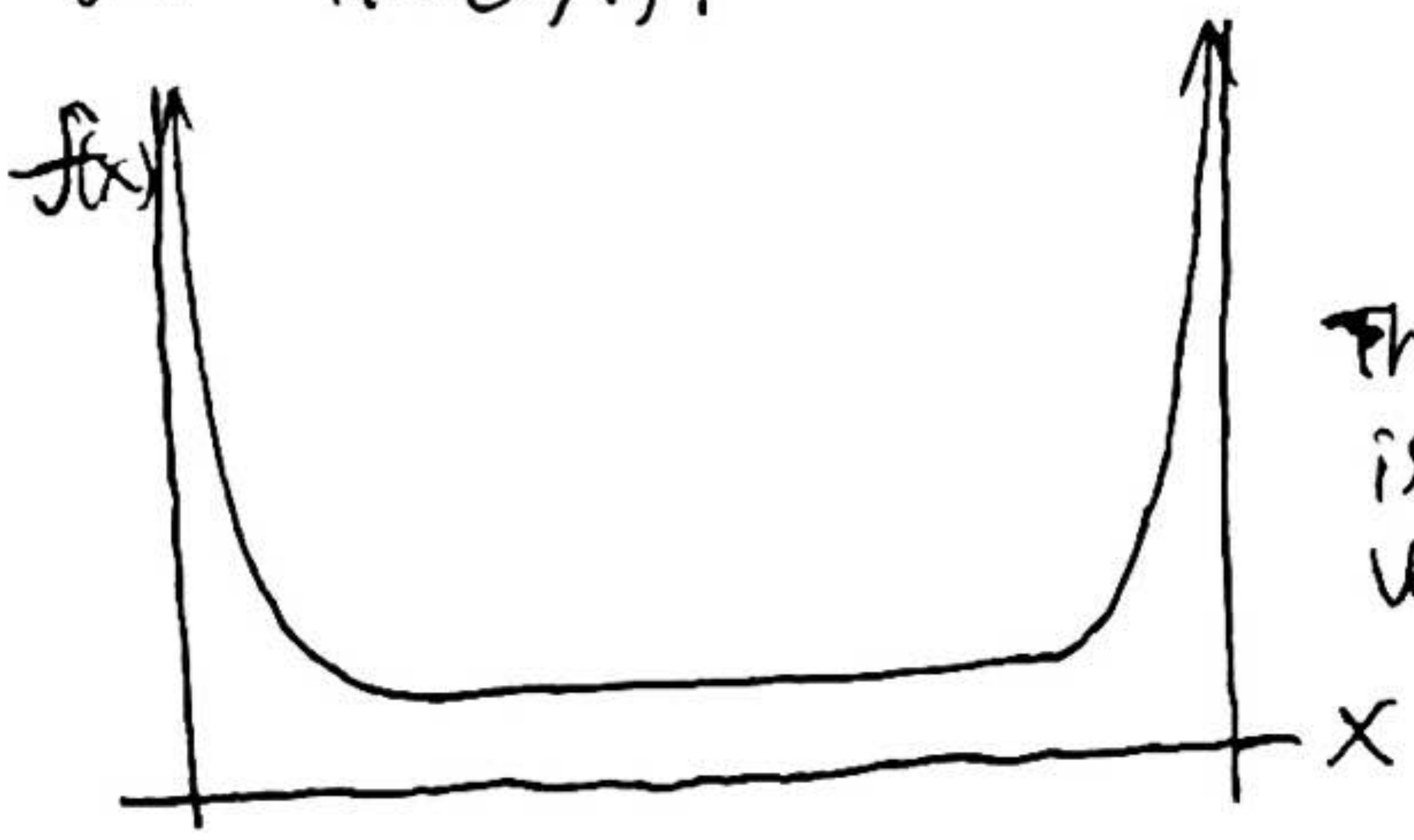
3 Arcsine Laws for Random Walks/Brownian Motion and their Relation to sports

Def The arcsine distribution on $(0,1)$ has density $f(x) = \frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}}$ and cdf $F(x) = P(X \le x) = \arcsin(\sqrt{x})$

for $x \in (0,1)$.



The density of the arcsine distribution is concentrated near the boundary values 0 and 1.

Arcsine laws for Brownian Motion

1. The last zero time of BM in $[0,1]$ $\quad L = \sup\{t \in [0,1] : B_t = 0\}$ is arcsine distributed.
2. The (unique) time that BM acheives its maximum in $[0,1]$, M such that $B_M = \max_{s \in [0,1]} B_s$, is arcsine distributed.
3. The positive occupation time of BM in $[0,1]$ $\quad T = \mathcal{L}\{t \in [0,1] : B_t > 0\}$ is arcsine distributed. ($\mathcal{L}$: lebesgue measure)

Note We find these distributions for BM by smart calculations, and then transfer these results to General Random Walks via Donsker's Invariance Principle and Portmanteau Thm.

Arcsine Laws for General Random Walks

Let $(X_k)_{k\geq 1}$ be iid with $\mathbb{E}X_k = 0$, $\mathbb{E}X_k^2 < \infty$ and let $S_n = \sum_{k=1}^{n} X_k$ be the associated general random walk.

1. The last time the random walk changes sign before time n, $N_n = \max\{1 \leq k \leq n : S_k S_{k-1} \leq 0\}$, satisfies $\frac{N_n}{n} \xrightarrow{d}$ Arcsine Distribution, i.e. $\mathbb{P}(\frac{N_n}{n} \leq x) \to \frac{2}{\pi}\arcsin(\sqrt{x})$.

2. The first time the random walk achieves its maximum before time n, $T_n = \min\{1 \leq k \leq n : S_k = \max_{1\leq j \leq n} S_j\}$, satisfies $\frac{T_n}{n} \xrightarrow{d}$ Arcsine Distribution.

3. The positive occupation time of the random walk $P_n = \#\{1 \leq k \leq n : S_k > 0\}$ satisfies $\frac{P_n}{n} \xrightarrow{d}$ Arcsine Distribution.

Question: Do These Arcsine Laws hold in Sports?

Examples

1. Is the time of the last lead change of a Lakers vs. Knicks game arcsine distributed?

2. Is the time of the maximum lead of a Lakers vs. Knicks game arcsine distributed?

3. Is the amount of time that the Lakers lead the Knicks arcsine distributed?

Note For the remainder of these notes, we will (mostly) go through the proofs of these arcsine laws for Brownian Motion / Random Walks.

I. Combinatorial Proof of 1st Arcsine Law for SRW

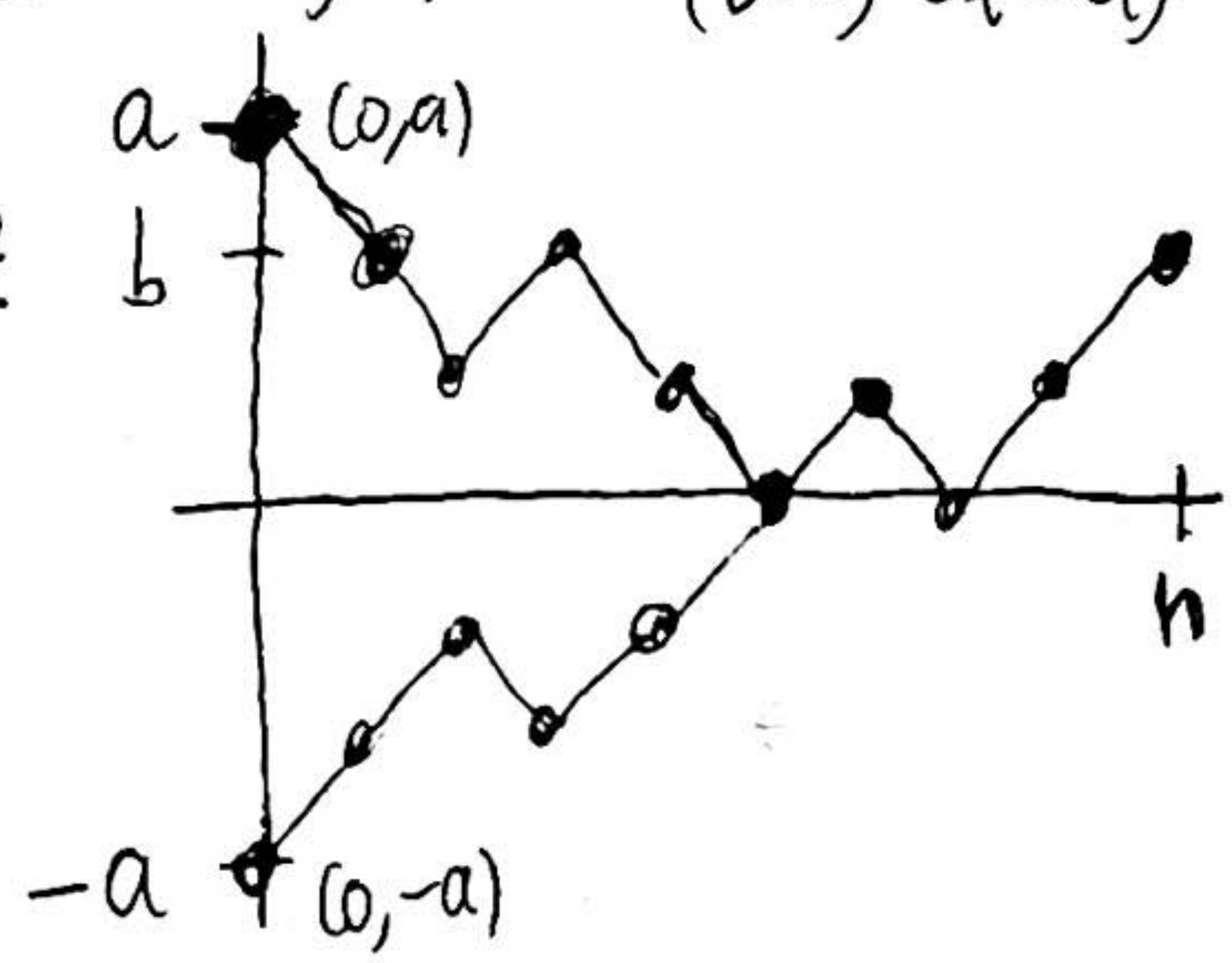
<u>Def</u> Let $\varepsilon_k \overset{iid}{\sim}$ Rademacher, so $\varepsilon_k = \begin{cases} +1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases}$

<u>Simple Random Walk (SRW)</u> $S_n = \sum_{k=1}^{n} \varepsilon_k$

> <u>Reflection Principle</u> If $a, b > 0$ then the number of SRW paths from $(0,a)$ to (n,b) that hit the x-axis ($S_t = 0$ at some point $0 \le t \le n$) is equal to the total number of SRW paths from $(0,-a)$ to (n,b).

<u>Note</u> $(0,a) = (t=0, S_t = a)$ and $(n,b) = (t=n, S_n = b)$.

<u>Pf</u>



The paths are in bijective correspondence, reflecting the segment up to the first point on the x-axis. □

No-Return Thm for SRW For a 1D simple Random Walk,

$$\mathbb{P}\begin{pmatrix}\text{Never Return to origin}\\ \text{up to time } 2n\end{pmatrix} = U_{2n} := 2^{-2n}\binom{2n}{n} \sim \frac{1}{\sqrt{\pi n}}$$

<u>Note</u> $\mathbb{P}\begin{pmatrix}\text{SRW returns to}\\ \text{origin at time } 2n\end{pmatrix} = \mathbb{P}(S_{2n}=0) = \mathbb{P}\begin{pmatrix}\text{exactly } n \ +1\text{'s}\\ \text{exactly } n \ -1\text{'s}\end{pmatrix} = \binom{2n}{n}2^{-2n} = U_{2n}$

<u>Note</u> $U_{2n} = \binom{2n}{n}2^{-2n} = \frac{(2n)!}{n!\,n!}2^{-2n} \sim \frac{\left(\frac{2n}{e}\right)^{2n}\sqrt{2\pi(2n)}}{\left(\frac{n}{e}\right)^{n}\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}\sqrt{2\pi n}} \cdot \frac{1}{2^{2n}}$ by Stirlings' $= \frac{1}{\sqrt{\pi n}}$

<u>Pf</u> # SRW paths that stay positive for $2n$ steps

$$= \sum_{\substack{\ell \in [2,2n]\\ \ell \text{ even}}} \left\{ \# \text{ paths } (1,1) \to (2n,\ell) \text{ not touching the x-axis} \right\}$$

$$= \sum_{k=1}^{n} \left\{ \begin{bmatrix}\text{Total \# paths}\\ (1,1) \to (2n,2k)\end{bmatrix} - \begin{bmatrix}\text{Total \# paths } (1,1) \to (2n,2k)\\ \text{that do touch x-axis}\end{bmatrix} \right\}$$

$$= \sum_{k=1}^{n} \left\{ [TNP\ (1,1) \to (2n,2k)] - [TNP\ (1,-1) \to (2n,2k)] \right\} \text{ by Reflection Principle}$$

$$= \sum_{k=1}^{n} \left\{ [TNP\ (0,0) \to (2n-1,2k-1)] - [TNP\ (0,0) \to (2n-1,2k+1)] \right\}$$

$$= \underbrace{[TNP\ (0,0) \to (2n-1,1)]}_{\substack{2n-1 \text{ timesteps}\\ n\ +1\text{'s},\ n-1\ -1\text{'s}}} - \underbrace{[TNP\ (0,0) \to (2n-1,2n+1)]}_{=0} \text{ by Telescoping Sum}$$

$= \binom{2n-1}{n}$, and total # SRW paths over $2n$ timesteps $= 2^{2n}$

and # SRW paths that stay negative over $2n$ steps $=$ # stay positive $= \binom{2n-1}{n}$.

Hence $\mathbb{P}\begin{pmatrix}\text{SRW no return}\\ \text{to 0 in } 2n \text{ steps}\end{pmatrix} = \frac{2\cdot\binom{2n-1}{n}}{2^{2n}} = 2^{-2n}\frac{(2n-1)!}{n!\,(n-1)!}\cdot\frac{2n}{n} = 2^{-2n}\binom{2n}{n} = U_{2n}$ $\square$

Arcsine Law for Last Time SRW hits 0

Let L_{2n} be the last time $2k \in \{2, 4, \dots, 2n\}$ that SRW hits 0.

$$\mathbb{P}\left(a \le \frac{L_{2n}}{2n} \le b\right) \longrightarrow \int_a^b \frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}}\, dx \qquad \forall\, a \le b \in [0,1]$$

i.e. $\frac{L_{2n}}{2n} \xrightarrow{d}$ Arcsin Distribution A, so $\mathbb{P}(A \le x) = \frac{2}{\pi} \arcsin(\sqrt{x})$.

Pf $L_{2n} = \max\{k : 1 \le k \le 2n,\ S_k = 0\}$.

$$\mathbb{P}(L_{2n} = 2k) = \mathbb{P}(S_{2k} = 0) \cdot \mathbb{P}\left(\begin{matrix}\text{SRW avoids zero}\\ \text{for } 2n-2k \text{ timesteps}\end{matrix}\right) = u_{2k}\, u_{2n-2k}$$

$$\sim \frac{1}{\sqrt{\pi k}} \frac{1}{\sqrt{\pi(n-k)}} = \frac{1}{\pi n} \frac{1}{\sqrt{\frac{k}{n}\left(1-\frac{k}{n}\right)}}$$

Hence $\mathbb{P}\left(\frac{L_{2n}}{2n} < x\right) = \mathbb{P}(L_{2n} < 2(nx)) = \sum_{k < nx} \mathbb{P}(L_{2n} = 2k)$

$$\sim \frac{1}{\pi} \sum_{\frac{k}{n} < x} \underbrace{\frac{1}{\sqrt{\frac{k}{n}\left(1-\frac{k}{n}\right)}}}_{\text{height}} \cdot \underbrace{\frac{1}{n}}_{\text{base}} \xrightarrow{n \to \infty} \frac{1}{\pi} \int_0^x \frac{1}{\sqrt{y(1-y)}}\, dy, \quad x \le 1 \qquad \square$$

Reimann sum

Arcsine Law for Last Time Brownian Motion hits 0 in $[0,1]$

$L = \sup\{t \in [0,1] : B_t = 0\}$ is arcsine distributed

Pf Let $M_t = \max_{0 \le s \le t} B_t$.

For $x \in [0,1]$,

$$\mathbb{P}(M \le x) = \mathbb{P}\left(\max_{0 \le u \le x} B_u > \max_{x \le v \le 1} B_v \right)$$

$$= \mathbb{P}\left(\max_{0 \le u \le x} B_u - B_x > \max_{x \le v \le 1} B_v - B_x \right)$$

$$= \mathbb{P}\left(M_x^{(1)} > M_{1-x}^{(2)} \right)$$

where $(M_t^{(1)})_{0 \le t \le 1}$ is the maximum process of the Brownian Motion $(B_t^{(1)})_{0 \le t \le 1}$ given by $B_t^{(1)} = B_{x-t} - B_x$

and $(M_t^{(2)})_{0 \le t \le 1}$ is the maximum process of the independent Brownian Motion $(B_t^{(2)})_{0 \le t \le 1}$ given by $B_t^{(2)} = B_{x+t} - B_x$.

By Reflection Principle, $(M_t)_{0 \le t \le 1} \overset{d}{=} (|B_t|)_{0 \le t \le 1}$.

Hence

$$P(M_x^{(1)} > M_{1-x}^{(2)}) = P(|B_x^{(1)}| > |B_{1-x}^{(2)}|)$$

$$= \mathbb{P}(\sqrt{x}\,|Z_1| > \sqrt{1-x}\,|Z_2|)$$ by scale invariance, $Z_1, Z_2 \overset{ind}{\sim} N(0,1)$

$$= P\left(\frac{|Z_2|}{\sqrt{Z_1^2 + Z_2^2}} < \sqrt{x} \right)$$

$$= P(|\sin(\theta)| < \sqrt{x})$$ where $\theta \sim unif[0, 2\pi]$.

$$= 4\,\mathbb{P}(\theta < \arcsin\sqrt{x})$$

$$= 4\left(\frac{\arcsin\sqrt{x}}{2\pi}\right)$$

$$= \frac{2}{\pi}\arcsin\sqrt{x} \quad \square$$

Skorokhod's Embedding Thm ($\Leftarrow$) Suppose $\mathbb{E}\xi = 0$, $\mathbb{E}\xi^2 < \infty$. Then $\exists$ stopping time T w/r $(B_t)_{t\geq 0}$ such that $\mathbb{E}T < \infty$ and $B_T \overset{d}{=} \xi$

Pf by Dubin

If $\xi \in \{a, b\}$ with $a < 0 < b$, then $T = T_a \wedge T_b$, where $T_a = \inf\{t : B_t = a\}$ and by Optional Stopping Thm, $B_T \overset{d}{=} \xi$.

If ξ takes 4 values, say $P(\xi=-1) = P(\xi=1) = P(\xi=-2) = P(\xi=2) = \frac{1}{4}$, then

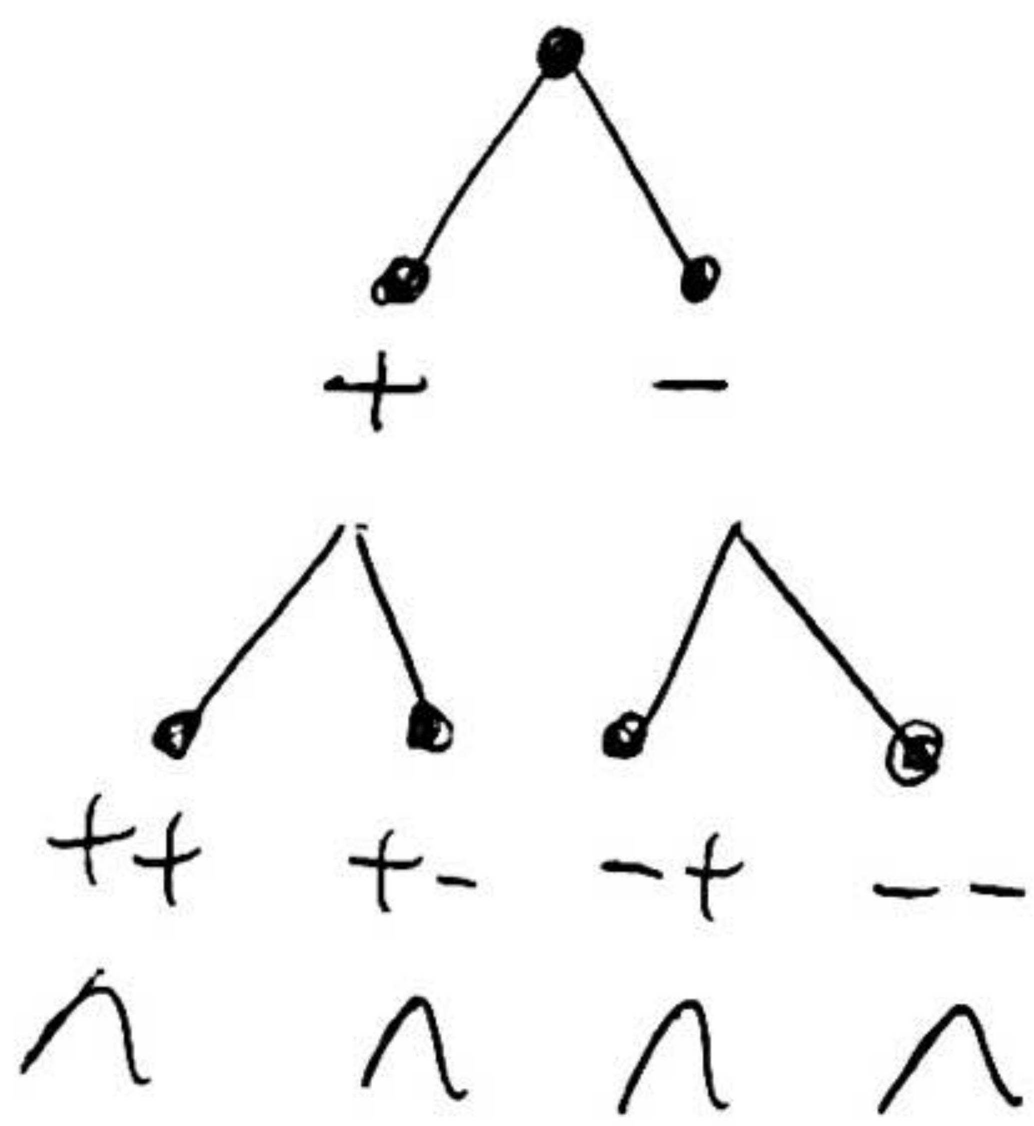
$B_T \in \{\pm 1, \pm 2\}$

General $\mathbb{E}\xi = 0$, $\mathbb{E}\xi^2 = 1$.
Use Binary Decision Tree to generate ξ.

$X_0 = \mathbb{E}\xi = 0$

$X_1 = \mathbb{E}(\xi | \mathcal{F}_1)$

$X_2 = \mathbb{E}(\xi | \mathcal{F}_2)$



$+$: $\xi \geq X_0$
$-$: $\xi < X_0$
$\mathcal{F}_1 = \sigma(+, -)$

$++$: $\xi \geq X_0$ and $\xi \geq X_1$
$+-$: $\xi \geq X_0$ and $\xi < X_1$

$X_0 = \mathbb{E}\xi = 0$

$$X_n = \mathbb{E}(\xi | \mathcal{F}_n) \quad n \geq 0.$$

$$Y_n = \begin{cases} +1 & \text{if } \xi \geq X_n \\ -1 & \text{if } \xi < X_n \end{cases}$$

$$\mathcal{F}_n = \sigma(Y_0, \dots, Y_{n-1})$$

<u>Claim</u> $X_n \xrightarrow[L^2]{a.s.} \xi$

<u>Pf</u> X_n is a L^2 martingale, so by Martingale Convergence Thm,
$X = \lim_{n\to\infty} X_n$ a.s. and in L^2. We claim $X = \xi$ a.s.

On the event $\{X < \xi\}$, we have $X_n < \xi$ for large enough n,
so $Y_n = 1$ for large n.

By def of Y_n, $Y_n \cdot (\xi - X_n) = |\xi - X_n|$
For large enough n,
$$\underbrace{Y_{n+1}}_{1} \cdot (\xi - X_n) = \underbrace{Y_n}_{\pm 1} \cdot (\xi - X_n) = |\xi - X_n|$$

Hence $\lim_{n\to\infty} Y_{n+1}(\xi - X_n) = |\xi - X|$ a.s.

But $\mathbb{E}\, Y_{n+1}(\xi - X_n) = \mathbb{E}\left[Y_{n+1}(\xi - \mathbb{E}(\xi | Y_n))\right] = 0$ by L^2 projection for conditional expectation.

Since $\sup_n \mathbb{E}\left[Y_{n+1}(\xi - X_n)\right]^2 < \infty$, $\mathbb{E}|\xi - X| = 0$,
so $\xi = X$ a.s.

Donsker's Invariance Principle / Functional CLT Let $(X_n)_{n\geq 0}$ be iid RV's with $\mathbb{E}X_n = 0$ and $\text{Var}(X_n) = 1$. Let $S_n = \sum_{k=0}^{n} X_k$ be the associated Random Walk $(S_n)_{n\geq 0}$, and interpolate linearly between integer points to get $(S_t)_{t\in\mathbb{R}^+}$, to define a random function $S \in C[0,\infty)$.

Define $S_t^{(n)} = \frac{S_{nt}}{\sqrt{n}}$, a sequence $\{(S^{(n)})_{0\leq t\leq 1}\}_{n\in\mathbb{N}}$ of random continuous functions on $[0,1]$.

On the space $C[0,1]$ of continuous functions on $[0,1]$ with metric induced by the sup-norm, the sequence $\{(S_t^{(n)})_{0\leq t\leq 1}\}_{n\in\mathbb{N}}$ converges in distribution to std. BM $(B_t)_{0\leq t\leq 1}$.

In other words, $\max_{0\leq t\leq 1} |S_t^{(n)} - B_t| \xrightarrow[P]{d} 0$

Pf By Skorokhod Embedding Thm, let $(T_k)_{k\geq 0}$ be the stopping times with $B_{T_k} \overset{d}{=} S_k$.

Let $\Delta_n = \max_{0\leq k\leq n} \left|\frac{S_k}{\sqrt{n}} - \frac{B_k}{\sqrt{n}}\right| = \max_{0\leq k\leq n} \left|\frac{B_{T_k}}{\sqrt{n}} - \frac{B_k}{\sqrt{n}}\right|$. Want $\forall\delta>0$, $\mathbb{P}(\Delta_n > \delta) \to 0$,

$B_t^{(n)} := \frac{B_{nt}}{\sqrt{n}}$ is a standard BM. $\Delta_n = \max_{0\leq k\leq n} \left|B^{(n)}\left(\frac{T_k}{n}\right) - B^{(n)}\left(\frac{k}{n}\right)\right|$.

Now, let event $E := \bigcap_{k=0}^{n} \left\{\left|\frac{T_k}{n} - \frac{k}{n}\right| < \varepsilon\right\}$. Note $0 \leq \frac{k}{n} \leq 1$.

$\lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} \mathbb{P}(E(n,\varepsilon)^c) = 0$ since $\frac{T_n}{n} \xrightarrow{a.s.} 1$

since $\frac{T_n}{n} = \frac{1}{n}\sum_{k=1}^{n}(T_k - T_{k-1}) \overset{d}{=} \frac{1}{n}\sum_{k=1}^{n} T_1^{(k)} \xrightarrow[a.s.]{SLLN} \mathbb{E}T_1 = 1$ by Skorokhod Embedding.

and $\frac{T_k}{n} = \frac{k}{n}\left(\frac{1}{k}\sum_{m=1}^{k}(T_m - T_{m-1})\right) \underset{\approx}{\xrightarrow{SLLN}} \frac{k}{n}\cdot 1 = \frac{k}{n}$.

Also, let event $F(n,\varepsilon,\delta) := \left\{\max_{\substack{|s-t|\leq\varepsilon \\ 0\leq s,t\leq 1}} |B_s^{(n)} - B_t^{(n)}| > \delta\right\}$.

$\mathbb{P}(F(n,\varepsilon,\delta))$ doesn't depend on n, and $\lim_{\varepsilon\downarrow 0} \mathbb{P}(F(n,\varepsilon,\delta)) \to 0$ as $\varepsilon\downarrow 0$ by a.s. continuity of BM.

Note $\{\Delta_n > \delta\} \subseteq E(n,\varepsilon)^c \cup F(n,\varepsilon,\delta)$.

Hence $\lim_{\varepsilon\to 0} \lim_{n\to\infty} \mathbb{P}(\Delta_n > \delta) \to 0$. Hence $\Delta_n \xrightarrow{P} 0$.

$\square$

Arcsine Law for Last Sign Change of a General Random Walk

Suppose $\{X_k\}_{k\geq 1}$ is iid with $\mathbb{E}X_1 = 0$ and $0 < \mathbb{E}X_1^2 = \sigma^2 < \infty$.
Let the associated Random Walk $S_n = \sum_{k=1}^{n} X_k$ be $\{S_n\}_{n\geq 1}$.
Let $N_n = \max\{1 \leq k \leq n : S_k S_{k-1} \leq 0\}$ be the last time the Random Walk changes sign before time n.

Then $\forall x \in (0,1)$, $\lim_{n\to\infty} \mathbb{P}\left(\frac{N_n}{n} \leq x\right) = \frac{2}{\pi}\arcsin(\sqrt{x})$,
i.e. $\frac{N_n}{n}$ converges in distribution to the Arcsine distribution.

Note We may assume $\sigma^2 = 1$ since N_n is unaffected by scaling.

Define a bounded function g on $C[0,1]$ by $g(f) = \max\{t \in [0,1] : f(t) = 0\}$.

Lemma 1 $\max_{0\leq t\leq 1} \left| \frac{N_n}{n} - g\left(S_t^{(n)}\right)\right| \leq \frac{1}{n} \longrightarrow 0$ as $n \to \infty$

Lemma 2 g is continuous on the set $\mathcal{C} = \left\{ f \in C[0,1] : \begin{array}{l} f \text{ takes positive and} \\ \text{negative values in} \\ \text{every nbhd of zero,} \\ \text{and } f(1) \neq 0 \end{array}\right\}$

Lemma 3 Brownian Motion $(B_t)_{0\leq t\leq 1}$ is almost surely in $\mathcal{C}$

Donsker's Invariance Principle $\max_{0\leq t\leq 1} \left|S_t^{(n)} - B_t\right| \xrightarrow{d} 0$,
i.e. $\left(S_t^{(n)}\right)_{0\leq t\leq 1} \xrightarrow{d} (B_t)_{0\leq t\leq 1}$ in the metric induced by the sup-norm on $[0,1]$.

Portmanteau Thm (V) (Given ~~the~~ our metric induced by the sup-norm on $[0,1]$),
$X_n \xrightarrow{d} X$ iff forall bounded measurable functions $g : E \to \mathbb{R}$ with $\mathbb{P}(g \text{ is discontinuous at } X) = 0$, we have $\mathbb{E}g(X_n) \to \mathbb{E}g(X)$.

Pf of Thm Forall continuous bounded $h : \mathbb{R} \to \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{E}h\left(\frac{N_n}{n}\right) = \lim_{n\to\infty} \mathbb{E}\left[h \circ g\left(\left(S_t^{(n)}\right)_{0\leq t\leq 1}\right)\right] \quad \text{by lemma 1}$$

$$= \mathbb{E}\left[h \circ g\left((B_t)_{0\leq t\leq 1}\right)\right] \quad \text{by Lemma 2,3, Donsker's, and Portmanteau (V)}$$

$$= \mathbb{E}\, h\left(\sup\{t\in[0,1] : B_t = 0\}\right) = \mathbb{E}h(A) \quad \text{where } A \text{ is arcsine distributed.}$$

Then, by definition of convergence in distribution, $\frac{N_n}{n} \xrightarrow{d} A$, i.e. $\lim_{n\to\infty} \mathbb{P}\left(\frac{N_n}{n} \leq x\right) = \arcsin\sqrt{x}$ □

Pf (Lemma 1) If $K = N_n$ is the last time the Random Walk $\{S_m\}_{m=1}^{n}$ changes sign, then the linear interpolation S_{nt} must hit 0 at some time $nt \in [N_n - 1, N_n]$, so $t \in \left[\frac{N_n}{n} - \frac{1}{n}, \frac{N_n}{n}\right]$, so $\left|g(S^{(n)}) - \frac{N_n}{n}\right| \leq \frac{1}{n}$

$\square$

Pf (Lemma 2) Let $\varepsilon > 0$ and $f \in C$.

Let $\delta_0 = \min\limits_{t \in [g(f)+\varepsilon, 1]} |f(t)|$.

Choose δ_1 so that $(-\delta_1, \delta_1) \subset f\left(g(f) - \varepsilon, g(f) + \varepsilon\right)$

Let $0 < \delta < \min(\delta_0, \delta_1)$.

If $\|h - f\|_\infty < \delta$ then h has no zero in $(g(f)+\varepsilon, 1)$, since $\delta < \delta_0$, but h has a zero in $(g(f) - \varepsilon, g(f)+\varepsilon)$ since $\delta < \delta_1$ and there are $s, t \in (g(f) - \varepsilon, g(f) + \varepsilon)$ with $h(t) < 0$ and $h(s) > 0$.

~~[illegible]~~

Hence $|g(h) - g(f)| < \varepsilon$. Hence g is continuous on C $\square$

Arcsine Law for Time of Maximum of BM the random variable $M \in [0,1]$, which is uniquely determined by $B_M = \max_{0 \le s \le 1} B_s$, is arcsine distributed

Pf We know B.M. has a unique local max on $[0,1]$, so M is well-defined.

- Can show $(M_t - B_t)_{t \ge 0}$ is a std. Brownian motion.

  M is the last zero of $(M_t - B_t)_{t \ge 0}$, and so is arcsine distributed. □

Arcsine Law for Positive Occupation Time of BM

$\mathcal{L}\{t \in [0,1] : B_t > 0\}$ is arcsine distributed

lebesgue measure

symmetric simple random walk

Richard's lemma SRW $\{S_k\}_{k=1}^n$, then

$$\#\{1 \le k \le n : S_k > 0\} \stackrel{d}{=} \min\{0 \le k \le n : S_k = \max_{0 \le j \le n} S_j\}$$

Pf HW

Pf Let $P_n = \#\{1 \le k \le n : S_k > 0\}$.

Let $T_n = \min\{0 \le k \le n : S_k = \max_{0 \le j \le n} S_j\}$.

Let $g : C[0,1] \to [0,1]$ by

$g(f) = \inf\{t \in [0,1] : f(t) = \sup_{s \in [0,1]} f(s)\}$.

Let $h : C[0,1] \to [0,1]$ by

$h(f) = \mathcal{L}\{t \in [0,1] : f(t) > 0\}$.

$$\frac{T_n}{n} = g\left(\left(S_t^{(n)}\right)_{0 \le t \le 1}\right) \xrightarrow{d} g\left((B_t)_{0 \le t \le 1}\right)$$ by Donsker's and Portmanteau (v)

because g is continuous in every $f \in C[0,1]$ which has a unique maximum, and $(B_t)_{0 \le t \le 1}$ has a unique max a.s.

Also, $$\left|\frac{P_n}{n} - h\left(\left(S_t^{(n)}\right)_{0 \le t \le 1}\right)\right| \le \frac{1}{n} \#\{1 \le k \le n : S_k = 0\} \longrightarrow 0$$

so $$\lim_{n \to \infty} \frac{P_n}{n} = \lim_{n \to \infty} h\left(\left(S_t^{(n)}\right)_{0 \le t \le 1}\right) \stackrel{d}{=} h\left((B_t)_{0 \le t \le 1}\right)$$ by Donsker's and Portmanteau (v)

because h is continuous in every $f \in C[0,1]$ such that $\lim_{\varepsilon \downarrow 0} \mathcal{L}\{t \in [0,1] : 0 \le f(t) \le \varepsilon\} = 0$ which B.M. satisfies a.s.

Thus $h\left((B_t)_{0 \le t \le 1}\right) \stackrel{d}{=} g\left((B_t)_{0 \le t \le 1}\right)$ and so is arcsine distributed. □

Arcsine Law for Positive Occupation Time of General Random Walks

<u>Pf</u> Same argument (Donsker's + Portmanteau ωs)