Kelly 56 Notes

Setup

- Consider a series of upcoming events that you would like to bet on. For instance, a series of N baseball games.
- For each event, suppose there are n outcomes
- Suppose you have an "information Channel" that sends you a predicted outcome $r \in \{1, \dots, n\}$ for each event/game. This info channel can be a real communication channel or represent the totality of inside information available to the gambler.
- We want to Know how much money we can make by betting, using the info from the Info Channel

Notation

α_s = odds paid if s^{th} outcome occurs
= # of dollars returned for a \$1 bet.

$p(s)$ = probability that s^{th} outcome occurs.

$a(s/r)$ = the fraction of the gambler's capital that she decides to bet on outcome s, given that she receives symbol r = our Gambling Strategy

V_N = gambler's capital after N bets
V_0 = gambler's initial capital
W_{rs} = the # of times you receive symbol r, and the outcome is s, in N games
G = exponential rate of growth of gambler's capital
$P(s,r)$ = probability that outcome s occurs, and receive symbol r
$q(r)$ = probability you receive symbol r
$q(s|r)$ = probability outcome s occurs, given that you receive r

1. case: Fair odds, No Rake/Vigorish

Fair odds

$$\boxed{\alpha_s = \frac{1}{p(s)}}$$

α_s = odds paid if outcome s occurs

$p(s)$ = probability outcome s occurs

"Fair" because if you bet \$B that outcome s will occur, then you're expected profit it

$$\mathbb{E}[\text{Profit}] = p(s)\cdot(\alpha_s B - B) + (1-p(s))(-B)$$
$$= B - p(s)\cdot B - B + p(s)\cdot B$$
$$= 0.$$

Fact

$$\boxed{1 = \sum \frac{1}{\alpha_s}}$$

Without loss of generality, impose the

Constraint

$$\boxed{1 = \sum_s a(s|r) \quad \forall r}$$

$a(s|r)$ = the fraction of gambler's capital bet on outcome s, given that receives symbol r = our Gambling Strategy

"Regardless of the symbol r the gambler receives, she will bet her entire capital"

We can assume this because the gambler can hold back some capital by placing cancelling bets; because the odds are fair, there is no rake to make cancelling bets unprofitable.

$$\boxed{V_N = \prod_{r,s} \left[a(s|r) \cdot \alpha_s\right]^{W_{rs}} \cdot V_0}$$

V_N = gambler's capital after N bets.
V_0 = gambler's initial capital.
W_{rs} = the # of times that you receive symbol r, and the outcome is s, in N games.

$$\boxed{G := \lim_{N \to \infty} \frac{1}{N} \log_2\left(\frac{V_N}{V_0}\right)}$$

G = Exponential rate of growth of gambler's capital

$$V_N \approx V_0 (e^G)^N$$

Goal Maximize G (maximize the logarithm of the gambler's capital)

Why G? "It is the logarithm which is additive in repeated bets, and to which the Law of Large Numbers applies"

"an essentially different criterion from the clasical gambler, who maximizes (expected) capital of each bet"

$$G = \lim_{N\to\infty} \frac{1}{N} \log_2\left(\frac{V_N}{V_0}\right)$$

$$= \lim_{N\to\infty} \sum_{rs} \left(\frac{W_{rs}}{N}\right) \log\left[\alpha_s \, a(s|r)\right]$$

$$= \sum_{rs} p(s,r) \log\left[\alpha_s \, a(s|r)\right]$$

P(s,r) = probability that outcome s occurs and receive symbol r

$$= \sum_{rs} p(s,r) \log\left[\frac{a(s|r)}{p(s)}\right]$$

$$= \sum_{rs} p(s,r) \log a(s|r) \underbrace{- \sum_{rs} p(s,r) \log p(s)}_{\text{Shannon's "Source Rate" } H(X)}$$

Goal Maximize G.

P(s,r) is fixed, but unknown (in practice, estimate it, but here, assume known).

a(s|r) is the Gambling Strategy, which the gambler has control over! The gambler chooses her strategy!

So, how can we choose the gambling strategy a(s|r) to maximize G?

Compute

$$\underset{a}{\operatorname{argmax}}\, G = \underset{a}{\operatorname{argmax}} \sum_{rs} p(s,r) \log a(s|r)$$

$$= \underset{a}{\operatorname{argmax}} \sum_{rs} q(r)\, q(s|r) \log a(s|r)$$

q(r) = probability that you receive symbol r

q(s|r) = probability that outcome s occurs given that you received symbol r

$$= \underset{a}{\operatorname{argmax}} \sum_r q(r) \sum_s q(s|r) \log a(s|r)$$

Equivalently, $\forall r$, $\underset{a}{\operatorname{argmax}} \sum_s q(s|r) \log a(s|r)$

- For notational convenience, we may drop the r and write $\underset{a}{\operatorname{argmax}} \sum_s q(s) \log a(s)$, knowing there is an implicit conditional r. Note that this is as if there is no information channel at all!

Task

$$\boxed{\begin{aligned} &\text{maximize} && f(a_1, \ldots, a_n) = \sum_{s=1}^{n} q_s \log a_s \\ &\text{subject to} && g(a_1, \ldots, a_n) = \sum_{s=1}^{n} a_s = 1 \end{aligned}}$$

"gambler bets entire capital" constraint.

↳ Lagrange Multipliers!

Treat $\vec{q}$ as known, and find the maximizing gambling strategy $\vec{a}$.

$$\vec{\nabla} f = \left(\frac{q_1}{a_1}, \ldots, \frac{q_n}{a_n}\right) \qquad \vec{\nabla} g = (1, \ldots, 1)$$

$$\vec{\nabla} f = \lambda \vec{\nabla} g \implies \left(\frac{q_1}{a_1}, \ldots, \frac{q_n}{a_n}\right) = \lambda(1, \ldots, 1)$$

$$\implies \lambda = \frac{q_1}{a_1} = \cdots = \frac{q_n}{a_n}$$

well, $\frac{q_1}{a_1} = \frac{q_s}{a_s} \Longrightarrow a_s = a_1 \frac{q_s}{q_1} \quad \forall s$

Constraint $1 = \sum' a_s = \frac{a_1}{q_1}\left(\sum' q_s\right) = \frac{a_1}{q_1}$

$\Longrightarrow a_1 = q_1 \Longrightarrow \boxed{a_s = q_s \quad \forall s}$

We know this critical point is a maximizer because

$$\begin{cases} f(a_1 = q_1, \dots, a_n = q_n) = \sum'_k q_k \log q_k \in \mathbb{R} \\ f(a_1 = 1, a_2 = 0, \dots, a_n = 0) = -\infty \end{cases}$$

- Re-incorporating the received symbols r, we have found the gambling strategy that maximizes G, in the fair odds case:

$$\boxed{a(s|r) = q(s|r) = \frac{p(s,r)}{q(r)} = \frac{p(s,r)}{\sum'_k p(k,r)}}$$

Therefore,

$$G_{max} = \sum_{rs} p(s,r) \log \underbrace{a(s|r)}_{= q(s|r)} - \sum'_{rs} p(s,r) \log p(s)$$

$$= -H(X|Y) + H(X)$$ "Shannon's Rate of Transmission"

(2) case: Unfair odds, No Rake/Vigorish

No Rake $\boxed{\sum \frac{1}{\alpha_s} = 1}$

unfair odds α_s is not necessarily $\frac{1}{P(s)}$

Without loss of generality we still have $\boxed{1 = \sum_s a(s|r) \quad \forall r}$ since the gambler can still hold back money by betting in proportion to the $1/\alpha_s$, since no Rake.

As before,

$$G = \sum_{rs} P(s,r) \log\left[\alpha_s a(s|r)\right]$$

$$= \underbrace{\sum_{rs} P(s,r) \log a(s|r)}_{\text{To maximize } G \text{ is to maximize this}} + \underbrace{\sum_s P(s) \log \alpha_s}_{:= H(\alpha)}.$$

To maximize G is to maximize this, which we already know is when $\boxed{a(s|r) = q(s|r).}$ It is the same since this term doesn't depend on the odds α_s

$$G_{max} = H(\alpha) - H(X|Y)$$

Interesting Facts

(A.) G is maximized as before by setting $\boxed{a(s|r) = q(s|r)}$. So, the gambler ignores the posted odds in placing his bets!

the better your information for an outcome, the more you should bet on that outcome.

(B.) Minimizing $H(\alpha) = \sum_s p(s) \log \alpha_s$ subject to $\sum_s \frac{1}{\alpha_s} = 1$ occurs when $\alpha_s = \frac{1}{p(s)}$, $H(X) = H(\alpha)$

(prove this using Lagrange multipliers).

So, any deviation from the fair odds helps the gambler!

this makes sense: the Gambler can exploit unfair odds

C.

$$G_{max} = H(\alpha) - H(X)$$ without information channel (ignore r)

$$G_{max} = H(\alpha) - H(X|Y)$$ with info

$$R = H(X) - H(X|Y)$$ "Shannon's Rate of Transmission"

= increase in G_{max} due to Information gain.

"If the odds are not fair (not consistent with the transmitted symbol probabilities but consistent with some other set of probabilities) then G_{max} is larger than it would have been with no Info Channel, by an amount equal to the Rate of Transmission of Information"

3. case : Unfair odds, Yes Rake/Vigorish

Gambler cannot make cancelling bets, due to the Rake.

Constraint $\underbrace{b_r}_{\text{some positive number}} + \sum_s a(s|r) = 1$

the fraction of capital not bet when Receive symbol r.

Goal
- Choose gambling strategy $a(s|r)$ to
- Maximize $G = \sum_{rs} p(s,r) \log\left[b_r + \alpha_s a(s|r)\right]$
- subject to $\sum_s a(s|r) = 1 - b_r \quad \forall r$

Math maximize each term over each r separately
Use Lagrange Multipliers/optimization ...