# Paraxial Wave Equation

Harmonic Wave obeys Helmholtz

$$(\nabla^2 + k^2) E(x,y,z) = 0$$

Take $\quad E(x,y,z) = \psi(x,y,z)\, e^{-ikz}$

Recall $\quad \nabla E = \frac{\partial E}{\partial x}\hat{x} + \frac{\partial E}{\partial y}\hat{y} + \frac{\partial E}{\partial z}\hat{z} \quad$ vector

$$\nabla^2 E = \nabla \cdot (\nabla E)$$

divergence of gradient
ie: vector transformed into a scalar

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial E}{\partial x}, \frac{\partial E}{\partial y}, \frac{\partial E}{\partial z}\right)$$

$$= \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2}$$

So $$\nabla^2 E = \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) e^{-ikz} + \frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial z} e^{-ikz} - ik\psi e^{-ikz}\right)$$

$$= \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) e^{-ikz} + \left(-ik\frac{\partial \psi}{\partial z} e^{-ikz} + \frac{\partial^2 \psi}{\partial z^2} e^{-ikz} - ik\left[\frac{\partial \psi}{\partial z} e^{-ikz} - ik\psi e^{-ikz}\right]\right)$$

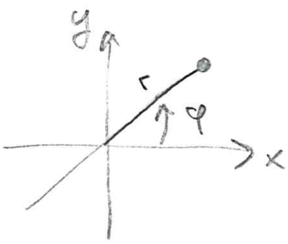
$$= \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right) e^{-ikz} - 2ik\frac{\partial \psi}{\partial z} e^{-ikz} - \underline{k^2 \psi e^{-ikz}}$$

So

$$0 = \nabla^2 E + k^2 E = \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}\right) e^{-ikz} - 2ik \frac{\partial \psi}{\partial z} e^{-ikz}$$

$$\Rightarrow \boxed{\begin{gathered}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) - 2ik \frac{\partial \psi}{\partial z} = 0 \\ \nabla_\perp^2 \psi - 2ik\, \partial\psi/\partial z = 0\end{gathered}}$$

Cylindrical Symmetry




cartesian → polar

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \psi}{\partial r}\right) + \underbrace{\frac{1}{r^2} \frac{\partial^2 \psi}{\partial \varphi^2}}_{=0}$$

$$\frac{\partial \psi}{\partial \varphi} = 0$$

(symmetry)

$$\boxed{\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \psi}{\partial r}\right) - 2ik \frac{\partial \psi}{\partial z} = 0}$$

Trial Gaussian beam solution.

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right)}_{1} \overbrace{- 2ik\frac{\partial \psi}{\partial z}}^{2} = 0 \text{ ——— } (*)$$

Assume $\psi = e^{-iP(z)} e^{-\frac{ikr^2}{2q(z)}}$.

Term 1

$$\frac{\partial \psi}{\partial r} = -\frac{ikr}{q(z)} e^{-ikr^2/2q(z)} e^{-iP(z)}$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) = \frac{\partial}{\partial r}\left(-\frac{ikr^2}{q(z)} e^{-ikr^2/2q(z)} e^{-iP(z)}\right)$$

$$= \left[-\frac{ikr^2}{q(z)}\right]\left[-\frac{ikr}{q(z)}\right] e^{-\frac{ikr^2}{2q(z)}} e^{-iP(z)} - \frac{2ikr}{q(z)} e^{-\frac{ikr^2}{2q(z)}} e^{-iP(z)}$$

$$= \left[-\frac{k^2r^3}{q^2(z)} - \frac{2ikr}{q(z)}\right] e^{-ikr^2/2q(z)} e^{-iP(z)}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) = -\left(\frac{k^2r^2}{q^2(z)} + \frac{2ik}{q(z)}\right) e^{-ikr^2/2q(z)} e^{-iP(z)}$$

Term 2

$$-2ik\frac{\partial \psi}{\partial z} = -2ik\frac{\partial}{\partial z}\left[e^{-iP(z)} e^{-ikr^2/2q(z)}\right]$$

$$= -2ik\left[\frac{+ikr^2 q'(z)}{2q^2(z)} - iP'(z)\right] e^{-iP(z)} e^{-ikr^2/2q(z)}$$

$$= \left[-2kP'(z) + k^2r^2\frac{q'(z)}{q^2(z)}\right] e^{-iP(z)} e^{-ikr^2/2q(z)}$$

So Eqn (*) becomes:

$$-\frac{k^2r^2}{q^2(z)} - \frac{2ik}{q(z)} - 2kP'(z) + \frac{k^2r^2q'(z)}{q^2(z)} = 0$$

$$\left(-\frac{k^2}{q^2} + \frac{k^2q'}{q^2}\right)r^2 - \left(\frac{2ik}{q} + 2kP'\right) = 0$$

(A) $$-\frac{k^2}{q^2} + \frac{k^2 q'}{q^2} = 0.$$

$$\frac{k^2}{q^2} = \frac{k^2}{q^2}\frac{dq}{dz}.$$

$$\frac{dq}{dz} = 1$$

$$\boxed{q(z) = z + q_0}$$

$q_0 = q(z=0)$
initial condition
[must keep this arbitrary
because it will decide
non-zero beam waist]

(B) $$\frac{2ik}{q} + 2kp' = 0.$$

$$\frac{2i}{q} = -2\frac{dP}{dz}.$$

$$\frac{dP}{dz} = \frac{-i}{z+q_0}$$

constant

$$P(z) = -i\ln(z+q_0) + c$$

$$= -i\ln\left(q_0\left[1+\frac{z}{q_0}\right]\right) + \text{cst}$$

$$= -i\ln q_0 - i\ln\left(1+\frac{z}{q_0}\right) + \text{cst}$$

now since $P(z)$ is a phase term we can choose

$P(z=0) = 0$ and cst $= +i\ln q_0$

so $$\boxed{P(z) = -i\ln\left(1+z/q_0\right)}$$