Problem Set I.3

1 Show that the nullspace of AB contains the nullspace of B. **If $Bx = 0$ then...**

2 Find a square matrix with rank $(A^2) <$ rank (A). Confirm that rank $(A^{\mathrm{T}}A) =$ rank (A).

3 How is the nullspace of C related to the nullspaces of A and B, if $C = \begin{bmatrix} A \\ B \end{bmatrix}$?

4 If row space of A = column space of A, and also $\mathbf{N}(A) = \mathbf{N}(A^{\mathrm{T}})$, is A symmetric ?

5 Four possibilities for the rank r and size m, n match four possibilities for $Ax = b$. Find four matrices A_1 to A_4 that show those possibilities:

$r = m = n$	$A_1x = b$ **has 1 solution for every** b
$r = m < n$	$A_2x = b$ **has 1 or** ∞ **solutions**
$r = n < m$	$A_3x = b$ **has 0 or 1 solution**
$r < m,\ r < n$	$A_4x = b$ **has 0 or** ∞ **solutions**

6 (*Important*) Show that $A^{\mathrm{T}}A$ **has the same nullspace as** A. Here is one approach: First, if Ax equals zero then $A^{\mathrm{T}}Ax$ equals _____. This proves $\mathbf{N}(A) \subset \mathbf{N}(A^{\mathrm{T}}A)$. Second, if $A^{\mathrm{T}}Ax = \mathbf{0}$ then $x^{\mathrm{T}}A^{\mathrm{T}}Ax = ||Ax||^2 = 0$. Deduce $\mathbf{N}(A^{\mathrm{T}}A) = \mathbf{N}(A)$.

7 Do A^2 and A always have the same nullspace ? A is a square matrix.

8 Find the column space $\mathbf{C}(A)$ and the nullspace $\mathbf{N}(A)$ of $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Remember that those are vector spaces, not just single vectors. This is an unusual example with $\mathbf{C}(A) = \mathbf{N}(A)$. It could not happen that $\mathbf{C}(A) = \mathbf{N}(A^{\mathrm{T}})$ because those two subspaces are orthogonal.

9 Draw a square and connect its corners to the center point: 5 nodes and 8 edges. Find the 8 by 5 incidence matrix A of this graph (rank $r = 5 - 1 = 4$). Find a vector x in $\mathbf{N}(A)$ and $8 - 4$ independent vectors y in $\mathbf{N}(A^{\mathrm{T}})$.

10 If $\mathbf{N}(A)$ is the zero vector, what vectors are in the nullspace of $B = [A \ \ A \ \ A]$?

11 For subspaces $\mathbf{S}$ and $\mathbf{T}$ of $\mathbf{R}^{10}$ with dimensions 2 and 7, what are all the possible dimensions of

(i) $\mathbf{S} \cap \mathbf{T} = \{$all vectors that are in both subspaces$\}$

(ii) $\mathbf{S} + \mathbf{T} = \{$all sums $s + t$ with s in $\mathbf{S}$ and t in $\mathbf{T}\}$

(iii) $\mathbf{S}^{\perp} = \{$all vectors in $\mathbf{R}^{10}$ that are perpendicular to every vector in $\mathbf{S}\}$.

I.4 Elimination and $A = LU$

The first and most fundamental problem of linear algebra is to solve $Ax = b$. We are given the n by n matrix A and the n by 1 column vector b. We look for the solution vector x. Its components $x_1, x_2, \ldots, x_n$ are the n unknowns and we have n equations. Usually a square matrix A means only one solution to $Ax = b$ (but not always). We can find x by geometry or by algebra.

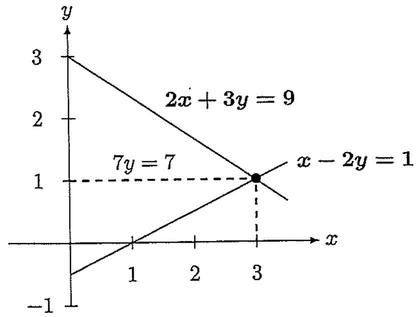
This section begins with the row and column pictures of $Ax = b$. Then we solve the equations by simplifying them—eliminate x_1 from $n-1$ equations to get a smaller system $A_2x_2 = b_2$ of size $n-1$. Eventually we reach the 1 by 1 system $A_nx_n = b_n$ and we know $x_n = b_n/A_n$. Working backwards produces x_{n-1} and eventually we know x_2 and x_1.

The point of this section is to see those elimination steps in terms of rank 1 matrices. **Every step (from A to A_2 and eventually to A_n) removes a matrix ℓu^*.** Then the original A is the sum of those rank one matrices. This sum is exactly the great factorization $A = LU$ into lower and upper triangular matrices L and U—as we will see.

$A = L$ times U is the matrix description of elimination without row exchanges. That will be the algebra. Start with geometry for this 2 by 2 example.

2 equations and 2 unknowns
2 by 2 matrix in $Ax = b$
$$\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix} \qquad \begin{aligned} x - 2y &= 1 \\ 2x + 3y &= 9 \end{aligned} \qquad (1)$$

Notice ! I multiplied Ax using inner products (dot products). Each row of the matrix A multiplied the vector x. That produced the two equations for x and y, and the two straight lines in Figure I.4. They meet at the solution $x = 3, y = 1$. Here is the **row picture.**



$$\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$$
$$A \qquad x \quad = \quad b$$

At the solution
$$\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$$

Figure I.4: **The row picture of** $Ax = b$: Two lines meet at the solution $x = 3, y = 1$.

Figure I.4 also includes the horizontal line $7y = 7$. I subtracted 2 (equation 1) from (equation 2). The unknown x has been eliminated from $7y = 7$. This is the algebra:

$$\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix} \quad \text{becomes} \quad \begin{bmatrix} 1 & -2 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix} \qquad \begin{aligned} x &= 3 \\ y &= 1 \end{aligned}$$

Orthogonal Basis = Orthogonal Axes in $\mathbf{R}^n$

Suppose the n by n orthogonal matrix Q has columns $q_1, \ldots, q_n$. Those unit vectors are a basis for n-dimensional space $\mathbf{R}^n$. Every vector v can be written as a combination of the basis vectors (the q's):

$$v = c_1 q_1 + \cdots + c_n q_n \tag{13}$$

Those $c_1 q_1$ and $c_2 q_2$ and $c_n q_n$ are the components of v along the axes. They are the projections of v onto the axes! There is a simple formula for each number c_1 to c_n:

Coefficients in an orthonormal basis $$\boxed{c_1 = q_1^{\mathrm{T}} v \qquad c_2 = q_2^{\mathrm{T}} v \quad \cdots \quad c_n = q_n^{\mathrm{T}} v} \tag{14}$$

I will give a vector proof and a matrix proof. Take dot products with q_1 in equation (13):

$$q_1^{\mathrm{T}} v = c_1 q_1^{\mathrm{T}} q_1 + \cdots + c_n q_1^{\mathrm{T}} q_n = c_1 \tag{15}$$

All terms are zero except $c_1 q_1^{\mathrm{T}} q_1 = c_1$. So $q_1^{\mathrm{T}} v = c_1$ and every $q_k^{\mathrm{T}} v = c_k$.

If we write (13) as a matrix equation $v = Qc$, **multiply by Q^{T} to see (14):**

$$Q^{\mathrm{T}} v = Q^{\mathrm{T}} Q c = c \quad \text{gives all the coefficients } c_k = q_k^{\mathrm{T}} v \text{ at once.}$$

This is the key application of orthogonal bases (for example the basis for Fourier series). When basis vectors are orthonormal, each coefficient c_1 to c_n can be found separately!

Householder Reflections

Here are neat examples of **reflection matrices** $Q = H_n$. Start with the identity matrix. Choose a unit vector u. Subtract the rank one symmetric matrix $2uu^{\mathrm{T}}$. Then $I - 2uu^{\mathrm{T}}$ is a "Householder matrix". For example, choose $u = (1, 1, \ldots, 1)/\sqrt{n}$.

Householder example $$\boxed{H_n = I - 2uu^{\mathrm{T}} = I - \frac{2}{n}\,\text{ones}\,(n, n).} \tag{16}$$

With uu^{T}, H_n is surely symmetric. Two reflections give $H^2 = I$ because $u^{\mathrm{T}} u = 1$:

$$H^{\mathrm{T}} H = H^2 = (I - 2uu^{\mathrm{T}})(I - 2uu^{\mathrm{T}}) = I - 4uu^{\mathrm{T}} + 4uu^{\mathrm{T}} uu^{\mathrm{T}} = I. \tag{17}$$

The 3 by 3 and 4 by 4 examples are easy to remember, and H_4 is like a Hadamard matrix:

$$H_3 = I - \frac{2}{3}\,\text{ones} = \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix} \qquad H_4 = I - \frac{2}{4}\,\text{ones} = \frac{1}{2}\begin{bmatrix} \mathbf{1} & -1 & -1 & -1 \\ -1 & \mathbf{1} & -1 & -1 \\ -1 & -1 & \mathbf{1} & -1 \\ -1 & -1 & -1 & \mathbf{1} \end{bmatrix}$$

Householder's n by n reflection matrix has $H_n u = (I - 2uu^{\mathrm{T}})u = u - 2u = -u$. And $H_n w = +w$ whenever w is perpendicular to u. The "eigenvalues" of H are -1 (once) and $+1$ ($n - 1$ times). All reflection matrices have eigenvalues -1 and 1.

Problem Set I.5

1 If u and v are orthogonal unit vectors, show that $u + v$ is orthogonal to $u - v$. What are the lengths of those vectors?

2 Draw unit vectors u and v that are *not* orthogonal. Show that $w = v - u(u^{\mathrm{T}} v)$ is orthogonal to u (and add w to your picture).

3 Draw any two vectors u and v out from the origin $(0, 0)$. Complete two more sides to make a parallelogram with diagonals $w = u + v$ and $z = u - v$. Show that $w^{\mathrm{T}} w + z^{\mathrm{T}} z$ is equal to $2u^{\mathrm{T}} u + 2v^{\mathrm{T}} v$.

4 Key property of every orthogonal matrix: $||Qx||^2 = ||x||^2$ for every vector x. More than this, show that $(Qx)^{\mathrm{T}}(Qy) = x^{\mathrm{T}} y$ for every vector x and y. So *lengths and angles are not changed by Q.* **Computations with Q never overflow!**

5 If Q is orthogonal, how do you know that Q is invertible and Q^{-1} is also orthogonal? If $Q_1^{\mathrm{T}} = Q_1^{-1}$ and $Q_2^{\mathrm{T}} = Q_2^{-1}$, show that $Q_1 Q_2$ is also an orthogonal matrix.

6 A **permutation matrix** has the same columns as the identity matrix (in some order). *Explain why this permutation matrix and every permutation matrix is orthogonal:*

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ has orthonormal columns so } P^{\mathrm{T}} P = \underline{\qquad} \text{ and } P^{-1} = \underline{\qquad}.$$

When a matrix is symmetric or orthogonal, **it will have orthogonal eigenvectors.** This is the most important source of orthogonal vectors in applied mathematics.

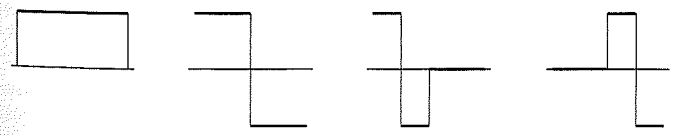
7 Four eigenvectors of that matrix P are $x_1 = (1, 1, 1, 1)$, $x_2 = (1, i, i^2, i^3)$, $x_3 = (1, i^2, i^4, i^6)$, and $x_4 = (1, i^3, i^6, i^9)$. Multiply P times each vector to find $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. The eigenvectors are the columns of the 4 by 4 **Fourier matrix F**.

$$\text{Show that } Q = \frac{F}{2} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & i^2 & 1 & -1 \\ 1 & i^3 & -1 & i \end{bmatrix} \text{ has orthonormal columns} : \overline{Q}^{\mathrm{T}} Q = I$$

8 **Haar wavelets are orthogonal vectors (columns of W) using only 1, -1, and 0.**

$$n = 4 \qquad W = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix}$$

Find $W^{\mathrm{T}} W$ and W^{-1} and the eight Haar wavelets for $n = 8$.



Nondiagonalizable Matrices (Optional)

Suppose λ is an eigenvalue of A. We discover that fact in two ways:

1. **Eigenvectors** (geometric) There are nonzero solutions to $Ax = \lambda x$.
2. **Eigenvalues** (algebraic) The determinant of $A - \lambda I$ is zero.

The number λ may be a simple eigenvalue or a multiple eigenvalue, and we want to know its ***multiplicity***. Most eigenvalues have multiplicity $M = 1$ (simple eigenvalues). Then there is a single line of eigenvectors, and $\det(A - \lambda I)$ does not have a double factor.

For exceptional matrices, an eigenvalue can be ***repeated***. Then there are two different ways to count its multiplicity. Always GM $\le$ AM for each λ:

1. **(Geometric Multiplicity = GM)** Count the **independent eigenvectors** for λ. Look at the dimension of the nullspace of $A - \lambda I$.
2. **(Algebraic Multiplicity = AM)** Count the **repetitions of** λ among the eigenvalues. Look at the roots of $\det(A - \lambda I) = 0$.

If A has $\lambda = 4, 4, 4$, then that eigenvalue has AM = 3 and GM = 1 or 2 or 3.

The following matrix A is the standard example of trouble. Its eigenvalue $\lambda = 0$ is repeated. It is a double eigenvalue (AM = 2) with only one eigenvector (GM = 1).

$$\begin{matrix}\textbf{AM} = \mathbf{2}\\ \textbf{GM} = \mathbf{1}\end{matrix} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ has } \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2. \quad \begin{matrix}\boldsymbol{\lambda = 0, 0} \textbf{ but}\\ \textbf{1 eigenvector}\end{matrix}$$

There "should" be two eigenvectors, because $\lambda^2 = 0$ has a double root. The double factor λ^2 makes AM = 2. But there is only one eigenvector $x = (1, 0)$. So GM = 1. ***This shortage of eigenvectors when* GM $<$ AM *means that* A *is not diagonalizable.*** There is no invertible eigenvector matrix. The formula $A = X\Lambda X^{-1}$ fails.

These three matrices all have the same shortage of eigenvectors. Their repeated eigenvalue is $\lambda = 5$. Traces are 10 and determinants are 25:

$$A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 6 & -1 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 7 & 2 \\ -2 & 3 \end{bmatrix}.$$

Those all have $\det(A - \lambda I) = (\lambda - 5)^2$. The algebraic multiplicity is AM = 2. But each $A - 5I$ has rank $r = 1$. The geometric multiplicity is GM = 1. There is only one line of eigenvectors for $\lambda = 5$, and these matrices are not diagonalizable.

Problem Set I.6

1 The rotation $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has complex eigenvalues $\lambda = \cos\theta \pm i\sin\theta$:

$$Q\begin{bmatrix} 1 \\ -i \end{bmatrix} = (\cos\theta + i\sin\theta)\begin{bmatrix} 1 \\ -i \end{bmatrix} \text{ and } Q\begin{bmatrix} 1 \\ i \end{bmatrix} = (\cos\theta - i\sin\theta)\begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Check that $\lambda_1 + \lambda_2$ equals the trace of Q (sum $Q_{11} + Q_{22}$ down the diagonal). Check that $(\lambda_1)(\lambda_2)$ equals the determinant. Check that those complex eigenvectors are orthogonal, using the complex dot product $\overline{x}_1 \cdot x_2$ (not just $x_1 \cdot x_2$!).

What is Q^{-1} and what are its eigenvalues?

2 Compute the eigenvalues and eigenvectors of A and A^{-1}. Check the trace!

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}.$$

A^{-1} has the ____ eigenvectors as A. When A has eigenvalues λ_1 and λ_2, its inverse has eigenvalues ____.

3 Find the eigenvalues of A and B (easy for triangular matrices) and $A + B$:

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A + B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

Eigenvalues of $A + B$ (*are equal to*)(*are not equal to*) eigenvalues of A plus eigenvalues of B.

4 Find the eigenvalues of A and B and AB and BA:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.$$

(a) Are the eigenvalues of AB equal to eigenvalues of A times eigenvalues of B?

(b) Are the eigenvalues of AB equal to the eigenvalues of BA?

5 (a) If you know that x is an eigenvector, the way to find λ is to ____.

(b) If you know that λ is an eigenvalue, the way to find x is to ____.

6 Find the eigenvalues and eigenvectors for both of these Markov matrices A and A^∞. Explain from those answers why A^{100} is close to A^∞:

$$A = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}.$$

7 **The determinant of A equals the product $\lambda_1\lambda_2\cdots\lambda_n$.** Start with the polynomial $\det(A-\lambda I)$ separated into its n factors (always possible). Then set $\lambda = 0$:

$$\det(A-\lambda I) = (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda) \quad \text{so} \quad \det A = \_\_\_\_.$$

Check this rule in Example 1 where the Markov matrix has $\lambda = 1$ and $\frac{1}{2}$.

8 The sum of the diagonal entries (the *trace*) equals the sum of the eigenvalues:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{has} \quad \det(A-\lambda I) = \lambda^2 - (a+d)\lambda + ad - bc = 0.$$

The quadratic formula gives the eigenvalues $\lambda = (a+d+\sqrt{})/2$ and $\lambda = \_\_\_\_$. Their sum is $\_\_\_\_$. If A has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $\det(A-\lambda I) = \_\_\_\_$.

9 If A has $\lambda_1 = 4$ and $\lambda_2 = 5$ then $\det(A-\lambda I) = (\lambda-4)(\lambda-5) = \lambda^2 - 9\lambda + 20$. Find three matrices that have trace $a+d = 9$ and determinant 20 and $\lambda = 4, 5$.

10 Choose the last rows of A and C to give eigenvalues 4, 7 and 1, 2, 3:

Companion matrices $$A = \begin{bmatrix} 0 & 1 \\ * & * \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{bmatrix}.$$

11 ***The eigenvalues of A equal the eigenvalues of A^{T}.*** This is because $\det(A-\lambda I)$ equals $\det(A^{\mathrm{T}}-\lambda I)$. That is true because $\_\_\_\_$. Show by an example that the eigenvectors of A and A^{T} are *not* the same.

12 This matrix is singular with rank one. Find three λ's and three eigenvectors:

$$A = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}.$$

13 Suppose A and B have the same eigenvalues $\lambda_1, \ldots, \lambda_n$ with the same independent eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$. Then $A = B$. *Reason:* Any vector $\boldsymbol{x}$ is a combination $c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n$. What is $A\boldsymbol{x}$? What is $B\boldsymbol{x}$?

14 Suppose A has eigenvalues 0, 3, 5 with independent eigenvectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

(a) Give a basis for the nullspace and a basis for the column space.

(b) Find a particular solution to $A\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$. Find all solutions.

(c) $A\boldsymbol{x} = \boldsymbol{u}$ has no solution. If it did then $\_\_\_\_$ would be in the column space.

15 (a) Factor these two matrices into $A = X\Lambda X^{-1}$:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}.$$

(b) If $A = X\Lambda X^{-1}$ then $A^3 = (\quad)(\quad)(\quad)$ and $A^{-1} = (\quad)(\quad)(\quad)$.

16 Suppose $A = X\Lambda X^{-1}$. What is the eigenvalue matrix for $A + 2I$? What is the eigenvector matrix? Check that $A + 2I = (\quad)(\quad)(\quad)^{-1}$.

17 True or false: If the columns of X (eigenvectors of A) are linearly independent, then

(a) A is invertible (b) A is diagonalizable

(c) X is invertible (d) X is diagonalizable.

18 Write down the most general matrix that has eigenvectors $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$.

19 True or false: If the eigenvalues of A are 2, 2, 5 then the matrix is certainly

(a) invertible (b) diagonalizable (c) not diagonalizable.

20 True or false: If the only eigenvectors of A are multiples of (1, 4) then A has

(a) no inverse (b) a repeated eigenvalue (c) no diagonalization $X\Lambda X^{-1}$.

21 $A^k = X\Lambda^k X^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every λ has absolute value less than $\_\_\_\_$. Which of these matrices has $A^k \to 0$?

$$A_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}.$$

22 Diagonalize A and compute $X\Lambda^k X^{-1}$ to prove this formula for A^k:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{has} \quad A^k = \frac{1}{2}\begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}.$$

23 The eigenvalues of A are 1 and 9, and the eigenvalues of B are -1 and 9:

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}.$$

Find a matrix square root of A from $R = X\sqrt{\Lambda}X^{-1}$. Why is there no real matrix square root of B?

24 Suppose the same X diagonalizes both A and B. They have the *same eigenvectors* in $A = X\Lambda_1 X^{-1}$ and $B = X\Lambda_2 X^{-1}$. Prove that $AB = BA$.

25 The transpose of $A = X\Lambda X^{-1}$ is $A^{\mathrm{T}} = (X^{-1})^{\mathrm{T}}\Lambda X^{\mathrm{T}}$. The eigenvectors in $A^{\mathrm{T}}\boldsymbol{y} = \lambda\boldsymbol{y}$ are the columns of that matrix $(X^{-1})^{\mathrm{T}}$. They are often called ***left eigenvectors of A***, because $\boldsymbol{y}^{\mathrm{T}}A = \lambda\boldsymbol{y}^{\mathrm{T}}$. How do you multiply matrices to find this formula for A?

$$\boxed{\textbf{Sum of rank-1 matrices} \quad A = X\Lambda X^{-1} = \lambda_1\boldsymbol{x}_1\boldsymbol{y}_1^{\mathrm{T}} + \cdots + \lambda_n\boldsymbol{x}_n\boldsymbol{y}_n^{\mathrm{T}}.}$$

26 **When is a matrix A similar to its eigenvalue matrix Λ ?**

A and Λ always have the same eigenvalues. But similarity requires a matrix B with $A = B\Lambda B^{-1}$. Then B is the $\_\_\_\_$ matrix and A must have n independent $\_\_\_\_$.