Nonparametrics and Local Methods: Polynomials

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Applied Econometrics

Polynomial Basis

Again consider the following relationship:

$$y_i = f(x_i) + \epsilon_i$$

One approach is to approximate $f(x_i)$ or $E[y_i|x_i]$ with a polynomial series.

$$y_i = a_0 + a_1 x_i + a_2 x_i^2 + a_3 x_i^3 + \cdots + a_M x_i^M + \varepsilon_i$$

An important choice is to choose polynomial order M (complexity)
Idea: we can approximate arbitrary (smooth) functions $f(x_i)$ with a high-order polynomial.

Polynomial Example

Let's suppose we want to approximate the following function:

$$\frac{1}{3}\sin(2x) \approx \frac{2}{3}x - \frac{4}{9}x^3 + \frac{4}{45}x^5 + O(x^7)$$

This should be easy since we have a Taylor Expansion that is polynomial in x (also it is odd).

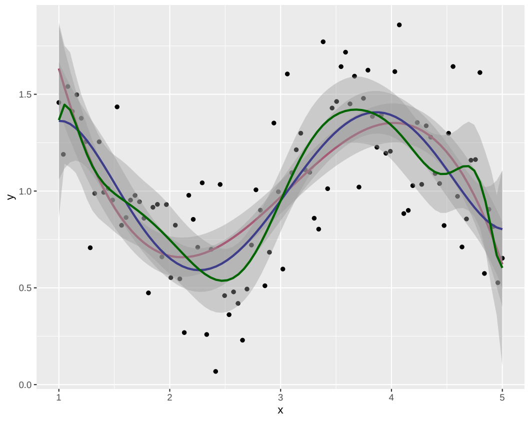
Polynomial Example

```
library(ggplot2)

# set the seed to make the results reproducible.
set.seed(3)

#### simulate some data ####
# epsilon = random error term
epsilon <- 0.25*rnorm(100)
x       <- seq(from=1, to=5, length.out=100)
y       <- 1 +sin(x*2)/3 + epsilon

# visualize the data (with a polynomial best-fit line)
ggplot(data=NULL,aes(x, y)) + geom_point() +
geom_smooth(method = "lm", formula = y ~ poly(x, 3,raw=T),color='maroon')+
geom_smooth(method = "lm", formula = y ~ poly(x, 5,raw=T),color='navy')+
geom_smooth(method = "lm", formula = y ~ poly(x, 17,raw=T),color='darkgreen')
```



Polynomial Example

- $M = 5$ order polynomial should fit the data well (it does).
- We are clearly overfitting at $M = 17$ since this doesn't look much like $y(x) = \sin(2x)/3 + \varepsilon$.
- We used raw polynomials: $x, x^2, x^3, \ldots$.
- By default R uses orthogonal polynomials when we drop `raw=TRUE`. (More on these later)

```
# visualize the data (with a polynomial best-fit line)
ggplot(data=NULL,aes(x, y)) + geom_point() +
geom_smooth(method = "lm", formula = y ~ poly(x, 3),color='maroon')+
geom_smooth(method = "lm", formula = y ~ poly(x, 5),color='navy')+
geom_smooth(method = "lm", formula = y ~ poly(x, 17),color='darkgreen')
```

"Raw" Polynomials

```
                         Estimate Std. Error t value Pr(>|t|)
(Intercept)             -5.421717   8.009814  -0.677    0.500
poly(x, 6, raw = T)1  18.072018  20.371055   0.887    0.377
poly(x, 6, raw = T)2 -17.392013  20.363051  -0.854    0.395
poly(x, 6, raw = T)3   7.503790  10.288970   0.729    0.468
poly(x, 6, raw = T)4  -1.561290   2.786268  -0.560    0.577
poly(x, 6, raw = T)5   0.148442   0.385418   0.385    0.701
poly(x, 6, raw = T)6  -0.004826   0.021378  -0.226    0.822
```

```
                         Estimate Std. Error t value Pr(>|t|)
(Intercept)              45.71304   19.95109   2.291  0.02423 *
poly(x, 7, raw = TRUE)1 -138.78101   59.74402  -2.323  0.02239 *
poly(x, 7, raw = TRUE)2  177.80275   72.90420   2.439  0.01665 *
poly(x, 7, raw = TRUE)3 -120.61050   47.13566  -2.559  0.01214 *
poly(x, 7, raw = TRUE)4   46.52356   17.50188   2.658  0.00926 **
poly(x, 7, raw = TRUE)5  -10.21667    3.74637  -2.727  0.00765 **
poly(x, 7, raw = TRUE)6    1.18842    0.42965   2.766  0.00686 **
poly(x, 7, raw = TRUE)7   -0.05682    0.02044  -2.780  0.00658 **
```

- 6th order polynomial: nothing significant
- 7th order polynomial: all terms significant (odd and even).
- Totally different coefficients!
- Both highly sensitive to small changes in x_i.

“Orthogonal” Polynomials

```
                        Estimate Std. Error t value Pr(>|t|)
(Intercept)              1.02278    0.02779  36.803  < 2e-16 ***
poly(x, 6, raw = FALSE)1  0.64772    0.27791   2.331  0.02193 *
poly(x, 6, raw = FALSE)2  0.48158    0.27791   1.733  0.08643 .
poly(x, 6, raw = FALSE)3 -2.47613    0.27791  -8.910 4.14e-14 ***
poly(x, 6, raw = FALSE)4 -0.16435    0.27791  -0.591  0.55571
poly(x, 6, raw = FALSE)5  0.79114    0.27791   2.847  0.00543 **
poly(x, 6, raw = FALSE)6 -0.06273    0.27791  -0.226  0.82190
```

```
                        Estimate Std. Error t value Pr(>|t|)
(Intercept)              1.02278    0.02684  38.111  < 2e-16 ***
poly(x, 7, raw = FALSE)1  0.64772    0.26837   2.414  0.01778 *
poly(x, 7, raw = FALSE)2  0.48158    0.26837   1.794  0.07602 .
poly(x, 7, raw = FALSE)3 -2.47613    0.26837  -9.227 9.68e-15 ***
poly(x, 7, raw = FALSE)4 -0.16435    0.26837  -0.612  0.54179
poly(x, 7, raw = FALSE)5  0.79114    0.26837   2.948  0.00405 **
poly(x, 7, raw = FALSE)6 -0.06273    0.26837  -0.234  0.81569
poly(x, 7, raw = FALSE)7 -0.74618    0.26837  -2.780  0.00658 **
```

- Odd terms are significant in both specifications
- Coefficients appear stable (!)
- Usually coefficients decline in (odd) polynomial order.
- But very high dimensional polynomials will still overfit.

Orthgonal Polynomials

Consider an arbitrary basis:

$$y(x_i) = a_0 + \sum_{j=1}^{M} a_j b_j(x_i) + \varepsilon_i$$

- $b_j(x_i) = x_i^j$ (regular polynomials)
- $\langle b_j(x), b_k(x) \rangle = 0$ for $j \neq k$ (orthogonal polynomials).
- Lots of options: Chebyshev, Legendre, Fourier, Gram Schmidt (discuss later).

What R does: Gram Schmidt

Let $a_j = x^j$ (the raw polynomial) and then orthogonalize as follows:

$$\hat{p}_j(x) = a_j(x) - \sum_{k=0}^{j-1} p_k(x) \frac{p_k \cdot a_j}{p_k \cdot p_k}, \quad p(x) = \frac{\hat{p}(x)}{\hat{p}(1)}$$

- We are forming the residual of x^j on $(x^{j-1}, x^{j-2}, \ldots, x, 1)$ (where each term has already been residualized).
- Notice, we don't need to know y_i, we can simply residualized x_i against powers of itself.
- We could do this for any matrix X! (doesn't need to be powers of x_i).

Orthogonal Polynomials

General Case

- Space: polynomials over domain D
- Weighting function: $w(x) >$ (positive everywhere)
- Inner product $\langle f, g \rangle = \int_D f(x)g(x)w(x)dx$
- Polynomials are orthogonal wrt to $w(x)$ IFF

$$\langle \phi_i, \phi_j \rangle = 0, \quad i \neq j$$

- Can compute orthogonal polynomials using recurrence formulas

$$\begin{aligned}
\phi_0(x) &= 1 \\
\phi_1(x) &= x \\
\phi_{k+1}(x) &= (a_{k+1}x + b_k)\phi_k(x) + c_{k+1}\phi_{k-1}(x)
\end{aligned}$$

Chebyshev Polynomials

- Can compute orthogonal polynomials using recurrence formulas
- $[a, b] = [-1, 1]$ and $w(x) = (1 - x^2)^{-1/2}$
- $T_n(x) = \cos(n \cos^{-1} x)$
- Recursive Definition

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x) \end{aligned}$$

General Intervals

- Most problems aren't on the $[-1, 1]$ interval so we need a COV

$$y = -1 + 2\frac{x-a}{b-a}$$

Chebyshev Approximation Algorithm

1. Compute the $m \geq n+1$ Chebyshev nodes on $[-1, 1]$

$$z_k = -\cos\left(\frac{2k-1}{2m}\pi\right), \quad k = 1, \ldots, m$$

2. Adjust the nodes to $[a, b]$ interval

$$x_k = (z_k + 1)\left(\frac{b-a}{2}\right) + a, \quad k = 1, \ldots, m$$

3. Evaluate f at the nodes $w_k = f(x_k)$ for $k = 1, \ldots, m$
4. Compute the coefficients a_i to get the approximation $p(x)$

$$a_i = \frac{\sum_{k=1}^{m} w_k T_i(z_k)}{\sum_{k=1}^{m} T_i(z_k)^2}, \quad p(x) = \sum_{i=0}^{n} a_i T_i\left(2\frac{x-a}{b-a} - 1\right)$$

Minimax Approximation

- Data: (x_i, y_i), $i = 1, \ldots, n$
- Objective: L^∞ fit

$$\min_{\beta \in R^m} \max_i \|y_i - f(x_i; \beta)\|$$

- Difficult to do (minimax problems are non-convex)
- Chebyshev Approximation satisfies this property, for C^2, C^3 functions but doesn't get $f'(x)$ right!

Theorem
Suppose $f : [-1, 1] \to R$ is C^k for some $k \geq 1$, and let I_n be the degree n polynomial interpolation of f based at the zeroes of $T_{n+1}(x)$ then

$$\|f - I_n\|_\infty \leq \left(\frac{2}{\pi}\log(n+1) + 1\right)\frac{(n-k)!}{n!}\left(\frac{\pi}{2}\right)^k \left(\frac{b-a}{2}\right)^k \|f^{(k)}\|_\infty$$

Recap: Polynomials

$$y(x_i) = a_0 + \sum_{j=1}^{M} a_j b_j(x_i) + \varepsilon_i$$

- Thus far have looked at global polynomial approximations.
- Global: basis $b_j(x)$ and coefficients a_j are the same for any value of x.
- Can choose polynomial order with cross validation, later we will explore penalization.
- Special case of sieve estimators: we let the order $M \to \infty$ as $n \to \infty$ but not as quickly!