A Foray into Stochastic Calculus

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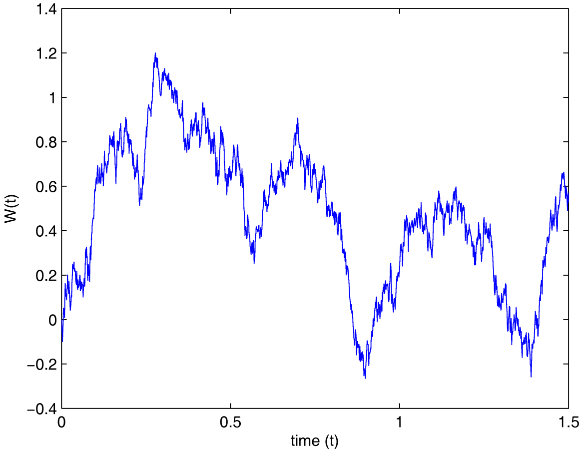
- **Nontrivial Fact** Brownian Motion exists (in fact, it can be constructred)
- **Theorem** With probability 1, $t \mapsto W(t)$ is continuous everywhere, but differentiable nowhere

Brownian Motion



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- One way to represent random fluctuations is via dW, or a small change in Brownian Motion. But how to rigorously define dW?
- It is easier to define the stochastic integral

$$\int_0^T G dW$$

i.e., the integral of a stochastic process G with respect to Brownian Motion

Defining the Ito Integral

- *Approximate by Step Processes*: G is a step process if there is a partition $0 = t_0 < t_1 < ... < t_m = T$ such that $G(t) \equiv G_k$ for $t_k \leq t_{k+1}$

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- The *Ito Integral* of G on $[0, T]$ is

$$\int_0^T GdW = \sum_{k=0}^{m-1} G_k(W(t_{k+1}) - W(t_k))$$

i.e., the Left Riemann Sum

The Chain Rule

- We say that $\{X(t) \big| t \geq 0\}$ satisfies the *stochastic differential*

$$dX = Fdt + GdW$$

if for all $s \geq r \geq 0$

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- *Ito's Chain Rule*: $Y(t) = u(X(t), t)$ satisfies the stochastic differential

$$dY = du(X, t) = u_t dt + u_x dX + \frac{1}{2} u_{xx} G^2 dt$$

An Example

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$$\begin{aligned} du(S, t) &= u_t dt + u_s dS + \frac{1}{2} u_{ss} G^2 dt \\ &= S^2 dt + 2St dS + \frac{1}{2} 2t\sigma^2 dt \\ &= (S^2 + 2St\mu + t\sigma^2) dt + (2St\sigma) dW \end{aligned}$$

Connection to (deterministic) PDEs

- Bounded open set $U \subset \mathbb{R}^n$, smooth boundary ∂U
- By standard PDE theory, there is a smooth solution u of

$$\begin{cases} -\frac{1}{2}\Delta u = 1 \text{ in } U \\ u = 0 \text{ on } \partial U \end{cases}$$

where Δu is the *Laplacian* $\Delta u = \sum_{i=1}^{n} u_{x_i x_i}$

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- Then for each $x \in U$,

$$u(x) = \mathbb{E}(\tau_x)$$