Deriving the Black-Scholes Formula

Ryan Brill

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Abstract

This paper details the derivation of the Black-Scholes Formula, a foundational result in options pricing. Chapters 2-5 take the reader through the math behind the original derivation of the Black-Scholes Formula, including Itô's Lemma, the Black-Scholes PDE, the Initial Value Problem for the Heat Equation on the Real Line, and solving the Black-Scholes PDE to find the Black-Scholes Formula for a call option. Chapter 6 covers the Black-Scholes Formula for a put option. Chapter 7 covers the probability approach to deriving the Black-Scholes Formula, which is quicker to read through and just as effective in producing the formula.

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1 Options

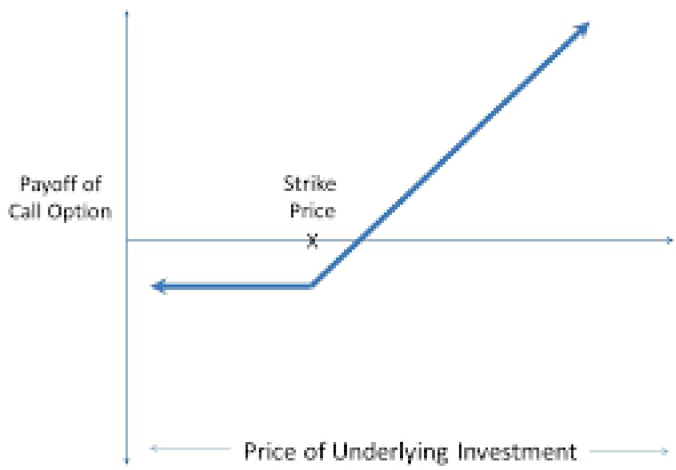
Definition 1.1. A *call option* is a contract between two parties in which the holder of the option has the right (not the obligation) to buy an asset at a specified time in the future, at a specified price.

- The asset in which the holder of the option has the right to buy is the *underlying asset*, whose value is denoted by S_t.

- The specified time T at which the holder of the option has the option to buy is called the *expiration date*.
- The specified price K at which the holder has the right to buy the underlying at is called the *strike price*.
- We will be dealing with *European-style* call options, which are characterized by the right to buy the underlying at the expiration date, as oppossed to *American-style* call options, which are characterized by the right to buy the underlying at any time prior to expiration.

Suppose you are the rational holder of a call option expiring today. If $K > S_T$, you would not exercise your right to buy the underlying at K when you could buy the underlying for S_T in the market; hence the option is worth nothing. However, if $K < S_T$, you would exercise your right to buy the underlying at K because you will be able to make a profit of $S_T - K$ by subsequently selling the underlying in the market. Therefore, a call option has payoff

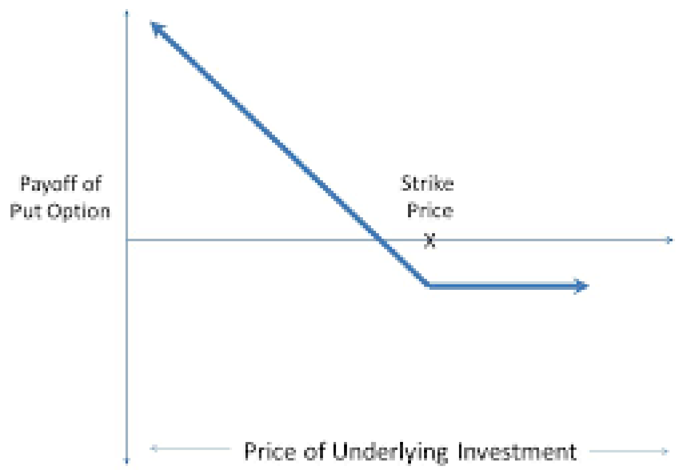
$$\max(S_T - K, 0)$$



Definition 1.2. A *put option* is a contract between two parties in which the holder of the option has the right (not the obligation) to sell an asset at a specified time in the future, at a specified price.

By similar logic, a put option has payoff

$$\max(K - S_T, 0)$$



Discussion 1.3. Because an option derives its value from an underlying asset whose future value is unknown, the value of the option is also unkown. However, in this paper we will accept certain assumptions which will allow us to derive a formula which produces the value of an option at any time t prior to the option's expiration. Specifically, we will derive a formula which gives us the time-t price of a European call and put option with strike price K and expiration T on an underlying asset whose value is given by S_t.

2 Itô's Lemma

Discussion 2.1. Because the future value of the underlying asset of an option is unkown, it is appropriate to model the value of the asset S_t as a stochastic (random) process. This section will discuss some mathematical consequences of the way we choose to model S_t.

Definition 2.2. A stochastic process W_t is a *Brownian Motion* if

- $W_0 = 0$
- W has independent increments
  For all $0 \leq s \leq t$, $W_t - W_s$ is independent of the previous history of W
- W has normally distributed increments
  For all $0 \leq s \leq t$, $W_t - W_s \sim$ Normal(mean $= 0$, var $= s - t$)
  For some time increment Δt, $\Delta W_t \sim$ Normal(mean $= 0$, var $= \Delta t$)
- W has continuous sample paths (trajectories)

Discussion 2.3. Let W_t be a Brownian Motion. We will need make sense of equations such as

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t \tag{*}$$

We should read the above equations as stating that at time t, X_t is evolving like a Brownian Motion with *drift* μ and *volatility* σ. But in terms of mathematical rigor, what does such an equation mean? In stochastic calculus, we define the derivative of a stochastic process dW_t in terms of the integral. In other words, we say that X_t is a solution to (*) if

$$X_t = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$$

The ds integral is the usual Reimann integral from calculus. We give meaning to the second term, and more generally to

$$\int_0^t A_s dW_s$$

using the *Itô integral.* To develop the Itô integral with complete mathematical rigor requires tools like measure theory which is outside of my domain of knowledge. However, I will be as rigorous as I can in order to provide a solid intuition behind some important results in stochastic calculus.

Definition 2.4. An *Itô integral* is an integral with respect to Brownian Motion, like

$$\int_0^T \sigma(t)dW_t$$

We divide the time period $[0, T]$ into N periods: for $0 \leq t \leq T$, let

$$\Delta t = \frac{T-t}{N} \ \text{ and } \ t_n = t + n\Delta t \ \text{ for } \ n \in \{0, 1, ..., N-1\}$$

We define

$$\int_t^T \sigma(s)dW_s = \lim_{N\to\infty} \sum_{n=0}^{N-1} \sigma(t_n)\Delta W_{t_n}$$

Discussion 2.5. Now, recall from multivariable calculus

$$\Delta f(x,y) = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + \frac{\partial^2 f}{\partial x \partial y}\Delta x \Delta y + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(\Delta y)^2 + ...$$

In the limit as Δx and Δy tend to 0, we have

$$df(x,y) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dx)^2 + \frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(dy)^2 + ... \quad (*)$$

If $f : \mathbb{R}^2 \to \mathbb{R}$ is smooth and X_t is a stochastic process, we may write

$$df(X_t,t) = \frac{\partial f}{\partial X_t}dX_t + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}(dX_t)^2 + \frac{\partial^2 f}{\partial X_t \partial t}dX_t dt + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + ... \quad (**)$$

Recalling the Chain Rule, in ordinary calculus (*), we drop all but the dx and dy terms in the expansion. In stochastic calculus (**), however, we keep an extra term, as detailed in the following lemmas.

Lemma 2.6. $\int_0^T dt^2 = 0$, $\int_0^T dt dW_t = 0$, and $\int_0^T dW_t^2 = dt$

Proof. From (2._) we let $\Delta t = \frac{T}{N}$.

- Let $p \in \mathbb{N}$. By definition of the Reimann Integral,

$$\int_0^T (dt)^p = \lim_{N\to\infty} \sum_{n=0}^{N-1} (\Delta t)^p = \lim_{N\to\infty} N(\Delta t)^p = \lim_{N\to\infty} (N\Delta t)(\Delta t)^{p-1}$$
$$= T \lim_{N\to\infty} (\Delta t)^{p-1} = T \lim_{N\to\infty} (\frac{T}{N})^{p-1} = \begin{cases} T & \text{if } p = 1 \\ 0 & \text{if } p > 1 \end{cases}$$

  Hence $\int_0^T dt^2 = 0$.

- Also,

$$\begin{aligned} \mathbb{E}\left[\int_0^T dW_t^2\right] &= \mathbb{E}\left[\lim_{N\to\infty} \sum_{n=0}^{N-1} (\Delta W_{t_n})^2\right] && \text{by (2.4)} \\ &= \lim_{N\to\infty} \sum_{n=0}^{N-1} \mathbb{E}\left[(\Delta W_{t_n})^2\right] \\ &= \lim_{N\to\infty} \sum_{n=0}^{N-1} \mathrm{Var}\left[\Delta W_{t_n}\right] \quad \text{since } \Delta W_{t_n} \text{ has mean 0} && \text{by (2.2)} \\ &= \lim_{N\to\infty} \sum_{n=0}^{N-1} \Delta t && \text{by (2.2)} \\ &= \lim_{N\to\infty} N\Delta t = \lim_{N\to\infty} T = T \end{aligned}$$

And

$$\begin{aligned}
\operatorname{Var}\left[\int_0^T dW_t^2\right] &= \operatorname{Var}\left[\lim_{N\to\infty}\sum_{n=0}^{N-1}(\Delta W_{t_n})^2\right] && \text{by (2.4)}\\
&= \lim_{N\to\infty}\sum_{n=0}^{N-1}\operatorname{Var}\left[(\Delta W_{t_n})^2\right]\\
&= \lim_{N\to\infty}\sum_{n=0}^{N-1}\mathbb{E}\left[(\Delta W_{t_n})^4\right]-\mathbb{E}\left[(\Delta W_{t_n})^2\right]^2\\
&= \lim_{N\to\infty}\sum_{n=0}^{N-1}\mathbb{E}\left[(\Delta W_{t_n})^4\right]-\operatorname{Var}\left[\Delta W_{t_n}\right]^2\\
& && \text{since } \Delta W_{t_n} \text{ has mean 0 by (2.2)}\\
&= \lim_{N\to\infty}\sum_{n=0}^{N-1}\mathbb{E}\left[(\Delta W_{t_n})^4\right]-(\Delta t)^2 && \text{by (2.2)}\\
&= \lim_{N\to\infty}\sum_{n=0}^{N-1}3(\Delta t)^2-(\Delta t)^2
\end{aligned}$$

because the fourth moment of a Normal(mean=m,variance=v) random variable is $3v^2$

$$\begin{aligned}
&= \lim_{N\to\infty}\sum_{n=0}^{N-1}2(\Delta t)^2\\
&= 2\lim_{N\to\infty}\sum_{n=0}^{N-1}(\frac{T}{N})^2 = 2T^2\lim_{N\to\infty}\frac{1}{N}=0
\end{aligned}$$

Because $\mathbb{E}\left[\int_0^T dW_t^2\right]=T$ and $\operatorname{Var}\left[\int_0^T dW_t^2\right]=0$, we say $\int_0^T dW_t^2 = dt$.

- $\int_0^T dt dW_t = 0$ is shown similarly.

$\square$

Lemma 2.7. *(Itô's Lemma)* If X_t a stochastic process satisfying

$$dX_t = \mu(X_t,t)dt + \sigma(X_t,t)dW_t$$

and $f(x,t)$ is twice-differentiable, then $f(X_t,t)$ is a stochastic process satisfying

$$df(X_t,t) = \frac{\partial f}{\partial t}(X_t,t)dt + \frac{\partial f}{\partial X_t}(X_t,t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial X_t^2}(X_t,t)(dX_t)^2$$

Proof. As discussed in (2.3), a stochastic derivative makes sense only in terms of the Itô integral. Clearly dt and dX_t may be nonzero when integrated. However, dt^2 integrates to 0 by the previous lemma, and therefore so do all higher order dt

terms. Likewise, $(dX_t)^3$ and all higher power terms will integrate to 0 because, by the previous lemma,

$$(dX_t)^3 = (dX_t)^2(dX_t) = (\sigma^2 X_t^2 dt)(\mu dt + \sigma dW_t) = \mu\sigma^2 dt^2 + \sigma^3 dt dW_t = 0$$

Also, $dtdX_t$ and all higher power mixed terms will integrate to 0 because, by the previous lemma,

$$dtdX_t = dt(\mu dt + \sigma dW_t) = \mu dt^2 + \sigma dt dW_t = 0$$

However, $(dX_t)^2$ may not intgrate to zero because, by the previous lemma,

$$(dX_t)^2 = \mu^2 dt^2 + 2\mu\sigma dt dW_t + \sigma^2 (dW_t)^2 = \sigma^2 dt$$

Hence the only terms of the taylor series expansion which survive are the dt, dX_t, and dX_t^2 terms. ☐

Definition 2.8. A stochastic process S_t is a *Geometric Brownian Motion* if it satisfies the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where μ and σ are constants. In deriving the Black-Scholes Formula, we will assume that the value of the underlying asset is given by a Geometric Brownian Motion S_t.

Corollary 2.9. If S_t is a Geometric Brownian Motion, then

$$(dS_t)^2 = \sigma^2 S_t^2 dt$$

Proof. As demonstrated in the proof of (2.7),

$$(dS_t)^2 = \mu^2 S_t^2 \cancelto{0}{dt^2} + 2\mu\sigma S_t^2 \cancelto{0}{dt dW_t} + \sigma^2 S_t^2 (dW_t)^2 = \sigma^2 S_t^2 dt$$

☐

3 The Black-Scholes PDE

Definition 3.1. The *Black-Scholes PDE* is a partial differential equation which describes the relationship between the partial derivatives of $C(S_t, t)$, the time-t value of a European call option with expiration T and strike price K on an underlying asset whose value is given by the Geometric Brownian Motion S_t.

Definition 3.2. The *risk-free rate* r is the theoretical rate of return of an investment with no risk. US Treasury bills are considered of the most risk-free investments, so the yield of a US Treasury bill is often used as the risk-free rate in practice.

Theorem 3.3. Let r denote the risk-free rate. Then the Black-Scholes PDE is

$$\frac{\partial C}{\partial t}(S_t,t)+rS_t\frac{\partial C}{\partial S}(S_t,t)+\frac{1}{2}\sigma^2S_t^2\frac{\partial^2 C}{\partial S_t^2}(S_t,t)=rC(S_t,t)$$

Proof. Since S_t is a Geometric Brownian Motion, note that dS_t and $(dS_t)^2$ are given by (2.8) and (2.9). We begin by applying Itô's Lemma (2.7) with $X_t \mapsto S_t, f \mapsto C$:

$$\begin{aligned}
dC(S_t,t) &= \frac{\partial C}{\partial t}(S_t,t)dt+\frac{\partial C}{\partial S_t}(S_t,t)dS_t+\frac{1}{2}\frac{\partial^2 C}{\partial S_t^2}(S_t,t)dS_t^2\\
&= \frac{\partial C}{\partial t}(S_t,t)dt+\frac{\partial C}{\partial S_t}(S_t,t)(\mu S_tdt+\sigma S_tdW_t)+\frac{1}{2}\frac{\partial^2 C}{\partial S_t^2}(S_t,t)(\sigma^2S_t^2dt)\\
&= (\frac{\partial C}{\partial t}(S_t,t)+\mu S_t\frac{\partial C}{\partial S_t}(S_t,t)+\frac{1}{2}\sigma^2S_t^2\frac{\partial^2 C}{\partial S_t^2}(S_t,t))dt+\sigma S_t\frac{\partial C}{\partial S_t}dW_t
\end{aligned}$$

Now consider the portfolio (1 call option, a shares), where the number of shares a that we hold will be chosen strategically later on. The value of the portfolio is

$$V_t = 1\cdot C + a\cdot S_t = C + aS_t$$

Then

$$\begin{aligned}
dV_t &= d(C+aS_t)\\
&= dC + a(dS_t)\\
&= (\frac{\partial C}{\partial t}(S_t,t)+\mu S_t\frac{\partial C}{\partial S_t}(S_t,t)+\frac{1}{2}\sigma^2S_t^2\frac{\partial^2 C}{\partial S_t^2}(S_t,t)+a\mu S_t)dt\\
&+\sigma S_t(\frac{\partial C}{\partial S_t}(S_t,t)+a)dW_t
\end{aligned}$$

Now, we let $a=-\frac{\partial C}{\partial S_t}(S_t,t)$ in order to hedge away the risk in our portfolio. In other words, with this choice of a, the random dW_t component disappears, leaving the portfolio void of risk/randomness. Hence

$$dV_t = (\frac{\partial C}{\partial t}(S_t,t)+\frac{1}{2}\sigma^2S_t^2\frac{\partial^2 C}{\partial S_t^2}(S_t,t))dt \tag{3.4}$$

On the other hand, now that the portfolio is now risk free, it must grow over time at the risk-free rate r, giving us another way to calculate the dynamics of the value of the portfolio:

$$\frac{dV_t}{dt}=rV_t$$

Hence

$$dV_t = rV_tdt = r(C(S_t,t)+aS_t)dt = r(C(S_t,t)-S_t\frac{\partial C}{\partial t}(S_t,t))dt \tag{3.5}$$

Because dV_t gives the dynamics of the value of the same portfolio in two ways - equations (4.4) and (4.5) - we can set them equal to each other:

$$\frac{\partial C}{\partial t}(S_t, t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2}(S_t, t) = rC(S_t, t) - rS_t \frac{\partial C}{\partial t}(S_t, t)$$

Rearranging gives us the Black-Scholes PDE:

$$\frac{\partial C}{\partial t}(S_t, t) + rS_t \frac{\partial C}{\partial S}(S_t, t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2}(S_t, t) = rC(S_t, t)$$

□

Discussion 3.6. In order to make the Black-Scholes PDE make sense in terms of options pricings, we must put some constraints on the PDE. Specifically, we know that the value of a European call option at its expiration date is given by its payoff function (1._):

$$C(S_T, T) = \max(S_T - K, 0)$$

Thus the above equation is a fitting terminal condition for the PDE. Also, we know that t represents time, so

$$t > 0$$

Further, S_t represents the value of the underlying, where the underlying could be anything as long as it is quantifiable, so

$$-\infty < S_t < \infty$$

Therefore, the Black-Scholes PDE with its constraints is given by

$$\boxed{\begin{array}{c} \frac{\partial C}{\partial t}(S_t, t) + rS_t \frac{\partial C}{\partial S}(S_t, t) + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial S_t^2}(S_t, t) = rC(S_t, t) \\[1em] \text{with terminal condition } C(S_T, T) = \max(S_T - K, 0) \\[1em] \text{for } t > 0 \text{ and } -\infty < S_t < \infty \end{array}}$$

Discussion 3.7. The Black-Scholes PDE follows directly from the assumption that the value of the underlying, S_t, is a Geometric Brownian Motion. So, why make this assumption? Well, the underlying on an option is often a stock, and stocks are volatile and often follow a drift (or trend), which on a qualitative level makes the assumption seem decent. Moreover, research has shown that in terms of predicing the value of options, it is a decent assumption, but I will not delve into that research here.

4 The Heat Equation

Discussion 4.1. We want to find a formula $C(S_t, t)$ for the time-t price of a European-style call option. We have found a partial differential equation describing $C(S_t, t)$, the Black-Scholes PDE (4.6), and we want to find a formula for C which satisfies this PDE. We plan on accomplishing this by reducing the PDE to the heat equation, since the heat equation has already been studied in depth by mathematicians.

Definition 4.2. The *initial value problem for the heat equation on the real line* is:

Find a function $u(x,t)$ which satisfies $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$
with initial condition $u(x,0) = f(x)$ (where $f(x)$ is some aribtrary function)
and constraints $-\infty < x < \infty$, $t > 0$

Mollifiers

Definition 4.3. A *mollifier* is a smooth, nonnegative function $\psi(x,t)$ defined for all $x \in \mathbb{R}$ and $t > 0$ which has the following properties:

$$1. \quad \int_{-\infty}^{\infty} \psi(x,t)dx = 1$$

For all $x \in \mathbb{R}$ and all bounded, continuous functions $f(x)$ defined for all $x \in \mathbb{R}$,

$$2. \quad \lim_{t\to 0^+} \int_{-\infty}^{\infty} \psi(y-x,t)f(y)dy = f(x)$$

Mollifiers are important for us because they will help us satisfy the initial condition in solving the heat equation PDE. So, let's find an example of a mollifier.

Lemma 4.4. Let ϕ be a smooth nonnegative function defined for all $x \in \mathbb{R}$ which decays to 0 at $-\infty$ and ∞ and satisfies

$$\int_{-\infty}^{\infty} \phi(x)dx = 1$$

Let $a > 0$. Then the following is a mollifier:

$$\psi(x,t) = t^{-a}\phi(xt^{-a})$$

Proof. (knowledge of basic real analysis is helpful in this proof). To show ψ is a mollifier, we must show ψ satisfies properties 1 and 2 of definition (4.3). Letting $u = xt^{-a}$, we see ψ clearly satisfies the first property:

$$\int_{-\infty}^{\infty} \psi(x,t)dx = \int_{-\infty}^{\infty} t^{-a}\phi(xt^{-a})dx = \int_{-\infty}^{\infty} \phi(u)du = 1$$

Now we show ψ satisfies the second property. Let $f(x)$ be a bounded (say by M) and continuous function defined for all $x \in \mathbb{R}$. Let $x \in \mathbb{R}$. Let $\epsilon > 0$. Since f is continuous, there is a $\delta > 0$ such that for all $x' \in \mathbb{R}$, $|x' - x| < \delta$ implies $|f(x') - f(x)| < \epsilon$.

Also, letting $u = (y-x)t^{-a}$, note that

$$\int_{-\infty}^{\infty} t^{-a}\phi((y-x)t^{-a})f(x)dy = f(x)\int_{-\infty}^{\infty}\phi(u)du = f(x) \qquad (!)$$

Then

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty}\psi(y-x,t)f(y)dy - f(x)\right| \\
&= \left|\int_{-\infty}^{\infty} t^{-a}\phi((y-x)t^{-a})f(y)dy - f(x)\right| \\
&= \left|\int_{-\infty}^{\infty} t^{-a}\phi((y-x)t^{-a})f(y)dy - \int_{-\infty}^{\infty} t^{-a}\phi((y-x)t^{-a})f(x)dy\right| && \text{by (!)} \\
&= \left|\int_{-\infty}^{\infty} t^{-a}\phi((y-x)t^{-a})(f(y)-f(x))dy\right| \\
&\leq \left|\int_{-\infty}^{-\delta} t^{-a}\phi((y-x)t^{-a})(f(y)-f(x))dy\right| \\
&\qquad + \left|\int_{\delta}^{\infty} t^{-a}\phi((y-x)t^{-a})(f(y)-f(x))dy\right| \\
&\qquad + \left|\int_{-\delta}^{\delta} t^{-a}\phi((y-x)t^{-a})(f(y)-f(x))dy\right| && \text{by the triangle inequality} \\
&\leq \left|\int_{-\infty}^{-\delta} t^{-a}\phi((y-x)t^{-a})(2M)dy\right| && \text{since } f \text{ is bounded} \\
&\qquad + \left|\int_{\delta}^{\infty} t^{-a}\phi((y-x)t^{-a})(2M)dy\right| && \text{since } f \text{ is bounded} \\
&\qquad + \left|\int_{-\delta}^{\delta} t^{-a}\phi((y-x)t^{-a})(\epsilon)dy\right| && \text{since } f \text{ is continuous} \\
&= 2M\int_{-\infty}^{-\delta} t^{-a}\phi((y-x)t^{-a})dy + 2M\int_{\delta}^{\infty} t^{-a}\phi((y-x)t^{-a})dy + \epsilon\int_{-\delta}^{\delta} t^{-a}\phi((y-x)t^{-a})dy \\
&\leq 2M\int_{-\infty}^{-\delta} t^{-a}\phi((y-x)t^{-a})dy + 2M\int_{\delta}^{\infty} t^{-a}\phi((y-x)t^{-a})dy + \epsilon && \text{by (!)} \\
&= \epsilon + 2M\int_{-\infty}^{(-\delta-x)t^{-a}}\phi(u)du + 2M\int_{(\delta-x)t^{-a}}^{\infty}\phi(u)du && \text{letting } u = (y-x)t^{-a}
\end{aligned}$$

Then

$$\lim_{t\to 0^+}\left|\int_{-\infty}^{\infty}\psi(y-x,t)f(y)dy - f(x)\right| \leq \lim_{t\to 0^+}\left[\epsilon + 2M\int_{-\infty}^{(-\delta-x)t^{-a}}\phi(u)du + 2M\int_{(\delta-x)t^{-a}}^{\infty}\phi(u)du\right] = \epsilon$$

Since $\epsilon > 0$ is aribtrary,

$$\lim_{t\to 0^+}\int_{-\infty}^{\infty}\psi(y-x,t)f(y)dy = f(x)$$

Since x is arbitrary, property 2 of (4.3) is satisfied. Hence ψ is a mollifier. $\square$

Solving the initial value problem for the heat equation on the real line requires one more lemma:

Lemma 4.5.

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

Proof. Let

$$I = \int_{-\infty}^{\infty} e^{-x^2}dx$$

Then

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy$$

Switching to polar coordinates,

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}rdrd\theta = 2\pi\int_0^{\infty} e^{-r^2}rdr = -\pi[e^{-r^2}]_0^{\infty} = \pi$$

Hence

$$I = \sqrt{\pi}$$

$\square$

Solving the Heat Equation

Theorem 4.6. The solution to the Initial Value Problem for the Heat Equation on the Real Line is

$$u(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} f(y)e^{\frac{-(x-y)^2}{4t}}dy$$

Proof. Suppose that $\psi(x,t)$ is a mollifier, so $\psi(x,t)$ is defined for $-\infty < x < \infty$ and $t > 0$, and

$$\int_{-\infty}^{\infty}\psi(x,t)dx = 1$$

and

$$\lim_{t\to 0^+}\int_{-\infty}^{\infty}\psi(y-x,t)f(y)dy = f(x)$$

for all bounded, continuous functions f. Define

$$u(x,t) = \int_{-\infty}^{\infty}\psi(y-x,t)f(y)dy \tag{4.7}$$

Then $u(x,0) = f(x)$ since $\lim_{t\to 0^+} u(x,t) = f(x)$, so the initial condition is satisfied. Suppose $\psi(x,t)$ satisfies the heat equation, so $\psi_t = \psi_{xx}$. Then $u(x,t)$ also satisfies the heat equation, for

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)u(x,t) = \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\int_{-\infty}^{\infty}\psi(y-x,t)f(y)dy \\
&= \int_{-\infty}^{\infty}(\psi_t(y-x,t) - \psi_{xx}(y-x,t))f(y)dy = 0
\end{aligned}$$

Solving the heat equation thus comes down to finding a mollifier $\psi(x,t)$ which satisfies the heat equation. Why don't we try

$$\psi(x,t) = t^{-a}\phi(xt^{-a})$$

where a and ϕ are defined as in lemma (4.4). Let's plug ψ into the heat equation:

$$\begin{aligned}
\psi_t - \psi_{xx} &= \frac{\partial}{\partial t}[t^{-a}\phi(xt^{-a})] - \frac{\partial^2}{\partial x^2}[t^{-a}\phi(xt^{-a})] \\
&= [-at^{-a-1}\phi(xt^{-a}) + t^{-a}\phi'(xt^{-a})(-axt^{-a-1})] - [t^{-3a}\phi''(xt^{-a})] \\
&= -at^{-a-1}\phi(xt^{-a}) - axt^{-2a-1}\phi'(xt^{-a}) - t^{-3a}\phi''(xt^{-a}) = 0 \qquad (**)
\end{aligned}$$

We will get rid of the argument to ϕ by defining

$$n(x) = \phi(xt^{-a})$$

Then

$$n'(x) = t^{-a}\phi(xt^{-a}) \;\; , \;\; n''(x) = t^{-2a}\phi(xt^{-a})$$

In (**), substitute ϕ and its derivatives for n and its derivatives, and solve the resulting ODE:

$$
\begin{aligned}
& -at^{-a-1}\phi(xt^{-a}) - axt^{-2a-1}\phi'(xt^{-a}) - t^{-3a}\phi''(xt^{-a}) = 0 \\
\implies & -at^{-1}\phi(xt^{-a}) - axt^{-a-1}\phi'(xt^{-a}) - t^{-2a}\phi''(xt^{-a}) = 0 \\
\implies & -at^{-1}n - axt^{-1}n' - n'' = 0 \\
\implies & at^{-1}n + axt^{-1}n' = -n'' \\
\implies & at^{-1}[n(x) + xn'(x)] = -n''(x) \\
\implies & at^{-1}\frac{d}{dx}[xn(x)] = -\frac{d}{dx}[n'(x)] \\
\implies & at^{-1}\int \frac{d}{dx}[xn(x)]dx = -\int \frac{d}{dx}[n'(x)]dx \\
\implies & at^{-1}xn(x) = -n'(x) + c_1 \\
\implies & n'(x) + at^{-1}xn(x) = c_1 \\
\implies & e^{\frac{ax^2}{2t}}(n'(x) + at^{-1}xn(x)) = e^{\frac{ax^2}{2t}}c_1 \\
\implies & \frac{d}{dx}[e^{\frac{ax^2}{2t}}n(x)] = e^{\frac{ax^2}{2t}}c_1 \\
\implies & \int \frac{d}{dx}[e^{\frac{ax^2}{2t}}n(x)]dx = \int e^{\frac{ax^2}{2t}}c_1 dx \\
\implies & e^{\frac{ax^2}{2t}}n(x) = c_1 \int e^{\frac{ax^2}{2t}}dx + c_2 \\
\implies & n(x) = c_1 e^{-\frac{ax^2}{2t}} \int e^{\frac{ax^2}{2t}}dx + c_2 e^{-\frac{ax^2}{2t}}
\end{aligned}
$$

Now substitue ϕ back in for n:

$$\phi(xt^{-a}) = c_1 e^{-\frac{ax^2}{2t}} \int e^{\frac{ax^2}{2t}}dx + c_2 e^{-\frac{ax^2}{2t}}$$

The right side is a function of $\frac{x^2}{t}$, or equivalently $\frac{x}{\sqrt{t}}$, so we will choose $a = \frac{1}{2}$ to make the argument to ϕ match the right side of the above equation. Also, we will choose $c_1 = 0$, which we can do since we are looking for only one mollifier ψ and hence one ϕ which satisfies the heat equation. Then

$$\phi(xt^{-1/2}) = c_2 e^{-\frac{x^2}{4t}}$$

Now substitue ψ back for ϕ:

$$\psi(x,t) = c_2 t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}}$$

This is our prospective mollifier. We will check that this mollifier satisfies the heat equation:

$$\psi_t = \frac{\partial}{\partial t}[c_2 t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}}] = c_2 t^{-\frac{1}{2}} (\frac{x^2}{4t^2}) e^{-\frac{x^2}{4t}} - c_2 \frac{1}{2} t^{-\frac{3}{2}} e^{-\frac{x^2}{4t}} = c_2 e^{-\frac{x^2}{4t}} [\frac{x^2}{4t^{5/2}} - \frac{1}{2} t^{-\frac{3}{2}}]$$

$$\begin{aligned}\psi_{xx} = \frac{\partial^2}{\partial x^2}[c_2 t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}}] = \frac{\partial}{\partial x}[\frac{-c_2}{2} x t^{-\frac{3}{2}} e^{-\frac{x^2}{4t}}] &= \frac{-c_2}{2} x t^{-\frac{3}{2}} e^{-\frac{x^2}{4t}} (\frac{-x}{2t}) - \frac{-c_2}{2} t^{-\frac{3}{2}} e^{-\frac{x^2}{4t}} \\ &= c_2 e^{-\frac{x^2}{4t}} [\frac{x^2}{4t^{5/2}} - \frac{1}{2} t^{-\frac{3}{2}}]\end{aligned}$$

Indeed $\psi_t - \psi_{xx} = 0$. Now, in order for ψ to be a mollifier, it must satisfy property 1 of definition (4.3). So, we will choose c_2 such that $\int_{-\infty}^{\infty} \psi(x,t)dx = 1$. Letting $u = \frac{1}{2} x t^{-\frac{1}{2}}$,

$$\begin{aligned}1 = \int_{-\infty}^{\infty} \psi(x,t)dx &= \int_{-\infty}^{\infty} c_2 t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} dx \\ &= 2c_2 \int_{-\infty}^{\infty} e^{-u^2} du \\ &= 2c_2 \sqrt{\pi} \qquad \text{by (4.4)}\end{aligned}$$

Hence $c_2 = \frac{1}{2\sqrt{\pi}}$, and we have found our mollifier:

$$\psi(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$$

Then, using (4.5), we have found a solution $u(x,t)$ satisfying the heat equation with the proper initial condition:

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-y)^2}{4t}} f(y)dy$$

□

5 The Black-Scholes Formula

Definition 5.1. The *Black-Scholes Formula* (for a call option) is a formula for $C(S_t, t)$ which solves the Black-Scholes PDE, given by (3.7). Specifically, the Black-Scholes Formula is the no-arbitrage time-t value of a European call option with maturity T and strike price K on an underlying asset whose value is given by the Geometric Brownian Motion S_t.

Theorem 5.2. This Black-Scholes Formula (for a call option) is

$$\boxed{\begin{array}{c} C(S_t,t) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \\ \\ \text{where } d_1 = \frac{\ln(S/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\ \\ \text{and } d_2 = \frac{\ln(S/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \end{array}}$$

Proof. We will use many changes of variable to reduce the Black-Scholes PDE to the heat equation, and then solve the heat equation using the solution derived in theorem (4.6).

We perform our first changes of variable:

$$t = T - \frac{\tau}{(1/2)\sigma^2} \quad ; \quad S = Ke^x \quad ; \quad C(S,t) = Kc(x,\tau) \tag{5.3}$$

which gives us

$$\tau = \frac{\sigma^2}{2}(T-t) \quad ; \quad x = \ln(\frac{S}{K}) \quad ; \quad c(x,\tau) = \frac{C(S,t)}{K} \tag{5.4}$$

Our choice for t reverses the sense of time - the heat equation requires an initial condition, but the framework of call options gives us a terminal condition.

Now take the partial derivatives of $C(S,t)$ relevant to the Black-Scholes PDE:

$$\frac{\partial C}{\partial t} = K\frac{\partial c}{\partial \tau}\frac{\partial \tau}{\partial t} = K\frac{\partial c}{\partial \tau}(\frac{-\sigma^2}{2}) = \frac{-\sigma^2 K}{2}\frac{\partial c}{\partial \tau}$$

$$\frac{\partial C}{\partial S} = K\frac{\partial c}{\partial x}\frac{\partial x}{\partial S} = K\frac{\partial c}{\partial x}(\frac{1}{S}) = \frac{K}{S}\frac{\partial c}{\partial x}$$

$$\begin{aligned} \frac{\partial^2 C}{\partial S^2} &= \frac{\partial}{\partial S}(\frac{\partial C}{\partial S}) = \frac{\partial}{\partial S}(K\frac{\partial c}{\partial x}\frac{1}{S}) = K\frac{\partial c}{\partial x}(\frac{-1}{S^2}) + \frac{K}{S}\frac{\partial}{\partial S}(\frac{\partial c}{\partial x}) \\ &= \frac{-K}{S^2}\frac{\partial c}{\partial x} + \frac{K}{S}\frac{\partial}{\partial x}(\frac{\partial c}{\partial x})(\frac{\partial x}{\partial S}) = \frac{-K}{S^2}\frac{\partial c}{\partial x} + \frac{K}{S^2}\frac{\partial^2 c}{\partial x^2} \\ &= \frac{K}{S^2}(\frac{\partial^2 c}{\partial x^2} - \frac{\partial c}{\partial x}) \end{aligned}$$

The terminal condition is

$$C(S_T,T) = \max(S_T - K, 0) = \max(Ke^x - K, 0)$$

Also, notice that

$$C(S_T,T) = Kc(x,\tau(T)) = Kc(x,0)$$

This gives us an initial condition for $c(x,\tau)$:

$$c(x,0) = \max\left(e^x - 1, 0\right) \tag{5.5}$$

Now substitute the above derivatives into the Black-Scholes PDE (6.2) and simplify:

$$\begin{aligned}
\frac{-\sigma^2 K}{2}\frac{\partial c}{\partial \tau} + rS(\frac{K}{S}\frac{\partial c}{\partial x}) + \frac{1}{2}\sigma^2 S^2(\frac{K}{S^2}(\frac{\partial^2 c}{\partial x^2} - \frac{\partial c}{\partial x})) - rKc = 0 \\
\implies \frac{-\sigma^2 K}{2}\frac{\partial c}{\partial \tau} + rK\frac{\partial c}{\partial x} + \frac{1}{2}\sigma^2 K(\frac{\partial^2 c}{\partial x^2} - \frac{\partial c}{\partial x}) - rKc = 0 \\
\implies \frac{-\sigma^2}{2}\frac{\partial c}{\partial \tau} + r\frac{\partial c}{\partial x} + \frac{\sigma^2}{2}(\frac{\partial^2 c}{\partial x^2} - \frac{\partial c}{\partial x}) - rc = 0 \\
\implies \frac{\sigma^2}{2}(\frac{\partial^2 c}{\partial x^2} - \frac{\partial c}{\partial x}) + r(\frac{\partial c}{\partial x} - c) = \frac{\sigma^2}{2}\frac{\partial c}{\partial \tau} \\
\implies (\frac{\partial^2 c}{\partial x^2} - \frac{\partial c}{\partial x}) + \frac{r}{(\sigma^2/2)}(\frac{\partial c}{\partial x} - c) = \frac{\partial c}{\partial \tau}
\end{aligned}$$

Now define

$$h = \frac{r}{(\sigma^2/2)} \tag{5.6}$$

This leaves us with the rescaled, constant coefficient equation

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial x^2} + (h-1)\frac{\partial c}{\partial x} - hc \tag{5.7}$$

We now perform our second changes of variable:

$$c(x,\tau) = e^{ax+b\tau}u(x,\tau) \quad ; \quad u = e^{-ax-b\tau}c \tag{5.8}$$

where a and b are to be determined strategically later on.

We now take the partial derivatives relevant to the rescaled, constant coefficient equation (6.9):

$$\begin{aligned}
c_\tau &= be^{ax+b\tau}u + e^{ax+b\tau}u_\tau \\
c_x &= ae^{ax+b\tau}u + e^{ax+b\tau}u_x \\
c_{xx} &= a^2 e^{ax+b\tau}u + 2ae^{ax+b\tau}u_x + e^{ax+b\tau}u_{xx}
\end{aligned}$$

Plug these partial derivatives into the rescaled, constant coefficient equation (6.9), and simplify:

$$\begin{aligned}
be^{ax+b\tau}u + e^{ax+b\tau}u_\tau = a^2 e^{ax+b\tau}u + 2ae^{ax+b\tau}u_x + e^{ax+b\tau}u_{xx} \\
+ (h-1)(ae^{ax+b\tau}u + e^{ax+b\tau}u_x) - he^{ax+b\tau}u
\end{aligned}$$

$$\implies bu + u_\tau = a^2u + 2au_x + u_{xx} + (h-1)(au + u_x) - hu$$
$$\implies u_\tau = u_{xx} + (2a + h - 1)u_x + (a^2 + (h-1)a - h - b)u$$

We now choose a and b so that the u and u_x terms in the above equation disappear:

$$a = \frac{-(h-1)}{2} \quad ; \quad b = a^2 + (h-1)a - h = \frac{-(h+1)^2}{4} \tag{5.9}$$

This leaves us with the heat equation

$$u_\tau = u_{xx}$$

We also need to adjust the initial condition from equation (6.8):

$$\begin{aligned} f(x) = u(x,0) = e^{-ax-b\cdot 0}c(x,0) = e^{(\frac{h-1}{2})x}\max\left(e^x - 1, 0\right) \\ = \max\left(e^{(\frac{h+1}{2})x} - e^{(\frac{h-1}{2})x}, 0\right) \end{aligned} \tag{5.10}$$

For future reference, we note that the above function is strictly positive when the argument x is strictly positive; that is,

$$f(x) > 0 \text{ when } x > 0, \text{ else } f(x) = 0 \tag{5.11}$$

We are now ready to use the solution to the heat equation (6.1):

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} f(y) e^{\frac{-(x-y)^2}{4\tau}}\, dy$$

We begin by making a change of variable in the integration:

$$z = \frac{y - x}{\sqrt{2\tau}}$$

which gives us

$$y = z\sqrt{2\tau} + x \quad ; \quad dz = \frac{1}{\sqrt{2\tau}} dy$$

Then

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z\sqrt{2\tau} + x) e^{\frac{-z^2}{2}}\, dy$$

We need only integrate over the domain where $f(y) > 0$, which by (6.12) is for $y > 0 \implies y = z\sqrt{2\tau} + x > 0 \implies z > -x/\sqrt{2\tau}$. On this domain,

$$f(z\sqrt{2\tau} + x) = e^{(\frac{h+1}{2})(z\sqrt{2\tau}+x)} - e^{(\frac{h-1}{2})(z\sqrt{2\tau}+x)}$$

Hence

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{(\frac{h+1}{2})(z\sqrt{2\tau}+x)} e^{\frac{-z^2}{2}} dz - \frac{1}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{(\frac{h-1}{2})(z\sqrt{2\tau}+x)} e^{\frac{-z^2}{2}} dz$$

We will first calculate the first integral, I_1. We begin by adjusting the exponent to e:

$$\begin{aligned}(\frac{h+1}{2})(z\sqrt{2\tau}+x) - \frac{z^2}{2} &= (-\frac{1}{2})(z^2 - \sqrt{2\tau}(h+1)z) + (\frac{h+1}{2})x \\ &= (-\frac{1}{2})(z - \sqrt{\tau/2}(h+1))^2 + (\frac{h+1}{2})x + \tau\frac{(h+1)^2}{4}\end{aligned}$$

Then

$$\begin{aligned}I_1 &= \frac{1}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{(-\frac{1}{2})(z-\sqrt{\tau/2}(h+1))^2 + (\frac{h+1}{2})x + \tau\frac{(h+1)^2}{4}} dz \\ &= \frac{1}{\sqrt{2\pi}} e^{(\frac{h+1}{2})x+\tau\frac{(h+1)^2}{4}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{(-\frac{1}{2})(z-\sqrt{\tau/2}(h+1))^2} dz\end{aligned}$$

We now employ a strategic change of variable to coax the integrand into the form of the probability density of the normal distribution:

$$y = z - \sqrt{\tau/2}(h+1)$$

which yields

$$dy = dz \quad ; \quad y(z = \frac{-x}{\sqrt{2\tau}}) = \frac{-x}{\sqrt{2\tau}} - \sqrt{\tau/2}(h+1) \quad ; \quad y(z=\infty) = \infty$$

Then

$$I_1 = \frac{1}{\sqrt{2\pi}} e^{(\frac{h+1}{2})x+\tau\frac{(h+1)^2}{4}} \int_{-x/\sqrt{2\tau}-\sqrt{\tau/2}(h+1)}^{\infty} e^{-y^2/2} dy$$

Because the integrand is the density of the normal distribution, we may change the limits as shown below, by the symmetry of the integral of the density of the normal distribution:

$$I_1 = \frac{1}{\sqrt{2\pi}} e^{(\frac{h+1}{2})x+\tau\frac{(h+1)^2}{4}} \int_{-\infty}^{x/\sqrt{2\tau}+\sqrt{\tau/2}(h+1)} e^{-y^2/2} dy$$

The above integral with the $\frac{1}{\sqrt{2\pi}}$ factor is the CDF (cumulative distribution function) of the normal distribution, denoted by Φ, which yeilds:

$$I_1 = e^{(\frac{h+1}{2})x+\tau\frac{(h+1)^2}{4}} \Phi(d_1) \text{ where } d_1 = \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(h+1)$$

This concludes our calculation of the first integral I_1. The calculation of the

second integral is identical, except that $(h+1) \mapsto (h-1)$ throughout.
Therefore

$$
\begin{aligned}
u(x,\tau) &= e^{(\frac{h+1}{2})x+\tau\frac{(h+1)^2}{4}}\Phi(d_1) - e^{(\frac{h-1}{2})x+\tau\frac{(h-1)^2}{4}}\Phi(d_2) \\
\text{where } d_1 &= \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(h+1) \quad ; \quad d_2 = \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(h-1)
\end{aligned}
\tag{5.12}
$$

Now we must unwind each of the changes of variables.

Remembering from (6.10) and (6.11), we have

$$c(x,\tau) = u(x,\tau)e^{-\frac{(h-1)}{2}x-\frac{(h+1)^2}{4}\tau}$$

Hence

$$c(x,\tau) = e^x\Phi(d_1) - e^{-h\tau}\Phi(d_2)$$

Remembering from (6.4), we have

$$C(S,t) = Kc(x,\tau)$$

Hence

$$C(S,t) = Ke^x\Phi(d_1) - Ke^{-h\tau}\Phi(d_2)$$

Remembering from (6.4) and (6.9), we have

$$\tau = \frac{\sigma^2}{2}(T-t) \quad ; \quad S = Ke^x \quad ; \quad h = \frac{r}{\sigma^2/2}$$

Hence

$$C(S,t) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

Remembering from (6.5) and (6.10), we have

$$x = \ln(S/K) \quad ; \quad \tau = \frac{\sigma^2}{2}(T-t) \quad ; \quad h = \frac{r}{\sigma^2/2}$$

These allow us to adjust d_1 and d_2:

$$
\begin{aligned}
d_1 = \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(h+1) &= \frac{\ln(S/K)}{\sigma\sqrt{(T-t)}} + \frac{\sigma\sqrt{(T-t)}}{2}(\frac{r}{\sigma^2/2}+1) \\
&= \frac{\ln(S/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\
\text{Similarly,} \quad d_2 &= \frac{\ln(S/K) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}
\end{aligned}
$$

$\square$

Discussion 5.13. Let's examine how the inputs to the Black-Scholes Formula are obtained in practice.

- Strike price, K (given in the option contract)
- Risk free rate of return, r (often given by US Treasury bonds, which are considered one of the most "risk-free" investments available)
- Time to expiration, $T-t$ (the maturity T is given in the option contract; $T-t$ is generally calculated in years, but the unit does not matter as long as it is consistent with the unit of volatility σ)
- Price of the underlying asset, S_t (can be easy or difficult to obtain/forecast, depending on the underlying)
- Volatility of the underlying, σ.

  In practice, volatility is generally calculated as a measure of the *standard deviation of log changes in value of the underlying.* For instance, suppose the underlying is the stock of a public company. The daily closing prices of the stock are known. We would calculate volatility of the stock by calculating the changes in the daily closing prices of the stock (the ratios S_{t+1}/S_t for each t) for the last 31 days (31 is arbitrary), taking the natural log of these values, taking the standard deviation of these values, and multiplying these numbers by $\sqrt{252}$ to annualize it. Succinctly,

  $$\sigma = \sqrt{252} * \text{standarddev}[\ln(\frac{S_1}{S_0}), \ln(\frac{S_2}{S_1}), ..., \ln(\frac{S_{31}}{S_{30}})]$$

  We take the natural log of the percent changes in prices because of continuous compounding. Suppose the stock grows overnight at rate r. Then at time 1 (tomorrow), $S_1 = S_0 e^{r*1}$ by (1._). Then $r = \ln(\frac{S_1}{S_0})$. Here, the changes in stock price play the role of r because they are the factor at which the stock price has grown or declined from one day to the next. Hence we should take the natural log of the changes in price. We annualize volatility to keep the units consistent with time $T-t$. We use 252 because there is an average of 252 trading days per year. We multiply by the square-root of time because volatility is meant to reflect the random fluctuations in the value of the underlying, which is why it scales dW_t in the definition of Geometric Brownian Motion, and by (2._) $dW_t^2 = dt$, so $dW_t = \sqrt{dt}$, so the random component is proportional to the square root of time.

6 Put-Call Parity

Discussion 6.1. In the previous section, we derived the Black-Scholes Formula for a European call option. We would like to find a similar formula for a put option. To do so, we employ Put-Call Parity.

Definition 6.2. A *portfolio* is a collection of assets.

Definition 6.3. The *value* of a portfolio $\boldsymbol{X}$ at time t, denoted $V_{\boldsymbol{X}}(t)$, is the sum of the value of each of the assets in the portfolio.

Definition 6.4. Let time $t = 0$ denote the current time. A portfolio $\boldsymbol{X}$ is an *arbitrage* if either

- $V_{\boldsymbol{X}}(0) = 0$ and $\mathrm{P}\left[V_{\boldsymbol{X}}(T) \geq 0\right] = 1$ and $\mathrm{P}\left[V_{\boldsymbol{X}}(T) > 0\right] \geq 0$
  (no initial investment, no chance of loss, some chance of gain)

  OR

- $V_{\boldsymbol{X}}(0) < 0$ and $\mathrm{P}\left[V_{\boldsymbol{X}}(T) \geq 0\right] = 1$
  (get paid to own the portfolio, don't have to pay anything back)

Aribtrage represents a severe mispricing.

Theorem 6.5. Suppose portfolio $\boldsymbol{X}$ is worth at least as much as portfolio $\boldsymbol{Y}$ at some future time T in an arbitrage-free world. Then at any time $t < T$, $\boldsymbol{X}$ will be worth at least as much as $\boldsymbol{Y}$.

Proof. Suppose $\mathrm{P}\left[V_{\boldsymbol{X}}(T) \geq V_{\boldsymbol{Y}}(T)\right] = 1$. Construct the portfolio $\boldsymbol{\Theta} = (1 \text{ unit of } \boldsymbol{X}, -1 \text{ unit of } \boldsymbol{Y})$. We have $\mathrm{P}\left[V_{\boldsymbol{\Theta}}(T) \geq 0\right] = \mathrm{P}\left[V_{\boldsymbol{X}}(T) - V_{\boldsymbol{Y}}(T) \geq 0\right] = 1$. At time $t < T$, if $V_{\boldsymbol{\Theta}}(t) = V_{\boldsymbol{X}}(t) - V_{\boldsymbol{Y}}(t) < 0$, then $V_{\boldsymbol{\Theta}}$ is an aribtrage by (6.3), contradicting that Θ exists in an arbitrage-free world. Hence $V_{\boldsymbol{X}}(t) - V_{\boldsymbol{Y}}(t) \geq 0$. Since t is arbitrary, the claim holds. □

Theorem 6.6. *(Law of One Price)* In an aribtrage-free world, if two static portfolios have identical future values, then they have identical current values.

Proof. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two static portfolios such that $\mathrm{P}\left[V_{\boldsymbol{X}}(T) = V_{\boldsymbol{Y}}(T)\right] = 1$ at some future time T. Then $\mathrm{P}\left[V_{\boldsymbol{X}}(T) \geq V_{\boldsymbol{Y}}(T)\right] = 1$, so $V_{\boldsymbol{X}}(0) \geq V_{\boldsymbol{Y}}(0)$ by the previous theorem. Similarly, $\mathrm{P}\left[V_{\boldsymbol{X}}(T) \leq V_{\boldsymbol{Y}}(T)\right] = 1$, so $V_{\boldsymbol{X}}(0) \leq V_{\boldsymbol{Y}}(0)$. Hence $V_{\boldsymbol{X}}(0) = V_{\boldsymbol{Y}}(0)$. □

Theorem 6.7. *(Put-Call Parity)* Let $C(S_t, t)$ be the value of a European call option with strike price K and expiration T on an underlying S_t. Let $P(S_t, t)$ be the value of a European put option on the same asset with the same strike price and expiration. Let $B(t, T)$ represent the value of a risk-free bond at time t with final value 1 at expiration time T. Suppose there is no arbitrage. Then

$$P(S_t, t) + S_t = C(S_t, t) + KB(t, T)$$

Proof. Consider the portfolio

$$\boldsymbol{X} = (1 \text{ put option }, 1 \text{ unit of underlying } S_t)$$

At time t, $\boldsymbol{X}$ has value

$$V_{\boldsymbol{X}}(t) = (1 \begin{array}{c}\text{put}\\ \text{option}\end{array}) \cdot (\begin{array}{c}\text{time t value}\\ \text{of put option}\end{array}) + (1 \begin{array}{c}\text{unit of}\\ \text{underlying}\end{array}) \cdot (\begin{array}{c}\text{timetT value}\\ \text{of underlying}\end{array}) = P(S_t, t) + S_t$$

Then at time T,

$$V_{\boldsymbol{X}}(T) = \begin{cases} K & \text{if } S_T \leq K \quad : \quad \text{option is worth } K - S_T\text{, underlying is worth } S_T \\ S_T & \text{if } S_T \geq K \quad : \quad \text{option is worth } 0\text{, underlying is worth } S_T \end{cases}$$

Now consider the portfolio

$$\boldsymbol{Y} = (1 \text{ call option }, K \text{ bonds that pay 1 at time } T\ )$$

At time t, $\boldsymbol{Y}$ has value

$$V_{\boldsymbol{Y}}(t) = (1 \begin{matrix}\text{call}\\ \text{option}\end{matrix}) \cdot (\begin{matrix}\text{time t value}\\ \text{of call option}\end{matrix}) + (K \text{ bonds }) \cdot (\begin{matrix}\text{time t value}\\ \text{of bond}\end{matrix}) = C(S_t, t) + KB(t, T)$$

Then at time T,

$$V_{\boldsymbol{Y}}(T) = \begin{cases} K & \text{if } S_T \leq K \quad : \quad \text{option is worth 0, each of } K \text{ bonds is worth 1} \\ S_T & \text{if } S_T \geq K \quad : \quad \text{option is worth } S_T - K\text{, each of } K \text{ bonds is worth 1} \end{cases}$$

Notice that $\mathrm{P}\left[V_{\boldsymbol{X}}(T) = V_{\boldsymbol{Y}}(T)\right] = 1$, so the portfolios $\boldsymbol{X}$ and $\boldsymbol{Y}$ will have the same value at time T. Then by the Law of One Price (6.5), the portfolios must also have the same value at time t, so $V_{\boldsymbol{X}}(t) = V_{\boldsymbol{Y}}(t)$. Therefore

$$P(S_t, t) + S_t = C(S_t, t) + KB(t, T)$$

$\square$

Definition 6.8. Let $P(S_t, t)$ be the time-t value of a European put option with strike price K and expiration T on an underlying S_t. Let $C(s_t, t)$ denote the valye of a European call option on the same underlying. Suppose there is no aribtrage, and S_t is a Geometric Brownian Motion, which means $C(S_t, t$ is given by the Black-Scholes Formula (for a call option) given by the previous theorem. Let $B(t, T)$ denote the value of a bond worth 1 at time T. Then $P(S_t, t)$ is given by the *Black-Scholes Formula (for a put option)*.

Corollary 6.9. The Black-Scholes Formula (for a put option) is given by

$$\boxed{\begin{array}{c} P(S_t, t) = -S\Phi(-d_1) + Ke^{-r(T-t)}\Phi(-d_2) \\[1em] \text{where } d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\[1em] \text{and } d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \end{array}}$$

Proof. By Put-Call Parity (1.8),

$$\begin{aligned}
P(S_t,t) &= C(S_t,t) + KB(t,T) - S_t \\
&= S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) + KB(t,T) - S_t && \text{by (5.15)} \\
&= S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) + Ke^{-r(T-t)} - S_t && \text{by (5.16)} \\
&= S_t(\Phi(d_1) - 1) + Ke^{-r(T-t)}(1 - \Phi(d_2)) \\
&= -S_t(1 - \Phi(d_1)) + Ke^{-r(T-t)}(1 - \Phi(d_2)) \\
&= -S_t\Phi(-d_1) + Ke^{-r(T-t)}\Phi(-d_2)
\end{aligned}$$

□

7 The Black-Scholes Formula: Probabilistic Approach

Discussion 7.1. Our goal is to find a formula $C(S_t, t)$ for the time-t value of a European call option with strike price K and maturity time T on an underlying asset whose value is given by the Geometric Brownian Motion S_t. In this section, we derive such a formula, the *Black-Scholes Formula*, using probability, as opposed to the PDE approach taken in Sections 4-6.

Definition 7.2. A stochastic process S_t is a *Geometric Brownian Motion* if it satisfies the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Theorem 7.3. Suppose S_t is a Geometric Brownian Motion. Then the following is a formula for S_t:

$$S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma W_t}$$

Proof. Apply Itô's Lemma (9.3) to $f(S_t) = \ln(S_t)$:

$$\begin{aligned}
d[\ln(S_t)] &= \frac{\partial}{\partial S_t}[\ln(S_t)] \cdot dS_t + \frac{1}{2}\frac{\partial^2}{\partial S_t^2}[\ln(S_t)] \cdot (dS_t)^2 \\
&= \frac{1}{S_t}(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2}(\frac{-1}{S_t^2})(\sigma^2 S_t^2 dt) && \text{by (9.1) and (9.2)} \\
&= \mu dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt \\
&= (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t
\end{aligned}$$

We now have

$$d[\ln(S_t)] = (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t$$

Then

$$
\begin{aligned}
\ln(S_t) - \ln(S_0) &= \int_0^t (\mu - \frac{\sigma^2}{2})dx + \int_0^t \sigma dW_x \\
\implies \ln(S_t) &= \ln(S_0) + (\mu - \frac{\sigma^2}{2})t + \sigma W_t \\
\implies e^{\ln(S_t)} &= e^{\ln(S_0) + (\mu - \frac{\sigma^2}{2})t + \sigma W_t}
\end{aligned}
$$

Therefore,

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$$

□

Discussion 7.4. One way of valuing an option is to forecast the value of the underlying and then calculate the resulting expected value of the option. With the formula for S_t given by the preceeding theorem, we may find a sample value for S_t many times - say, 30,000 times - and use the average of the sample values as our projected time-T value for S_t. A single value for S_t given by the preceeding formula is called a *sample path* because the random component W_t gives it a different value each time it is calculated. Simulating the value of S_t thousands of times and using the average as the expected value of S_t is known as running a *Monte Carlo Simulation*, a common method in valuing complex derivatives.

Definition 7.5. A random variable X is *log-normally distributed* if $\ln(X)$ is normally distributed.

Lemma 7.6. A Geometric Brownian Motion S_t is log-normally distributed. Specifically, $\ln(\frac{S_t}{S_0})$ is normally distributed with mean $(\mu - \frac{\sigma^2}{2})t$ and variance $\sigma^2 t$. Succintly,

$$\ln(\frac{S_t}{S_0}) \sim \text{Normal}(\ (\mu - \frac{\sigma^2}{2})t\ ,\ \sigma^2 t\ )$$

Proof. By the previous theorem,

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$$

Hence

$$\ln(\frac{S_t}{S_0}) = (\mu - \frac{\sigma^2}{2})t + \sigma W_t$$

Then by the properties of Brownian Motion (2.__),

$$\begin{aligned}
\mathbb{E}\left[\ln(\frac{S_t}{S_0})\right] &= \mathbb{E}\left[(\mu - \frac{\sigma^2}{2})t + \sigma W_t\right] \\
&= (\mu - \frac{\sigma^2}{2})t + \sigma \mathbb{E}\left[W_t\right] \\
&= (\mu - \frac{\sigma^2}{2})t \\
\mathrm{Var}\left[\ln(\frac{S_t}{S_0})\right] &= \mathrm{Var}\left[(\mu - \frac{\sigma^2}{2})t + \sigma W_t\right] \\
&= \sigma^2 \mathrm{Var}\left[W_t\right] \\
&= \sigma^2 t
\end{aligned}$$

□

Construction 7.7. With conciseness in mind, for the following proofs we will say that for a Geometric Brownian Motion S_t, $\ln(\frac{S_t}{S_0})$ has mean m and variance v, so that $\ln(\frac{S_t}{S_0}) \sim \mathrm{Normal}(m, v)$:

$$m := (\mu - \frac{\sigma^2}{2})t \quad ; \quad v := \sigma^2 t$$

Lemma 7.8. The probability density of a Geometric Brownian Motion S_t such that $\ln(\frac{S_t}{S_0}) \sim \mathrm{Normal}(m, v)$ is given by

$$f_S(x) = \frac{1}{\sqrt{2\pi v}x} e^{\frac{-(\ln(\frac{x}{S_0})-m)^2}{2v}}$$

Proof. First find the cumulative distribution function (CDF) of S_t:

$$\begin{aligned}
F_S(x) = \mathrm{P}\left[S_t \leq x\right] = \mathrm{P}\left[\frac{S_t}{S_0} \leq \frac{x}{S_0}\right] &= \mathrm{P}\left[\ln(\frac{S_t}{S_0}) \leq \ln(\frac{x}{S_0})\right] \\
&= \Phi(\frac{\ln(\frac{x}{S_0}) - m}{\sqrt{v}}) \qquad (\Phi \text{ is the CDF of the std. normal distribution}) \\
&= \int_{-\infty}^{\frac{\ln(\frac{x}{S_0})-m}{\sqrt{v}}} \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} dz
\end{aligned}$$

Then differentiate to obtain the probability density function for S_t:

$$\begin{aligned}
f_S(x) &= \frac{d}{dx} \int_{-\infty}^{\frac{\ln(\frac{x}{S_0})-m}{\sqrt{v}}} \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} dz \\
&= \frac{1}{\sqrt{2\pi}} e^{\frac{-(\ln(\frac{x}{S_0})-m)^2}{2v}} \cdot (\frac{1}{x\sqrt{v}}) \\
&= \frac{1}{\sqrt{2\pi v}x} e^{\frac{-(\ln(\frac{x}{S_0})-m)^2}{2v}}
\end{aligned}$$

□

Lemma 7.9. *(Continuous Compounding)* Suppose an asset A grows at a constant rate r. If A_0 is the current value of the asset and A_t is the value of the asset at time t, then

$$A_t = A_0 e^{rt}$$

Proof. The compound interest formula is

$$A_t = A_0(1 + \frac{r}{n})^{nt}$$

In continuous time, the discrete time periods defined by n are infenitesimal, and so we take the limit as $n \to \infty$:

$$\begin{aligned}
A_t &= \lim_{n\to\infty} A_0(1 + \frac{r}{n})^{nt} \\
&= \lim_{x\to\infty} A_0(1 + \frac{1}{x})^{xrt} && \text{letting } x = \tfrac{n}{r} \\
&= \lim_{x\to\infty} A_0[(1 + \frac{1}{x})^{x}]^{rt} \\
&= A_0[\lim_{x\to\infty} (1 + \frac{1}{x})^{x}]^{rt} \\
&= A_0 e^{rt} && \text{by definition of } e
\end{aligned}$$

□

Definition 7.10. A universe is *risk-neutral* if for all assets A and future times t in our universe, A is expected to grow at the risk-free rate r:

$$\mathbb{E}[A_t] = A_0 e^{rt}$$

Lemma 7.11. In a risk-neutral universe, the drift μ of a Geometric Brownian Motion S_t is the risk-free rate r.

Proof. By definition (9.11), in a risk-neutral universe,

$$\mathbb{E}[S_t] = S_0 e^{rt} \qquad (*)$$

Also,

$$\mathbb{E}[S_t] = \int_S x f_S(x)dx = \int_0^\infty x \frac{1}{\sqrt{2\pi v}x} e^{\frac{-(\ln(\frac{x}{S_0})-m)^2}{2v}} dx \qquad \text{by (9.9)}$$

Let $z = \frac{(\ln(\frac{x}{S_0})-m)}{\sqrt{v}}$
So that $dz = \frac{dx}{x\sqrt{v}}$, $x = S_0 e^{z\sqrt{v}+m}$, $z(0) = -\infty$, $z(\infty) = \infty$

Then

$$
\begin{aligned}
E[S_t] &= \frac{1}{\sqrt{2\pi v}} \int_{-\infty}^{\infty} e^{\frac{-z^2}{2}} x\sqrt{v} dz \\
&= \frac{S_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-z^2}{2}+z\sqrt{v}+m} dz \\
&= \frac{S_0}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(z-\sqrt{v})^2}{2}+\frac{v}{2}+m} dz \\
&= \frac{S_0}{\sqrt{2\pi}} e^{\frac{v}{2}+m} \int_{-\infty}^{\infty} e^{\frac{-(z-\sqrt{v})^2}{2}} dz
\end{aligned}
$$

Let $y = (z-\sqrt{v})/\sqrt{2}$. Then

$$
\begin{aligned}
E[S_t] &= \frac{S_0}{\sqrt{\pi}} e^{\frac{v}{2}+m} \int_{-\infty}^{\infty} e^{-y^2} dy \\
&= S_0 e^{\frac{v}{2}+m} && \text{by } (4.\_\_) \\
&= S_0 e^{\frac{\sigma^2 t}{2}+(\mu-\frac{\sigma^2}{2})t} && \text{by } (9.8) \\
&= S_0 e^{\mu t}
\end{aligned}
$$

Therefore, recalling (*),

$$\mathbb{E}\left[S_t\right] = S_0 e^{\mu t} = S_0 e^{rt}$$

Hence

$$\mu = r$$

□

Discussion 7.12. As a result of the previous theorem, we may use $\mu \mapsto r$ with regard to the underlying asset S_t, since the Black-Scholes Formula describes the value of a European call option in a risk-neutral universe. This is consistent with our initial derivation of the Black-Scholes PDE in chapter 4, for at (4.4) we hedged out the risk in our portfolio with our smart choice for a, the number of shares in our hedging portfolio.

Lemma 7.13. The time-0 value of a European call option with time to maturity T and strike price K on an underlying Geometric Brownian Motion S_t is given by

$$C(S_0, 0) = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2)$$

where

$$d_{1,2} = d_{+,-} = \frac{\ln(\frac{S_0}{K}) + (r \pm \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

Proof. Because we're valuing a call option in a risk-neutral universe, by definition (9.10) we have

$$\begin{aligned}
C(S_0,0) &= e^{-rT}\mathbb{E}\left[C(S_T,T)\right] \\
&= e^{-rT}\mathbb{E}\left[\max\left(S_t-K,0\right)\right] && \text{by } (1.__) \\
&= e^{-rT}\mathbb{E}\int_S \max\left(x-K,0\right) f_S(x)dx && \text{by } (9.9) \\
&= e^{-rT}\int_K^\infty (x-K)\frac{1}{\sqrt{2\pi v}x}e^{\frac{-(\ln(\frac{x}{S_0})-m)^2}{2v}}dx \\
&= e^{-rT}\int_K^\infty \frac{1}{\sqrt{2\pi v}}e^{\frac{-(\ln(\frac{x}{S_0})-m)^2}{2v}}dx - e^{-rT}\int_K^\infty \frac{K}{\sqrt{2\pi v}x}e^{\frac{-(\ln(\frac{x}{S_0})-m)^2}{2v}}dx \\
&=: I_1 + I_2
\end{aligned}$$

The first integral is similar to what we encountered in lemma (9.11):

$$I_1 = \frac{e^{-rT}}{\sqrt{2\pi v}}\int_K^\infty e^{\frac{-(\ln(\frac{x}{S_0})-m)^2}{2v}}dx$$

Let $z = \frac{\ln(\frac{x}{S_0})-m}{\sqrt{v}}$
So that $dz = \frac{dx}{x\sqrt{v}}$, $x = S_0e^{z\sqrt{v}+m}$, $z(\infty)=\infty$, $A := z(K) = \frac{\ln(\frac{K}{S_0})-m}{\sqrt{v}}$
Then

$$\begin{aligned}
I_1 &= \frac{e^{-rT}}{\sqrt{2\pi v}}\int_A^\infty e^{\frac{-z^2}{2}}S_0e^{z\sqrt{v}+m}\sqrt{v}dz \\
&= \frac{S_0e^{-rT}}{\sqrt{2\pi}}\int_A^\infty e^{\frac{-z^2}{2}+z\sqrt{v}+m}dz \\
&= \frac{S_0e^{-rT}}{\sqrt{2\pi}}\int_A^\infty e^{\frac{-(z-\sqrt{v})^2}{2}+m+\frac{v}{2}}dz \\
&= \frac{S_0e^{-rT+m+\frac{v}{2}}}{\sqrt{2\pi}}\int_A^\infty e^{\frac{-(z-\sqrt{v})^2}{2}}dz
\end{aligned}$$

But $-rT+m+\frac{v}{2} = -rT+(r-\frac{\sigma^2}{2})T+\frac{\sigma^2T}{2} = 0$ by (9.8) and (9.11), so

$$I_1 = \frac{S_0}{\sqrt{2\pi}}\int_A^\infty e^{\frac{-(z-\sqrt{v})^2}{2}}dz$$

Now let $y = z - \sqrt{v}$ so that $dy = dz$, $y(\infty) = \infty$, and

$$
\begin{aligned}
B := y(A) = A - \sqrt{v} &= \frac{\ln(\frac{K}{S_0}) - m}{\sqrt{v}} - \sqrt{v} \\
&= \frac{\ln(\frac{K}{S_0}) - m - v}{\sqrt{v}} \\
&= \frac{\ln(\frac{K}{S_0}) - (r - \frac{\sigma^2}{2})T - \sigma^2 T}{\sigma\sqrt{T}} \qquad \text{by (9.8) and (9.11)} \\
&= \frac{\ln(\frac{K}{S_0}) - (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}
\end{aligned}
$$

Then

$$I_1 = S_0 \int_B^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy$$

Recognizing the integrand as the density of the standard normal distribution,

$$I_1 = S_0(1 - \Phi(B)) = S_0\Phi(-B) = S_0\Phi(d_1)$$

Where Φ is the CDF of the standard normal distribution, and

$$d_1 := -B = \frac{-\ln(\frac{K}{S_0}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = \frac{\ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

Now we solve the second integral:

$$I_2 = -e^{-rT} \int_K^\infty \frac{K}{\sqrt{2\pi v}x} e^{\frac{-(\ln(\frac{x}{S_0}) - m)^2}{2v}} dx$$

Let $z = \frac{\ln(\frac{x}{S_0}) - m}{\sqrt{v}}$

So that $dz = \frac{dx}{x\sqrt{v}}$, $x = S_0 e^{z\sqrt{v} + m}$, $z(\infty) = \infty$, $A := z(K) = \frac{\ln(\frac{K}{S_0}) - m}{\sqrt{v}}$

Then

$$I_2 = -Ke^{-rT} \int_A^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} dz = -Ke^{-rT}(1 - \Phi(A)) = -Ke^{-rT}\Phi(-A) = -Ke^{-rT}\Phi(d_2)$$

Where

$$d_2 := -A = \frac{-\ln(\frac{K}{S_0}) + m}{\sqrt{v}} = \frac{\ln(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

Hence

$$C(S_0, 0) = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

Where

$$d_{1,2} = d_{+,-} = \frac{\ln(\frac{S_0}{K}) + (r \pm \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

□

Corollary 7.14. The risk-neutral time-t value of a European call option with maturity T and strike price K on an underlying Geometric Brownian Motion S_t (with drift r, the risk-free rate, and volatility σ) is given by the Black-Scholes Formula:

$$\begin{aligned} C(S,t) &= S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \\ \text{where } d_1 &= \frac{\ln(S/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\ \text{and } d_2 &= \frac{\ln(S/K) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

Proof. In the previous lemma, use $T \mapsto T-t$. This is logically consistent because in the previous lemma, T acted as the "time until maturity", but when valuing an option at an arbitrary time t, the $T-t$ is the "time until maturity". □

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