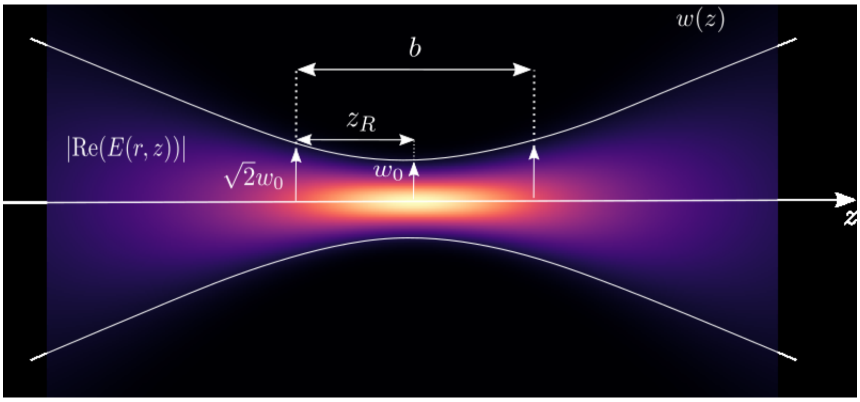
THE UNIVERSITY OF FRANCHE-COMTÉ

MASTER COMPUPHYS - LASER

Homework Problem 2



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Contents

1 Plano-Convex Laser Cavity Stability Condition

1.1 Question 1:

We consider the following system :

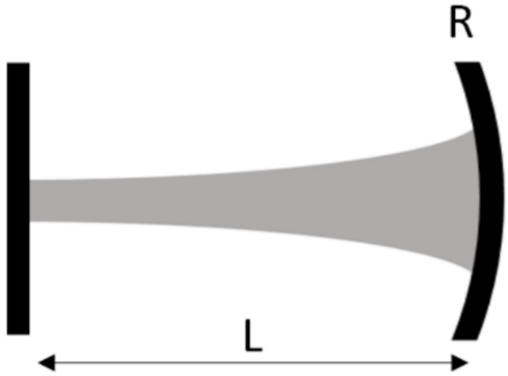


Figure 1: Picture of a Laser cavity consisting of a plane and a concave mirror

We want determine the location of the beam waist in this cavity, we have to show that is given by :

$$w_0{}^2 = \frac{\lambda}{\pi}\sqrt{L(R-L)} \tag{1}$$

We know that the radius of curvature of a Gaussian beams is given by the formula :

$$R(z) = z\left(1 + \left(\frac{z_r}{z}\right)^2\right) \tag{2}$$

With $z_r = \frac{\pi w_0{}^2}{\lambda}$. When the Beam is in $z = L$ the radius of curvature of it is equal to R :

$$R(L) = L\left(1 + \left(\frac{z_r}{L}\right)^2\right) \tag{3}$$

$$R = L\frac{z_r{}^2}{L} \tag{4}$$

$$L(R-L) = z_r{}^2 \tag{5}$$

$$\sqrt{(}L(R-L)) = \frac{\pi w_0{}^2}{\lambda} \tag{6}$$

$$w_0{}^2 = \frac{\lambda}{\pi}\sqrt{L(R-L)} \tag{7}$$

$$\tag{8}$$

1.2 Question 2:

Now we want to verify the stability condition of the previous cavity, we know the stability condition for a Gaussian Beam is $q_s = \frac{Aq_s+B}{Cq_s+D}$ we can find A, B, C, D is our optical system under matrix forms, we start from the concave mirror and we propagate our system until we go back to it :

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix} \begin{pmatrix} 1 & 2L \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 2L \\ -2/R & -4L/R+1 \end{pmatrix}$$

We can easily see that the determinant of it is equal to 1. Now we can transform our stability condition to fin an inequality :

$$q_s = \frac{Aq_s + B}{Cq_s + D} \tag{9}$$
$$C{q_s}^2 + q_s D = Aq_s + D \tag{10}$$
$$C + \frac{D}{q_s} = \frac{A}{q_s} + \frac{B}{{q_s}^2} \tag{11}$$
$$\frac{B}{{q_s}^2} + \frac{A-D}{q_s} - C = 0 \tag{12}$$
$$\tag{13}$$

We have a second degree equation with $x = \frac{1}{q_s}$ so we can solve it :

$$\frac{1}{q_s} = \frac{(D-A) \pm \sqrt{(A-D)^2 + 4BC}}{2B} \tag{14}$$
$$\frac{1}{q_s} = \frac{(D-A) \pm \sqrt{A^2 - 2AD + D^2 + 4AD - 4}}{2B} \tag{15}$$
$$\frac{1}{q_s} = \frac{(D-A) \pm \sqrt{(A+D)^2 - 4}}{2B} \tag{16}$$
$$\tag{17}$$

We can easily do the comparison with :

$$\frac{1}{q_s} = \frac{1}{R} - i\frac{\lambda}{\pi w^2} \tag{18}$$

We have to get a complex number on the right of the equation so :

$$(A+D)^2 = 4 \tag{19}$$
$$\frac{A+D}{2} < 1 \text{ and } \frac{A+D}{2} > -1 \tag{20}$$
$$-1 < \frac{A+D}{2} < 1 \tag{21}$$
$$0 < \frac{A+D}{2} + 1 < 2 \tag{22}$$
$$0 < \frac{A+D+2}{4} < 1 \tag{23}$$
$$0 < \frac{1 - 4L/R + 1 + 2}{4} < 1 \tag{24}$$
$$0 < 1 - L/R < 1 \tag{25}$$
$$-1 < -L/R < 0 \tag{26}$$
$$-R < -L < 0 \tag{27}$$
$$R > L > 0 \tag{28}$$
$$\tag{29}$$

1.3 Question 3 and 4:

Now we can plot the evolution of w_0 over the stability range of L :

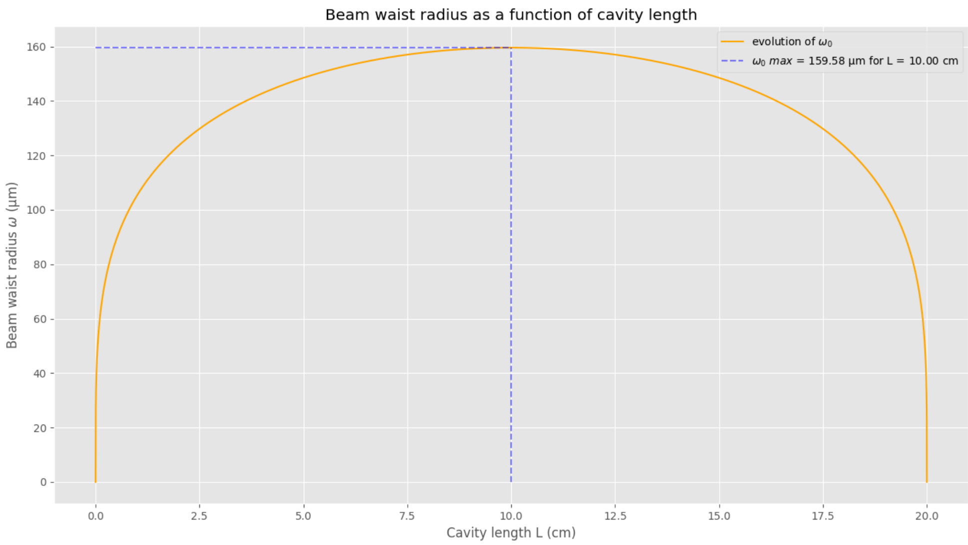
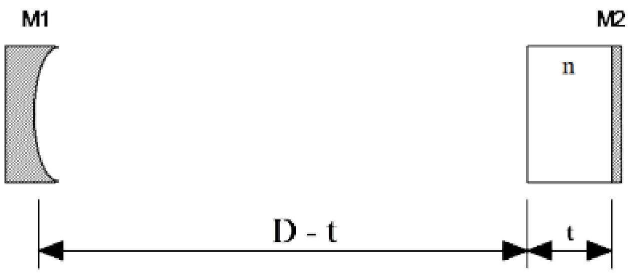


Figure 2: Graph of the evolution of the waist size over the stability range of L

We can see the evolution start with a waist equal to 0 and pass trough a maximum at $L = R/2$ and finally collapse to 0 when L goes closer to R. At the maximum the waist size is equal to $w_0 = 24.56\ nm$ at $L = R/2$.

1.4 Question 5:

Now we want study the stability condition for :



We write our matrix system :

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix} \begin{pmatrix} 1 & (D-t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t/n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t/n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & (D-t) \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & (D-t) \\ -2/R & -\frac{2(D-t)}{R} + 1 \end{pmatrix} \begin{pmatrix} 1 & 2t/n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & (D-t) \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 2t/n + (D-t) \\ -2/R & -\frac{4t}{Rn} - \frac{2(D-t)}{R} + 1 \end{pmatrix} \begin{pmatrix} 1 & (D-t) \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 2t/n + 2(D-t) \\ -2/R & -\frac{4t}{Rn} - \frac{4(D-t)}{R} + 1 \end{pmatrix}$$

We can easily see that $AD - BC = 1$, now we can show the stability condition :

$$
\begin{aligned}
0 &< \frac{A+D+2}{4} < 1 \\
0 &< \frac{1-\frac{4t}{Rn}-\frac{4(D-t)}{R}+1+2}{4} < 1 \\
0 &< \frac{-\frac{4t}{Rn}-\frac{4(D-t)}{R}+4}{4} < 1 \\
0 &< -\frac{t}{Rn}-\frac{(D-t)}{R}+1 < 1 \\
-1 &< -\frac{t}{Rn}-\frac{(D-t)}{R} < 0 \\
R &> \frac{t}{n}+(D-t) > 0 \\
R-\frac{t}{n} &> (D-t) > 0 \\
(D-t) &< R-\frac{t}{n}
\end{aligned}
$$

2 Classical Electron Oscillator Model

2.1 Question 1:

In the classical electron oscillator model of an atomic transition, the polarization is defined like $P(\omega) = \epsilon_0 \chi(\omega) E(\omega)$ where $\chi(\omega)$ is the atomic susceptibility defined like :

$$
\chi(\omega) = -i\left(\frac{Ne^2}{m\omega_0\sigma\varepsilon_0}\right)\frac{1}{1+i2\frac{(\omega-\omega_0)}{\sigma}} \tag{30}
$$

We can write it like $\chi(\omega) = \chi'(\omega) - i\chi''(\omega)$:

$$
\begin{aligned}
\chi(\omega) &= \left(\frac{Ne^2}{m\omega_0\sigma\varepsilon_0}\right)\frac{-i}{1+i2\frac{(\omega-\omega_0)}{\sigma}} = \left(\frac{Ne^2}{m\omega_0\sigma\varepsilon_0}\right)\frac{-i\left(1-i2\frac{(\omega-\omega_0)}{\sigma}\right)}{\left(1+i2\frac{(\omega-\omega_0)}{\sigma}\right)\left(1-i2\frac{(\omega-\omega_0)}{\sigma}\right)} \\
&= \left(\frac{Ne^2}{m\omega_0\sigma\varepsilon_0}\right)\frac{-i-2\frac{(\omega-\omega_0)}{\sigma}}{1-i2\frac{\omega-\omega_0}{\sigma}+i2\frac{\omega-\omega_0}{\sigma}+4\frac{(\omega-\omega_0)^2}{\sigma^2}} = \left(\frac{Ne^2}{m\omega_0\sigma\varepsilon_0}\right)\frac{-i-2\frac{(\omega-\omega_0)}{\sigma}}{1+4\frac{(\omega-\omega_0)^2}{\sigma^2}} \\
&= \left(\frac{Ne^2}{m\omega_0\sigma\varepsilon_0}\right)\frac{-i\sigma^2-2\frac{(\omega-\omega_0)}{\sigma}\sigma^2}{\sigma^2+4(\omega-\omega_0)^2} = \left(\frac{Ne^2}{m\omega_0\varepsilon_0}\right)\frac{-i\sigma-2(\omega-\omega_0)}{\sigma^2+4(\omega-\omega_0)^2} \\
&= \left(\frac{Ne^2}{m\omega_0\varepsilon_0}\right)\frac{-i\sigma-2(\omega-\omega_0)}{\sigma^2+4(\omega-\omega_0)^2}
\end{aligned}
$$

We know $\nu = \omega/2\pi$, so $\nu_0 = \omega_0/2\pi$ and $\Delta v = \sigma/2\pi$

$$
\begin{aligned}
&= \left(\frac{Ne^2}{m\nu_0 2\pi\varepsilon_0}\right)\frac{-i\Delta\nu 2\pi - 2\left(\nu 2\pi - \nu_0 2\pi\right)}{\left(\Delta\nu 2\pi\right)^2 + 4\left(\nu 2\pi - \nu_0 2\pi\right)^2} \\
&= \left(\frac{Ne^2}{m\nu_0\varepsilon_0}\right)\frac{1}{32\pi^3}\frac{-i\Delta\nu 2\pi + 4\pi\left(\nu_0 - \nu\right)}{\left(\Delta\nu/2\right)^2 + \left(\nu - \nu_0\right)^2} \\
&= \left(\frac{Ne^2}{m\nu_0\varepsilon_0}\right)\frac{1}{32\pi^3}\frac{-i\Delta\nu 2\pi + 4\pi\left(\nu_0 - \nu\right)}{\left(\Delta\nu/2\right)^2 + \left(\nu - \nu_0\right)^2} \\
\chi'(v) &= \left(\frac{Ne^2}{m\nu_0\varepsilon_0}\right)\frac{1}{32\pi^3}\frac{4\pi\left(\nu_0 - \nu\right)}{\left(\Delta\nu/2\right)^2 + \left(\nu - \nu_0\right)^2} = \left(\frac{Ne^2}{m\nu_0 8\pi^2\varepsilon_0}\right)\frac{\left(\nu_0 - \nu\right)}{\left(\Delta\nu/2\right)^2 + \left(\nu - \nu_0\right)^2} \\
\chi''(v) &= \left(\frac{Ne^2}{m\nu_0\varepsilon_0}\right)\frac{1}{32\pi^3}\frac{\Delta\nu 2\pi}{\left(\Delta\nu/2\right)^2 + \left(\nu - \nu_0\right)^2} = \left(\frac{Ne^2}{m\nu_0 16\pi^2\varepsilon_0}\right)\frac{\Delta\nu}{\left(\Delta\nu/2\right)^2 + \left(\nu - \nu_0\right)^2}
\end{aligned}
$$

We can see that we can rewrite $\chi(\nu)$ like $\chi'(\nu) - i\chi''(\nu)$

2.2 Question 2:

Now we want to show that $\Delta\nu$ is the FWHM, first we will evaluate $\chi''(\nu_0)/2$ to have the half maximum, and we will show that we can reach this value in $\nu_0 + \Delta\nu/2$ and $\nu_0 - \Delta\nu/2$:

$$
\begin{aligned}
\chi''(\nu_0)/2 &= \left(\frac{Ne^2}{m\nu_0 16\pi^2\varepsilon_0}\right)\frac{\Delta\nu}{2\left(\left(\Delta\nu/2\right)^2 + \left(\nu_0 - \nu_0\right)^2\right)} \\
\chi''(\nu_0)/2 &= \left(\frac{Ne^2}{m\nu_0 16\pi^2\varepsilon_0}\right)\frac{\Delta\nu}{2\Delta\nu^2/4} \\
\chi''(\nu_0)/2 &= \left(\frac{Ne^2}{m\nu_0 16\pi^2\varepsilon_0}\right)\frac{2}{\Delta\nu}
\end{aligned}
$$

We can evaluate $\chi''(\nu_0 + \Delta\nu/2)$:

$$
\begin{aligned}
\chi''(\nu_0 + \Delta\nu/2) &= \left(\frac{Ne^2}{m\nu_0 16\pi^2\varepsilon_0}\right)\frac{\Delta\nu}{\left(\Delta\nu/2\right)^2 + \left(\nu_0 + \Delta\nu/2 - \nu_0\right)^2} \\
\chi''(\nu_0 + \Delta\nu/2) &= \left(\frac{Ne^2}{m\nu_0 16\pi^2\varepsilon_0}\right)\frac{\Delta\nu}{\left(\Delta\nu/2\right)^2 + \left(\Delta\nu/2\right)^2} \\
\chi''(\nu_0 + \Delta\nu/2) &= \left(\frac{Ne^2}{m\nu_0 16\pi^2\varepsilon_0}\right)\frac{\Delta\nu}{2\Delta\nu^2/4} \\
\chi''(\nu_0 + \Delta\nu/2) &= \left(\frac{Ne^2}{m\nu_0 16\pi^2\varepsilon_0}\right)\frac{2}{\Delta\nu}
\end{aligned}
$$

We can evaluate $\chi''(\nu_0 - \Delta\nu/2)$:

$$\chi''(\nu_0 - \Delta\nu/2) = \left(\frac{Ne^2}{m\nu_0 16\pi^2\varepsilon_0}\right)\frac{\Delta\nu}{(\Delta\nu/2)^2 + (\nu_0 - \Delta\nu/2 - \nu_0)^2}$$
$$\chi''(\nu_0 - \Delta\nu/2) = \left(\frac{Ne^2}{m\nu_0 16\pi^2\varepsilon_0}\right)\frac{\Delta\nu}{(\Delta\nu/2)^2 + (-\Delta\nu/2)^2}$$
$$\chi''(\nu_0 - \Delta\nu/2) = \left(\frac{Ne^2}{m\nu_0 16\pi^2\varepsilon_0}\right)\frac{\Delta\nu}{2\Delta\nu^2/4}$$
$$\chi''(\nu_0 - \Delta\nu/2) = \left(\frac{Ne^2}{m\nu_0 16\pi^2\varepsilon_0}\right)\frac{2}{\Delta\nu}$$

We can see $\chi''(\nu_0)/2 = \chi''(\nu_0 + \Delta\nu/2) = \chi''(\nu_0 - \Delta\nu/2)$
We have show that $\Delta\nu$ is the FWHM.

2.3 Question 3:

We can study the effect of susceptibility on an incident plan wave because this modify the expression of the wave vector :

$$\vec{E} = \vec{E}_0 e^{i(\omega t - kz)} \text{ with } k' = k\left(1 + \frac{\chi'(\omega)}{2n^2}\right) - ik\frac{\chi''(\omega)}{2n^2} \tag{31}$$

We can replace k by k' :

$$\vec{E} = \vec{E}_0 e^{i(\omega - k'z)} \tag{32}$$
$$= \vec{E}_0 e^{i\left(\omega t - z\left(k\left(1+\frac{\chi'(\omega)}{2n^2}\right) - ik\frac{\chi''(\omega)}{2n^2}\right)\right)} \tag{33}$$
$$= \vec{E}_0 e^{iwt} e^{-izk\left(1+\frac{\chi'(\omega)}{2n^2}\right)} e^{-zk\frac{\chi''(\omega)}{2n^2}} \tag{34}$$
$$\tag{35}$$

we can see this leads to an additional phase delay (dispersion) that depends on $\chi'(\omega)$ and an absorption term that depends on $\chi''(\omega)$

2.4 Question 4:

Now we have a fours levels system like :

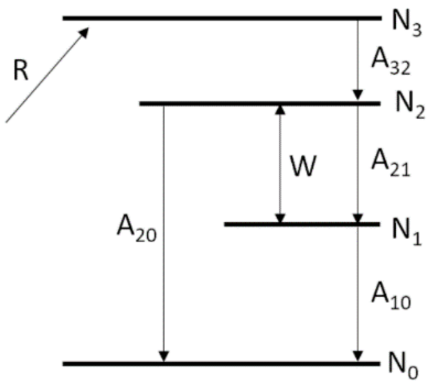


Figure 3: Picture of a fourth system laser

The rate equation for N_1, N_2 are :

$$
\frac{dN_2}{dt} = R - W\left(N_2 - N_1\right) - N_2\left(A_{21} + A_{20}\right) \tag{36}
$$

$$
\frac{dN_1}{dt} = W\left(N_2 - N_1\right) + N_2 A_{21} - N_1 A_{10} \tag{37}
$$

We want know the steady state inversion $N_2 - N_1$, we are in steady state so $\frac{dN_i}{dt} = 0$:

$$
0 = R - W\left(N_2 - N_1\right) - N_2\left(A_{21} + A_{20}\right) \Longrightarrow R = W\left(N_2 - N_1\right) + N_2\left(A_{21} + A_{20}\right) \tag{38}
$$

$$
0 = W\left(N_2 - N_1\right) + N_2 A_{21} - N_1 A_{10} \tag{39}
$$

We can put that under matrix form :

$$
\begin{pmatrix} -W & W + A_{21} + A20 \\ -W - A_{10} & W + A_{21} \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} R \\ 0 \end{pmatrix}
$$

We can easily solve that by Cramer's rule :

for N_1 :

$$
\begin{aligned}
N_1 &= \frac{\begin{vmatrix} R & W + A_{21} + A20 \\ 0 & W + A_{21} \end{vmatrix}}{\begin{vmatrix} -W & W + A_{21} + A20 \\ -W - A_{10} & W + A_{21} \end{vmatrix}} \\
N_1 &= \frac{R\left(W + A_{21}\right)}{-W\left(W + A_{21}\right) - \left[\left(-W - A_{10}\right)\left(W + A_{21} + A_{20}\right)\right]} \\
N_1 &= \frac{R\left(W + A_{21}\right)}{-W^2 - WA_{21} - \left[-W^2 - WA_{21} - WA_{20} - WA_{10} - A_{10}\left(A_{21} + A_{20}\right)\right]} \\
N_1 &= \frac{R\left(W + A_{21}\right)}{-W^2 - WA_{21} + W^2 + WA_{21} + WA_{20} + WA_{10} + A_{10}\left(A_{21} + A_{20}\right)} \\
N_1 &= \frac{R\left(W + A_{21}\right)}{WA_{20} + WA_{10} + A_{10}\left(A_{21} + A_{20}\right)}
\end{aligned}
$$

for N_2 :

$$
\begin{aligned}
N_2 &= \frac{\begin{vmatrix} -W & R \\ -W - A_{10} & 0 \end{vmatrix}}{\begin{vmatrix} -W & W + A_{21} + A20 \\ -W - A_{10} & W + A_{21} \end{vmatrix}} \\
N_2 &= \frac{R\left(W + A_{10}\right)}{WA_{20} + WA_{10} + A_{10}\left(A_{21} + A_{20}\right)}
\end{aligned}
$$

Now We can express $N_2 - N_1$:

$$
\begin{aligned}
N_2 - N_1 &= \frac{R\left(W + A_{10}\right)}{WA_{20} + WA_{10} + A_{10}\left(A_{21} + A_{20}\right)} - \frac{R\left(W + A_{21}\right)}{WA_{20} + WA_{10} + A_{10}\left(A_{21} + A_{20}\right)} \\
N_2 - N_1 &= \frac{R\left(W + A_{10}\right) - R\left(W + A_{21}\right)}{WA_{20} + WA_{10} + A_{10}\left(A_{21} + A_{20}\right)} \\
N_2 - N_1 &= \frac{RW + RA_{10} - RW - RA_{21}}{A_{10}\left(A_{21} + A_{20}\right)\left(1 + \frac{WA_{20} + WA_{10}}{A_{10}(A_{21} + A_{20})}\right)} \\
N_2 - N_1 &= \frac{\frac{R(A_{10} - A_{21})}{A_{10}(A_{21} + A_{20})}}{1 + W\frac{(A_{20} + A_{10})}{A_{10}(A_{21} + A_{20})}} = \frac{\Delta N_0}{1 + \frac{W}{W_{sat}}}
\end{aligned}
$$

We have $\Delta N_0 = \frac{R(A_{10} - A_{21})}{A_{10}(A_{21} + A_{20})}$ and $W_{sat} = \frac{A_{10}(A_{21} + A_{20})}{(A_{20} + A_{10})}$