Chapter 4. Possibilities and probabilities

Fold

Table of Contents

This chapter deals with events for which we do not know the outcome in advance. We call such events *random event*. For example, when we toss a die, we do not know in advance what number will show up on the top face, or we do not know in advance if it will rain a month from today.

Probability is the branch of mathematics that deals with these uncertain events.

The first part of this chapter deals with what is *possible*, that is we list or count the possible outcomes of an event. The second part deals with what is *probable*. Namely, we will define what the probability of an event is.

Counting

We want to count the possible outcomes of an event. For example, if a quiz consists of one true/false question and 1 multiple choice questions with 5 options (a,b,c,d,e), then there is 2x5 = 10 possible ways of answering the quiz, namely

True (a)
False (a)

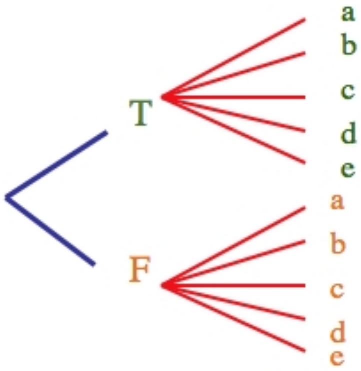
True (b)
False (b)

True (c)
False (c)

True (d)
False (d)

True (e)
False (e)

A more concise way to represent these options is by using a tree diagram, like the one below:



In general, we have the following rule

Multiplication of choices (Pigeonhole principle)

> if a choice consists of two steps, where the first one can be made in m ways and the second one in n ways, then the two together can be made in ***m* x *n*** ways.

A generalized version of this principle says that

> if a choice consists of *k* steps, where the first step can be made in n_1 ways, the second step in n_2 ways, the third in n_3, and so on, then the whole choice can be made in $\boldsymbol{n_1 n_2 n_3 \ldots n_k}$ ways.

Examples:

1. A combination lock with 3 dials, each with 10 digits, has 10x10x10 possible combinations.

2. For a license plate with 2 letters and 4 digits has 26x26x10x10x10x10 possible choices.

Factorials

A mathematical tool/notation that will be very useful is the concept of *factorial*.

$$n! = n(n-1)(n-2)\ldots(3)(2)(1) \qquad (1)$$

For example,
1! = (1)
2! = (2)(1) = 2
3! = (3)(2)(1) = 6
4!= (4)(3)(2)(1) = 24
5! = (5)(4)(3)(2)(1) = 120

By definition,
0! = 1.

Also notice that when a factorial is divided by another

Permutations

If however, repetitions are not allowed, then the answer to example 1 is different. Namely then, the option "999" is not allowed, and neither is "242", since the number 2 is repeated.

In that case, one way of finding how many possible choices we have is to fix the positions in the lock and say:

- For the first position we have 10 choices.
- For the second position we have 9 choices (since one number has been taken already)
- For the third position we have 8 choices (2 numbers have been taken).

Therefore, the number of possible choices is 10x9x8 = 720.

Similarly, if in a race with 20 people you will award different prizes to the first 4 to arrive, then the number of ways in which you could award those prizes to 4 people is 20x19x18x17. Why?

> In general, this is represented by the number of permutations of r objects selected from a set of n distinct objects, which is denoted by
> $_n P_r$ = n (n-1) (n-2) … (n-r+1)
> In factorial notation, this is $_nP_r = \frac{n!}{(n-r)!}$.

where the factorial *n!* is defined above.

Combinations

If repetitions are not allowed (like in the case of permutations), **and the order does not matter**, then the answer to example 1 is different again. Namely in that case, the option "235" is allowed, but it is the same one as "253", "325", "352", "523" and "532", since the the order does not matter.

For that reason, all this 6 options collapse into one. Therefore, the number of combinations is the number obtained earlier for permutations (in the example above that would be $_10P_3$ divided by the number of ways in which we can arrange 3 symbols in 3 spots, which is 3!

Therefore, for this case we obtain that the number of combinations (denoted $_{10}C_3$ or $\binom{10}{3}$) of choosing 3 items out of 10 different objects is

$$_{10}C_3 = \frac{_{10}P_3}{3!} = \frac{\frac{10!}{(10-3)!}}{3!} = \frac{10!}{(10-3)!3!} \tag{2}$$

> In general, the number of ways to choose *k* objects out of a set of *n* distinct objects, where the order does not matter, is given by the number of combinations of n choose k, which is denoted by
> $_n C_k$ or $\binom{n}{k}$

and given by the formula

$$_nC_k = \binom{n}{k} = \frac{n!}{(n-k)!k!} \tag{3}$$

For example, the number of ways that a person can invite 3 out of her eight closest friends to a concert for is

$$\binom{8}{3} = \frac{8!}{(8-3)!3!} = \frac{8!}{5!3!} = \frac{6 \cdot 7 \cdot 8}{1 \cdot 2 \cdot 3} = 56 \qquad (4)$$

Probability

There are 3 definitions of probability

- the classical concept of probability
- the frequency approach
- the subjective approach

We will mostly deal with the two above.

The classical concept of probability

> If there are *n* equally likely possibilities (events), of which one must occur and *s* are regarded as favorable (called success *S*), then the probability of success is denoted by
> P(*S*) = s/n.

Example:

A manufacturer of cell-phones is shipping 12 cell-phones to a customer. As a precaution, 4 random cell-phones are checked, and they are ok, but in the actual shipping of 12, 2 cell-phones are defective.

Find the probability of selecting 4 phones with no defects in a sample of 4 out of 12.

The probability of selecting 4 out of 12 is $\binom{12}{4}$ since the order of the selection does not matter. The probability of selecting 4 non-defective ones is the combinations of 4 out of 10, i.e $\binom{10}{4}$.

Therefore

$$P(\text{4 cell-phones not defective}) = \frac{s}{n} = \frac{\binom{10}{4}}{\binom{12}{4}} = \frac{210}{495} \approx 0.424 \qquad (5)$$

Frequency interpretation of probability

> The probability of an event is the proportion of time that events of that kind occur in the long run.

For example, if we want to find out the probability that a random adult in the US is in favor of the death penalty, we could take a survey/poll and find the frequency interpretation of the probability.

Suppose that in a Gallup poll 611 people say they are in favor of the death penalty, out of a total of 954 people surveyed. Find the probability that a random American is in favor of the death penalty.

Using the frequency interpretation

(6)

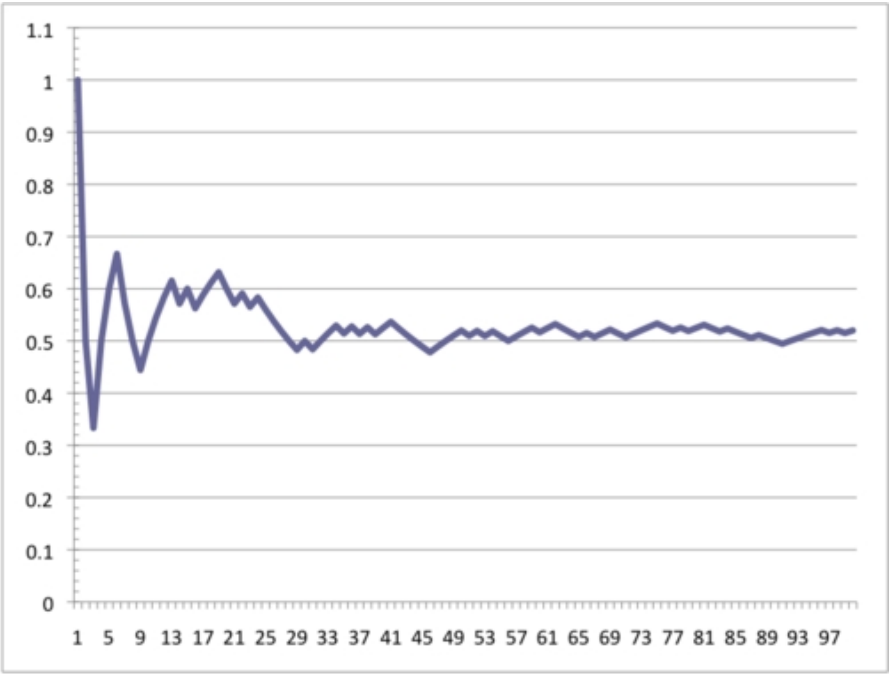
$$P(\text{person in favor of death penalty}) \approx \frac{\text{people in favor in poll}}{\text{total number of people surveyed}},$$

therefore

$$P(\text{person in favor of death penalty}) \approx = \frac{611}{954} \approx 0.64 = 64\% \qquad (7)$$

The Law of Large Numbers

> If a trial is repeated over and over again, then the proportion of successes will tend to approach to the probability of success.



http://statistics.wikidot.com/local--files/ch4/HeadsTails.xls

Experiment with this notion by using the applet available athttp://users.ece.gatech.edu/gtz/java/cointoss/index.html or the one available at http://www.stat.wvu.edu/SRS/Modules/ProbDef/probdef.html

Mathematical Expectation

> If the probabilities of obtaining amount a_1 , a_2 , …, a_k are p_1 , p_2 , …, p_k respectively, then the **mathematical expectation** is
>
> $E = a_1p_1 + a_2p_2 + \ldots a_kp_k$

This can be expressed using the sigma-notation, namely

$$E = \sum a \cdot p \qquad (8)$$

For example, suppose that you play roulette and you pay $1 to play but get $30 when the ball falls on your number, say 17. What is the expected value of your gains/losses.

This can be summarized in the following table, where success denotes when the ball falls on 17, failure otherwise:

event	**net gain/loss**	**probability**	***ap***
success	29	1/38	29/38
failure	-1	37/38	- 37/38
Expectation	**…**	**=**	**-8/38**