Numerical Dynamic Programming

Chris Conlon

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Grad IO

Reading

- Rust 1994 "Numerical Dynamic Programming in Economics" (Handbook Chapter)
- SLP Chapter 9: Stochastic Dynamic Programming (Theory)
- Judd Chapter 6: Approximation Methods
- Judd Chapter 7: Numerical Integration
- Judd Chapter 11: Projection Methods
- Judd Chapter 12: Numerical Dynamic Progrmamming

MDP Definition

- A time index $t \in \{0, 1, 2, \ldots, T\}$, with $T \leq \infty$
- A state space S
- A decision space A
- A family of constraint sets $A_t(s_t) \subseteq A$
- A family of transition probabilities $p_{t+1}(\cdot|s_t, a_t)$, and $\mathfrak{B}(S) \to [0, 1]$[1]
- A family of discount $\beta_t(s_t, a_t) \geq 0$ and a single period utility function $u_t(s_t, a_t)$ such that the utility functional U has the additively separable decomposition

$$\begin{aligned} U(s, a) = & \quad \textstyle\sum_{t=0}^{T} \left[\prod_{j=0}^{t-1} \beta_j(s_j, a)\right] u_t(s_t, a_t) \\ \alpha(s) = & \quad \arg\max_{\delta=(a_0, \ldots, a_T)} E_a U(s, a) \end{aligned}$$

Characterizations of MDPs

Finite Horizon have $T < \infty$. Can compute a^* by backward induction starting in the terminal period T.

Infinite Horizon $T = \infty$ use a recursive definition of the value function

Discrete State Space solve problems up to machine precision. (Good for estimation). How realistic?

Infinite State Space most of economics, numerical approximation $\rightarrow$ errors and interactions of approximation errors.

Discrete Decision Process D takes on a discrete set of values $\{a_1, \ldots, a_J\}$

Continuous Decision Process (CDP) These are tricky – there are results about how well they can be approximated by DDPs and they are not overwhelmingly positive. (See Rust 1994).

Examples of MDPs

- Entry/Exit decisions of firms
- R&D Investment
- Replacement of Durables
- Consumer Search
- Advertising?

Bellman's Equation

It is helpful to consider $V(s)$ as the solution to the MDP.

$$V(s) = \max_{a \in A(s)} [u(s,a) + \beta \int V(s')p(ds'|s,a)]$$

This is a functional equation and V represents a fixed point to the functional equation.

We are interested in existence/uniqueness of a solution to Bellman's Equation:

1. A and S are complete metric spaces
2. $u(s,a)$ is jointly continuous and bounded in (s,a)
3. $s \to A(s)$ is a continuous correspondence

Bellman Operator

Helpful to rewrite Bellmans equation as an operator $V = \Gamma(V)$ where $\Gamma : \mathfrak{B}(S) \to \mathfrak{B}(S)$ (Measurable, bounded)

The Bellman operator has the contraction mapping property which guarantees a unique fixed point point in B.

$$\|\Gamma(V) - \Gamma(W)\| \leq \beta\|V - W\| \quad \forall V, W \in B$$

Blackwell's Theorem
The stationary, Markovian, infinite horizon policy given by $\alpha(s)$ (the solution to Bellman's equation) constitutes an optimal decision rule for the infinite horizon MDP.

Simple Case

Start by thinking about *Discrete Decision Problems* with a *Discrete State Space*. We can generalize to harder problems later.

Solution Method: Finite Horizon Problems

- The optimal decision depends only on the state $a_t = a(s_t)$ not the whole history because p is markovian flow utilities are additively separable.
- See this by thinking about choosing a unique $a_T(s_t)$ in the ultimate period (unless there are ties) hence randomization can only make the agent worse off.
- All prior d_t's are now deterministic functions of s_t and so on
- At time $t = 0$ the value function $V_0^T(s_0)$ represents the conditional expectation of utility over all future periods> it follows that $V_0^T(s) = \max_a E_a\{U(\tilde{s}, \tilde{d}) | s_0 = s\}$.

Solution Method: Value Function Iteration

- Well known, basic algorithm of dynamic programming
- Based on our backward induction method of solving finite-horizon problems.
- We have tight convergence properties and bounds on errors
- Well suited for parallelization
- It is linearly convergent (slow depends on modulus of contraction mapping).

Value Function Iteration: Finite Horizon

Begin with the Bellman operator:

$$\Gamma(V^t)(s) = \max_{a \in A(s)} \left[u(s,a) + \beta \int V^{t+1}(s') p(ds'|s,a) \right]$$

Specify V^T and apply the Bellman operator:

$$\Gamma(V^{T-1})(s) = \max_{a \in A(s)} \left[u(s,a) + \beta \int V^{T}(s') p(ds'|s,a) \right]$$

Iterate to first period:

$$\Gamma(V^1)(s) = \max_{a \in A(s)} \left[u(s,a) + \beta \int V^{2}(s') p(ds'|s,a) \right]$$

Value Function Iteration: Infinite Horizon

Begin with the Bellman operator:

$$\Gamma(V^k)(s) = \max_{a \in A(s)} \left[u(s,a) + \beta \int V^{k+1}(s') p(ds'|s,a) \right]$$

Specify V^0 and apply the Bellman operator:

$$V^1(s) = \max_{a \in A(s)} \left[u(s,a) + \beta \int V^{0}(s') p(ds'|s,a) \right]$$

Iterate until convergence $\sup_s \|\Gamma(V^k)(s) - V^k(s)\| \leq \epsilon$

Value Function Iteration: Bounds

- Suppose we set $V^0 = 0$ then the value function iteration approach is just like solving the finite horizon problem by backward induction.
- The CMT guarantees consistency at a geometric rate or linear convergence with modulus β
- We can derive an expression for the number of steps we need to get an ϵ-approximation.

$$T(\epsilon, \beta) = \frac{1}{|\log(\beta)|} \log\left(\frac{1}{(1-\beta)\epsilon}\right)$$

Initial Value in Finite Horizon

- Economics provides natural choices (sometimes)
- Example: value of not being in the market is zero
- There are subtle issues: What is the value of dying? Bequests? OLG?

Initial Guesses in Infinite Horizon

- Theorems tell us we will converge from any initial guess
- Doesn't mean we should choose bad guesses!
- Try the steady state.
- Reduce dimensions: K/L ratio in Solow model

Policy Iteration (Howard 1960)

An alternative to value function iteration is policy function iteration.

- Make a guess for an initial policy, call it $a^k(s) = \arg\max_a U(a,s)$ that maps each state into an action
- Assume the guess is stationary; compute the implied $V^{a^k}(a,s)$ (solve linear system)
- Improvement Step: improve on policy a^k:

$$a^{k+1} = \arg\max_a U(a,s) + \beta \sum_{s'} V^{a^k}(a,s')p(s'|s,a)$$

- Determine if $\|a^k - a^{k-1}\| < \epsilon$. If yes then we have found the optimal policy a^* otherwise we need to recompute $V^{a^k}(s)$.

Policy Iteration (Howard 1960)

Policy Iteration is even easier if choices AND states are discrete.

- For Markov transition matrix $\sum_j P_{ij} = 1$,we want $\pi P = \pi$
- $\lim_{t \to \infty} P^t = \pi$ where the jth element of π represents the long run probability of state j.
- We want the eigenvalue for which $\lambda = 1$.

Now updating the value function is easy for kth iterate of PI

$$\begin{aligned} V^k(s) &= Eu(a^k(s), s) + \beta \tilde{P}^k V^k(s) \\ \Rightarrow V^k(s) &= [1 - \beta \tilde{P}^k]^{-1} Eu(a^k(s), s) \end{aligned}$$

- Very fast when $\beta > 0.95$ and s is relatively small. (Rust says 500 more like 5000).
- Inverting a large matrix is tricky

More Comments

- In the DDP case convergence of PI easy to verify: $a^{k+1} = a^k$.
- It is helpful to exploit the monotonicity of policy or value function: (S-s Rules, bus replacement, etc.)
- Exploit Concavity of value and/or policy functions (decreasing return to R&D investment, etc.)

Linear Programming

When everything is discrete (DDP, w/ discrete state space) we can write the dual instead

$$\min_V \sum_{s \in S} V(s) \text{ s.t.}$$
$$V(s_i) \quad \geq \quad Eu(a, s_i, \epsilon_t) + \beta \sum_j \tilde{p}_{ij}(a, s_i,)V_j \quad \forall s_i, a$$

This problem is now linear in V and can be solved with linear programming techniques (may be large since we need to find a, V).

Collocation Method (Judd 1992)

These methods are specifically designed for problems with continuous state space.

- Use a polynomial representation of the value function $\tilde{V}(s,c)$ with coefficients c that approximate the true value function
- Successively approximate until $\|c^k - c\| < \epsilon$

$$
\begin{aligned}
\tilde{V}(s,c) &= \sum_{i=1} c_i \phi_i(x) \\
\sum_{j=1} c_j \phi_j(s_i) &- \max_z E\left[u(s_i, z) + \beta \sum_{i=1}^{m} \tilde{p}_{il}(s) \sum_{j=1}^{n} c_j \phi_j(s_i) \right] = 0
\end{aligned}
$$

- Nonlinear system of equations (at each grid point s_i).
- Tricky because it involves a $\max$
- Very fast (Christiano and Fisher)

Discretization

Many problems we are interested in have continuous states or continuous decisions.

- In the case where we have a continuous state space, we need to discretize it into a grid
- How do we do that?
- Dealing with the curse of dimensionality.
- Do we let future states outside the grid?

New Approximation Problem

Exact Problem

$$V(s) = \max_{a \in A(s)} \left[(1-\beta)u(s,a) + \beta \int V(s')p(ds'|s,a) \right]$$

Approximation to the problem:

$$\hat{V}(s) = \max_{a \in \hat{A}(s)} \left[(1-\beta)u(s,a) + \beta \sum_{k=1}^{N} \hat{V}(s')p_N(s'_k|s,a) \right]$$

How to Approximate on Grids

I. Huge literature on numerical analysis on how to efficiently generate grids
II. Two main issues:
    A. How to select grid points s_k
    B. How to approximate transition matrix p with p_N.
III. Answer to second issue follows from answer to first problem
IV. We can combine strategies to generate grids

Uniform Grid

- Decide how many grid points
- Distribute them uniformly on the state space
- What if the state space is unbounded?
- Advantages and disadvantages (bounded errors)

Non-Uniform Grid

- Economic theory or error analysis to evaluate where to accumulate points
- Standard argument: close to curvature of value function: nobody replaces bus at 10 miles.
- Problem: this is a heuristic argument
- Self-confirming equilibria in computations (Citation)

Quadrature Grids

- Tauchen and Hussey (Econometrica, 1991)
- Motivation quadrature points in integrals

$$\int f(s)p(s)ds \approx \sum_{k=1}^{N} f(s_k)w_k$$

- Gaussian quadrature: choose nodes and weights so that the equation holds exactly for all polynomials of degree less than or equal to $2N-1$.

Stochastic Grids

- Randomly chosen grids
- Rust (1995): Breaks the Curse of Dimensionality – you don't need exponentially more points as the dimension increases for a fixed level of accuracy
- Downside: accuracy is low to start with
- How do we generate random numbers?

Interpolation

- Discretization means we need to interpolate the function at intermediate values for CDPs
- Simple: Linear interpolation
- Problem: In more that one dimension, linear interpolation does not preserve concavity
- Cubic Splines?
- Shape-preserving splines: Schumacher (1983).

Multigrid Algorithms

- Old tradition in numerical literature
- Idea: Solve a problem in coarser grid and use it as a guess for a more refined solution (and Iterate)
- Examples: Differential Equations, Projection Methods, Dynamic Programming (Chow and Tsisiklis, 1991).

Applying the Algorithm

After deciding initialization and discretization we need to implement each step

$$V^T(s) = \max_{a \in \hat{A}(s)} \left[u(s,a) + \beta \sum_{k=1}^{N} V^{T-1}(s') p_N(s'_k | s, a) \right]$$

Two numerical operations

1. Maximization
2. Integration

Maximization

Maximization is the costly step of the value function iteration (Especially for CDPs)

- Brute force: check all possible choices in the grid
- Sensibly: using quasi-Newton algorithm

Brute Force

- Only alternative: discrete choices, constraints, non-differentiabilities
- Tricks: Previous solution, Exploit Monotonicity, Concavity of $V(\cdot), A(\cdot)$.

Newton's Method

- Fast but we need to compute derivatives

Accelerator

For many algorithms maximization is the most expensive part

- Often we update $V(\cdot)$ and choice that are not optimal.
- Trick: Don't always apply the $\max$ operator at every iteration.
- How do we choose the best timing of $\max$ operator.

Convergence Assessment

For many algorithms maximization is the most expensive part

- How do we assess convergence?
- By the contraction mapping property:

$$\|V - V^k\|_\infty \leq \frac{1}{1-\beta} \|V^{k+1} - V^k\|_\infty$$

- Relates the error in VFI to Euler Equation error
- Can use this to refine the grid

Integration Methods

- Exact
- Approximate and integrate that: Taylor's Rule, Laplace's Method.
- Quadrature
- Monte Carlo simulation

Approximation Methods (Judd Ch 6)

Objective: Given a complicated $f(x)$ at some points, can we construct a simpler approximation $g(x)$?

- What data should be produced and used?
- What family of "simpler" functions? What kind of approximation
- Like regression but not "natural" data.

Interpolation Methods (Judd Ch 6)

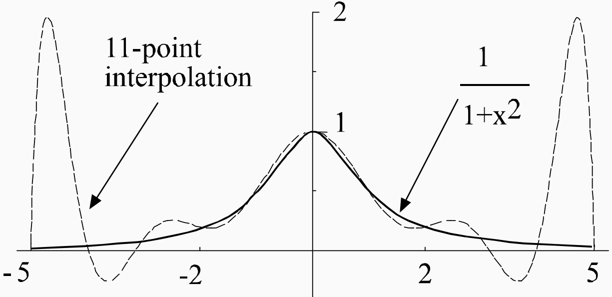
Find $g(x)$ from n-dimensional family of functions to fit n data points exactly.

Lagrange Polynomial Interpolation

- Data (x_i, y_i), $i = 1, \ldots, n$
- Objective, find an $n-1$ degree polynomial $p_n(x)$ which agrees with the data $y_i = f(x_i)$
- Result: If x_i are distinct there is a unique interpolating polynomial
- Does $p_n(x)$ converge to $f(x)$ as we use more points? *No!*

Figure 1: (from Judd)

$$f(x) = \frac{1}{1+x^2}$$
$$x_i = -5, -4, ..., 3, 4, 5$$



Interpolation Methods (Judd Ch 6)

Hermite Polynomial Interpolation

- Data (x_i, y_i, y_i'), $i = 1, \ldots, n$
- Objective, find an $2n-1$ degree polynomial $p_n(x)$ which agrees with the data $y_i = p(x_i)$ AND $y_i' = p'(x_i)$
- Result: If x_i are distinct there is a unique interpolating polynomial

Least Squares Approx

- Data the function $f(x)$
- Objective: find a function $g(x)$ in class G that best approximates $f(x)$ ie:

$$g = \arg\min_{g \in G} \|f - g\|^2$$

Orthogonal Polynomials

General Case

- Space: polynomials over domain D
- Weighting function: $w(x) >$ (positive everywhere)
- Inner product $\langle f, g \rangle = \int_D f(x)g(x)w(x)dx$
- Polynomials are orthogonal wrt to $w(x)$ IFF

$$\langle \phi_i, \phi_j \rangle = 0, \quad i \neq j$$

- Can compute orthogonal polynomials using recurrence formulas

$$\begin{aligned}
\phi_0(x) &= 1 \\
\phi_1(x) &= x \\
\phi_{k+1}(x) &= (a_{k+1}x + b_k)\phi_k(x) + c_{k+1}\phi_{k-1}(x)
\end{aligned}$$

Chebyshev Polynomials

- Can compute orthogonal polynomials using recurrence formulas
- $[a, b] = [-1, 1]$ and $w(x) = (1 - x^2)^{-1/2}$
- $T_n(x) = \cos(n \cos^{-1} x)$
- Recursive Definition

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x) \end{aligned}$$

General Intervals

- Most problems aren't on the $[-1, 1]$ interval so we need a COV

$$y = -1 + 2\frac{x - a}{b - a}$$

Chebyshev Approximation Algorithm

1. Compute the $m \geq n+1$ Chebyshev nodes on $[-1, 1]$

$$z_k = -\cos\left(\frac{2k-1}{2m}\pi\right), \quad k = 1, \ldots, m$$

2. Adjust the nodes to $[a, b]$ interval

$$x_k = (z_k + 1)\left(\frac{b-a}{2}\right) + a, \quad k = 1, \ldots, m$$

3. Evaluate f at the nodes $w_k = f(x_k)$ for $k = 1, \ldots, m$
4. Compute the coefficients a_i to get the approximation $p(x)$

$$a_i = \frac{\sum_{k=1}^{m} w_k T_i(z_k)}{\sum_{k=1}^{m} T_i(z_k)^2}$$

$$p(x) = \sum^{n} a_i T_i \left(2\frac{x-a}{} - 1\right)$$

Minimax Approximation

- Data: (x_i, y_i), $i = 1, \ldots, n$
- Objective: L^∞ fit

$$\min_{\beta \in R^m} \max_i \|y_i - f(x_i; \beta)\|$$

- Difficult to do (minimax problems are non-convex)
- Chebyshev Approximation satisfies this property, for C^2, C^3 functions but doesn't get $f'(x)$ right!

Theorem
Suppose $f : [-1, 1] \to R$ is C^k for some $k \geq 1$, and let I_n be the degree n polynomial interpolation of f based at the zeroes of $T_{n+1}(x)$ then

$$\|f - I_n\|_\infty \leq \left(\frac{2}{\pi}\log(n+1) + 1\right) \frac{(n-k)!}{n!} \left(\frac{\pi}{2}\right)^k \left(\frac{b-a}{2}\right)^k \|f^{(k)}\|_\infty$$

Splines are piecewise interpolating functions

Definition
A function $s(x)$ on $[a, b]$ is a spline of order n IFF

1. s is C^{n-2} on $[a, b]$ and
2. there is a grid of points (nodes) $a = x_0 < x_1 < \cdots < x_m = b$ such that $s(x)$ is a polynomial of degree $n - 1$ on each subinterval $[x_i, x_{i+1}]$, $i = 0, \ldots, m - 1$

Second order plane is piecewise linear.

We usually use cubic splines.

Splines

Cubic Splines

- Lagrange data set (x_i, y_i) for $i = 0, \ldots n$.
- Nodes: the x_i are the nodes of the spline
- Functional form $s(x) = a_i + b_i x + c_i x^2 + d_i x^3$ on $[x_{i-1}, x_i]$
- Unknowns $4n$ unknown coefficients
- $2n$ interpolation and continuity conditions:

$$\begin{aligned} y_i &= a_i + b_i x_i + c_i x_i^2 + d_i x_i^3 \quad i = 1, \ldots, n \\ y_i &= a_{i+1} + b_{i+1} x_i + c_{i+1} x_i^2 + d_{i+1} x_i^3 \quad i = 0, \ldots, n-1 \end{aligned}$$

- $2n - 2$ conditions from C^2 at the interior for $i = 1, \ldots, n-1$

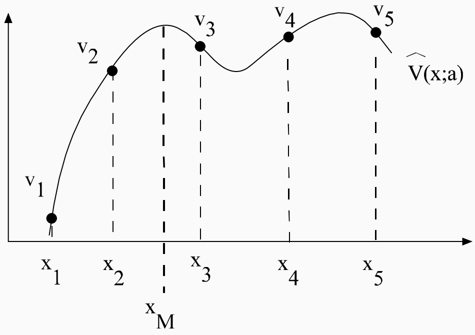
$$\begin{aligned} b_i + 2c_i x_i + 3d_i x_i^2 &= b_{i+1} + 2c_{i+1} x_i + 3d_{i+1} x_i^2 \\ 2c_i + 6d_i x_i &= 2c_{i+1} + 6d_{i+1} x_i \end{aligned}$$

Side Conditions

We have $4n-2$ linear equations and $4n$ unknowns we need two side conditions to identify the system

- Natural spline: $s''(x_0) = s''(x_n) = 0$ minimizes the total curvature $\int_{x_0}^{x_n} s''(x)^2 dx$
- Hermite spline: $s'(x_0) = y_0'$ and $s'(x_n) = y_n'$ (with extra data)
- Secant Hermite: $s'(x_0) = \frac{s(x_1)-s(x_0)}{x_1-x_0}$, $s'(x_n) = \frac{s(x_n)-s(x_{n-1})}{x_n-x_{n-1}}$
- Solvers are built in to packages like MATLAB (check documentation for which method).

Shape Issues



- Concave (monotone) data may lead to non concave (non monotone) approximations
- Shape problems stabilize VFI.

Schumaker Procedure (Shape Preserving Splines)

1. Take level (and maybe slope) data at nodes x_i
2. Add intermediate nodes $z_i^+ \in [x_i, x_{i+1}]$
3. Run quadratic spline with nodes at the x and z nodes which interpolate data and preserves shape
4. Schumaker formulas tell you how to choose the z and spline coefficient
5. Detail in Judd and in companion paper (Judd and Solnick)

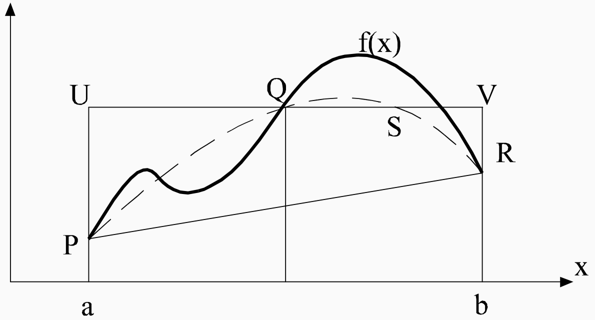
Numerical Integration

We are interested in lots of problems that require computing difficult integrals (e.g.: evaluating expectations)

1. Midpoint/Trapezoid Rules
2. Simpson's Rule
3. Gaussian Rules
4. Higher-Dimensional Rules

Quadrature Rules

Basic idea of quadrature is to approximate complicated functions with something easier to integrate, and then integrate that exactly.



- Constant $f(x)$ at midpoint of $[a, b]$ $aUQVb$ for box.
- Linear: Trapezoid $aPRb$
- Parabola through $f(x)$ at a, b and $\frac{a+b}{2}$ for $aPQRb$

Simpsons Rule (Newton-Cotes)

Piecewise Quadratic Approximation at some $\xi \in [a, b]$

$$\int_a^b f(x)dx \approx \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}(\xi)$$

With approximation error

$$\frac{1}{90}\left(\frac{b-a}{2}\right)^5 |f^{(4)}(\xi)|$$

Works well when quadratic approximation is good $f^{(4)}$ is small or interval is small.

Gaussian Quadrature

Formulas of the form

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

for some quadrature nodes $x_i \in [a, b]$ and weights w_i.

- Let $\mathcal{F}_k$ be the space of degree k polynomials
- Quadrature formulas are exact of degree k if it correctly integrates each function in $\mathcal{F}_k$
- Gaussian quadrature formulas use n points and are exact of degree $2n - 1$.

Approximation Error

$$(f, g) = \int_a^b w(x) f(x)dx - \sum_{i=1}^{n} w_i f(x_i) = \frac{f^{(2n)}(\xi)}{(2n)!}(p_n, p_n)$$

Gaussian Quadrature

Legendre Domain: $[-1,1]$, $w(x) = 1$

Chebyshev Domain: $[-1,1]$, $w(x) = \frac{1}{\sqrt{1-x^2}}$

Laguerre Domain: $[0,\infty]$, $w(x) = \exp[-x]$ (useful for present value)

Hermite Domain: $[-\infty,\infty]$, $w(x) = \exp[-x^2]$ (useful for normal)

Helpful if function is C^∞ or analytic.

Gauss Herrmite

Let $Y \sim N(\mu, \sigma^2)$ and apply COV $x = (y - \mu)/\sqrt{2}\sigma$

$$E[f(Y)] = (2\pi\sigma^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(y) \exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right] dy$$

$$\int_{-\infty}^{\infty} f(y) \exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right] dy = \int_{-\infty}^{\infty} f(\sqrt{2}\sigma x + \mu) e^{-x^2} \sqrt{2}\sigma dx$$

Gives the quadrature formula using Gauss Hermite (x_i, w_i).

$$E[f(Y)] = \frac{1}{\sqrt{\pi}} \sum_{i=1}^{n} w_i f(\sqrt{2}\sigma x_i + \mu)$$

Higher Dimensional Integration

- In higher dimension we can use product rules of 1-D integrals.
- This grows exponentially in dimension D (Curse of Dimensionality)
- Monte Carlo is not cused but slow to converge $\frac{1}{\sqrt{n}}$ vs $\frac{1}{2n!}f^{(2n)}$
- Some monomial rules (Judd), (Skrainka and Judd) aren't cursed
- Sparse Grids show how to combine 1-D rules more efficiently (www.sparse-grids.de)