


Setup

- Consider a series of upcoming events that you would like to bet on.
For instance, a series of N baseball games.
- For each event, suppose there are n outcomes
- Suppose you have an "information channel" that sends you a predicted outcome $r \in \{1, \dots, n\}$ for each event/game
- We want to know how much money we can make by betting, using the info from the Info Channel

Notation

α_s = odds paid if s^{th} outcome occurs
= # of dollars returned for a \$1 bet.

$p(s)$ = probability that s^{th} outcome occurs.

$a(s|r)$ = the fraction of the gambler's capital that she decides to bet on outcome s , given that she receives symbol r = our gambling strategy

V_N = gambler's capital after N bets

V_0 = gambler's initial capital

W_{rs} = the # of times you receive symbol r and the outcome is s , in N games

G = exponential rate of growth of gambler's capital

$P(s|r)$ = probability that outcome s occurs given that you receive symbol r

$q(r)$ = probability you receive symbol r

$q(s|r)$ = probability outcome s occurs, given that you receive r

① Case: Fair Odds, No Rake/Vigorish

Fair odds

$$\alpha_s = \frac{1}{p(s)}$$

α_s = odds paid
if outcome
s occurs

$p(s)$ = probability
outcome s
occurs

"Fair" because if you bet \$B that outcome S will occur, then you're expected profit it

$$\begin{aligned} E[\text{Profit}] &= p(s) \cdot (\alpha_s B - B) + (1-p(s))(-B) \\ &= B - p(s) \cdot B - B + p(s) \cdot B \\ &= 0. \end{aligned}$$

Fact

$$1 = \sum \frac{1}{\alpha_s}$$

without loss of generality, impose the

Constraint

$$1 = \sum_s a(s|r) \quad \forall r$$

"Regardless of the symbol r the gambler receives, she will bet her entire capital"

$a(s|r)$ = the
fraction of
gambler's
capital
bet on
outcome s,
given that
reciever
symbol r
= our
Gambling
Strategy

We can assume this because the gambler can hold back some capital by placing cancelling bets; because the odds are fair, there is no Rake to make cancelling bets unprofitable.

$$V_N = \prod_{r,s} [a(s|r) \cdot d_s]^{W_{rs}} \cdot V_0$$

V_N = gambler's capital after N bets.

V_0 = gambler's initial capital.

W_{rs} = the # of times that you receive symbol r , and the outcome is S , in N games.

$$G := \lim_{N \rightarrow \infty} \frac{1}{N} \log_2 \left(\frac{V_N}{V_0} \right)$$

G = Exponential rate of growth of gambler's capital

$$V_N \approx V_0 (e^G)^N$$

Motivation for G

Suppose our information channel is perfect, so we know the game outcomes before they happen. Suppose there are only 2 outcomes (think Dodgers vs. Yankees), with even odds, and say the teams are evenly matched.

The gambler will bet her entire capital each time,

$$\text{so } V_N = 2^N V_0,$$

$$\text{so } G = 1 \text{ is the exponential growth rate of capital.}$$

Goal

Even when information is imperfect,

how can we **Maximize G?**

$$G = \lim_{N \rightarrow \infty} \frac{1}{N} \log_2 \left(\frac{V_N}{V_0} \right)$$

$$= \lim_{N \rightarrow \infty} \sum_{rs} \left(\frac{w_{rs}}{N} \right) \log \left[ds \alpha(s/r) \right]$$

$$= \sum_{rs} P(s, r) \log [a_s a(s|r)]$$

$P(s, r)$ = probability that outcome s occurs and receive symbol r

$$= \sum_{rs} P(s, r) \log \left[\frac{a(s|r)}{P(s)} \right]$$

$$= \sum_{rs} P(s, r) \log a(s|r) - \underbrace{\sum_{rs} P(s, r) \log P(s)}$$

Shannon's "Source Rate" $H(X)$

Goal Maximize G .

$P(s, r)$ is fixed, but unknown (in practice, estimate it, but here, assume known)

$a(s|r)$ is the Gambling Strategy, which the gambler has control over! The gambler chooses her strategy!

So, how can we choose the gambling strategy $a(s|r)$ to maximize G ?

$$\text{Compute } \underset{a}{\operatorname{argmax}} G = \underset{a}{\operatorname{argmax}} \sum_{rs} P(s, r) \log a(s|r)$$

$$= \underset{a}{\operatorname{argmax}} \sum_{rs} q_r(r) q_s(s|r) \log a(s|r)$$

$q_r(r)$ = probability that you receive symbol r

$q_s(s|r)$ = probability that outcome s occurs given that you received symbol r

$$= \underset{a}{\operatorname{argmax}} \sum_r q(r) \sum_s q(s|r) \log a(s|r)$$

Equivalently, \forall r, $\underset{a}{\operatorname{argmax}} \sum_s q(s|r) \log a(s|r)$

- For notational convenience, we may drop the r and write $\underset{a}{\operatorname{argmax}} \sum_s q(s) \log a(s)$, knowing there is an implicit conditional r . Note that this is as if there is no information channel at all!

Task

$$\begin{aligned} & \text{Maximize} && f(a_1, \dots, a_n) = \sum_{s=1}^n q_s \log a_s \\ & \text{Subject to} && g(a_1, \dots, a_n) = \sum_{s=1}^n a_s = 1 \end{aligned}$$

"gambler bets entire capital" constraint.

↳ Lagrange Multipliers!

Treat \vec{q} as known, and find the maximizing gambling strategy \vec{a} .

$$\vec{\nabla} f = \left(\frac{a_1}{a_1}, \dots, \frac{a_n}{a_n} \right)$$

$$\vec{\nabla} g = (1, \dots, 1)$$

$$\vec{\nabla} f = \lambda \vec{\nabla} g \Rightarrow$$

$$\left(\frac{a_1}{a_1}, \dots, \frac{a_n}{a_n} \right) = \lambda (1, \dots, 1)$$

$$\Rightarrow \lambda = \frac{a_1}{a_1} = \dots = \frac{a_n}{a_n}$$

$$\text{well, } \frac{q_1}{a_1} = \frac{q_S}{a_S} \Rightarrow a_S = a_1, \frac{q_S}{a_1} \quad \forall s$$

$$\begin{aligned} \text{Constraint} \quad 1 &= \sum a_s = \frac{a_1}{q_1} (\sum q_s) = \frac{a_1}{q_1}, \\ \Rightarrow a_1 &= q_1 \Rightarrow \boxed{a_s = q_s \quad \forall s} \end{aligned}$$

We know this critical point is a Maximizer because

$$\begin{cases} f(a_1=q_1, \dots, a_n=q_n) = \sum_k q_k \log q_k \in \mathbb{R} \\ f(a_1=1, a_2=0, \dots, a_n=0) = -\infty \end{cases}$$

- Re-incorporating the received symbols r , we have found the Gambling strategy that maximizes G_f , in the fair odds case:

$$a(S|r) = q(S|r) = \frac{p(S,r)}{q(r)} = \frac{p(S,r)}{\sum_k p(k,r)}$$

Therefore,

$$\begin{aligned} G_{\max} &= \sum_{rs} p(s,r) \log \underbrace{a(S|r)}_{= q(S|r)} - \sum_{rs} p(s,r) \log p(s) \\ &= -H(X|Y) + H(X) \end{aligned}$$

"Shannon's Rate of Transmission"

② case: Unfair Odds, No Rake/Vigorish

No Rake

$$\sum \frac{1}{\alpha_s} = 1$$

unfair odds α_s is not necessarily $\frac{1}{P(s)}$

Without loss of generality we still have

$$1 = \sum_s a(s|r) \quad \forall r$$

Since the gambler can still hold back money by betting in proportion to the $1/\alpha_s$, since no Rake.

As before,

$$G = \sum_{rs} P(s,r) \log [\alpha_s a(s|r)]$$

$$= \underbrace{\sum_{rs} P(s,r) \log a(s|r)} + \underbrace{\sum_s P(s) \log \alpha_s}_{}.$$

To maximize G is to maximize this,

which we already know is when

$$a(s|r) = q(s|r). \quad \text{It is the same since}$$

this term doesn't depend on the odds α_s

$$:= H(\alpha)$$

$$G_{\max} = H(\alpha) - H(X|Y)$$

Interesting Facts

① G is maximized as before by setting $a(s|r) = q(s|r)$. So, the gambler ignores the posted odds in placing his bets!

Prove

↑ this please,

② Minimizing $H(\alpha) = \sum_s p(s) \log \alpha_s$ subject to $\sum_s \alpha_s = 1$ occurs when $\alpha_s = \frac{1}{p(s)}$, $H(\alpha) = H(\alpha)$

So, any deviation from the fair odds helps the gambler!

③.

$$\left. \begin{array}{l} G_{\max} = H(\alpha) - H(X) \\ G_{\max} = H(\alpha) - H(X|Y) \\ R = H(X) - H(X|Y) \\ = \text{increase in } G_{\max} \text{ due to Information Gain.} \end{array} \right\}$$

without information channel (ignore r)
with info
"Shannon's Rate of Transmission"