Late Stage Review

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1 Introduction

The goal of my research is to eventually investigate sparse spectral methods on triangulations of the sphere as an alternative to the spherical harmonics approach currently in use at the European Centre for Medium-range Weather Forecasts (ECMWF) in their weather and climate model [17]. The aim would be to preserve the predictive skill of the current approach whilst avoiding the parallel scalability bottleneck from the global spectral transform, which is expected to inhibit the future performance of the ECMWF model.

This document is to provide a brief outline of my work to date, with a closer look at our most recent area of research (solving PDEs on the half disk using sparse spectral methods). The work outlined in this report serves as initial steps to pave the way to solving PDEs on spherical triangles. All code used thus far and from hence forth will mainly be written in the Julia programming language.

2 Professional Skills Development Requirement

I have met the initial part of the professional skills development requirement which is to attend a minimum of 2 further professional development workshops by the Late Stage Review (LSR). The 3 further courses taken are:

- Maximising Management Skills 1: Becoming an Effective Researcher
- Maximising Management Skills 2: Time Management Strategies for your PhD
- Maximising Management Skills 4: Put Project Management into Action

The 100 hours of PG courses/activities have been met by virtue of the courses taken in the first year of my studies with the Mathematics of Planet Earth CDT.

3 Orthogonal polynomials on the sphere (spherical harmonics)

The first step was to develop a small code library so that we can use spherical harmonics as orthogonal polynomials in three variables x, y, z to evaluate functions on the sphere.

We can the define the spherical harmonics, orthogonal on the unit sphere as [7, 5], [10, 14.30.1, 14.3.1, 14.3.6]:

$$
\begin{aligned}
Y_l^m(\theta, \varphi) &:= c_l^m e^{im\varphi} P_l^m(\cos\theta) && (1)\\
&= \alpha_l^m (1 - (\cos\theta)^2)^{\frac{|m|}{2}} e^{im\varphi} P_{l-|m|}^{(|m|,|m|)}(\cos\theta), \quad \text{where } 0 \le |m| \le l,\ l \in \mathbb{N}_0. && (2)
\end{aligned}
$$

Note that we can then express Y_l^m in terms of x, y, z instead of θ, φ by noting that $\cos\theta = z$ and that $e^{im\varphi}$ can be expressed in terms of x, y, z for any $m \in \mathbb{Z}$. Indeed, they are polynomials in x, y, z which we denote $Y_l^m(x, y, z)$. They span all polynomials modulo the ideal generated by $x^2 + y^2 + z^2 - 1$. While this is equivalent to standard techniques in the litrature concerning spherical harmonics, we approach this with our goal in mind of developing sparse spectral methods on spherical triangles. Thus, we try to use language and formulations that will translate as we move through our steps towards that goal.

[6] (p75-85) provided a framework for gaining a recurrence relation for each $\mathbb{P}_{l+1}$ in terms of the previous two sub-vectors $\mathbb{P}_l$ and $\mathbb{P}_{l-1}$, utilising the sub matrices that make up the block-tridiagonal Jacobi operator matrices for multiplication by x, y and z. Clenshaw's algorithm then gives us a way of evaluating a function given its expansion in the spherical harmonic basis (up to a finite order). This provides an optimal complexity method for evaluating spherical harmonic expansions; an alternative to using the explicit expressions that may have benefits, though further investigation is required.

3.1 Tangent space

By noting that the tangent space is then spanned by $\{\nabla Y_l^m, \nabla^\perp Y_l^m\}$, and that these gradients and perpendicular gradients are orthogonal polynomials themselves, we can use the same ideas to create a framework for functions in the tangent space as expansions in this OP basis. Note that the perpendicular gradient is related to the regular surface gradient by

$$\nabla^\perp Y_l^m(x, y, z) = \hat{\boldsymbol{k}} \times \nabla Y_l^m(x, y, z), \tag{3}$$

where $\hat{\boldsymbol{k}}$ is the unit vector normal to the surface of the sphere at the point (x, y, z), i.e. as we are looking at the unit sphere, $\hat{\boldsymbol{k}}$ is simply given by

$$\hat{\boldsymbol{k}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{4}$$

For any scalar spherical harmonic (SSH) Y_l^m, the triad of ∇Y_l^m, $\hat{\boldsymbol{k}} \times \nabla Y_l^m$, $\hat{\boldsymbol{k}} Y_l^m$ can be referred to as vector spherical harmonics [1].

Since the spherical harmonics are a basis for the surface of the sphere, and the tangent space of the sphere is spanned by the gradient and perpendicular gradient of a scalar function, we have that the gradients and perpendicular gradients of the spherical harmonics are a basis for the tangent space, namely ∇Y_l^m, $\nabla^\perp Y_l^m$.

Importantly, ∇Y_l^m, $\nabla^\perp Y_l^m$ are in fact vector-valued orthogonal polynomials, that span all vector-valued polynomials modulo the vanishing ideal $\{\boldsymbol{p} \in (\Pi_3)^3 : \hat{\boldsymbol{k}} \cdot \boldsymbol{p} = 0\}$. A simple calculation shows that such orthogonal polynomials must still have block-tridiagonal Jacobi operators, as multiplication by x, y, or z remains inside the ideal. In the following, we determine the coefficients of these Jacobi operators, as well as other important operators needed.

3.1.1 Spin-1 tensor spherical harmonics

The gradient and perpendicular gradient of a spherical harmonic $Y_l^m(x, y, z)$ can be expressed in terms of spin-1 tensor spherical harmonics, which in turn can each be expressed as a vector-weighted sum of spherical harmonics.

For any $l \in \mathbb{N}_0$, $m \in \mathbb{Z}$ s.t. $0 \leq |m| \leq l$ we have that

$$\nabla Y_l^m = \beta_{l,-1} \boldsymbol{\mathcal{Y}}_{l,m}^{l-1,1} + \beta_{l,1} \boldsymbol{\mathcal{Y}}_{l,m}^{l+1,1}, \tag{5}$$

$$\nabla^\perp Y_l^m = \beta_{l,0} \boldsymbol{\mathcal{Y}}_{l,m}^{l,1}, \tag{6}$$

where

$$\beta_{l,-1} := (l+1) \left(\frac{l}{2l+1} \right)^{\frac{1}{2}}, \quad \beta_{l,0} := i \left(l(l+1) \right)^{\frac{1}{2}}, \quad \beta_{l,1} := l \left(\frac{l+1}{2l+1} \right)^{\frac{1}{2}}. \tag{7}$$

We can then gain similar (vector analogue) derivations for recurrences, Jacobi matrices and the Clenshaw algorithm implementation for the tangent space vector-valued orthogonal polynomial basis.

The aim is to, given a PDE say, work with the coefficients of the functions involved (coefficients of their expansion in the OP bases), reframe the PDE in terms of the OP coefficients utilising a suitable timestepping method. For derivative, gradient, divergence etc. operators, we can formulate matrix operators to act on the coefficients, thus making our system to simulate a simple matrix vector system.

3.2 Example: linear shallow water equations

Let $\mathbf{u}(x, y, z)$ be the tangential velocity of the flow and $h(x, y, z)$ be the height of the water from some constant reference height $\mathcal{H}$. The linear SWEs are

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + f\hat{\boldsymbol{k}} \times \mathbf{u} - \nabla h = \mathbf{0} \\ \frac{\partial h}{\partial t} + \mathcal{H}\nabla \cdot \mathbf{u} = 0 \end{cases} \tag{8}$$

where $f = 2\Omega\cos(\theta) = 2\Omega z$ is the Coriolis parameter (note that the operator on the coefficients vector for multiplication by this parameter would be $F = 2\Omega(J^z_\nabla)^T$).

Using backward Euler:

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \Delta t \left(\nabla h_{n+1} - f\hat{\boldsymbol{k}} \times \mathbf{u}_{n+1}\right) \tag{9}$$

$$h_{n+1} = h_n - \Delta t\, \mathcal{H}\nabla \cdot \mathbf{u}_{n+1} \tag{10}$$

Then using the operators on the coefficient vectors, the timestepping method is:

$$\mathbf{u}^c_{n+1} = \mathbf{u}^c_n + \Delta t \left(G\mathbf{h}^c_{n+1} - FK\mathbf{u}^c_{n+1}\right) \tag{11}$$

$$\mathbf{h}^c_{n+1} = \mathbf{h}^c_n - \Delta t\, \mathcal{H}D\mathbf{u}^c_{n+1} \tag{12}$$

$\iff$

$$\mathbf{u}^c_{n+1} = \left(I + \Delta t FK + \Delta t^2\mathcal{H}GD\right)^{-1} \left(\mathbf{u}^c_n + \Delta t\, G\mathbf{h}^c_n\right) \tag{13}$$

$$\mathbf{h}^c_{n+1} = \mathbf{h}^c_n - \Delta t\, \mathcal{H}D\mathbf{u}^c_{n+1} \tag{14}$$

where F, K, G, D are all operators corresponding to multiplying by the Coriolis parameter, outward normal vector cross product, gradient and divergence operations respectively. It is important to note that these can be easily constructed due to the relationships between the vector spherical harmonics and scalar spherical harmonics we have outlined, and have a sparse Block-Banded structure.

These ideas will also be used to gain similar relationships for the circle arc, half disk, and spherical triangle orthogonal polynomials.

4 Orthogonal polynomials on the circle arc

As a stepping stone to working on spherical triangles, we will try to construct orthogonal polynomials on a circle arc, given by $x \in [h, 1]$, $y = (1 - x^2)^{\frac{1}{2}}$. We will use the Lanczos algorithm as implemented in the ApproxFun package [11] to obtain 1D OPs $\{T^h_n\}$ (orthogonal wrt $w(x) = (1 - x^2)^{-\frac{1}{2}}$) and $\{U^h_n\}$ (orthogonal wrt $w(x) = (1 - x^2)^{\frac{1}{2}}$) on the interval

$[h, 1]$, and create our 2D OPs according to [8]:

$$\mathbb{P}_0(x,y) := T_0^h(x) \equiv T_0, \qquad \mathbb{P}_n(x,y) := \begin{bmatrix} T_n^h(x) \\ y\, U_{n-1}^h(x) \end{bmatrix} \in \mathbb{R}^2 \quad \forall n \in \mathbb{N}, \qquad \mathbb{P} := \begin{bmatrix} \mathbb{P}_0 \\ \hline \mathbb{P}_1 \\ \hline \mathbb{P}_2 \\ \hline \vdots \end{bmatrix}. \tag{15}$$

5 Orthogonal Polynomials on the disk-slice (2D)

Sparse spectral methods for solving partial differential equations have been derived in recent years using hierarchies of classical orthogonal polynomials on intervals, disks, and triangles. In the work in this section we extend this methodology to a hierarchy of non-classical orthogonal polynomials on disk slices and trapeziums. This builds on the observation that sparsity is guaranteed due to the boundary being defined by an algebraic curve, and that the entries of partial differential operators can be determined using formulae in terms of (non-classical) univariate orthogonal polynomials. We apply the framework to solving the Poisson, variable coefficient Helmholtz, and Biharmonic equations. For now, we have focused on constant Dirichlet boundary conditions, as well as zero Dirichlet and Neumann boundary conditions, with other types of boundary conditions requiring future work.

5.1 Explicit construction

Here, we have developed sparse spectral methods for solving linear partial differential equations on a special class of geometries that includes disk slices and trapeziums. More precisely, we consider the solution of partial differential equations on the domain

$$\Omega := \{(x,y) \in \mathbb{R}^2 \quad | \quad \alpha < x < \beta,\ \gamma\rho(x) < y < \delta\rho(x)\} \tag{16}$$

where either of the following conditions hold:

Condition 1. *ρ is a degree 1 polynomial.*

Condition 2. *ρ is the square root of a non-negative degree ≤ 2 polynomial, $-\gamma = \delta > 0$.*

For simplicity of presentation we focus on the disk-slice, where $\rho(x) = \sqrt{1-x^2}$, $(\alpha, \beta) \subset (0, 1)$, and $(\gamma, \delta) = (-1, 1)$, and discuss an extension to other the end-disk-slice in the appendix (note we omit detailed discussion on the extension to trapezium in this document).

We can construct 2D orthogonal polynomials on Ω from 1D orthogonal polynomials on the intervals $[\alpha, \beta]$ and $[\gamma, \delta]$.

Proposition 1 ([6, p55–56])**.** *Let $w_1 : (\alpha, \beta) \to \mathbb{R}$, $w_2 : (\gamma, \delta) \to \mathbb{R}$ be weight functions with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, and let $\rho : (\alpha, \beta) \to (0, \infty)$ be such that either Condition 1 or Condition 2 with w_2 being an even function hold. $\forall$, $n = 0, 1, 2, \ldots$, let $\{p_{n,k}\}$ be polynomials orthogonal with respect to the weight $\rho(x)^{2k+1} w_1(x)$ where $0 \le k \le n$, and $\{q_n\}$ be polynomials orthogonal with respect to the weight $w_2(x)$. Then the 2D polynomials defined on Ω*

$$H_{n,k}(x, y) := p_{n-k,k}(x)\, \rho(x)^k\, q_k\left(\frac{y}{\rho(x)}\right) \qquad \textit{for} \qquad 0 \le k \le n,\ n = 0, 1, 2, \ldots$$

are orthogonal polynomials with respect to the weight $W(x, y) := w_1(x)\, w_2\left(\frac{y}{\rho(x)}\right)$ on Ω.

For disk slices and trapeziums, we specialise Proposition 1 in the following definition. First we introduce notation for two families of univariate OPs.

Definition 1. *Let $w_R^{(a,b,c)}(x)$ and $w_P^{(a,b)}(x)$ be two weight functions on the intervals (α, β) and (γ, δ) respectively, given by:*

$$\begin{cases} w_R^{(a,b,c)}(x) & := (\beta - x)^a\, (x - \alpha)^b\, \rho(x)^c \\ w_P^{(a,b)}(x) & := (\delta - x)^a\, (x - \gamma)^b \end{cases}$$

and define the associated inner products by:

$$\langle p,\ q \rangle_{w_R^{(a,b,c)}} := \frac{1}{\omega_R^{(a,b,c)}} \int_\alpha^\beta p(x)\, q(x)\, w_R^{(a,b,c)}(x)\, \mathrm{d}x \tag{17}$$

$$\langle p, q \rangle_{w_P^{(a,b)}} := \frac{1}{\omega_P^{(a,b)}} \int_\gamma^\delta p(y)\, q(y)\, w_P^{(a,b)}(y)\, \mathrm{d}y \tag{18}$$

where

$$\omega_R^{(a,b,c)} := \int_\alpha^\beta w_R^{(a,b,c)}(x)\, \mathrm{d}x, \quad \omega_P^{(a,b)} := \int_\gamma^\delta w_P^{(a,b)}(y)\, \mathrm{d}y. \tag{19}$$

Denote the three-parameter family of orthonormal polynomials on $[\alpha, \beta]$ by $\{R_n^{(a,b,c)}\}$, orthonormal with respect to the inner product defined in (17), and the two-parameter family of orthonormal polynomials on $[\gamma, \delta]$ by $\{P_n^{(a,b)}\}$, orthonormal with respect to the inner product defined in (18).

Definition 2. *Define the four-parameter 2D orthogonal polynomials via:*

$$H_{n,k}^{(a,b,c,d)}(x, y) := R_{n-k}^{(a,b,c+d+2k+1)}(x)\, \rho(x)^k\, P_k^{(d,c)}\left(\frac{y}{\rho(x)}\right), \quad (x, y) \in \Omega,$$

$\{H_{n,k}^{(a,b,c,d)}\}$ are orthogonal with respect to the weight

$$W^{(a,b,c,d)}(x,y) := w_R^{(a,b,c+d)}(x)\, w_P^{(d,c)}\left(\frac{y}{\rho(x)}\right), \quad (x,y) \in \Omega,$$

assuming that either Condition 1 or Condition 2 with $w_P^{(a,b)}$ being an even function (i.e. $a = b$, and we can hence denote the weight as $w_P^{(a)}(x) = w_P^{(a,a)}(x) = (\delta - x^2)^a$) hold. That is,

$$\left\langle H_{n,k}^{(a,b,c,d)},\, H_{m,j}^{(a,b,c,d)} \right\rangle_{W^{(a,b,c,d)}} = \omega_R^{(a,b,c+d+2k+1)}\, \omega_P^{(d,c)}\, \delta_{n,m}\, \delta_{k,j},$$

where for $f, g : \Omega \to \mathbb{R}$ the inner product is defined as

$$\langle f,\, g \rangle_{W^{(a,b,c,d)}} := \iint_\Omega f(x,y)\, g(x,y)\, W^{(a,b,c,d)}(x,y)\, \mathrm{d}y\, \mathrm{d}x.$$

We can see that they are indeed orthogonal using the change of variable $t = \frac{y}{\rho(x)}$, for the following normalisation:

$$\begin{aligned}
&\left\langle H_{n,k}^{(a,b,c,d)},\, H_{m,j}^{(a,b,c,d)} \right\rangle_{W^{(a,b,c,d)}} \qquad (20)\\
&\quad = \iint_\Omega \Big[R_{n-k}^{(a,b,c+d+2k+1)}(x)\, R_{m-j}^{(a,b,c+d+2j+1)}(x)\, \rho(x)^{k+j} \\
&\qquad\qquad \cdot P_k^{(d,c)}\left(\frac{y}{\rho(x)}\right)\, P_j^{(d,c)}\left(\frac{y}{\rho(x)}\right)\, W^{(a,b,c,d)}(x,y) \Big]\, \mathrm{d}y\, \mathrm{d}x \\
&\quad = \Big(\int_\alpha^\beta R_{n-k}^{(a,b,c+d+2k+1)}(x)\, R_{m-j}^{(a,b,c+d+2j+1)}(x)\, w_R^{(a,b,c+d+k+j+1)}(x)\, \mathrm{d}x \Big) \\
&\qquad\qquad \cdot \Big(\int_\gamma^\delta P_k^{(d,c)}(t)\, P_j^{(d,c)}(t)\, w_P^{(d,)}(t)\, \mathrm{d}t \Big) \\
&\quad = \omega_P^{(d,c)}\, \delta_{k,j} \int_\alpha^\beta R_{n-k}^{(a,b,c+d+2k+1)}(x)\, R_{m-k}^{(a,b,c+d+2k+1)}(x)\, w_R^{(a,b,c+d+2k+1)}(x)\, \mathrm{d}x \\
&\quad = \omega_R^{(a,b,c+d+2k+1)}\, \omega_P^{(d,c)}\, \delta_{n,m}\, \delta_{k,j}. \qquad (21)
\end{aligned}$$

For the disk-slice, the weight $W^{(a,b,c)}(x,y) = (\beta - x)^a\, (x - \alpha)^b\, (1 - x^2 - y^2)^c$ results from setting:

$$\begin{cases} (\alpha, \beta) & \subset (0,1) \\ (\gamma, \delta) & := (-1, 1) \\ \rho(x) & := (1 - x^2)^{\frac{1}{2}} \end{cases}$$

so that

$$\begin{cases} w_R^{(a,b,c)}(x) := (\beta - x)^a \, (x-\alpha)^b \, \rho(x)^c \\ w_P^{(c)}(x) := (1-x)^c \, (1+x)^c = (1-x^2)^c. \end{cases}$$

Note here we can simply remove the need for including a fourth parameter d. The 2D OPs orthogonal with respect to the weight above on the disk-slice Ω are then given by:

$$H_{n,k}^{(a,b,c)}(x,y) := R_{n-k}^{(a,b,2c+2k+1)}(x) \, \rho(x)^k \, P_k^{(c,c)}\left(\frac{y}{\rho(x)}\right), \quad (x,y) \in \Omega \tag{22}$$

In this case the weight $w_P(x)$ is an ultraspherical weight, and the corresponding OPs are the normalized Jacobi polynomials $\{P_n^{(b,b)}\}$, while the weight $w_R(x)$ is non-classical (it is in fact semi-classical, and is equivalent to a generalized Jacobi weight [9, §5]).

5.2 Jacobi matrices

We can express the three-term recurrences associated with $R_n^{(a,b,c)}$ and $P_n^{(d,c)}$ as

$$xR_n^{(a,b,c)}(x) = \beta_n^{(a,b,c)} R_{n+1}^{(a,b,c)}(x) + \alpha_n^{(a,b,c)} R_n^{(a,b,c)}(x) + \beta_{n-1}^{(a,b,c)} R_{n-1}^{(a,b,c)}(x) \tag{23}$$

$$yP_n^{(d,c)}(y) = \delta_n^{(d,c)} P_{n+1}^{(d,c)}(y) + \gamma_n^{(d,c)} P_n^{(d,c)}(y) + \delta_{n-1}^{(d,c)} P_{n-1}^{(d,c)}(y). \tag{24}$$

Of course, for the disk-slice case, we have that $c = d$ and $\gamma_n^{(c,c)} = 0 \quad \forall n = 0, 1, 2, \ldots$. We can use (23) and (24) to determine the 2D recurrences for $H_{n,k}^{(a,b,c,d)}(x,y)$. Importantly, we can deduce sparsity in the recurrence relationships:

Lemma 1. *$H_{n,k}^{(a,b,c,d)}(x,y)$ satisfy the following 3-term recurrences:*

$$\begin{aligned} xH_{n,k}^{(a,b,c,d)}(x,y) &= \alpha_{n,k,1}^{(a,b,c,d)} \, H_{n-1,k}^{(a,b,c,d)}(x,y) + \alpha_{n,k,2}^{(a,b,c,d)} \, H_{n,k}^{(a,b,c,d)}(x,y) + \alpha_{n+1,k,1}^{(a,b,c,d)} \, H_{n+1,k}^{(a,b,c)}(x,y), \\ yH_{n,k}^{(a,b,c,d)}(x,y) &= \beta_{n,k,1}^{(a,b,c,d)} \, H_{n-1,k-1}^{(a,b,c,d)}(x,y) + \beta_{n,k,2}^{(a,b,c,d)} \, H_{n-1,k}^{(a,b,c,d)}(x,y) + \beta_{n,k,3}^{(a,b,c,d)} \, H_{n-1,k+1}^{(a,b,c,d)}(x,y) \\ &\quad + \beta_{n,k,4}^{(a,b,c,d)} \, H_{n,k-1}^{(a,b,c,d)}(x,y) + \beta_{n,k,5}^{(a,b,c,d)} \, H_{n,k}^{(a,b,c,d)}(x,y) + \beta_{n,k,6}^{(a,b,c,d)} \, H_{n,k+1}^{(a,b,c,d)}(x,y) \\ &\quad + \beta_{n,k,7}^{(a,b,c,d)} \, H_{n+1,k-1}^{(a,b,c,d)}(x,y) + \beta_{n,k,8}^{(a,b,c,d)} \, H_{n+1,k}^{(a,b,c,d)}(x,y) + \beta_{n,k,9}^{(a,b,c,d)} \, H_{n+1,k+1}^{(a,b,c,d)}(x,y), \end{aligned}$$

for $(x, y) \in \Omega$*, where*

$$
\begin{aligned}
\alpha_{n,k,1}^{(a,b,c,d)} &:= \beta_{n-k-1}^{(a,b+k+\frac{1}{2})} \\
\alpha_{n,k,2}^{(a,b,c,d)} &:= \alpha_{n-k}^{(a,b+k+\frac{1}{2})} \\
\beta_{n,k,1}^{(a,b,c,d)} &:= \delta_{k-1}^{(d,c)} \left\langle R_{n-k}^{(a,b,c+d+2k+1)},\ R_{n-k}^{(a,b,c+d+2k-1)} \right\rangle_{w_R^{(a,b,c+d+2k+1)}} \\
\beta_{n,k,2}^{(a,b,c,d)} &:= \gamma_{k}^{(d,c)} \left\langle R_{n-k}^{(a,b,c+d+2k+1)},\ \rho(x) R_{n-k-1}^{(a,b,c+d+2k+1)} \right\rangle_{w_R^{(a,b,c+d+2k+1)}} \\
\beta_{n,k,3}^{(a,b,c,d)} &:= \delta_{k}^{(d,c)} \left\langle R_{n-k}^{(a,b,c+d+2k+1)}, R_{n-k-2}^{(a,b,c+d+2k+3)} \right\rangle_{w_R^{(a,b,c+d+2k+3)}} \\
\beta_{n,k,4}^{(a,b,c,d)} &:= \delta_{k-1}^{(d,c)} \left\langle R_{n-k}^{(a,b,c+d+2k+1)},\ R_{n-k+1}^{(a,b,c+d+2k-1)} \right\rangle_{w_R^{(a,b,c+d+2k+1)}} \\
\beta_{n,k,5}^{(a,b,c,d)} &:= \gamma_{k}^{(d,c)} \left\langle R_{n-k}^{(a,b,c+d+2k+1)},\ \rho(x) R_{n-k}^{(a,b,c+d+2k+1)} \right\rangle_{w_R^{(a,b,c+d+2k+1)}} \\
\beta_{n,k,6}^{(a,b,c,d)} &:= \delta_{k}^{(d,c)} \left\langle R_{n-k}^{(a,b,c+d+2k+1)}, R_{n-k-1}^{(a,b,c+d+2k+3)} \right\rangle_{w_R^{(a,b,c+d+2k+3)}} \\
\beta_{n,k,7}^{(a,b,c,d)} &:= \delta_{k-1}^{(d,c)} \left\langle R_{n-k}^{(a,b,c+d+2k+1)},\ R_{n-k+2}^{(a,b,c+d+2k-1)} \right\rangle_{w_R^{(a,b,c+d+2k+1)}} \\
\beta_{n,k,8}^{(a,b,c,d)} &:= \gamma_{k}^{(d,c)} \left\langle R_{n-k}^{(a,b,c+d+2k+1)},\ \rho(x) R_{n-k+1}^{(a,b,c+d+2k+1)} \right\rangle_{w_R^{(a,b,c+d+2k+1)}} \\
\beta_{n,k,9}^{(a,b,c,d)} &:= \delta_{k}^{(d,c)} \left\langle R_{n-k}^{(a,b,c+d+2k+1)}, R_{n-k}^{(a,b,c+d+2k+3)} \right\rangle_{w_R^{(a,b,c+d+2k+3)}}.
\end{aligned}
$$

Proof. The 3-term recurrence for multiplication by x follows from equation (23). For the recurrence for multiplication by y, since $\{H_{m,j}^{(a,b,c,d)}\}$ for $m = 0, \ldots, n+1$, $j = 0, \ldots, m$ is an orthogonal basis for any degree $n+1$ polynomial, we can expand $y\, H_{n,k}^{(a,b,c,d)}(x, y) = \sum_{m=0}^{n+1} \sum_{j=0}^{m} c_{m,j}\, H_{m,j}^{(a,b,c,d)}(x, y)$. These coefficients are given by

$$c_{m,j} = \left\langle y\, H_{n,k}^{(a,b,c,d)}, H_{m,j}^{(a,b,c,d)} \right\rangle_{W^{(a,b,c,d)}} \left\| H_{m,j}^{(a,b,c,d)} \right\|_{W^{(a,b,c,d)}}^{-2}.$$

Recall from equation (21) that $\left\| H_{m,j}^{(a,b,c,d)} \right\|_{W^{(a,b,c,d)}}^{2} = \omega_R^{(a,b,c+d+2j+1)}\, \omega_P^{(d,c)}$. Then for $m =$

$0,\dots,n+1$, $j=0,\dots,m$, using the change of variable $t=\frac{y}{\rho(x)}$:

$$\begin{aligned}
&\left\langle yH_{n,k}^{(a,b,c,d)}, H_{m,j}^{(a,b,c,d)}\right\rangle_{W^{(a,b,c,d)}} \\
&\quad= \iint_\Omega H_{n,k}^{(a,b,c,d)}(x,y)\,H_{m,j}^{(a,b,c,d)}(x,y)\,y\,W^{(a,b,c,d)}(x,y)\,dy\,dx \\
&\quad= \Big(\int_\alpha^\beta R_{n-k}^{(a,b,c+d+2k+1)}(x)\,R_{m-j}^{(a,b,c+d+2j+1)}(x)\,\rho(x)^{k+j+2}\,w_R^{(a,b,c+d)}(x)\,\mathrm{d}x\Big) \\
&\qquad\quad\cdot\Big(\int_\gamma^\delta P_k^{(d,c)}(t)\,P_j^{(d,c)}(t)\,t\,w_P^{(d,c)}(t)\,\mathrm{d}t\Big) \\
&\quad= \Big(\int_\alpha^\beta R_{n-k}^{(a,b,c+d+2k+1)}(x)\,R_{m-j}^{(a,b,c+d+2j+1)}(x)\,w_R^{(a,b,c+d+k+j+2)}(x)\,\mathrm{d}x\Big) \\
&\qquad\quad\cdot\Big(\int_\gamma^\delta P_k^{(d,c)}(t)\,P_j^{(d,c)}(t)\,t\,w_P^{(d,c)}(t)\,\mathrm{d}t\Big)
\end{aligned}$$

$$=\begin{cases}
\delta_k^{(d,c)}\,\omega_P^{(d,c)}\,\omega_R^{(a,b,c+d+2k+3)}\,\left\langle R_{n-k}^{(a,b,c+d+2k+1)}, R_{m-k-1}^{(a,b,c+d+2k+3)}\right\rangle_{w_R^{(a,b,c+d+2k+3)}} & \text{if } j=k+1\\
\gamma_k^{(d,c)}\,\omega_P^{(d,c)}\,\omega_R^{(a,b,c+d+2k+1)}\,\left\langle R_{n-k}^{(a,b,c+d+2k+1)},\ \rho(x)R_{m-k}^{(a,b,c+d+2k+1)}\right\rangle_{w_R^{(a,b,c+d+2k+1)}} & \text{if } j=k\\
\delta_{k-1}^{(d,c)}\,\omega_P^{(d,c)}\,\omega_R^{(a,b,c+d+2k-1)}\,\left\langle R_{n-k}^{(a,b,c+d+2k-1)},\ \rho(x)^2R_{m-k+1}^{(a,b,c+d+2k-1)}\right\rangle_{w_R^{(a,b,c+d+2k-1)}} & \text{if } j=k-1\\
0 & \text{otherwise}
\end{cases}$$

where, by orthogonality,

$$\begin{aligned}
\left\langle R_{n-k}^{(a,b,c+d+2k+1)}, R_{m-k-1}^{(a,b,c+d+2k+3)}\right\rangle_{w_R^{(a,b,c+d+2k+3)}} &= 0 \quad\text{for}\quad m<n-1,\\
\left\langle R_{n-k}^{(a,b,c+d+2k+1)},\ \rho(x)^2R_{m-k+1}^{(a,b,c+d+2k-1)}\right\rangle_{w_R^{(a,b,c+d+2k-1)}} &= 0 \quad\text{for}\quad m<n-1.
\end{aligned}$$

Finally, if Condition 1 holds we have that

$$\left\langle R_{n-k}^{(a,b,c+d+2k+1)},\ \rho(x)R_{m-k}^{(a,b,c+d+2k+1)}\right\rangle_{w_R^{(a,b,c+d+2k+1)}} = 0 \quad\text{for}\quad m<n-1.$$

If Condition 2 holds we have that $\gamma_k^{(d,c)}=\gamma_k^{(c,c)}\equiv 0$ for any k.

□

Three-term recurrences lead to Jacobi operators that correspond to multiplication by x

and y. Define, for $n = 0, 1, 2, \dots$:

$$\mathbb{H}_n^{(a,b,c,d)} := \begin{pmatrix} H_{n,0}^{(a,b,c,d)}(x,y) \\ \vdots \\ H_{n,n}^{(a,b,c,d)}(x,y) \end{pmatrix} \in \mathbb{R}^{n+1}, \qquad \mathbb{H}^{(a,b,c,d)} := \begin{pmatrix} \mathbb{H}_0^{(a,b,c,d)} \\ \mathbb{H}_1^{(a,b,c,d)} \\ \mathbb{H}_2^{(a,b,c,d)} \\ \vdots \end{pmatrix}$$

and set $J_x^{(a,b,c,d)}, J_y^{(a,b,c,d)}$ as the Jacobi matrices corresponding to

$$J_x^{(a,b,c,d)}\, \mathbb{H}^{(a,b,c,d)}(x,y) = x\, \mathbb{H}^{(a,b,c,d)}(x,y), \quad J_y^{(a,b,c,d)}\, \mathbb{H}^{(a,b,c,d)}(x,y) = y\, \mathbb{H}^{(a,b,c,d)}(x,y). \tag{25}$$

The matrices $J_x^{(a,b,c,d)}, J_y^{(a,b,c,d)}$ act on the coefficients vector of a function's expansion in the $\{H_{n,k}^{(a,b,c,d)}\}$ basis. For example, let a, b be general parameters and a function $f(x,y)$ defined on Ω be approximated by its expansion $f(x,y) = \mathbb{H}^{(a,b,c,d)}(x,y)^\top \mathbf{f}$. Then $x\, f(x,y)$ is approximated by $\mathbb{H}^{(a,b,c,d)}(x,y)^\top J_x^{(a,b,c,d)\top} \mathbf{f}$. In other words, $J_x^{(a,b,c,d)\top} \mathbf{f}$ is the coefficients vector for the expansion of the function $(x,y) \mapsto x\, f(x,y)$ in the $\{H_{n,k}^{(a,b,c,d)}\}$ basis. Further, note that $J_x^{(a,b,c,d)}, J_y^{(a,b,c,d)}$ are banded-block-banded matrices:

Definition 3. *A block matrix A with blocks $A_{i,j}$ has block-bandwidths (L, U) if $A_{i,j} = 0$ for $-L \leq j - i \leq U$, and sub-block-bandwidths (λ, μ) if all blocks $A_{i,j}$ are banded with bandwidths (λ, μ). A matrix where the block-bandwidths and sub-block-bandwidths are small compared to the dimensions is referred to as a banded-block-banded matrix.*

For example, $J_x^{(a,b,c,d)}, J_y^{(a,b,c,d)}$ are block-tridiagonal (block-bandwidths $(1,1)$):

$$J_{x/y}^{(a,b,c,d)} = \begin{pmatrix} B_0^{x/y} & A_0^{x/y} & & & & \\ C_1^{x/y} & B_1^{x/y} & A_1^{x/y} & & & \\ & C_2^{x/y} & B_2^{x/y} & A_2^{x/y} & & \\ & & C_3^{x/y} & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}$$

where the blocks themselves are diagonal for $J_x^{(a,b,c,d)}$ (sub-block-bandwidths $(0,0)$),

$$
\begin{aligned}
A_n^x &:= \begin{pmatrix} \alpha_{n+1,0,1}^{(a,b,c,d)} & 0 & \dots & 0 \\ & \ddots & & \vdots \\ & & \alpha_{n+1,n,1}^{(a,b,c,d)} & 0 \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+2)}, \quad n = 0,1,2,\dots \\
B_n^x &:= \begin{pmatrix} \alpha_{n,0,2}^{(a,b,c,d)} & & \\ & \ddots & \\ & & \alpha_{n,n,2}^{(a,b,c,d)} \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)} \quad n = 0,1,2,\dots \\
C_n^x &:= \left(A_n^x\right)^\top \in \mathbb{R}^{(n+1)\times n}, \quad n = 1,2,\dots
\end{aligned}
$$

and tridiagonal for $J_y^{(a,b,c,d)}$ (sub-block-bandwidths $(1,1)$),

$$
\begin{aligned}
A_n^y &:= \begin{pmatrix} \beta_{n,0,8}^{(a,b,c,d)} & \beta_{n,0,9}^{(a,b,c,d)} & & & \\ \beta_{n,1,7}^{(a,b,c,d)} & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{n,n,7}^{(a,b,c,d)} & \beta_{n,n,8}^{(a,b,c,d)} & \beta_{n,n,9}^{(a,b,c,d)} \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+2)}, \quad n = 0,1,2,\dots \\
B_n^y &:= \begin{pmatrix} \beta_{n,0,5}^{(a,b,c,d)} & \beta_{n,0,6}^{(a,b,c,d)} & & \\ \beta_{n,1,4}^{(a,b,c,d)} & \ddots & \ddots & \\ & \ddots & \ddots & \beta_{n,n-1,6}^{(a,b,c,d)} \\ & & \beta_{n,n,4}^{(a,b,c,d)} & \beta_{n,n,5}^{(a,b,c,d)} \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)} \quad n = 0,1,2,\dots \\
C_n^y &:= \begin{pmatrix} \beta_{n,0,2}^{(a,b,c,d)} & \beta_{n,0,3}^{(a,b,c,d)} & & \\ \beta_{n,1,1}^{(a,b,c,d)} & \ddots & \ddots & \\ & \ddots & \ddots & \beta_{n,n-2,3}^{(a,b,c,d)} \\ & & \ddots & \beta_{n,n-1,2}^{(a,b,c,d)} \\ & & & \beta_{n,n,1}^{(a,b,c,d)} \end{pmatrix} \in \mathbb{R}^{(n+1)\times n}, \quad n = 1,2,\dots
\end{aligned}
$$

Note that the sparsity of the Jacobi matrices (in particular the sparsity of the sub-blocks) comes from the natural sparsity of the three-term recurrences of the 1D OPs, meaning that the sparsity is not limited to the specific disk-slice case.

5.3 Building the OPs

We can combine each system in (25) into a block-tridiagonal system:

$$\begin{pmatrix} 1 & & & \\ B_0 - G_0(x,y) & A_0 & & \\ C_1 & B_1 - G_1(x,y) & A_1 & \\ & C_2 & \ddots & \ddots \\ & & \ddots & \end{pmatrix} \mathbb{H}^{(a,b,c,d)}(x,y) = \begin{pmatrix} H_{0,0}^{(a,b,c,d)} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where we note $H_{0,0}^{(a,b,c,d)}(x,y) \equiv R_0^{(a,b,c+d+1)}\, P_0^{(d,c)}$, and for each $n = 0, 1, 2 \dots$,

$$A_n := \begin{pmatrix} A_n^x \\ A_n^y \end{pmatrix} \in \mathbb{R}^{2(n+1)\times(n+2)}, \quad C_n := \begin{pmatrix} C_n^x \\ C_n^y \end{pmatrix} \in \mathbb{R}^{2(n+1)\times n} \quad (n \neq 0),$$
$$B_n := \begin{pmatrix} B_n^x \\ B_n^y \end{pmatrix} \in \mathbb{R}^{2(n+1)\times(n+1)}, \quad G_n(x,y) := \begin{pmatrix} xI_{n+1} \\ yI_{n+1} \end{pmatrix} \in \mathbb{R}^{2(n+1)\times(n+1)}.$$

For each $n = 0, 1, 2 \dots$ let $D_n^\top$ be any matrix that is a left inverse of A_n, i.e. such that $D_n^\top A_n = I_{n+2}$. Multiplying our system by the preconditioner matrix that is given by the block diagonal matrix of the $D_n^\top$'s, we obtain a lower triangular system [6, p78], which can be expanded to obtain the recurrence:

$$\begin{cases} \mathbb{H}_{-1}^{(a,b,c,d)}(x,y) := 0 \\ \mathbb{H}_0^{(a,b,c,d)}(x,y) := H_{0,0}^{(a,b,c,d)} \\ \mathbb{H}_{n+1}^{(a,b,c,d)}(x,y) = -D_n^\top (B_n - G_n(x,y)) \mathbb{H}_n^{(a,b,c,d)}(x,y) - D_n^\top C_n\, \mathbb{H}_{n-1}^{(a,b,c,d)}(x,y), \quad n = 0, 1, 2, \dots . \end{cases}$$

Note that we can define an explicit $D_n^\top$ as follows:

$$D_n^\top := \begin{pmatrix} \frac{1}{\alpha_{n+1,0,1}^{(a,b,c,d)}} & & & & & & & \\ & \ddots & & & & & & \\ & & \ddots & & & & & \\ & & & \frac{1}{\alpha_{n+1,n,1}^{(a,b,c,d)}} & & & & \\ \eta_0 & 0 & \dots & 0 & \eta_1 & \dots & \dots & \eta_{n+1} \end{pmatrix},$$

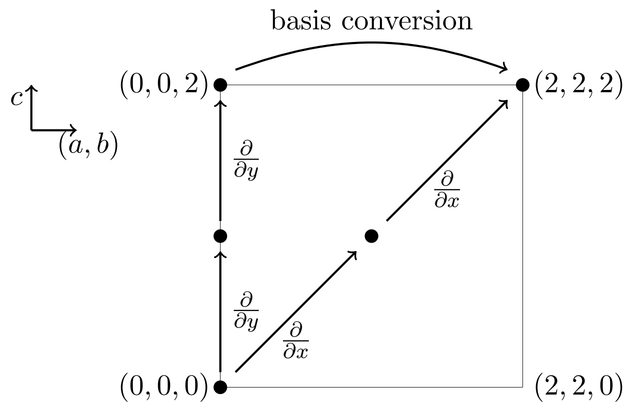

Figure 1: The Laplace operator acting on vectors of $H^{(0,0,0)}_{n,k}$ coefficients has a sparse matrix representation if the range is represented as vectors of $H^{(2,2,2)}_{n,k}$ coefficients. Here, the arrows indicate that the corresponding operation has a sparse matrix representation when the domain is $H^{(a,b,c)}_{n,k}$ coefficients, where (a,b,c) is at the tail of the arrow, and the range is $H^{(\tilde{a},\tilde{b},\tilde{c})}_{n,k}$ coefficients, where $(\tilde{a},\tilde{b},\tilde{c})$ is at the head of the arrow.

where

$$\begin{aligned}
\eta_{n+1} &= \frac{1}{\beta^{(a,b,c,d)}_{n,n,9}}, \\
\eta_n &= -\frac{1}{\beta^{(a,b,c,d)}_{n,n-1,9}}\big(\beta^{(a,b,c,d)}_{n,n,8}\ \eta_{n+1}\big), \\
\eta_j &= \frac{1}{\beta^{(a,b,c,d)}_{n,j-1,9}}\big(\beta^{(a,b,c,d)}_{n,n+j+1,7}\ \eta_{j+2} + \beta^{(a,b,c,d)}_{n,n+j,8}\ \eta_{j+1}\big) \quad \text{for} \quad j = n-1, n-2, \dots, 1, \\
\eta_0 &= \frac{1}{\alpha^{(a,b,c,d)}_{n+1,0,1}}\big(\beta^{(a,b,c,d)}_{n,1,7}\ \eta_2 + \beta^{(a,b,c,d)}_{n,0,8}\ \eta_1\big).
\end{aligned}$$

It follows that we can apply $D_n^\top$ in $O(n)$ complexity, and thereby calculate $\mathbb{H}^{(a,b,c,d)}_0(x,y)$ through $\mathbb{H}^{(a,b,c,d)}_n(x,y)$ in optimal $O(n^2)$ complexity.

For the disk-slice, $\beta^{(a,b,c)}_{n,k,2} = \beta^{(a,b,c)}_{n,k,5} = \beta^{(a,b,c)}_{n,k,8} \equiv 0$ for any n, k.

5.4 Sparse partial differential operators

In this section, we concentrate on the disk-slice case, and simply note that similar arguments apply for the trapezium case. Recall that, for the disk-slice,

$$\Omega := \{(x,y) \in \mathbb{R}^2 \quad | \quad \alpha < x < \beta,\ \gamma\rho(x) < y < \delta\rho(x)\}$$

where

$$\begin{cases} (\alpha, \beta) & \subset (0,1) \\ (\gamma, \delta) & := (-1,1) \\ \rho(x) & := (1-x^2)^{\frac{1}{2}} \end{cases}.$$

The 2D OPs on the disk-slice Ω, orthogonal with respect to the weight

$$\begin{aligned} W^{(a,b,c)}(x,y) &:= w_R^{(a,b,2c)}(x)\, w_P^{(c)}\left(\frac{y}{\rho(x)}\right) \\ &= (\beta - x)^a\, (x-\alpha)^b\, (1-x^2-y^2)^c, \quad (x,y) \in \Omega, \end{aligned}$$

are then given by:

$$H_{n,k}^{(a,b,c)}(x,y) := R_{n-k}^{(a,b,2c+2k+1)}(x)\, \rho(x)^k\, P_k^{(c,c)}\left(\frac{y}{\rho(x)}\right), \quad (x,y) \in \Omega$$

where the 1D OPs $\{R_n^{(a,b,c)}\}$ are orthogonal on the interval (α, β) with respect to the weight

$$w_R^{(a,b,c)}(x) := (\beta - x)^a\, (x - \alpha)^b\, \rho(x)^c$$

and the 1D OPs $\{P_n^{(c,c)}\}$ are orthogonal on the interval $(\gamma, \delta) = (-1,1)$ with respect to the weight

$$w_P^{(c)}(x) := (1-x)^c\, (1+x)^c = (1-x^2)^c.$$

Denote the weighted OPs by

$$\mathbb{W}^{(a,b,c)}(x,y) := W^{(a,b,c)}(x,y)\, \mathbb{H}^{(a,b,c)}(x,y),$$

and recall that a function $f(x,y)$ defined on Ω is approximated by its expansion $f(x,y) = \mathbb{H}^{(a,b,c)}(x,y)^\top \mathbf{f}$.

Definition 4. *Define the operator matrices $D_x^{(a,b,c)}$, $D_y^{(a,b,c)}$, $W_x^{(a,b,c)}$, $W_y^{(a,b,c)}$ according to:*

$$\begin{aligned} \frac{\partial f}{\partial x} &= \mathbb{H}^{(a+1,b+1,c+1)}(x,y)^\top\, D_x^{(a,b,c)}\, \mathbf{f}, \\ \frac{\partial f}{\partial y} &= \mathbb{H}^{(a,b,c+1)}(x,y)^\top\, D_y^{(a,b,c)}\, \mathbf{f}, \\ \frac{\partial}{\partial x}[W^{(a,b,c)}(x,y)\, f(x,y)] &= \mathbb{W}^{(a-1,b-1,c-1)}(x,y)^\top\, W_x^{(a,b,c)}\, \mathbf{f}, \\ \frac{\partial}{\partial y}[W^{(a,b,c)}(x,y)\, f(x,y)] &= \mathbb{W}^{(a,b,c-1)}(x,y)^\top\, W_y^{(a,b,c)}\, \mathbf{f}. \end{aligned}$$

The incrementing and decrementing of parameters as seen here is analogous to other well known orthogonal polynomial families' derivatives, for example the Jacobi polynomials on the interval, as seen in the DLMF [10, (18.9.3)], and on the triangle [12].

Theorem 1. *The operator matrices* $D_x^{(a,b,c)}$, $D_y^{(a,b,c)}$, $W_x^{(a,b,c)}$, $W_y^{(a,b,c)}$ *from Definition 4 are sparse, with banded-block-banded structure. More specifically:*

- $D_x^{(a,b,c)}$ *has block-bandwidths* $(-1,3)$*, and sub-block-bandwidths* $(0,2)$*.*
- $D_y^{(a,b,c)}$ *has block-bandwidths* $(-1,1)$*, and sub-block-bandwidths* $(-1,1)$*.*
- $W_x^{(a,b,c)}$ *has block-bandwidths* $(3,-1)$*, and sub-block-bandwidths* $(2,0)$*.*
- $W_y^{(a,b,c)}$ *has block-bandwidths* $(1,-1)$*, and sub-block-bandwidths* $(1,-1)$*.*

There exist conversion matrix operators that increment/decrement the parameters, transforming the OPs from one (weighted or non-weighted) parameter space to another.

Definition 5. *Define the operator matrices* $T^{(a,b,c)\to(a+1,b+1,c)}$, $T^{(a,b,c)\to(a,b,c+1)}$ *and* $T^{(a,b,c)\to(a+1,b+1,c+1)}$ *for conversion between non-weighted spaces, and* $T_W^{(a,b,c)\to(a-1,b-1,c)}$, $T_W^{(a,b,c)\to(a,b,c-1)}$ *and* $T_W^{(a,b,c)\to(a-1,b-1,c-1)}$ *for conversion between weighted spaces, according to:*

$$
\begin{aligned}
\mathbb{H}^{(a,b,c)}(x,y) &= \left(T^{(a,b,c)\to(a+1,b+1,c)}\right)^\top \mathbb{H}^{(a+1,b+1,c)}(x,y)\\
\mathbb{H}^{(a,b,c)}(x,y) &= \left(T^{(a,b,c)\to(a,b,c+1)}\right)^\top \mathbb{H}^{(a,b,c+1)}(x,y)\\
\mathbb{H}^{(a,b,c)}(x,y) &= \left(T^{(a,b,c)\to(a+1,b+1,c+1)}\right)^\top \mathbb{H}^{(a+1,b+1,c+1)}(x,y)\\
\mathbb{W}^{(a,b,c)}(x,y) &= \left(T_W^{(a,b,c)\to(a-1,b-1,c)}\right)^\top \mathbb{W}^{(a-1,b-1,c)}(x,y)\\
\mathbb{W}^{(a,b,c)}(x,y) &= \left(T_W^{(a,b,c)\to(a,b,c-1)}\right)^\top \mathbb{W}^{(a,b,c-1)}(x,y)\\
\mathbb{W}^{(a,b,c)}(x,y) &= \left(T_W^{(a,b,c)\to(a-1,b-1,c-1)}\right)^\top \mathbb{W}^{(a-1,b-1,c-1)}(x,y).
\end{aligned}
$$

Lemma 2. *The operator matrices in Definition 5 are sparse, with banded-block-banded structure. More specifically:*

- $T^{(a,b,c)\to(a+1,b+1,c)}$ *has block-bandwidth* $(0,2)$*, with diagonal blocks.*
- $T^{(a,b,c)\to(a,b,c+1)}$ *has block-bandwidth* $(0,2)$ *and sub-block-bandwidth* $(0,2)$*.*
- $T^{(a,b,c)\to(a+1,b+1,c+1)}$ *has block-bandwidth* $(0,4)$ *and sub-block-bandwidth* $(0,2)$*.*
- $T_W^{(a,b,c)\to(a-1,b-1,c)}$ *has block-bandwidth* $(2,0)$ *with diagonal blocks.*

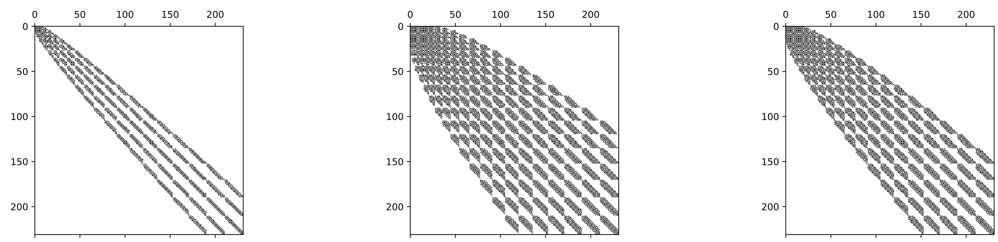

Figure 2: "Spy" plots of (differential) operator matrices, showing their sparsity. Left: the Laplace operator $\Delta_W^{(1,1,1)\to(1,1,1)}$. Centre: the weighted variable coefficient Helmholtz operator $\Delta_W^{(1,1,1)\to(1,1,1)} + k^2\, T^{(0,0,0)\to(1,1,1)}\, V(J_x^{(0,0,0)^\top}, J_y^{(0,0,0)^\top})\, T_W^{(1,1,1)\to(0,0,0)}$ for $v(x,y) = 1-(3(x-1)^2+5y^2)$ and $k=200$. Right: the biharmonic operator ${}_2\Delta_W^{(2,2,2)\to(2,2,2)}$.

- $T_W^{(a,b,c)\to(a,b,c-1)}$ *has block-bandwidth* $(2,0)$ *and sub-block-bandwidth* $(2,0)$.
- $T_W^{(a,b,c)\to(a-1,b-1,c-1)}$ *has block-bandwidth* $(4,0)$ *and sub-block-bandwidth* $(2,0)$.

General linear partial differential operators with polynomial variable coefficients can be constructed by composing the sparse representations for partial derivatives, conversion between bases, and Jacobi operators. As a canonical example, we can obtain the matrix operator for the Laplacian Δ, that will take us from coefficients for expansion in the weighted space

$$\mathbb{W}^{(1,1,1)}(x,y) = W^{(1,1,1)}(x,y)\, \mathbb{H}^{(1,1,1)}(x,y)$$

to coefficients in the non-weighted space $\mathbb{H}^{(1,1,1)}(x,y)$. Note that this construction will ensure the imposition of the Dirichlet zero boundary conditions on Ω. The matrix operator for the Laplacian we denote $\Delta_W^{(1,1,1)\to(1,1,1)}$ acting on the coefficients vector is then given by

$$\Delta_W^{(1,1,1)\to(1,1,1)} := D_x^{(0,0,0)}\, W_x^{(1,1,1)} + T^{(0,0,1)\to(1,1)}\, D_y^{(0,0,0)}\, T_W^{(1,1,0)\to(0,0,0)}\, W_y^{(1,1,1)}.$$

Importantly, this operator will have banded-block-banded structure, and hence will be sparse, as seen in Figure 2.

Another important example is the Biharmonic operator Δ^2, where we assume zero Dirichlet and Neumann conditions. To construct this operator, we first note that we can obtain the matrix operator for the Laplacian Δ that will take us from coefficients for expansion in the space $\mathbb{H}^{(0,0,0)}(x,y)$ to coefficients in the space $\mathbb{H}^{(2,2)}(x,y)$. We denote this matrix operator that acts on the coefficients vector as $\Delta^{(0,0,0)\to(2,2,2)}$, and is given by

$$\Delta^{(0,0,0)\to(2,2,2)} := D_x^{(1,1,1)}\, D_x^{(0,0,0)} + T^{(1,1,2)\to(2,2,2)}\, D_y^{(1,1,1)}\, T^{(0,0,1)\to(1,1,1)}\, D_y^{(0,0,0)}.$$

Further, we can represent the Laplacian as a map from coefficients in the space $\mathbb{W}^{(2,2)}$ to coefficients in the space $\mathbb{H}^{(0,0,0)}$. Note that a function expanded in the $\mathbb{W}^{(2,2)}$ basis will satisfy both zero Dirichlet and Neumann boundary conditions on Ω. We denote this matrix operator as $\Delta_W^{(2,2,2)\to(0,0,0)}$, and is given by

$$\Delta_W^{(2,2,2)\to(0,0,0)} := W_x^{(1,1,1)}\, W_x^{(2,2,2)} + T_W^{(1,1,0)\to(0,0,0)}\, W_y^{(1,1,1)}\, T_W^{(2,2,1)\to(1,1,1)}\, W_y^{(2,2,2)}.$$

We can then construct a matrix operator for Δ^2 that will take coefficients in the space $\mathbb{W}^{(2,2,2)}$ to coefficients in the space $\mathbb{H}^{(2,2,2)}$. Note that any function expanded in the $\mathbb{W}^{(2,2,2)}$ basis will satisfy both zero Dirichlet and zero Neumann boundary conditions on Ω. The matrix operator for the Biharmonic operator is then given by

$${}_2\Delta_W^{(2,2,2)\to(2,2,2)} = \Delta^{(0,0,0)\to(2,2,2)}\, \Delta_W^{(2,2,2)\to(0,0,0)}.$$

The sparsity and structure of this biharmonic operator are seen in Figure 2.

5.5 Computational aspects

In this section we discuss how to take advantage of the proposed basis and sparsity structure in partial differential operators in practical computational applications.

5.5.1 Constructing $R_n^{(a,b,c)}(x)$

It is possible to obtain the recurrence coefficients for the $\{R_n^{(a,b,c)}\}$ OPs in (23), by careful application of the Christoffel-Darboux formula [10, 18.2.12]. We explain the process here for the disk-slice case, however we note that a similar but simpler argument holds for the trapezium case. We thus first need to define a new set of 'interim' 1D OPs.

Definition 6. *Let $w_{\tilde{R}}^{(a,b,c,d)}(x) := (\beta - x)^a\, (x-\alpha)^b\, (1-x)^c\, (1+x)^d$ be a weight function on the interval (α,β), and define the associated inner product by:*

$$\langle p,\, q\rangle_{w_{\tilde{R}}^{(a,b,c,d)}} := \frac{1}{\omega_{\tilde{R}}^{(a,b,c,d)}} \int_\alpha^\beta p(x)\, q(x)\, w_{\tilde{R}}^{(a,b,c,d)}(x)\, \mathrm{d}x \tag{26}$$

where

$$\omega_{\tilde{R}}^{(a,b,c,d)} := \int_\alpha^\beta w_{\tilde{R}}^{(a,b,c,d)}(x)\, \mathrm{d}x \tag{27}$$

Denote the four-parameter family of orthonormal polynomials on $[\alpha,\beta]$ by $\{\tilde{R}_n^{(a,b,c,d)}\}$, orthonormal with respect to the inner product defined in (26).

Note that the OPs $\{R_n^{(a,b,2c)}\}$ are then equivalent to the OPs $\{\tilde{R}_n^{(a,b,c,c)}\}$. Let the recurrence coefficients for the OPs $\{\tilde{R}_n^{(a,b,c,d)}\}$ be given by:

$$x\,\tilde{R}_n^{(a,b,c,d)}(x) = \tilde{\beta}_n^{(a,b,c,d)}\,\tilde{R}_{n+1}^{(a,b,c,d)}(x) + \tilde{\alpha}_n^{(a,b,c,d)}\,\tilde{R}_n^{(a,b,c,d)}(x) + \tilde{\beta}_{n-1}^{(a,b,c,d)}\,\tilde{R}_{n-1}^{(a,b,c,d)}(x) \tag{28}$$

Proposition 2. *There exist constants $\mathcal{C}_n^{(a,b,c,d)}$, $\mathcal{D}_n^{(a,b,c,d)}$ such that*

$$\tilde{R}_n^{(a,b,c+1,d)}(x) = \mathcal{C}_n^{(a,b,c,d)}\sum_{k=0}^{n}\tilde{R}_k^{(a,b,c,d)}(1)\,\tilde{R}_k^{(a,b,c,d)}(x) \tag{29}$$

$$\tilde{R}_n^{(a,b,c,d+1)}(x) = \mathcal{D}_n^{(a,b,c,d)}\sum_{k=0}^{n}\tilde{R}_k^{(a,b,c,d)}(-1)\,\tilde{R}_k^{(a,b,c,d)}(x). \tag{30}$$

The constants $\mathcal{C}_n^{(a,b,c,d)}$, $\mathcal{D}_n^{(a,b,c,d)}$ are given by

$$\mathcal{C}_n^{(a,b,c,d)} = \left(\frac{\omega_{\tilde{R}}^{(a,b,c+1,d)}}{\omega_{\tilde{R}}^{(a,b,c,d)}\,\tilde{R}_n^{(a,b,c,d)}(1)\,\tilde{R}_{n+1}^{(a,b,c,d)}(1)\,\tilde{\beta}_n^{(a,b,c,d)}}\right)^{\frac{1}{2}}, \tag{31}$$

$$\mathcal{D}_n^{(a,b,c,d)} = (-1)^n\left(\frac{-\omega_{\tilde{R}}^{(a,b,c,d+1)}}{\omega_{\tilde{R}}^{(a,b,c,d)}\,\tilde{R}_n^{(a,b,c,d)}(-1)\,\tilde{R}_{n+1}^{(a,b,c,d)}(-1)\,\tilde{\beta}_n^{(a,b,c,d)}}\right)^{\frac{1}{2}}. \tag{32}$$

Proposition 3. *The recurrence coefficients for the OPs $\{\tilde{R}_n^{(a,b,c+1,d)}\}$ are given by:*

$$\tilde{\alpha}_n^{(a,b,c+1,d)} = \frac{\tilde{R}_{n+2}^{(a,b,c,d)}(1)}{\tilde{R}_{n+1}^{(a,b,c,d)}(1)}\,\tilde{\beta}_{n+1}^{(a,b,c,d)} - \frac{\tilde{R}_{n+1}^{(a,b,c,d)}(1)}{\tilde{R}_n^{(a,b,c,d)}(1)}\,\tilde{\beta}_n^{(a,b,c,d)} + \tilde{\alpha}_{n+1}^{(a,b,c,d)}, \tag{33}$$

$$\tilde{\beta}_n^{(a,b,c+1,d)} = \frac{\mathcal{C}_n^{(a,b,c,d)}}{\mathcal{C}_{n+1}^{(a,b,c,d)}}\,\frac{\tilde{R}_n^{(a,b,c,d)}(1)}{\tilde{R}_{n+1}^{(a,b,c,d)}(1)}\,\tilde{\beta}_n^{(a,b,c,d)}. \tag{34}$$

The recurrence coefficients for the OPs $\{\tilde{R}_n^{(a,b,c,d+1)}\}$ are given by:

$$\tilde{\alpha}_n^{(a,b,c,d+1)} = \frac{\tilde{R}_{n+2}^{(a,b,c,d)}(-1)}{\tilde{R}_{n+1}^{(a,b,c,d)}(-1)}\,\tilde{\beta}_{n+1}^{(a,b,c,d)} - \frac{\tilde{R}_{n+1}^{(a,b,c,d)}(-1)}{\tilde{R}_n^{(a,b,c,d)}(-1)}\,\tilde{\beta}_n^{(a,b,c,d)} + \tilde{\alpha}_{n+1}^{(a,b,c,d)}, \tag{35}$$

$$\tilde{\beta}_n^{(a,b,c,d+1)} = \frac{\mathcal{D}_n^{(a,b,c,d)}}{\mathcal{D}_{n+1}^{(a,b,c,d)}}\,\frac{\tilde{R}_n^{(a,b,c,d)}(-1)}{\tilde{R}_{n+1}^{(a,b,c,d)}(-1)}\,\tilde{\beta}_n^{(a,b,c,d)}. \tag{36}$$

Corollary 1. *The recurrence coefficients for the OPs* $\{\tilde{R}_n^{(a,b,c+1,d)}\}$ *can be written as:*

$$\tilde{\alpha}_n^{(a,b,c+1,d)} = \frac{\tilde{\beta}_{n-1}^{(a,b,c,d)}}{\chi_{n-1}^{(a,b,c,d)}(1)} - \frac{\tilde{\beta}_n^{(a,b,c,d)}}{\chi_n^{(a,b,c,d)}(1)} + \tilde{\alpha}_n^{(a,b,c,d)}, \tag{37}$$

$$\tilde{\beta}_n^{(a,b,c+1,d)} = \left(\frac{1 - \tilde{\alpha}_{n+1}^{(a,b,c,d)} - \frac{\tilde{\beta}_n^{(a,b,c,d)}}{\chi_n^{(a,b,c,d)}(1)}}{1 - \tilde{\alpha}_n^{(a,b,c,d)} - \frac{\tilde{\beta}_{n-1}^{(a,b,c,d)}}{\chi_{n-1}^{(a,b,c,d)}(1)}} \right)^{\frac{1}{2}} \tilde{\beta}_n^{(a,b,c,d)}. \tag{38}$$

The recurrence coefficients for the OPs $\{\tilde{R}_n^{(a,b,c,d+1)}\}$ *can be written as:*

$$\tilde{\alpha}_n^{(a,b,c,d+1)} = \frac{\tilde{\beta}_{n-1}^{(a,b,c,d)}}{\chi_{n-1}^{(a,b,c,d)}(-1)} - \frac{\tilde{\beta}_n^{(a,b,c,d)}}{\chi_n^{(a,b,c,d)}(-1)} + \tilde{\alpha}_n^{(a,b,c,d)}, \tag{39}$$

$$\tilde{\beta}_n^{(a,b,c,d+1)} = \left(\frac{-1 + \tilde{\alpha}_{n+1}^{(a,b,c,d)} + \frac{\tilde{\beta}_n^{(a,b,c,d)}}{\chi_n^{(a,b,c,d)}(-1)}}{-1 + \tilde{\alpha}_n^{(a,b,c,d)} + \frac{\tilde{\beta}_{n-1}^{(a,b,c,d)}}{\chi_{n-1}^{(a,b,c,d)}(-1)}} \right)^{\frac{1}{2}} \tilde{\beta}_n^{(a,b,c,d)}. \tag{40}$$

where

$$\chi_n^{(a,b,c,d)}(y) := \frac{\tilde{R}_{n+1}^{(a,b,c,d)}(y)}{\tilde{R}_n^{(a,b,c,d)}(y)} \tag{41}$$

$$= \frac{1}{\tilde{\beta}_n^{(a,b,c,d)}} \left(y - \tilde{\alpha}_n^{(a,b,c,d)} - \frac{\tilde{\beta}_{n-1}^{(a,b,c,d)}}{\chi_{n-1}^{(a,b,c,d)}(y)} \right), \quad y \in \{-1, 1\}. \tag{42}$$

These two propositions allow us to recursively obtain the recurrence coefficients for the OPs $\{R_{n-k}^{(a,b,2c+2k+1)}\}$ as k increases to be large.

Remark: The Corollary demonstrates that in order to obtain the recurrence coefficients $\{\alpha_m^{(a,b,2c+2k+1)}\}$, $\{\beta_m^{(a,b,2c+2k+1)}\}$ for some m and k, we require that we obtain the recurrence coefficients $\{\alpha_{m+2}^{(a,b,2c+2(k-1)+1)}\}$, $\{\beta_{m+2}^{(a,b,2c+2(k-1)+1)}\}$. Thus, for large N, this recursive method of obtaining the recurrence coefficients requires a large initialisation (i.e. using the Lanczos algorithm to compute the recurrence coefficients $\{\alpha_n^{(a,b,2c+1)}\}$, $\{\beta_n^{(a,b,2c+1)}\}$ – however, we only need to compute these once, and can store and save this initialisation to disk once computed, for the given values of a, b, c).

5.5.2 Quadrature rule on the disk-slice

In this section we construct a quadrature rule exact for polynomials in the disk-slice Ω that can be used to expand functions in $H_{n,k}^{(a,b,c)}(x, y)$ when Ω is a disk-slice.

Theorem 2. *Denote the Gauss quadrature nodes and weight on $[\alpha, \beta]$ with weight $(\beta - s)^a\,(s-\alpha)^b\,\rho(s)^{2c+1}$ as $(s_k, w_k^{(s)})$, and on $[-1, 1]$ with weight $(1-t^2)^c$ as $(t_k, w_k^{(t)})$. Define*

$$\begin{aligned} x_{i+(j-1)N} &:= s_j, \quad i, j = 1, \dots, \left\lceil \frac{N+1}{2} \right\rceil, \\ y_{i+(j-1)N} &:= \rho(s_j)\, t_i, \quad i, j = 1, \dots, \left\lceil \frac{N+1}{2} \right\rceil, \\ w_{i+(j-1)N} &:= w_j^{(s)} w_i^{(t)}, \quad i, j = 1, \dots, \left\lceil \frac{N+1}{2} \right\rceil. \end{aligned}$$

Let $f(x, y)$ be a polynomial on Ω. The quadrature rule is then

$$\iint_\Omega f(x,y)\, W^{(a,b)}(x,y)\,\mathrm{d}A \approx \frac{1}{2}\sum_{j=1}^{M} w_j\,\big[f(x_j, y_j) + f(x_j, -y_j)\big],$$

where $M = \left\lceil \frac{1}{2}(N+1) \right\rceil^2$, and the quadrature rule is exact if $f(x, y)$ is a polynomial of degree $\leq N$.

5.6 Obtaining the coefficients for expansion of a function on the disk-slice

Fix $a, b, c \in \mathbb{R}$. Then for any function $f : \Omega \to \mathbb{R}$ we can express f by

$$f(x,y) \approx \sum_{n=0}^{N} \mathbb{H}_n^{(a,b,c)}(x,y)^\top\, \mathbf{f}_n$$

for N sufficiently large, where

$$\mathbb{H}_n^{(a,b,c)}(x,y) := \begin{pmatrix} H_{n,0}^{(a,b,c)}(x,y) \\ \vdots \\ H_{n,n}^{(a,b,c)}(x,y) \end{pmatrix} \in \mathbb{R}^{n+1} \quad \forall n = 0, 1, 2, \dots, N,$$

and where

$$\mathbf{f}_n := \begin{pmatrix} f_{n,0} \\ \vdots \\ f_{n,n} \end{pmatrix} \in \mathbb{R}^{n+1} \quad \forall n = 0, 1, 2, \dots, N, \quad f_{n,k} := \frac{\left\langle f,\ H_{n,k}^{(a,b,c)} \right\rangle_{W^{(a,b,c)}}}{\left\| H_{n,k}^{(a,b,c)} \right\|_{W^{(a,b,c)}}}$$

Recall from (21) that $\left\| H_{n,k}^{(a,b,c)} \right\|^2_{W^{(a,b,c)}} = \omega_R^{(a,b,2c+2k+1)}\ \omega_P^{(c)}$. Using the quadrature rule detailed in Section 4.2 for the inner product, we can calculate the coefficients $f_{n,k}$ for each

$n = 0, \ldots, N,\ k = 0, \ldots, n$:

$$f_{n,k} = \frac{1}{2\,\omega_R^{(a,b,2c+2k+1)}\ \omega_P^{(c)}} \sum_{j=1}^{M} w_j \left[f(x_j, y_j)\, H_{n,k}^{(a,b,c)}(x_j, y_j) + f(x_j, -y_j)\, H_{n,k}^{(a,b,c)}(x_j, -y_j) \right]$$

where $M = \left\lceil \frac{1}{2}(N+1) \right\rceil^2$.

5.6.1 Calculating non-zero entries of the operator matrices

The proofs of Theorem 1 and Lemma 2 provide a way to calculate the non-zero entries of the operator matrices given in Definition 4 and Definition 5. We can simply use quadrature to calculate the 1D inner products, which has a complexity of $\mathcal{O}(N^3)$. This proves much cheaper computationally than using the 2D quadrature rule to calculate the 2D inner products, which has a complexity of $\mathcal{O}(N^4)$.

5.7 Examples on the disk-slice with zero Dirichlet conditions

We now demonstrate how the sparse linear systems constructed as above can be used to efficiently solve PDEs with zero Dirichlet conditions. We consider Poisson, inhomogeneous variable coefficient Helmholtz equation and the Biharmonic equation, demonstrating the versatility of the approach.

5.7.1 Poisson

The Poisson equation is the classic problem of finding $u(x, y)$ given a function $f(x, y)$ such that:

$$\begin{cases} \Delta u(x,y) = f(x,y) & \text{in } \Omega \\ u(x,y) = 0 & \text{on } \partial\Omega \end{cases} . \tag{43}$$

noting the imposition of zero Dirichlet boundary conditions on u.

We can tackle the problem as follows. Denote the coefficient vector for expansion of u in the $\mathbb{W}^{(1,1,1)}$ OP basis up to degree N by $\mathbf{u}$, and the coefficient vector for expansion of f in the $\mathbb{H}^{(1,1,1)}$ OP basis up to degree N by $\mathbf{f}$. Since f is known, we can obtain $\mathbf{f}$ using the quadrature rule above. In matrix-vector notation, our system hence becomes:

$$\Delta_W^{(1,1,1)\to(1,1,1)} \mathbf{u} = \mathbf{f}$$

which can be solved to find $\mathbf{u}$.

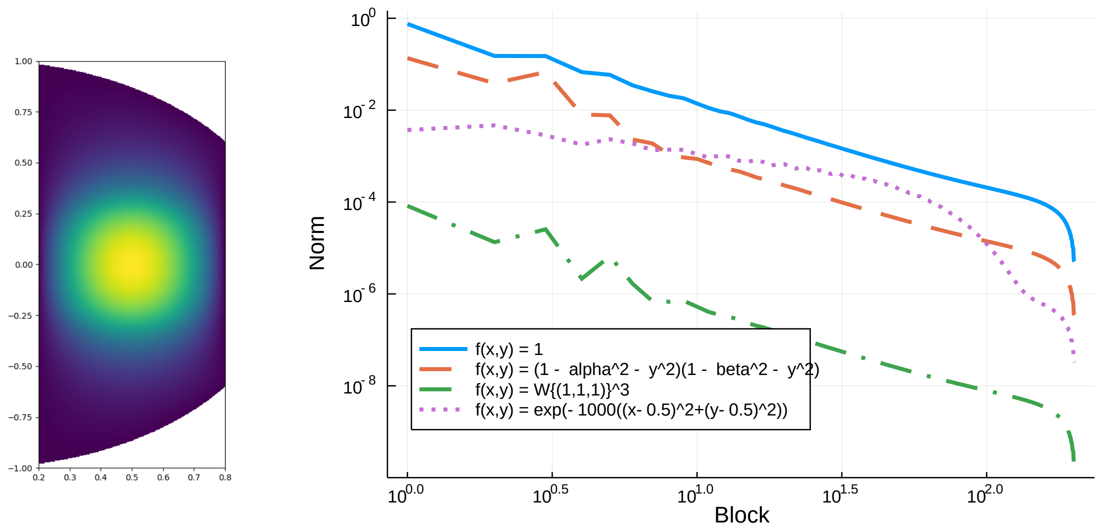

Figure 3: Left: The computed solution to $\Delta u = f$ with zero boundary conditions with $f(x, y) = 1 + \mathrm{erf}(5(1 - 10((x - 0.5)^2 + y^2)))$. Right: The norms of each block of the computed solution of the Poisson equation with the given right hand side functions. This demonstrates algebraic convergence with the rate dictated by the decay at the corners, with spectral convergence observed when the right-hand side vanishes to all orders.

5.7.2 Inhomogeneous variable-coefficient Helmholtz

Find $u(x, y)$ given functions $v,\ f : \Omega \to \mathbb{R}$ such that:

$$\begin{cases} \Delta u(x,y) + k^2\, v(x,y)\, u(x,y) = f(x,y) & \text{in } \Omega \\ u(x,y) = 0 \quad \text{on } \partial\Omega \end{cases} . \tag{44}$$

where $k \in \mathbb{R}$, noting the imposition of zero Dirichlet boundary conditions on u.

We can tackle the problem as follows. Denote the coefficient vector for expansion of u in the $\mathbb{W}^{(1,1,1)}$ OP basis up to degree N by $\mathbf{u}$, and the coefficient vector for expansion of f in the $\mathbb{H}^{(1,1,1)}$ OP basis up to degree N by $\mathbf{f}$. Since f is known, we can obtain the coefficients $\mathbf{f}$ using the quadrature rule above. We can obtain the matrix operator for the variable-coefficient function $v(x, y)$ by using the Clenshaw algorithm with matrix inputs as the Jacobi matrices ${J_x^{(0,0,0)}}^\top, {J_y^{(0,0,0)}}^\top$, yielding an operator matrix of the same dimension as the input Jacobi matrices a la the procedure introduced in [13]. We can denote the resulting operator acting on coefficients in the $\mathbb{H}^{(0,0,0)}$ space by $V({J_x^{(0,0,0)}}^\top, {J_y^{(0,0,0)}}^\top)$. In matrix-vector notation, our system hence becomes:

$$(\Delta_W^{(1,1,1)\to(1,1,1)} + k^2 T^{(0,0,0)\to(1,1,1)}\, V({J_x^{(0,0,0)}}^\top, {J_y^{(0,0,0)}}^\top)\, T_W^{(1,1,1)\to(0,0,0)})\mathbf{u} = \mathbf{f}$$

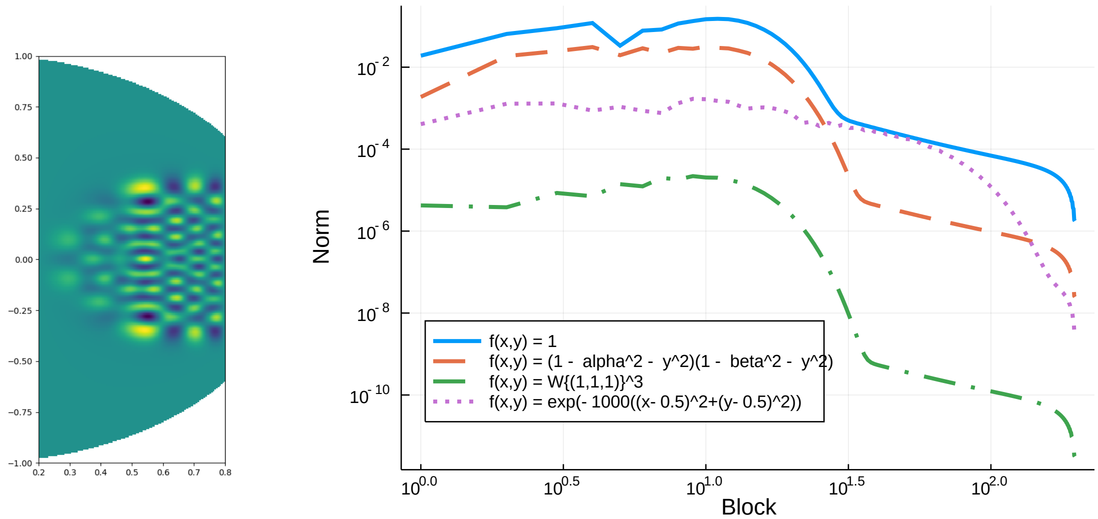

Figure 4: Left: The computed solution to $\Delta u + k^2\, v\, u = f$ with zero boundary conditions with $f(x, y) = x(1 - x^2 - y^2)e^x$, $v(x, y) = 1 - (3(x - 1)^2 + 5y^2)$ and $k = 100$. Right: The norms of each block of the computed solution of the Helmholtz equation with the given right hand side functions, with $k = 20$ and $v(x, y) = 1 - (3(x - 1)^2 + 5y^2)$.

which can be solved to find $\mathbf{u}$.

We can extend this to constant non-zero boundary conditions by simply noting that the problem

$$\begin{cases} \Delta u(x, y) + k^2\, v(x, y)\, u(x, y) = f(x, y) & \text{in } \Omega \\ u(x, y) = c \in \mathbb{R} \quad \text{on } \partial\Omega \end{cases}$$

is equivalent to letting $u = \tilde{u} + c$ and solving

$$\begin{cases} \Delta \tilde{u}(x, y) + k^2\, v(x, y)\, \tilde{u}(x, y) = f(x, y) - c\, k^2\, v(x, y) \;\; =: g(x, y) & \text{in } \Omega \\ \tilde{u}(x, y) = 0 \quad \text{on } \partial\Omega. \end{cases} .$$

6 Orthogonal Polynomials on spherical caps, spherical strips and spherical triangles (3D)

Moving forward, our aim is to try and develop a similar framework of using 3D OPs to expand functions and obtaining sparse differential operators on these spherical sub-domains, in order to solve partial differential equations and even lead to finite element methods on the sphere using these spherical sub-domains as elements. Developing these

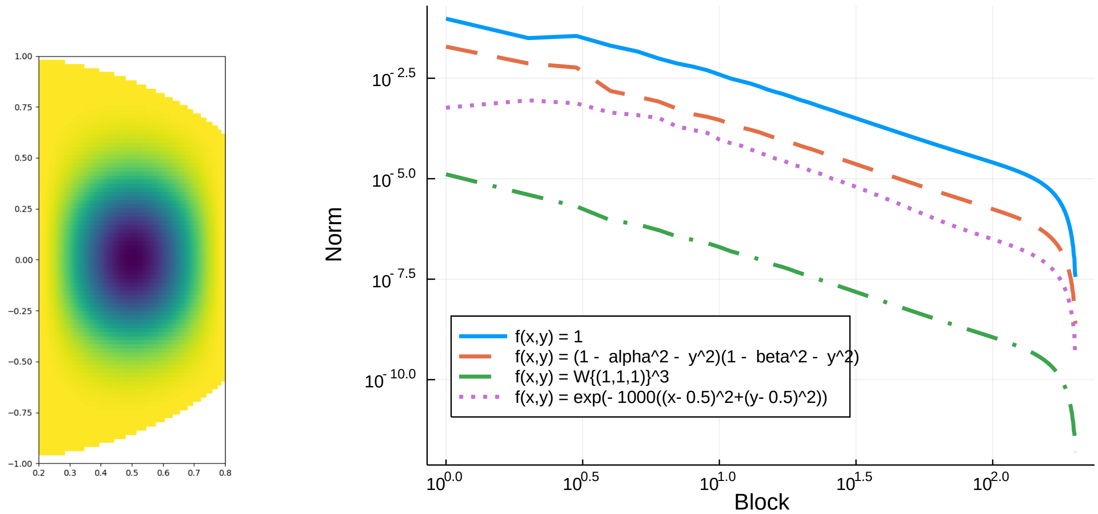

Figure 5: Left: The computed solution to $\Delta^2 u = f$ with zero Dirichlet and Neumann boundary conditions with $f(x, y) = 1 + \mathrm{erf}(5(1 - 10((x - 0.5)^2 + y^2)))$. Right: The norms of each block of the computed solution of the biharmonic equation with the given right hand side functions.

methods on the spherical cap is the natural first step from the work completed on the disk-slice to 3D domains, and where our work will head moving forward.

One approach we will look at will be to build the 3D OPs on the spherical cap by in a similar vein as to how we could build the circle arc OPs, by projecting up from the 2D orthogonal polynomials on the disk to 3D OPs on the sphere.

Define the end-disk-slice, denoted by ω, by

$$\omega := \{(x, y) \in \mathbb{R}^2 \quad | \quad \alpha \leq x \leq 1, \ -\rho(x) \leq y \leq \rho(x)\}. \tag{45}$$

where $\alpha \geq 0$ and $\rho(x) := (1 - x^2)^{\frac{1}{2}}$, $x \in (\alpha, 1)$. Further, define the spherical cap, denoted by Ω, by

$$\Omega := \{(x, y, z) \in \mathbb{R}^3 \quad | \quad (x, y) \in \omega, \ x^2 + y^2 + z^2 = 1\}. \tag{46}$$

We can define the 3D OP family on Ω as the set of polynomials $\mathbb{Q}^{(a,b)}(x, y, z)$ where

$$\mathbb{Q}_n^{(a,b)}(x, y, z) := \begin{pmatrix} \mathbb{H}_n^{(a,b)}(x, y) \\ z\, \mathbb{H}_{n-1}^{(a,b+1)}(x, y) \end{pmatrix} \in \mathbb{R}^{2n+1}, \qquad \mathbb{Q}^{(a,b)} := \begin{pmatrix} \mathbb{Q}_0^{(a,b)} \\ \mathbb{Q}_1^{(a,b)} \\ \vdots \\ \mathbb{Q}_N^{(a,b)} \end{pmatrix}. \tag{47}$$

The relations for multiplication by x, y are then trivial. For multiplication by z, note that $z^2 = W^{(0,1)}(x,y)$ for $(x,y,z) \in \Omega$ and consider that (as the 2D OPs for any parameters form a 2D basis of ω):

$$z\, H^{(a,b)}_{n,k}(x,y) = \sum_{m,j} c^{(a,b)\to(a,b+1)}_{n,k,m,j}\; z\, H^{(a,b+1)}_{m,j}(x,y), \tag{48}$$

$$z^2\, H^{(a,b+1)}_{n,k}(x,y) = \sum_{m,j} d^{(a,b+1)\to(a,b)}_{n,k,m,j}\; H^{(a,b)}_{m,j}(x,y). \tag{49}$$

The coefficients $c^{(a,b)\to(a,b+1)}_{n,k,m,j}$ and $d^{(a,b+1)\to(a,b)}_{n,k,m,j}$ are given by

$$c^{(a,b)\to(a,b+1)}_{n,k,m,j} = \frac{\left\langle H^{(a,b)}_{n,k},\; H^{(a,b+1)}_{m,j} \right\rangle_{W^{(a,b+1)}}}{\left\| H^{(a,b+1)}_{m,j} \right\|^2_{W^{(a,b+1)}}} \tag{50}$$

$$\begin{aligned}
d^{(a,b+1)\to(a,b)}_{n,k,m,j} &= \frac{\left\langle W^{(0,1)} H^{(a,b+1)}_{n,k},\; H^{(a,b)}_{m,j} \right\rangle_{W^{(a,b)}}}{\left\| H^{(a,b)}_{m,j} \right\|^2_{W^{(a,b)}}} \\
&= \frac{\left\langle H^{(a,b+1)}_{n,k},\; H^{(a,b)}_{m,j} \right\rangle_{W^{(a,b+1)}}}{\left\| H^{(a,b)}_{m,j} \right\|^2_{W^{(a,b)}}} \\
&= \frac{\left\| H^{(a,b+1)}_{n,k} \right\|^2_{W^{(a,b+1)}}}{\left\| H^{(a,b)}_{m,j} \right\|^2_{W^{(a,b)}}}\; c^{(a,b)\to(a,b+1)}_{m,j,n,k}.
\end{aligned} \tag{51}$$

The coefficients $c^{(a,b)\to(a,b+1)}_{n,k,m,j}$ are just the entries to the parameter transformation operator matrix $T^{(a,b)\to(a,b+1)}$, defined by

$$\mathbb{H}^{(a,b)}(x,y) = \left(T^{(a,b)\to(a,b+1)}\right)^T \mathbb{H}^{(a,b+1)}(x,y), \tag{52}$$

which we have shown to be sparse with banded-block-banded structure, meaning the resulting operator matrix for the Jacobi operator for multiplication by z here is sparse.

7 Conclusions

In this document I have presented the two main components of my PhD thus far, namely the introductory work I did using spherical harmonics in cartesian coordinates to solve multivariate (3D) PDEs on the sphere. From here, this allowed me to use my newly gained knowledge to begin developing similar sparse spectral methods on the 2D domains as defined by equation (16) with Conditions (1, 2).

We have shown that bivariate orthogonal polynomials can lead to sparse discretizations of general linear PDEs on specific domains whose boundary is specified by an algebraic curve—notably here the disk-slice—with Dirichlet boundary conditions. This work is related to the triangle case [13], and forms a building block in developing an $hp-$finite element method to solve PDEs on other polygonal domains by using suitably shaped elements, for example, by dividing the disk into disk slice elements.

This work serves as a stepping stone to constructing similar methods to solve partial differential equations on 3D sub-domains of the sphere, such as spherical caps and spherical triangles. In particular, orthogonal polynomials (OPs) in cartesian coordinates (x, y, and z) on a half-sphere can be represented using two families of OPs on the half-disk, see [14, Theorem 3.1] for a similar construction of OPs on an arc in 2D, and it is clear from the construction in this paper that discretizations of spherical gradients and Laplacian's are sparse on half-spheres and other suitable sub-components of the sphere. The resulting sparsity in high-polynomial degree discretizations presents an attractive alternative to methods based on bijective mappings (e.g., [3, 15, 4]). Constructing these sparse spectral methods for surface PDEs on half-spheres, spherical caps, and spherical triangles is the next steps of my PhD, which has applications in weather prediction [16]. Other extensions to the disk-slice work could include a full hp-finite element method on sections of a disk, which has applications in turbulent pipe flow.

A P-finite element methods using sparse operators

We follow the method of [2] to construct a sparse p-finite element method in terms of the operators constructed above, with the benefit of ensuring that the resulting discretisation is symmetric. Consider the 2D Dirichlet problem on a domain Ω:

$$\begin{cases} -\Delta u(x,y) = f(x,y) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{cases}$$

This has the weak formulation for any test function $v \in V := H_0^1(\Omega) = \{v \in H^1(\Omega) \quad | \quad v|_{\partial\Omega} = 0\}$,

$$L(v) := \int_\Omega f\, v\, d\mathbf{x} = \int_\Omega \nabla u \cdot \nabla v\, d\mathbf{x} =: a(u,v).$$

In general, we would let $\mathcal{T}$ be the set of elements τ that make up our finite element discretisation of the domain, where each τ is a trapezium or disk slice for example.

In this section, we limit our discretisation to a single element, that is we let $\tau = \Omega$ for a disk-slice domain. We can choose our finite dimensional space $V_p = \{v_p \in V \quad | \quad \deg(v_p|_\tau) \leq p\}$ for some $p \in \mathbb{N}$.

We seek $u_p \in V_p$ s.t.

$$L(v_p) = a(u_p, v_p) \quad \forall\, v_p \in V_p. \tag{53}$$

Define $\Lambda^{(a,b,c)} := \left\langle \mathbb{H}^{(a,b,c)},\, \mathbb{H}^{(a,b,c)\top} \right\rangle_{W^{(a,b,c)}}$ where $W^{(a,b,c)}$ is the weight with which the OPs in $\mathbb{H}^{(a,b,c)}$ are orthogonal with respect to. Note that due to orthogonality this is a diagonal matrix. We can choose a basis for V_p by using the weighted orthogonal polynomials on τ with parameters $a = b = 1$:

$$\mathbb{W}^{(1,1,1)}(x,y) := \begin{pmatrix} \mathbb{W}^{(1,1,1)}{}_0(x,y) \\ \mathbb{W}^{(1,1,1)}{}_1(x,y) \\ \mathbb{W}^{(1,1,1)}{}_2(x,y) \\ \vdots \\ \mathbb{W}^{(1,1,1)}{}_p(x,y) \end{pmatrix},$$

$$\mathbb{W}^{(1,1,1)}{}_n(x,y) := W^{(1,1,1)}(x,y) \begin{pmatrix} H^{(1,1,1)}_{n,0}(x,y) \\ \vdots \\ H^{(1,1,1)}_{n,n}(x,y) \end{pmatrix} \in \mathbb{R}^{n+1} \quad \forall n = 0,1,2,\ldots,p,$$

and rewrite (53) in matrix form:

$$\begin{aligned}
a(u_p, v_p) &= \int_\tau \nabla u_p \cdot \nabla v_p \,\mathrm{d}\mathbf{x} \\
&= \int_\tau \begin{pmatrix} \partial_x v_p \\ \partial_y v_p \end{pmatrix}^\top \begin{pmatrix} \partial_x u_p \\ \partial_y u_p \end{pmatrix} \mathrm{d}\mathbf{x} \\
&= \int_\tau \begin{pmatrix} \mathbb{H}^{(0,0,0)\top} W_x^{(1,1,1)} \mathbf{v} \\ \mathbb{H}^{(0,0,0)\top} T_W^{(1,1,0)\to(0,0,0)} W_y^{(1,1,1)} \mathbf{v} \end{pmatrix}^\top \begin{pmatrix} \mathbb{H}^{(0,0,0)\top} W_x^{(1,1,1)} \mathbf{u} \\ \mathbb{H}^{(0,0,0)\top} T_W^{(1,1,0)\to(0,0,0)} W_y^{(1,1,1)} \mathbf{u} \end{pmatrix} \mathrm{d}\mathbf{x} \\
&= \int_\tau \Big(\mathbf{v}^\top W_x^{(1,1,1)\top} \mathbb{H}^{(0,0,0)} \mathbb{H}^{(0,0,0)\top} W_x^{(1,1,1)} \mathbf{u} \\
&\qquad + \mathbf{v}^\top (T_W^{(1,1,0)\to(0,0,0)} W_y^{(1,1,1)})^\top \mathbb{H}^{(0,0,0)} \mathbb{H}^{(0,0,0)\top} T_W^{(1,1,0)\to(0,0,0)} W_y^{(1,1,1)} \mathbf{u} \Big) \mathrm{d}\mathbf{x} \\
&= \mathbf{v}^\top \Big(W_x^{(1,1,1)\top} \Lambda^{(0,0,0)} W_x^{(1,1,1)} \\
&\qquad + (T_W^{(1,1,0)\to(0,0,0)} W_y^{(1,1,1)})^\top \Lambda^{(0,0,0)} T_W^{(1,1,0)\to(0,0,0)} W_y^{(1,1,1)} \Big) \mathbf{u}
\end{aligned}$$

where $\mathbf{u}, \mathbf{v}$ are the coefficient vectors of the expansions of $u_p, v_p \in V_p$ respectively in the V_p

basis ($\mathbb{W}^{(1,1,1)}$ OPs), and

$$\begin{aligned} L(v_p) &= \int_\tau v_p \, f \, \mathrm{d}\mathbf{x} \\ &= \int_\tau \mathbf{v}^\top \, \mathbb{W}^{(1,1,1)} \, {\mathbb{H}^{(1,1,1)}}^\top \, \mathbf{f} \, \mathrm{d}\mathbf{x} \\ &= \mathbf{v}^\top \, \left\langle \mathbb{H}^{(1,1,1)}, {\mathbb{H}^{(1,1,1)}}^\top \right\rangle_{W^{(1,1,1)}} \, \mathrm{d}\mathbf{x} \\ &= \mathbf{v}^\top \Lambda^{(1,1,1)} \, \mathbf{f}, \end{aligned}$$

where $\mathbf{f}$ is the coefficient vector for the expansion of the function $f(x, y)$ in the $\mathbb{H}^{(1,1,1)}$ OP basis.

Since (53) is equivalent to stating that

$$L(W^{(1,1,1)} H_{n,k}^{(1,1,1)}) = a(u_p, W^{(1,1,1)} H_{n,k}^{(1,1,1)}) \quad \forall \, n = 0, \dots, p, \; k = 0, \dots, n,$$

(i.e. holds for all basis functions of V_p) by choosing v_p as each basis function, we can equivalently write the linear system for our finite element problem as:

$$A\mathbf{u} = \tilde{\mathbf{f}}.$$

where the (element) stiffness matrix A is defined by

$$A = {W_x^{(1,1,1)}}^\top \Lambda^{(0,0,0)} W_x^{(1,1,1)} + (T_W^{(1,1,0)\to(0,0,0)} W_y^{(1,1,1)})^\top \Lambda^{(0,0,0)} T_W^{(1,1,0)\to(0,0,0)} W_y^{(1,1,1)}$$

and the load vector $\tilde{\mathbf{f}}$ is given by

$$\tilde{\mathbf{f}} = \Lambda^{(1,1,1)} \, \mathbf{f}.$$

Note that since we have sparse operator matrices for partial derivatives and basis-transform, we obtain a symmetric sparse (element) stiffness matrix, as well as a sparse operator matrix for calculating the load vector (rhs).

B End-Disk-Slice (e.g. the Half-Disk)

The work in this paper on the disk-slice can be easily transferred to the special-case domain of the end-disk-slice by which we mean

$$\Omega := \{(x, y) \in \mathbb{R}^2 \quad | \quad \alpha < x < \beta, \; \gamma\rho(x) < y < \delta\rho(x)\}$$

with

$$\begin{cases} \alpha & \in (0,1) \\ \beta & := 1 \\ (\gamma,\delta) & := (-1,1) \\ \rho(x) & := (1-x^2)^{\frac{1}{2}}. \end{cases}$$

Our 1D weight functions on the intervals (α,β) and (γ,δ) respectively are then given by:

$$\begin{cases} w_R^{(a,b)}(x) & := (x-\alpha)^a \, \rho(x)^b \\ w_P^{(a)}(x) & := (1-x^2)^b. \end{cases}$$

Note here how we can remove the need for third parameter, which is why we consider this a special case. This will make some calculations easier, and the operator matrices more sparse. The weight $w_P^{(b)}(x)$ is a still the same ultraspherical weight (and the corresponding OPs are the Jacobi polynomials $\{P_n^{(b,b)}\}$). $w_R^{(a,b)}(x)$ is the (non-classical) weight for the OPs denoted $\{R_n^{(a,b)}\}$. Thus we arrive at the two-parameter family of 2D orthogonal polynomials $\{H_{n,k}^{(a,b)}\}$ on Ω given by, for $0 \le k \le n$, $n = 0,1,2,\ldots$,

$$H_{n,k}^{(a,b)}(x,y) := R_{n-k}^{(a,2b+2k+1)}(x)\, \rho(x)^k \, P_k^{(b,b)}\left(\frac{y}{\rho(x)}\right), \quad (x,y) \in \Omega,$$

orthogonal with respect to the weight

$$\begin{aligned} W^{(a,b)}(x,y) &:= w_R^{(a,2b)}(x) w_P^{(b)}\left(\frac{y}{\rho(x)}\right) \\ &= (x-\alpha)^a \, (\rho(x)^2 - y^2)^b \\ &= (x-\alpha)^a \, (1-x^2-y^2)^b, \quad (x,y) \in \Omega. \end{aligned}$$

The sparsity of operator matrices for partial differentiation by x, y as well as for parameter transformations generalise to such end-disk-slice domains. For instance, if we inspect the proof of Lemma 1, we see that it can easily generalise to the weights and domain Ω for an end-disk-slice.

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