Appendix[33]

A.1 Partial Identification: Example

Let's assume that: (a) $f(\Delta\nu_j|X_i^*, Z_i, X_i) = f(\Delta\nu_j)$; (b) $f(X_i^*) = (1/\sigma)\phi(X_i^*/\sigma)$; (c) $f(Z_i|X_i, X_i^*) = (1/\sigma_Z)\phi((Z_i - X_i^*)/\sigma_Z)$; (d) $f(X_i|X_i^*) = (1/\sigma_X)\phi((X_i - X_i^*)/\sigma_X)$. It is obvious that restrictions (a) to (d) are sufficient but not necessary for Assumptions 1 to 3 to hold. Under these restrictions, we can rewrite the individual likelihood function as

$$F(d_i, X_i, Z_i; \beta, \sigma, \sigma_X, \sigma_Z) = \int_{x^*} F_\nu(\beta \Delta X_j^*) \frac{1}{\sigma_X}\phi(\frac{X_i - X_i^*}{\sigma_X}) \frac{1}{\sigma_Z}\phi(\frac{Z_i - X_i^*}{\sigma_Z}) \frac{1}{\sigma}\phi(\frac{X^*}{\sigma}) dX_i^*.$$

By substitution of variables, we can show that we can reparameterize the likelihood as a function of $(\beta\sigma$, (σ_X/σ), $(\sigma_Z/\sigma))$. Let $X_j^* = \sigma\tilde{X}_i^*$, $Z = \sigma\tilde{Z}_i$, and $X_i = \sigma\tilde{X}_i$. Therefore, $f(X_i^*) = (1/\sigma)\phi(\tilde{X}_i^*)$, $f(Z_i|X_i, \tilde{X}_i^*) = (\sigma/\sigma_Z)\phi((\sigma(\tilde{Z}_i - \tilde{X}_i^*)/\sigma_Z)$, and $f(X_i|\tilde{X}_i^*) = (\sigma/\sigma_X)\phi((\sigma(\tilde{X}_i - \tilde{X}_i^*)/\sigma_X)$. Therefore, we can rewrite the likelihood function as:

$$F(d_i, \tilde{X}_i, \tilde{Z}_i; \beta\sigma, \sigma_X/\sigma, \sigma_Z/\sigma) = \int_{\tilde{x}^*} F_\nu((\beta\sigma)\Delta\tilde{X}_{ijj'}^*)) \frac{\sigma}{\sigma_X}\phi(\frac{\tilde{X}_i - \tilde{X}_i^*}{(\sigma_X/\sigma)}) \frac{\sigma}{\sigma_Z}\phi(\frac{\tilde{Z}_i - \tilde{X}_i^*}{(\sigma_Z/\sigma)}) \phi(\tilde{X}_i^*) d\tilde{X}_i^*.$$

Therefore, the only three parameters we can identify are $\beta\sigma$, (σ_X/σ), (σ_Z/σ). Consequently, the parameter β is not point identified. If $(\beta^*$, σ^*, σ_Z^*, $\sigma_X^*)$ denotes the true parameter vector, then the vector $(\beta^a$, σ^a, σ_X^a, $\sigma_Z^a)$, with $\beta^a = \beta^*/\gamma$, $\sigma^a = \sigma^*\gamma$, $\sigma_X^a = \sigma_X^*\gamma$ and $\sigma_Z^a = \sigma_Z^*\gamma$, generates the same value of the likelihood function as the true parameter vector.

A.2 Proof of Theorem 1

Lemma A.2.1 *For any distribution function F_ν, the score function of the model defined in equations* (9) *to* (**??**)*, conditional on the vector (X^*, z), is*

$$\mathbb{E}\Big[d_j \frac{F_\nu\big(-\beta\Delta X_{jj'}^*\big)}{1 - F_\nu\big(-\beta\Delta X_{jj'}^*\big)} - d_{j'}\Big| z, X^*\Big] = 0. \tag{66}$$

Proof: Let $\mathcal{L}(d|X^*, z)$ denote the log-likelihood function of the model in equations (9) to (**??**) conditional on the vector (X^*, z):

$$\mathcal{L}(d|X^*, z) = \mathbb{E}\Big[d_j \log\big(F_\nu(-\beta\Delta X_{jj'}^*)\big) + d_{j'} \log\big((1 - F_\nu(-\beta\Delta X_{jj'}^*))\big)|z, X^*\Big].$$

The corresponding score function is:

$$\frac{\partial\mathcal{L}(d|X^*, z)}{\partial\beta} = \mathbb{E}\Bigg[d_j \frac{1}{F_\nu(-\beta\Delta X_{jj'}^*)} \frac{\partial F_\nu(-\beta\Delta X_{jj'}^*)}{\partial\beta} + d_{j'} \frac{1}{1 - F_\nu(-\beta\Delta X_{jj'}^*)} \frac{\partial\big(1 - F_\nu(-\beta\Delta X_{jj'}^*)\big)}{\partial\beta}\Bigg| z, X^*\Bigg] = 0.$$

Reordering terms:

$$\frac{\partial\mathcal{L}(d|X^*, z)}{\partial\beta} = \mathbb{E}\Bigg[d_j \frac{1 - F_\nu(-\beta\Delta X_{jj'}^*)}{F_\nu(-\beta\Delta X_{jj'}^*)} \frac{\frac{\partial F_\nu(-\beta\Delta X_{jj'}^*)}{\partial\beta}}{\frac{\partial\left(1 - F_\nu(-\beta\Delta X_{jj'}^*)\right)}{\partial\beta}} - d_{j'}\Bigg| z, X^*\Bigg] = 0.$$

[33] This Appendix corresponds to a previous draft, and the notation might be slightly different from that used in the main text. A new version of the draft will be posted soon at https://sites.google.com/site/edumoralescasado/.

As, for any F_ν,

$$\frac{\frac{\partial F_\nu(-\beta\Delta X^*_{jj'})}{\partial\beta}}{\frac{\partial\left(1-F_\nu(-\beta\Delta X^*_{jj'})\right)}{\partial\beta}} = -1,$$

the score function may be rewritten as:

$$\mathbb{E}\left[d_j \frac{F_\nu\big(-\beta\Delta X^*_{jj'}\big)}{1-F_\nu\big(-\beta\Delta X^*_{jj'}\big)} - d_{j'}\middle| z, X^*\right] = 0. \quad \blacksquare$$

Lemma A.2.2 *For any log concave distribution function F_ν, the function*

$$\frac{F_\nu(y)}{1-F_\nu(y)}$$

is convex in y for any $y \in R$.

Proof: The first derivative of $F_\nu(y)/(1-F_\nu(y))$ can be written as:

$$\frac{\partial[F_\nu(y)/(1-F_\nu(y))]}{\partial y} = \frac{1}{1-F_\nu(y)}\frac{F'_\nu(y)}{1-F_\nu(y)}.$$

For any distribution function F_ν, it holds

$$\frac{1}{1-F_\nu(y)} \geq 0, \quad \frac{F'_\nu(y)}{1-F_\nu(y)} \geq 0, \quad \frac{\partial 1/(1-F_\nu(y))}{\partial y} \geq 0.$$

As Heckman and Honoré (1990), and Bagnoli and Bergstrom (2005) show, for any log concave distribution function F_ν, it holds

$$\frac{\partial[F'_\nu(y)/(1-F_\nu(y))]}{\partial y} \geq 0.$$

Therefore,

$$\frac{\partial^2[F_\nu(y)/(1-F_\nu(y))]}{\partial y^2} \geq 0. \quad \blacksquare$$

Lemma A.2.3 *For any log concave distribution function F_ν and any random variable η such that*

$$\mathbb{E}[\eta|X^*, z] = 0,$$

it holds that

$$\mathbb{E}\Big[\frac{F_\nu(y+\eta)}{1-F_\nu(y+\eta)}\Big| X^*, z\Big] \geq \frac{F_\nu(y)}{1-F_\nu(y)}$$

Proof: From Lemma A.4 and Jensen's Inequality. $\blacksquare$

Proof of Theorem 1 By the Law of Iterated Expectations (LIE), we can write the left hand side of equation (22) as:

$$\mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[d_j \frac{F_\nu\big(-(\beta_1\Delta Z_{1j}+\beta_2\Delta X_j)\big)}{1-F_\nu\big(-(\beta_1\Delta Z_{1j}+\beta_2\Delta X_j)\big)} - d_{j'}\middle| z, X^*, \nu\right]\middle| z, X^*\right]\middle| z\right].$$

Proof of Result 2 From derivations in Section A.2, we know we can rewrite the left hand side of equation (22) as:

$$\mathbb{E}\Bigg[\mathbb{E}\Bigg[d_j\mathbb{E}\Bigg[\frac{F_\nu\big(-(\beta_1\Delta Z_{1j}+\beta_2\Delta X_j^*+\Delta\varepsilon_j)\big)}{1-F_\nu\big(-(\beta_1\Delta Z_{1j}+\beta_2\Delta X_j^*+\Delta\varepsilon_j)\big)}\Bigg|z,X^*,\nu\Bigg]-d_{j'}\Bigg|z,X^*\Bigg]\Bigg|z\Bigg].$$

Given Assumptions 2 and 3 and Lemmas and A.4.1, for any (z, X^*, ν) and $\beta \in \Gamma_\beta$

$$\frac{\partial\Bigg\{\mathbb{E}\Bigg[\frac{F_\nu\left(-(\beta_1\Delta Z_{1j}+\beta_2\Delta X_j^*+\Delta\varepsilon_j)\right)}{1-F_\nu\left(-(\beta_1\Delta Z_{1j}+\beta_2\Delta X_j^*+\Delta\varepsilon_j)\right)}\Bigg|z,X^*,\nu\Bigg]\Bigg\}}{\partial\sigma_\varepsilon}\geq 0,$$

and, therefore, for any $z \in \mathcal{Z}$ and $\beta \in \Gamma_\beta$

$$\frac{\partial\Bigg\{\mathbb{E}\Bigg[\mathbb{E}\Bigg[d_j\mathbb{E}\Bigg[\frac{F_\nu\left(-(\beta_1\Delta Z_{1j}+\beta_2\Delta X_j^*+\Delta\varepsilon_j)\right)}{1-F_\nu\left(-(\beta_1\Delta Z_{1j}+\beta_2\Delta X_j^*+\Delta\varepsilon_j)\right)}\Bigg|z,X^*,\nu\Bigg]-d_{j'}\Bigg|z,X^*\Bigg]\Bigg|z\Bigg]\Bigg\}}{\partial\sigma_\varepsilon}\geq 0.$$

Therefore, if $\bar{\bar{\sigma}}_\varepsilon \geq \bar{\sigma}_\varepsilon$, then, for any given value of $\tilde{\beta} \in \Gamma_\beta$ such that $\mathcal{M}_s(\tilde{\beta}, z|\bar{\sigma}_\varepsilon) \geq 0$, it will be true that $\mathcal{M}_s(\tilde{\beta}, z|\bar{\bar{\sigma}}_\varepsilon) \geq 0$. ■

A.5 Proof of Theorem 2

Lemma A.5.1 *For any distribution function F_ν such that $\mathbb{E}[\Delta\nu_j|Z_i, X_i^*] = 0$, the expectation of equation* (12)*, conditional on the vector (x^*, z), is*

$$\mathbb{E}\big[d_j\beta\Delta x^*_{ijj'} + d_{j'}\mathbb{E}[\Delta\nu_{j'}|\Delta\nu_{j'} \geq -\beta\Delta x^*_{ij'j}]\big|z_i, x_i^*\big] \geq 0. \tag{71}$$

Proof: Applying the LIE to equation (12),

$$\begin{aligned}\mathbb{E}[d_j\beta\Delta x^*_{ijj'} + d_j\Delta\nu_j|x_i^*, z_i] &\geq 0,\\ \mathbb{E}\Big[\mathbb{E}\big[d_j\beta\Delta x^*_{ijj'} + d_j\Delta\nu_j\big|x_i^*, z_i, d_j\big]\Big|x_i^*, z_i\Big] &\geq 0,\\ \mathbb{E}\Big[d_j\beta\Delta x^*_{ijj'} + d_j\mathbb{E}\big[\Delta\nu_j\big|x_i^*, z_i, d_j = 1\big]\Big|x_i^*, z_i\Big] &\geq 0.\end{aligned}$$

Using Assumption 2 and the LIE,

$$\begin{aligned}\mathbb{E}\big[\Delta\nu_j\big|x_i^*, z_i\big] &= 0,\\ \mathbb{E}\big[\mathbb{E}[\Delta\nu_j|x_i^*, z_i, d_j]\big|x^*, z\big] &= 0,\\ \mathbb{E}[d_j|x_i^*, z_i]\mathbb{E}[\Delta\nu_j|x_i^*, z_i, d_j = 1] &= -\mathbb{E}[d_{j'}|x_i^*, z_i]\mathbb{E}[\Delta\nu_j|x_i^*, z_i, d_{j'} = 1],\\ \mathbb{E}[d_j|x_i^*, z_i]\mathbb{E}[\Delta\nu_j|x_i^*, z_i, d_j = 1] &= \mathbb{E}[d_{j'}|x_i^*, z_i]\mathbb{E}[\Delta\nu_{j'}|x_i^*, z_i, d_{j'} = 1],\\ \mathbb{E}\big[d_j\mathbb{E}[\Delta\nu_j|x_i^*, z_i, d_j = 1]\big|x_i^*, z_i\big] &= \mathbb{E}\big[d_{j'}\mathbb{E}[\Delta\nu_{j'}|x_i^*, z_i, d_{j'} = 1]\big|x_i^*, z_i\big].\end{aligned}$$

Plugging this equality into the previous inequality,

$$\mathbb{E}\Big[d_j\beta\Delta x^*_{ijj'} + d_{j'}\mathbb{E}\big[\Delta\nu_{j'}\big|x_i^*, z_i, d_{j'} = 1\big]\Big|x_i^*, z_i\Big] \geq 0.$$

Using again the individual revealed preference inequality in equation (12) and Assumption 1,

$$\mathbb{E}\Big[d_j\beta\Delta x^*_{ijj'} + d_{j'}\mathbb{E}\big[\Delta\nu_{j'}\big|\Delta\nu_{j'} \geq -\beta\Delta x^*_{ij'j}\big]\Big|x^*_i, z_i\Big] \geq 0. \quad \blacksquare$$

Proof of Theorem 2 By the LIE, we can write the left hand side of equation (24) as:

$$\mathbb{E}\Bigg[\mathbb{E}\Big[\mathbb{E}\Big[d_j(\beta_1\Delta Z_{1j} + \beta_2\Delta X_j) + d_{j'}\mathbb{E}\big[\Delta\nu_{j'}|\Delta\nu_{j'} \geq -(\beta_1\Delta Z_{1j'} + \beta_2\Delta X_{j'})\big]\Big|z, X^*, \nu\Big]\Big|z, X^*\Big]\Bigg|z\Bigg].$$

Using equation (**??**), we can rewrite:

$$\mathbb{E}\Bigg[\mathbb{E}\Big[d_j\mathbb{E}\Big[\beta_1\Delta Z_{1j} + \beta_2\Delta X_j\Big|z, X^*, \nu\Big]\Big|z, X^*\Big]\Bigg|z\Bigg] + \mathbb{E}\Bigg[\mathbb{E}\Big[d_{j'}\mathbb{E}\Big[\mathbb{E}\big[\Delta_{ij'j}|\Delta_{ij'j} \geq -(\beta_1\Delta Z_{1j'} + \beta_2\Delta X_{j'})\big]\Big|z, X^*, \nu\Big]\Big|z, X^*\Big].\Bigg|z\Bigg]$$

Using the notation in equation (14) and $\Delta\varepsilon_j = -\beta_2\Delta\epsilon_{ijj'}$,

$$\mathbb{E}\Bigg[\mathbb{E}\Big[d_j\mathbb{E}\Big[\beta_1\Delta Z_{1j} + \beta_2\Delta X^*_j + \Delta\varepsilon_j\Big|z, X^*, \nu\Big]\Big|z, X^*\Big]\Bigg|z\Bigg] + \mathbb{E}\Bigg[\mathbb{E}\Big[d_{j'}\mathbb{E}\Big[\mathbb{E}\big[\Delta_{ij'j}|\Delta_{ij'j} \geq -(\beta_1\Delta Z_{1j'} + \beta_2\Delta X^*_{ij'j} + \Delta\varepsilon_j)\big]\Big|z, X^*, \nu\Big]\Big|z, X^*\Big].\Bigg|z\Bigg].$$

With respect to the first summatory, Assumption 3 imposes

$$\mathbb{E}\Big[\beta_1\Delta Z_{1j} + \beta_2\Delta X^*_j + \Delta\varepsilon_j\Big|z, X^*, \nu\Big] = \mathbb{E}\Big[\beta_1\Delta Z_{1j} + \beta_2\Delta X^*_j\Big|z, X^*, \nu\Big]. \tag{72}$$

Concerning the second term, from Remark 1, Assumptions 2 and 3 and Jensen's Inequality, we know that, for every (z, X^*, ν),

$$\mathbb{E}\Big[\mathbb{E}\big[\Delta_{ij'j}|\Delta_{ij'j} \geq -(\beta_1\Delta Z_{1j'} + \beta_2\Delta X^*_{ij'j} + \Delta\varepsilon_j)\big]\Big|z, X^*, \nu\Big] \geq \mathbb{E}\Big[\mathbb{E}\big[\Delta_{ij'j}|\Delta_{ij'j} \geq -(\beta_1\Delta Z_{1j'} + \beta_2\Delta X^*_{ij'j})\big]\Big|z, X^*, \nu\Big]. \tag{73}$$

Combining the equality in equation (72) and the inequalities in equations (71) and (73), we obtain

$$\mathbb{E}\Bigg[\mathbb{E}\Big[d_j\mathbb{E}\Big[\beta_1\Delta Z_{1j} + \beta_2\Delta X_j\Big|z, X^*, \nu\Big]\Big|z, X^*\Big]\Bigg|z\Bigg] + \mathbb{E}\Bigg[\mathbb{E}\Big[d_{j'}\mathbb{E}\Big[\mathbb{E}\big[\Delta_{ij'j}|\Delta_{ij'j} \geq -(\beta_1\Delta Z_{1j'} + \beta_2\Delta X_{j'})\big]\Big|z, X^*, \nu\Big]\Big|z, X^*\Big].\Bigg|z\Bigg] \geq 0.$$