Econometrics I

José Luis Montiel Olea

Introduction to Probability and Statistics for Economists
(Ph.D. in Economics, 1st year)

Lectures 1 and 2

Two Main Blocks

1. Probability Theory

2. Mathematical Statistics

Probability Theory

How to model 'randomness'?

Things we will learn in Lecture 1:

- ⋆ What is a probability space?
    1. What is measurable space?
    2. What is a probability space?
- ⋆ What is a random variable?
- ⋆ What is the 'distribution' or 'law' of a random variable?

What is a probability space?

$$(\Omega, \mathcal{F}, \mathbb{P})$$

Probability Space: Two components

$$(\Omega, \mathcal{F}, \mathbb{P})$$

1. $(\Omega, \mathcal{F})$ measurable space.

$$\Omega: \text{ States of the world,} \quad \mathcal{F}: \text{ Set of events } (\subseteq \Omega).$$

2. $\mathbb{P} : \mathcal{F} \to [0, 1]$. Probability Measure

'How likely is an event in $\mathcal{F}$'

Measurable Space

See notes.

Probability Measures

- $\mathbb{P}(\phi) = 0$, $\mathbb{P}(\Omega) = 1$ (Normalization)
- For any finite collection $A_1, A_2, \ldots A_m$ such that $A_i \cap A_j = \emptyset$

$$\mathbb{P}\Big(\cup_{i=1}^{m} A_i \Big) = \sum_{i=1}^{m} \mathbb{P}(A_i)$$

  This property is called additivity.
- If you replace finite by *countably infinite*, Property 2 is called σ-additivity.

Important

Normalization and σ-additivity define a probability measure

What is a random variable?

$$X : \Omega \to S$$

Random Variable

$$X : \Omega \to S$$

- $\star$ Ω : Set of states of the world.
- $\star$ S : Image Space
- $\star$ X : Random Variable

What is the distribution or law of a random variable?

'Induced' Probability of a Random Variable

⋆ The probability $\mathbb{P}$ on Ω induces a probability on subsets of S:

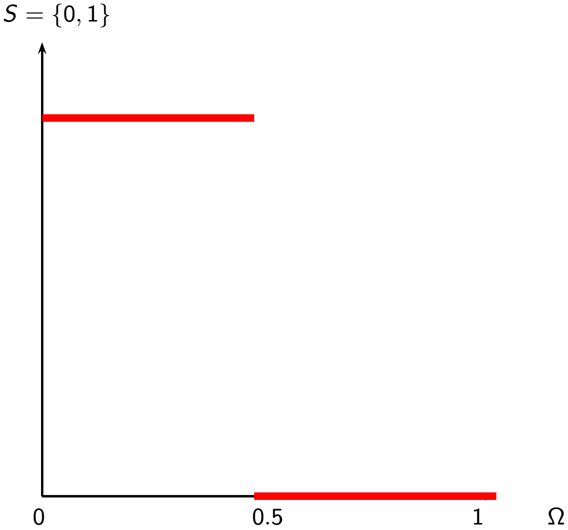
$$\mathbb{P}_X[F] \equiv \mathbb{P}[\,\{\omega \mid X(\omega) \in F\}\,], \quad F \subseteq S$$

⋆ How likely are the states of the world in which F occurs?

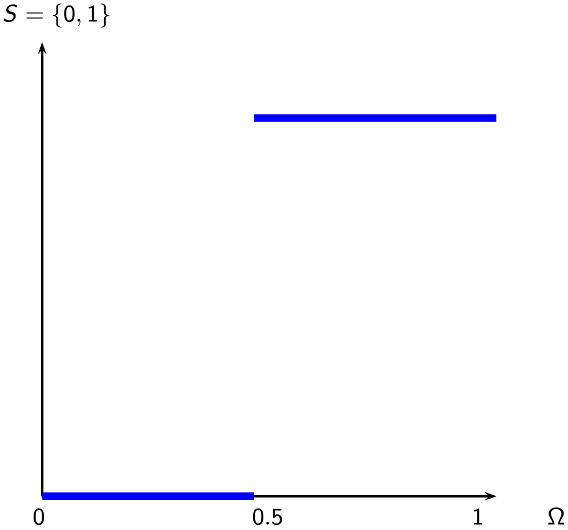
The induced probability of a random variable is usually called its DISTRIBUTION OR LAW

Different random variables can induce the same probability on S.

$$P_{X_2}(1) = .5$$



$$P_{X_2}(1) = .5$$



The Cumulative Distribution Function of Real-valued Random Variables

C.D.F OF AN $\mathbb{R}$-VALUED RANDOM VARIABLE

- ⋆ How likely is a realization of the random variable X below x?
- ⋆ The c.d.f. summarizes this information

$$F_X : \mathbb{R} \to [0, 1]$$

$$F_X(x) \equiv \mathbb{P}\Big\{\omega \in \Omega \mid X(\omega) \leq x\Big\}$$

Examples of c.d.f.

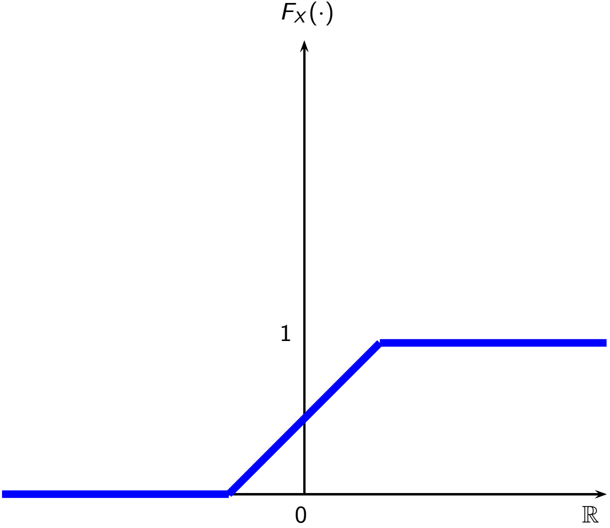
Uniform distribution

Bernoulli distribution

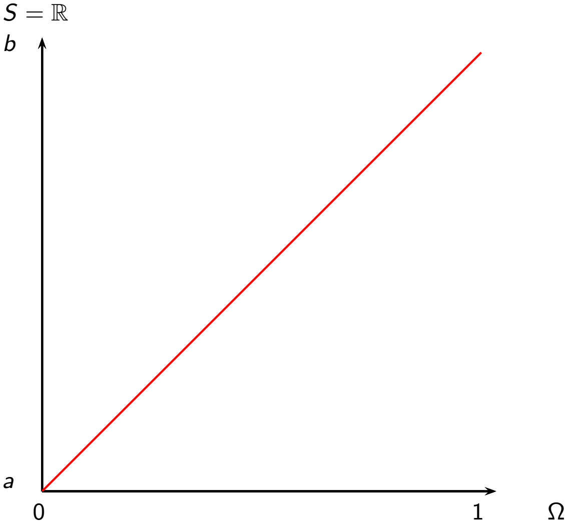
Skip to Properties

$$F_X(x) = \begin{cases} 0 & \text{if} \quad x < a \\ (x-a)/b-a & \text{if} \quad x \in [a,b) \\ 1 & \text{if} \quad x \geq b \end{cases}$$

Uniform Distribution on $[a, b]$

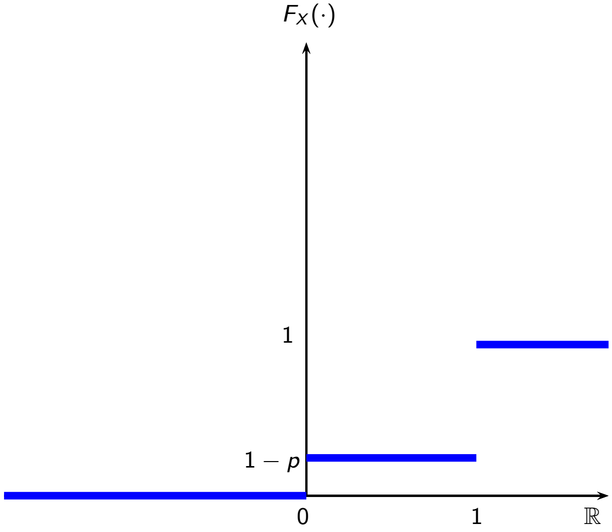


$$X(\omega) = a + \omega[b - a]$$

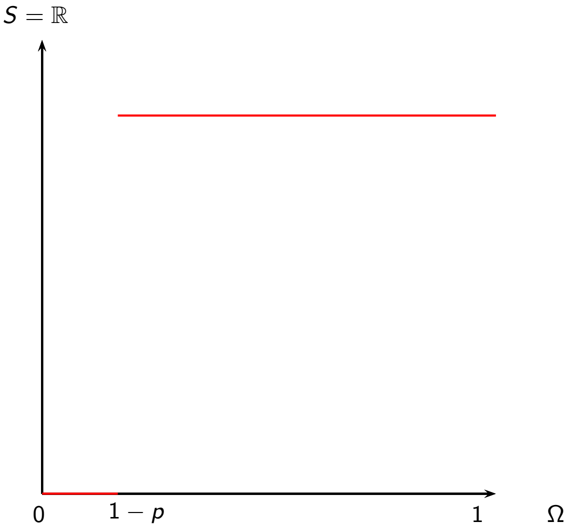


$$F_X(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ 1 - p & \text{if} \quad x \in [0, 1) \\ 1 & \text{if} \quad x \geq 1 \end{cases}$$

Bernoulli Distribution with parameter p



$$X(\omega) = \mathbf{1}[\omega \geq 1 - p]$$



What are the common properties of c.d.f?

Characterization

1. F_X is non-decreasing
2. $\lim_{x \to \infty} F_X(x) = 1$
3. $\lim_{x \to -\infty} F_X(x) = 0$
4. $\lim_{h \to 0^+} F_X(x+h) = F_X(x)$

In fact, these 4 properties characterize the induced c.d.f. of a real-valued random variable!

Discrete and Continuous Type of Real-Valued Random Variables

Discrete Distributions / Discrete r.v.s

Distribution: Disributions for which $\exists$ a countable set

$$\text{Supp} = \{x_1, x_2, \ldots\}, x_i \in \mathbb{R},$$

such that

a) $\mathbb{P}_X(X = x_i) > 0 \quad \forall \quad x_i \in \text{Supp}$

b) $\sum_{x_i \in \text{Supp}} \mathbb{P}_X(X = x_i) = 1$

are called discrete.

P.M.F.

We will identify discrete distributions/r.v.s by its support and its p.m.f.

(Absolutely) Continuous Distributions

Random Variables for which

$$F_X(x) = \int_{-\infty}^{x} f(z)dz$$

for some $f(z) \geq 0 \ \forall z \in \mathbb{R}$ are called:

(Absolutely) Continuous

The function $f(z)$ is called

Probability Density Function (p.d.f.)

We will identify continuous distributions/r.v.s by its p.d.f.

The set

$$\{z \in \mathbb{R} \mid f(z) > 0\} \subseteq \mathbb{R}$$

is called the support of the continuous r.v.

From p.d.f. to probabilities

Let X be a real-valued random variable with p.d.f. $f(z)$

- ⋆ $\mathbb{P}_X[X \leq a] = \int_{-\infty}^{a} f(z)dz$
- ⋆ $\mathbb{P}_X[a \leq X \leq b] = \int_{a}^{b} f(z)dz$
- ⋆ $\mathbb{P}_X[X > a] = \int_{a}^{\infty} f(z)dz$

Examples of (Univariate) Discrete Distributions

▸ Bernoulli ▸ Binomial

⏩ Skip to Continuous Distributions

Bernoulli Distribution (p), $p \in (0,1)$

$\star$ The Bernoulli distribution with parameters p has support:

$$\text{Supp} = \{0, 1\}$$

and p.m.f. given by:

$$\mathbb{P}_X(X = x) = p^x (1-p)^{1-x} \quad x \in \{0, 1\}$$

How do we know it is a p.m.f.?

Two parts:

a) $\mathbb{P}_X(X = x) > 0 \quad \forall\, x \in \{0, 1\}$. Easy to verify:

$$p^x(1-p)^{1-x} > 0$$

b)

$$\sum_{x \in \{0,1\}} p^x(1-p)^{1-x} = (1-p) + p$$

Examples

BINOMIAL DISTRIBUTION $(n, p),\ n \in \mathbb{N}, p \in (0, 1)$

⋆ The binomial distribution with parameters (n, p) has support:

$$\text{Supp} = \{0, 1, 2, \ldots n\}$$

and p.m.f. given by:

$$\mathbb{P}_X(X = x) \equiv \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} \quad x \in \text{Supp}$$

How do we know it is a p.m.f.?

Two parts:

a) $\mathbb{P}_X(X = x) > 0 \quad \forall\, x \in \{0, 1, 2, \ldots n\}$. Easy to verify:

$$\frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} > 0$$

b)

$$\sum_{x \in \{0,1,\ldots n\}} \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} = 1?$$

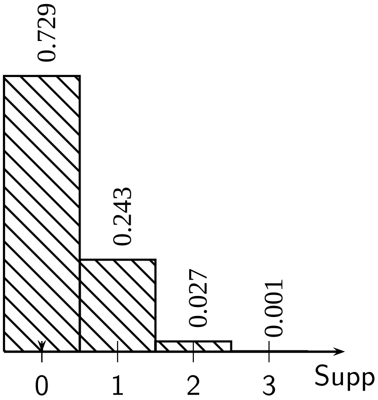
How do we know it is a p.m.f.?

Use the Binomial Theorem

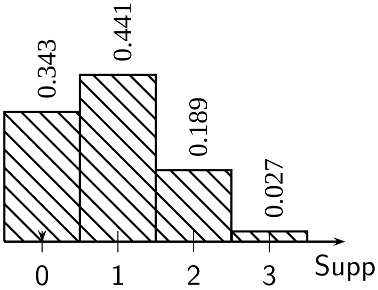
$$(a+b)^n = \sum_{x \in \{0,1,\ldots n\}} \frac{n!}{(n-x)!x!} a^x b^{n-x} = 1$$

$$a = p, \quad b = 1 - p$$

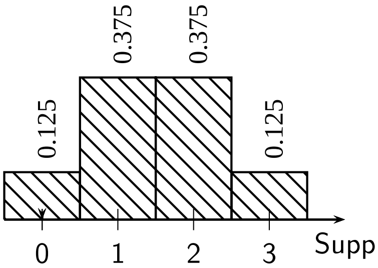
Binomial (3, .1)



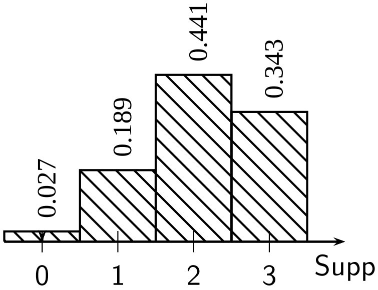
Binomial (3, .3)



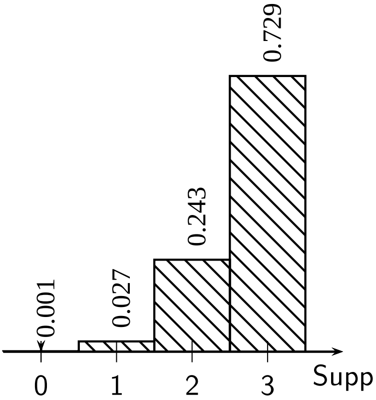
Binomial (3, .5)



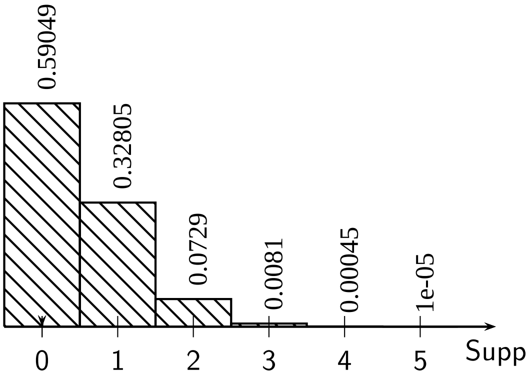
Binomial (3, .7)



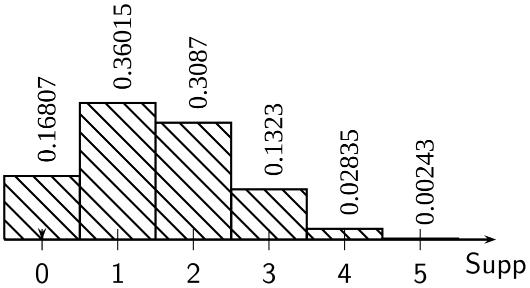
Binomial (3, .9)



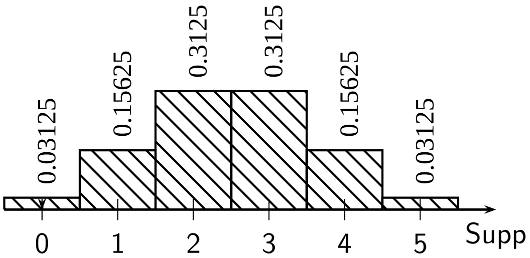
Binomial $(5, .1)$



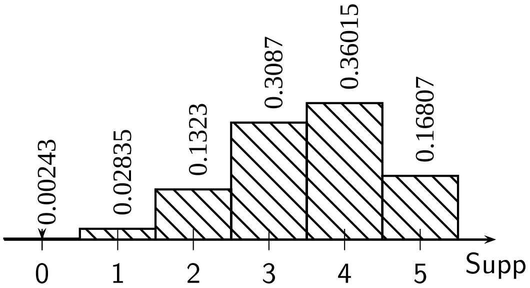
Binomial (5, .3)



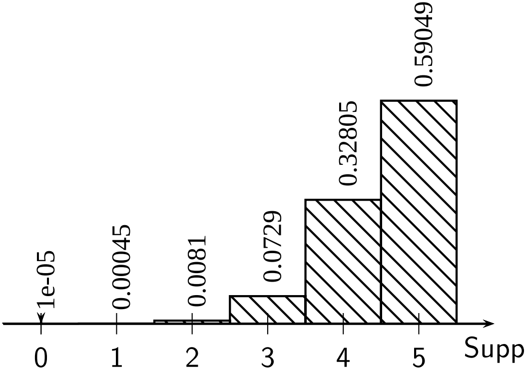
BINOMIAL (5, .5)



Binomial (5, .7)



Binomial (5, .9)



◂ Examples

Examples of (Univariate) Continuous Distributions

Uniform Normal

Skip to Moments

Uniform [a,b]

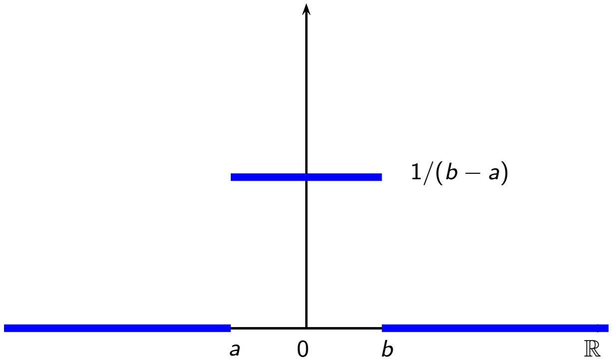
- ⋆ The uniform distribution with parameters $[a, b]$ has p.d.f.

$$f(z) = \frac{1}{b-a}\mathbf{1}\{z \in [a, b]\}$$

- ⋆ and support $[a, b]$

p.d.f. of the Uniform Distribution on $[a, b]$

$$f(z) = \begin{cases} 0 & \text{if} \quad z < a \\ 1/(b-a) & \text{if} \quad z \in [a, b] \\ 0 & \text{if} \quad z > b \end{cases}$$



Normal Distribution (μ, σ^2)

- $\star$ The normal distribution with parameters (μ, σ^2) has p.d.f.

$$f(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\Big(-\frac{1}{2\sigma^2}(z-\mu)^2\Big)$$

- $\star$ and support $\mathbb{R}$

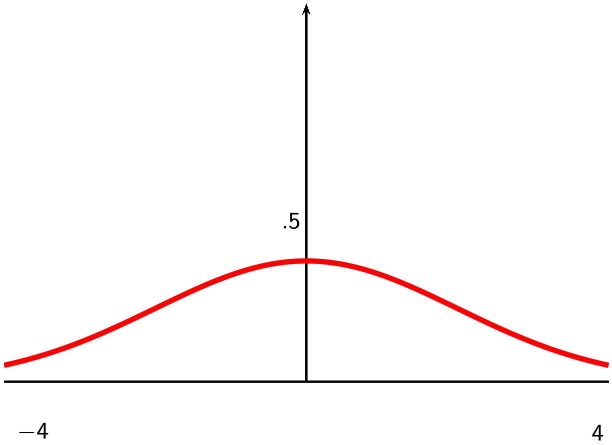
Normal Distribution (μ, σ^2)

- $\star$ $f(z) > 0\checkmark$
- $\star$ How do we know that:

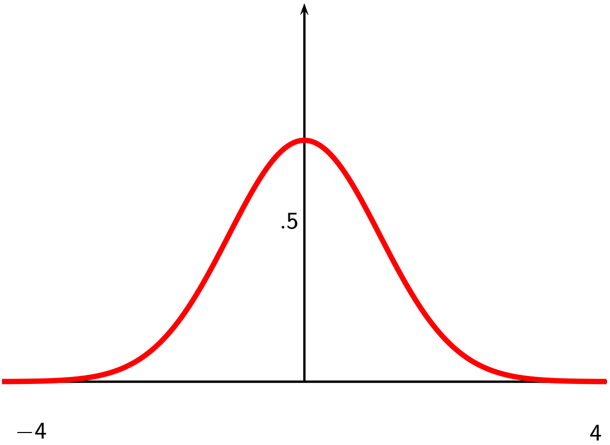
$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(z-\mu)^2 \right) = 1?$$

Euler-Poisson Integral/Gaussian Integral

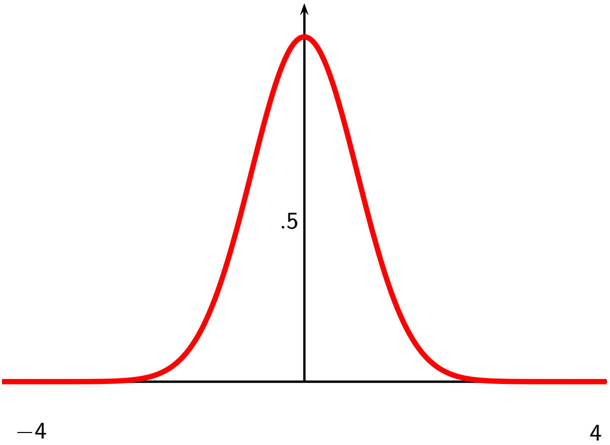
$N(0,1)$



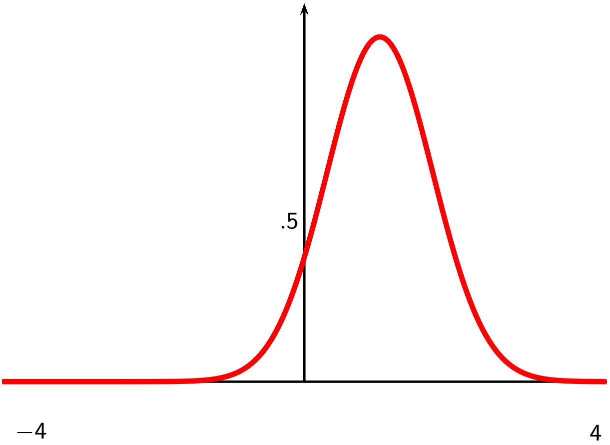
$N(0, .5)$



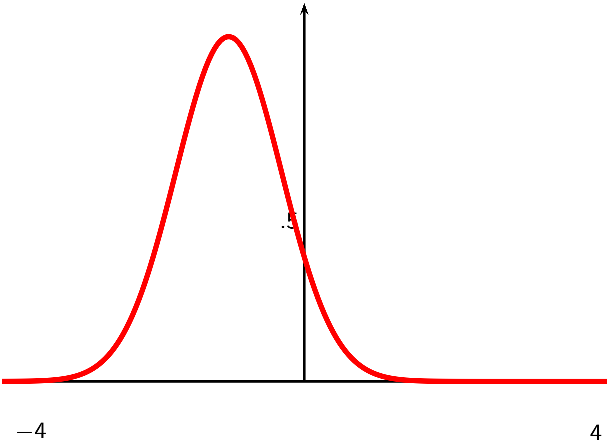
$N(0, .35)$



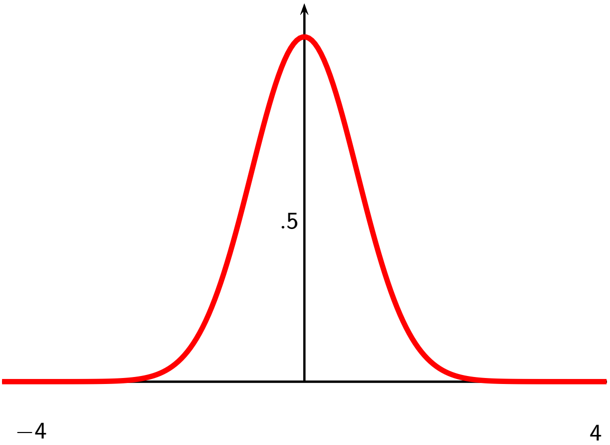
$N(.5, .35)$



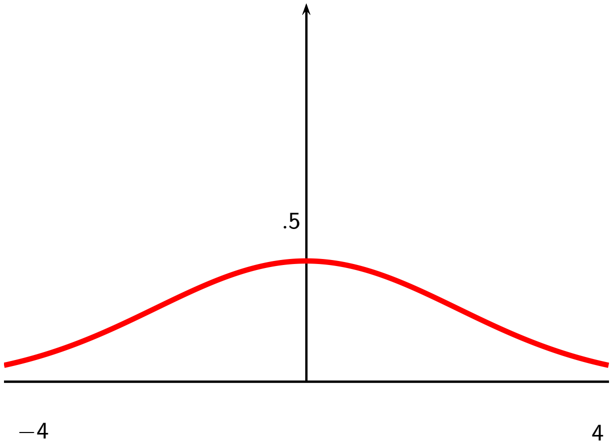
$N(-5, .35)$



$N(0, .35)$



N(0,1)



Moments

Mean of a discrete r.v.

- ⋆ Let X be a discrete r.v. with support S and p.m.f. $\mathbb{P}_X$
- ⋆ The mean or expected value of X is defined as:

$$\mathbb{E}_{\mathbb{P}_X}[X] \equiv \mu \equiv \sum_{x_n \in S} x_n \, \mathbb{P}_X[X = x_n]$$

- ⋆ The variance of X is defined as:

$$\mathbb{E}_{\mathbb{P}_X}[(X-\mu)^2] \equiv \sigma^2 \equiv \sum_{x_n \in S} (x_n - \mu)^2 \, \mathbb{P}_X[X = x_n]$$

Mean of a Continuous Type r.v.

⋆ For a continuous-type random variable $g(X)$, where $X \sim f_X$:

$$\mathbb{E}_{f_X}[g(X)] \equiv \int_{-\infty}^{\infty} g(z) f_X(z) dz$$

⋆ Therefore, the mean and the variance of X are given by:

$$\begin{aligned} \mu &= \int_{-\infty}^{\infty} z f_X(z) dz \\ \sigma^2 &= \int_{-\infty}^{\infty} (z-\mu)^2 f_X(z) dz \end{aligned}$$

Examples

▸ Mean of a Uniform [a,b]

▸ Second Moment of a Uniform [a,b]

⏩ Skip to Moment Generating Function

U[A,B]: μ

- $\mu = \frac{b+a}{2}$
- $f(z) = \frac{1}{b-a}\mathbf{1}\{z \in [a,b]\}$
- Derivation:

$$\begin{aligned} \mu = \int_{-\infty}^{\infty} z f(z) dz &= \frac{1}{b-a}\int_a^b z dz \\ &= \frac{1}{2}\frac{1}{b-a} z^2 \Big]_a^b \\ &= \frac{1}{2}\frac{1}{b-a}(b^2 - a^2) \\ &= \frac{b+a}{2} \end{aligned}$$

U[A,B]: $\mathbb{E}[X^2]$

- ⋆ $\mathbb{E}[X^2] = \frac{b^2+ab+a^2}{3}$
- ⋆ Derivation:

$$\begin{aligned}
\mathbb{E}[X^2] = \int_{-\infty}^{\infty} z^2 f(z)dz &= \frac{1}{b-a}\int_a^b z^2 dz \\
&= \frac{1}{3}\frac{1}{b-a}z^3\Big]_a^b \\
&= \frac{1}{3}\frac{1}{b-a}(b^3-a^3) \\
&= \frac{b^2+ab+a^2}{3}
\end{aligned}$$

Hence, σ^2 is given by:

$$\mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{b^2 + ab + a^2}{3} - \frac{b^2 + 2ab + a^2}{4} = \frac{(b-a)^2}{12}$$

Moment Generating Function

MGF

- ⋆ The $\mathbb{R}$-valued r.v. X has moment generating function $m_X(\cdot)$ if

$$m_x(t) \equiv E_F\Big[\exp\Big(tX\Big)\Big] < \infty, \quad \forall\, t \in (-\epsilon, \epsilon).$$

- ⋆ The k-th moment of X is the k-th derivative of $m_x(t)$ at $t = 0$.

Example

⋆ $X \sim \text{Bernoulli}(p)$

$$\begin{aligned} m_X(t) &= E_F\left[\exp\left(tX\right)\right] \\ &= p\exp(t) + (1-p)\exp(0) \end{aligned}$$

⋆ Moments:

$E_F[X] = p$

$E_F[X^2] = p$

$\vdots$

$E_F[X^k] = p$