $P=1$ (guaranteed)
then $f=1$ (bet entire bankroll)
- If -110 bet,
decimal odds $\alpha = 1 + \frac{100}{110} = \frac{210}{110} = 1.909$
for f to be positive (to bet something)
we need $\frac{P\alpha - 1}{\alpha - 1} > 0 \Rightarrow P > \frac{1}{\alpha} = 0.524$
- Fair odds is $\alpha = \frac{1}{P}$
Suppose $\alpha = \frac{1}{P} + \delta$, so $\delta \geq 0$ is your "edge"
Then $\frac{P\alpha - 1}{\alpha - 1} = \frac{P(\frac{1}{P} + \delta) - 1}{\frac{1}{P} + \delta - 1} = \frac{\delta}{\delta + \frac{1}{P} - 1}$
- Desmos $f = \max(0, \frac{P\alpha - 1}{\alpha - 1})$

* In practice, the win probability p of a horse or team is not an observable OR known quantity (with a deck of cards it is, but in Real life sports it's not); it needs to be estimated from data $\rightarrow \hat{p}$.

How does Kelly betting change under this?

* Ideally our estimator \hat{p} of p is unbiased $E\hat{p} = p$ but subject to some uncertainty $Var(\hat{p}) = T^2$. The more uncertain we are in our estimate, the less we should bet.

* Fractional Kelly says bet a fraction $K \in [0, 1]$ of the Kelly betting fraction f , $f < K \cdot f$.

* K is some function $K = K(\tau)$

such that $\lim_{\tau \downarrow 0} K(\tau) = 1$

(if $E\hat{P} = p$ and $\text{var}(\hat{P}) = 0$, we know the true win probabilities, so use original Kelly formula)

and

$\lim_{\tau \uparrow \infty} K(\tau) = 0$

(fully uncertain about win prob)

* K -Fractional Kelly with \hat{P} is equivalent to full Kelly with a shrinkage estimator for p , shrinking more if τ^2 larger

Lab: Kelly Betting

I. A smart -EV Bet

Kelly (1956) extended his analysis to horses

m horses $1, \dots, m$
true (known) win probabilities p_1, \dots, p_m
decimal odds $\alpha_1, \dots, \alpha_m$
payroll fractions $f = (f_1, \dots, f_m)$ to bet on each horse.
make N sequential bets, solve for f .

Use Lagrange Multipliers / KKT conditions to maximize expected log bankroll after N bets,
yielding the Kelly allocations f via this algorithm:

- (a) Permute indices so that $p(s)\alpha_s \geq p(s+1)\alpha_{s+1}$
- (b) Set b equal to the minimum positive value of

$$\frac{1 - p_t}{1 - \sigma_t} \quad \text{where } p_t = \sum_1^t p(s), \quad \sigma_t = \sum_1^t \frac{1}{\alpha_s}$$

- (c) Set $f(s) = p(s) - b/\alpha_s$ or zero, whichever is larger. (The $f(s)$ will sum to $1 - b$.)

Now, Consider a horse race with $m=3$ horses.
Find an example of true horse win probabilities (P_1, P_2, P_3) and decimal odds (d_1, d_2, d_3) such that it is Kelly optimal to bet on a -EV horse.

The Kelly bet allocations (f_1, f_2, f_3) according to the above algorithm must consist of $f_i > 0$ for some horse i , where betting on horse i is -EV : $P_i d_i - 1 < 0$.

Then, $M=100$ times, simulate $N=1000$ bets on this horse race and record your bankroll at the end using allocations (f_1, f_2, f_3) according to ① Kelly and ② separately letting $f_i = 0$ (where i is the -EV horse). Which yields more wealth on average?

Why is it possible that making a -EV bet is smart?

If the game is profitable on the whole, you should put money into the game.

If you only bet on the +EV horse you can lose more than if you hedge.