Kelly Betting

Card game demo

- split up into pairs
- each pair grab a deck of cards
- each person starts with, say, $100
- let person A in the pair be the dealer and person B be the bettor.

Person B bets as much as he wants of his bankroll on whether the next card will be Red or Black at Fair Odds. Person A flips a card. Record the new bankroll after the card is flipped. Play until 40 cards have been flipped.

- Then Reverse the roles A and B and replay
- The person who makes the most money wins!

We want to bet on Basketball games over the course of the entire NBA season and beyond.

Initial Bankroll $B = \$100$.

P_i = true prob. that team A_i wins (assume known) vs. team B_i in game i

Decimal odds α_{A_i} = # dollars returned for a \$1 bet on team A_i

$\quad\quad = 1 +$ profit from \$1 bet on team A_i

α_{B_i}

Q How should we bet?

* Someone's going to say: maximize EV

* EV of a \$1 bet on team A: ignoring subscript i,

$$EV_A = \mathbb{E}(\text{profit}_A) = \begin{pmatrix}\text{pr. } A\\ \text{wins}\end{pmatrix} \cdot \begin{pmatrix}\text{profit}\\ \text{if } A\\ \text{wins}\end{pmatrix} + \begin{pmatrix}\text{pr. } A\\ \text{loses}\end{pmatrix} \cdot \begin{pmatrix}\text{profit}\\ \text{if } A\\ \text{loses}\end{pmatrix}$$

$$= p(\alpha_A - 1) + (1-p)(-1)$$

$$= p\alpha_A - 1$$

$$\mathbb{E}(\text{profit}_B) = (1-p)\alpha_B - 1$$

* Let's say $EV_A < 0$. How much will you bet?

→ they'll say ZERO. Fair enough.
Come back to this

* Let's say $EV_A > 0$. How much will you bet?

→ they'll say make a bet!

* Even if $EV_A > 0$, if you bet you're entire bankroll on A and lose, then you're out of money :(

How to account for this?

→ What about, over N bets, just split up you're money evenly betting B/N on team A each time?

→ Make money on average
but can we make more money?
What are we missing?

* Not taking advantage of the sequential nature of the bets.
If I make money on the first bet, I can use that profit to bet more on the second bet! <u>compounding</u>!

* How do we actually acheive compounding?
$\longrightarrow$ bet a <u>fraction</u> $f \in (0,1)$ of your bankroll

Bet size $B \cdot f_1$ on team A_1 in game 1
Make $(\alpha_{A_1} - 1) B f_1$ w.p. p
~~Lose~~ make $-B f_1$ w.p. $1-p$
Profit $B \cdot f_1 (\alpha_{A_1} X_1 - 1)$
where $X_1 = \begin{cases} 1 & \text{if } A_1 \text{ wins in game 1} \\ 0 & \text{if } A_1 \text{ loses in game 1} \end{cases}$

* Is this what we want to maximize though? Profit? What do we actually want to have at the end?
A high bankroll!

Bankroll $B + Bf_1(\alpha_{A_1}X_1 - 1) = B\left[1 + f_1(\alpha_{A_1}X_1 - 1)\right]$

After first bet, we have $B\left[1 + f_1(\alpha_{A_1}X_1 - 1)\right]$

Bet size $B\left[1 + f_1(\alpha_{A_1}X_1 - 1)\right] \cdot f_2$ on tm A_2 in game 2

Profit $B\left[1 + f_1(\alpha_{A_1}X_1 - 1)\right] \cdot f_2(\alpha_{A_2}X_2 - 1)$ after game 2

Bankroll $B\left[1 + f_1(\alpha_{A_1}X_1 - 1)\right]\left[1 + f_2(\alpha_{A_2}X_2 - 1)\right]$ by same logic

After N games,

$$\boxed{\text{Bankroll} = B \prod_{i=1}^{N} \left[1 + f_i(\alpha_{A_i}X_i - 1)\right]}$$

B = initial bankroll (say, \$100)

α_{A_i} = decimal odds for betting on team A in game i (known)

$X_i = 1$ if team A wins in game i, else 0 (random variable)

f_i = fraction of bankroll on game i (want to find)

* Want to Maximize Bankroll

but, bankroll is a Random Variable

⟶ Maximize Expected Bankroll (a number)

$$\underset{f}{\operatorname{argmax}}\ \mathbb{E}\ \text{BankRoll}$$

$$= \underset{f}{\operatorname{argmax}}\ \mathbb{E}\left(\mathbf{B} \prod_{i=1}^{N} \left[1 + f_i\left(\alpha_{A_i} X_i - 1\right)\right]\right)$$

the random variables $X_1, \ldots, X_N$

* Good luck doing this!!

Too hard due to the product.

So, we're stuck because of the product.

How to get rid of a product?

⟶ log

Kelly's brilliant idea:

TRY $$\underset{f}{\operatorname{argmax}} \mathbb{E} \log \left(B \prod_{i=1}^{N} \left[1 + f_i (\alpha_{A_i} X_i - 1) \right] \right)$$

Shannon-McMillan-Breiman 1950s:

the f that maximizes the log bankroll has more money asymptotically as N goes to ∞ than any other allocation f!

$$= \underset{f}{\operatorname{argmax}} \mathbb{E} \sum_{i=1}^{n} \log \left(1 + f_i (\alpha_{A_i} X_i - 1) \right)$$

$$= \sum \mathbb{E} \log$$

$$= \underset{f}{\operatorname{argmin}} \sum_{i=1}^{n} \log \left(1 + f_i (\alpha_{A_i} - 1) \right) \cdot P_i + \log (1 - f_i) \cdot (1 - P_i)$$

Same minimization for each i, due to the $\sum$

$$\underset{f}{\operatorname{argmin}} \quad \log(1+f(\alpha-1))p + \log(1-f)(1-p)$$

* How to solve this? Calculus!

$$\frac{d}{df}\left[\log(1+f(\alpha-1))p + \log(1-f)(1-p)\right]$$

$$= \frac{\alpha-1}{1+f(\alpha-1)} \cdot p + \frac{-1}{1-f} \cdot (1-p) = 0$$

$$\Rightarrow \frac{1-p}{1-f} = \frac{p(\alpha-1)}{1+f(\alpha-1)}$$

$$\Rightarrow (1-p)(1+f(\alpha-1)) = p(\alpha-1)(1-f)$$

$$\Rightarrow f(1-p)(\alpha-1) + (1-p) = -fp(\alpha-1) + p(\alpha-1)$$

$$\Rightarrow f((1-p)(\alpha-1) + p(\alpha-1)) = p(\alpha-1) - (1-p)$$

$$\Rightarrow f(\alpha-1) = p\alpha \cancel{-p} - 1 \cancel{+p}$$

$$\Rightarrow f = \frac{p\alpha-1}{\alpha-1} \qquad \boxed{f = \max\left(0, \frac{p\alpha-1}{\alpha-1}\right)}$$

Kelly Fraction

Exs

- If $p=1$ (guaranteed)
  then $f=1$ (bet entire bankroll)

- If -110 bet,
  decimal odds $\alpha = 1 + \frac{100}{110} = \frac{210}{110} = 1.909$
  for f to be positive (to bet something)
  we need $\frac{p\alpha - 1}{\alpha - 1} > 0 \Rightarrow p > \frac{1}{\alpha} = 0.524$

- Desmos $f = \max\left(0, \frac{p\alpha - 1}{\alpha - 1}\right)$

- Kelly (1956) extended this analysis to horses

m horses $1, \dots, m$
true (known) win probabilities $p_1, \dots, p_m$
decimal odds $\alpha_1, \dots, \alpha_m$
payroll fractions $f = (f_1, \dots, f_m)$ to bet on each horse.
make N sequential bets, solve for f.

Use Lagrange Multipliers/KKT conditions to optimize expected log bankroll after N bets.

(a) Permute indices so that $p(s)\alpha_s \geqq p(s+1)\alpha_{s+1}$

(b) Set b equal to the minimum positive value of

$$\frac{1-p_t}{1-\sigma_t} \quad \text{where} \quad p_t = \sum_1^t p(s), \;\; \sigma_t = \sum_1^t \frac{1}{\alpha_s}$$

(c) Set $f(s) = p(s) - b/\alpha_s$ or zero, whichever is larger. (The $f(s)$ will sum to $1 - b$.)

Q. Should you ever make a negative EV bet?

Yes there are scenarios (p, α) for which Kelly says you should bet on minus EV horse(s).

Intuition

- if it's a profitable game you should put money in!
- if you just bet on the, say, one +EV horse you'll lose a lot; need to hedge

Interesting Fact (case: 3 teams, $(P_1+P_2<1,\ P_3>0)$)

One of the results in the paper is that there exist horses with true probabilities and odds that make bets on the horse negative EV. Yet Kelly betting on the horse suggests wagering some amount of your bankroll anyway.

$$\begin{cases} P_1+P_2<1 \\ P_1, P_2 \geq 0 \end{cases}$$ probabilities

$$\begin{cases} P_1\alpha_1 - 1 > 0 \\ P_2\alpha_2 - 1 < 0 \end{cases}$$

option 1 is +EV $\Rightarrow$ we will bet on some team

option 2 is −EV

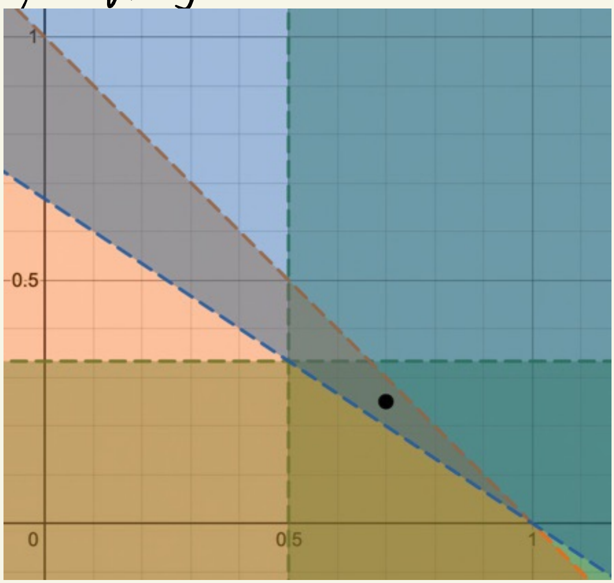
$$\mathrm{EV}(\$1\text{ bet}) = \begin{pmatrix}\text{profit}\\ \text{amount}\end{pmatrix}\cdot\begin{pmatrix}P\\ \text{win}\end{pmatrix} - \begin{pmatrix}\text{bet}\\ \text{size}\end{pmatrix}\begin{pmatrix}P\\ \text{lose}\end{pmatrix}$$
$$= (\alpha-1)P - 1(1-P)$$
$$= \alpha P - P - 1 + P$$
$$= P\alpha - 1$$

$$\begin{cases} 0 < \dfrac{1-P_1-P_2}{1-\frac{1}{\alpha_1}-\frac{1}{\alpha_2}} < \dfrac{1-P_1}{1-\frac{1}{\alpha_1}} \\ \text{Equivalently, } 0 < f_2 < f_1 \end{cases}$$

make a nonzero bet on team 2!

$\alpha_1 \mapsto a,\quad \alpha_2 \mapsto b,\quad p \mapsto x,\quad q \mapsto y$

1. $a = 2$ (slider −10 to 10)
2. $b = 3$ (slider −10 to 10)
3. $x \cdot a - 1 > 0$
4. $y \cdot b - 1 < 0$
5. $x + y < 1$
6. $0 < \dfrac{(1-x-y)}{\left(1-\frac{1}{a}-\frac{1}{b}\right)}$ → Kelly's "b"
7. $\dfrac{(1-x-y)}{\left(1-\frac{1}{a}-\frac{1}{b}\right)} < \dfrac{(1-x)}{\left(1-\frac{1}{a}\right)}$
8. $(.7, .25)$ Label



Check $a_2 = \max\left(P_2 - \dfrac{b}{\alpha_2}, 0\right) = 0.1$, so we make a −EV bet as desired!

* In practice, the win probability p of a horse or team is not an observable or known quantity (with a deck of cards it is, but in real life sports it's not); it needs to be estimated from data $\to \hat{p}$.
How does Kelly betting change under this?

* Ideally our estimator $\hat{p}$ of p is unbiased $\mathbb{E}\hat{p} = p$
but subject to some uncertainty
$\mathrm{Var}(\hat{p}) = \tau^2$.
The more uncertain we are

in our estimate, the less we should bet.

* Fractional Kelly says bet a fraction $K \in [0,1]$ of the Kelly betting fraction f, $f \leftarrow K \cdot f$.

* K is some function $K = K(\tau)$ such that $\lim_{\tau \downarrow 0} K(\tau) = 1$

(if $\mathbb{E}\hat{p} = p$ and $\mathrm{var}(\hat{p}) = 0$, we know the true win probabilities, so use original Kelly fraction)

and $\lim_{\tau \uparrow \infty} K(\tau) = 0$

(fully uncertain about win prob)

* K-Fractional Kelly with $\hat{p}$ is equivalent to full Kelly with a shrinkage estimator for $\hat{p}$, shrinking more if τ^2 larger