Statistical exploitation of measurements

Tutorial Booklet

MASTER
FUNDAMENTAL PHYSICS AND APPLICATIONS

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CHAPTER 1 : BAYES THEOREM

Background The general form of Bayes Theorem giving the conditional probability of event A_i given event B is expressed as follows:

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{n} P(A_i)P(B|A_i)}$$

Where $A_1, A_2, \cdots A_i, \cdots A_n$ are set of n mutually exclusive and exhaustive events.

PROBLEM 1.1

3 urns contain 6 red, 4 black; 4 red, 6 black, and 5 red, 5 black balls respectively. One of the urns is selected at random and a ball is drawn from it. If the ball drawn is red, find the probability that it is drawn from the first urn. Let E_1, E_2, E_3, and A be the events defined as follows:

- E_1: urn first is chosen
- E_2: urn second is chosen
- E_3: urn third is chosen
- A = ball drawn is red

1. Considering that one of the three urns is chosen at random, what are the probabilities $P(E_i)$ of the events E_i?
2. If E_1 occurred, what is the conditional probability $P(A|E_1)$ of drawing a red ball from it ?
3. Similarly, what are the conditional probabilities $P(A|E_2)$ and $P(A|E_3)$?
4. What the probability of drawing a red ball $P(A)$?
5. Applying the Bayes theorem, given that the ball drawn is red, what is the conditional probability $P(E_1|A)$ that it is drawn from the first urn.

PROBLEM 1.2

An insurance company insured 2000 scooter drivers, 4000 car drivers, and 6000 truck drivers. The probability of an accident involving a scooter driver, car driver, and a truck is 0.01, 0.03, and 0.015 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?

1. According to the same principle as in problem 1.1, define the events E_1, E_2, E_3, and A.
2. What are the probabilities $P(E_i)$ of the events E_i?
3. According to the data of the problem, quantify and state the meaning of the following conditional probabilities: $P(A|E_1)$, $P(A|E_2)$ and $P(A|E_3)$.
4. Express and calculate the conditional probability corresponding to the following question:
   "One of the insured persons meets with an accident. What is the probability that he is a scooter driver? "

CHAPTER 2 : ESTIMATION

PROBLEM 2.1: DEMONSTRATION OF THE CENTRAL-LIMIT THEOREM

Using Matab, Octave or Python software, reproduce the results from Section 5 of Chapter 2 of the lecture booklet, showing that the frequency distribution of the sum of N Bernouilli trials $\{X_1, \ldots, X_i, \ldots\ldots, X_N\}$ tends toward a Gaussian law centered at $\frac{N}{2}$ and with a variance $\frac{N}{4}$. Show that the probability $p(Z_N)$ of random variable Z_N (Eq. 2.8, pp. 10) tends toward a Gaussian law of mean 0 and variance 1.

CHAPTER 4 : HYPOTHESIS TESTING

PROBLEM 4.1 : CHI-2 TEST OF FAIRNESS OF A COIN TOSS

Consider the following coin-toss experiment. We flip a coin 20 times, getting 12 "heads" and 8 "tails."

1. Using the binomial distribution (you can use *pdf* function with Matlab, Octave or Python), calculate the exact probability of getting 12H/8T and any of the other possible outcomes (from 0H/20T to 20H/0T). Remember, for the binomial distribution, we must define k (the number of successes), N (the number of Bernoulli trials) and p (the probability of success). Here, $N = 20$ and $p = 0.5$ (if our hypothesis is that the coin is "fair").
2. Now, let's test the hypothesis that the coin is fair. To do this, calculate the probability (*p-value*) of seeing our observed result (12 heads/8 tails) or any other result that is as far or farther from the expected result (10 heads/10 tails). Following the convention of failing to reject a hypothesis if *p-value* > 0.05 (corresponding to a confidence of 95%), do we fail or succeed to reject the hypothesis that the coin is fair?

You should note that many statistical packages for computers can calculate exact *p*-values for chi-square distributed test statistics. However, it is common for people to simply refer to chi-square tables. Table 4.1 gives some typical critical value of the chi-square function with different degree of freedom.

d.f.	*p*=0.9	*p*=0.5	*p*=0.1	*p*=0.05	*p*=0.01
1	0.016	0.455	2.706	3.841	6.635
2	0.211	1.386	4.605	5.991	9.210
3	0.584	2.366	6.251	7.815	11.345

Table 4.1 : chi-square table. The first column lists degrees of freedom. The top row shows the *p*-value in question. The cells of the table give the **critical value** of chi-square for a given *p*-value and a given number of degrees of freedom.

The formula for the chi-square test statistic is: $\sum_{i=1}^{N} \frac{(obs_i - exp_i)^2}{exp_i}$, where N is the number of possible outcomes. In the coin-flipping experiment, $N = 2$. When $i = 1$, we could be talking about "heads." Therefore, when $i = 2$, we'd be talking about "tails." For each outcome, there is an observed value (obs_i) and an expected value (exp_i). We are summing $(obs_i - exp_i)^2/exp_i$ for each outcome.

3. According to the result of the coin-toss experiment describe in this problem, fill in the table with the appropriate numerical values

Outcome Class	Observed Number of Occurrences of Outcome	Probability of Outcome Class	Expected Number of Occurrences of Outcome	$(obs_i - exp_i)^2 / exp_i$
"heads"				
"tails"				
Sum				

4. For this type of test, the number of degrees of freedom is simply the number of outcome classes minus one. Since we have two outcome classes ("heads" and "tails"), we have 1 degree of freedom. Going to the chi-square table 4.1, look in the row for 1 *d.f.* to see where the sum of last column lies and deduce the range including p (or calculate with a software the corresponding exact value of p). Conclude about the hypothesis that the coin is fair.
5. Repeat the chi-square goodness-of-fit test for the larger sample size (4865 heads/135 tails) and conclude again about the hypothesis that the coin is fair. Compare the result of this test with the previous one performed with 20 flip of the coin.