The 2 Envelopes Problem.

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Suppose you have 2 envelopes containing money. One envelope contains twice as much money as the other envelope.

The "Paradox".

1. Suppose you pick up the first envelope. It has some amount of money inside it, call it x
2. Then other envelope contains $2x$ dollars with probability $\frac{1}{2}$
3. And the other envelope contains $\frac{x}{2}$ dollars with probability $\frac{1}{2}$
4. Therefore the expected value of switching envelopes is

$$\mathbb{E}\left[switching\right] = \frac{1}{2} \cdot 2x + \frac{1}{2} \cdot \frac{x}{2} = \frac{5x}{4} > x$$

5. Since the expected value of switching envelopes is greater than the expected value of sticking with the current envelope, it is more profitable on average to switch envelopes
6. But when you switch envelopes, it has some amount of money in it, call it y
7. Then by symmetric logic, we should switch back to the first envelope
8. Thus the optimal strategy is to switch envelopes forever, which is clearly absurd

Which step(s) in the proof are wrong, if any?

Step (4) is wrong.

The amount of money in both envelopes is fixed: one has z dollars, the other has $2z$ dollars. However, the amount of money in the first envelope is a random variable, say X, and the amount of money in the second envelope is a random variable, Y, since we don't know which envelope has z dollars and which has $2z$ dollars. Then the expected value of switching is $E[Y]$, which is

$$\begin{aligned} E[Y] &= E[Y|Y < X]P[Y < X] + E[Y|Y > X]P[Y > X] \\ &= \frac{1}{2}E[Y|Y < X] + \frac{1}{2}E[Y|Y > X] = \frac{1}{2}z + \frac{1}{2}2z = \frac{3}{2}z = E[X] \end{aligned}$$

So the paradox is resolved: you gain nor lose nothing by switching. The issue with step (4) is that it fails to use conditional expectation correctly.