Econometrics 1

Practice Exam

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- Answer **all** of the following exercises in either German or English.
- Explain your answers and derivations.
- If you prefer a notation different from the one used in the course, define it.
- Always use significance level $a = 5\%$ (if not otherwise stated).
- Please report 3 decimal places in numerical answers.
- If not otherwise stated, assume the validity of the assumption A, B and C given in the lecture.

Exercise 1 (Understanding)

a. Consider the following confidence set for a parameter β_i:

$$Pr(0.5 < \beta_i < 1.5) = 0.95$$

Now test the following hypothesis without a concrete calculation:

i. $H_0 : \beta_i = 1.6$ vs. $H_1 : \beta_i \neq 1.6$ for $a = 5\%$
ii. $H_0 : \beta_i = 0.6$ vs. $H_1 : \beta_i \neq 0.6$ for $a = 5\%$
iii. $H_0 : \beta_i = 0.5$ vs. $H_1 : \beta_i \neq 0.5$ for $a = 10\%$

b. Formalize the optimization problem which is solved by the ML estimator and give its first order conditions (you need not solve the resulting system of equations).

Answer of exercise 1

a. i. As 1.6 is not in the 95% confidence set (0.5;1.5), we can reject the null hypothesis.
ii. As 0.6 is wihtin the 95% confidence set (0.5;1.5), we cannot reject the null hypothesis.
iii. Note that for $a = 10\%$ the confidence set will become smaller. Therefore, 0.5 will not lie in the confidence set anymore.

b. Log-Likelihood:

$$\ln L(\beta, \sigma) = \frac{-T}{2} \ln(2\pi) - \frac{T}{2} \ln(\sigma^2) - \frac{(y - X\beta)'(y - X\beta)}{2\sigma^2}$$

The necessary conditions are:

$$\frac{\partial \ln L}{\partial \beta} = \frac{X'(y - X\beta)}{\sigma^2} = 0$$

$$\frac{\partial \ln L}{\partial \sigma^2} = \frac{-T}{2\sigma^2} + \frac{(y - X\beta)'(y - X\beta)}{2\sigma^4} = 0$$

Exercise 2 (Cobb-Douglas Production Function)

Consider the following Cobb-Douglas production function

$$Y = \tau K^{\beta_1} L^{\beta_2}$$

where Y denotes total production, which is dependent on the used capital stock K and labor input L. Assume that technology τ is constant. The observation period contains quarterly data from 1990Q1 to 2009Q4

a. Derive the econometric model, which can be used to estimate the parameters of the production function.

b. Which parameters are elasticities?

c. Estimate the parameters of the model and test their significance. For this the following results are available:

$$(\mathbf{X}'\mathbf{X})^{-1} = \begin{pmatrix} 0.021 & 0.005 & 0.007 \\ 0.005 & 0.014 & -0.004 \\ 0.007 & -0.004 & 0.012 \end{pmatrix}, \qquad \mathbf{X}'\mathbf{y} = \begin{pmatrix} -552 \\ 381 \\ 496 \end{pmatrix}, \qquad \mathbf{y}'\mathbf{y} = 4012.214$$

where the first column of X contains ones, the second column the values of the capital stock in logs, the third column the labor input in logs and $\mathbf{y}$ is the vector with the values for production in logs.

d. Compute and interpret the coefficient of determination.

e. Test whether the model is statistically significant, i.e. $H_0 : R^2 = 0$ vs. $H_1 : R^2 \neq 0$.

f. Test the hypothesis that the Cobb-Douglas production function has constant elasticities to scale, i.e. $\beta_1 + \beta_2 = 1$

g. Determine a 98% and 95% confidence set for β_1. Do confidence sets get smaller or larger for increasing significance level a? Why?

Answer of exercise 2

a. Linearization yields:

$$\ln(F(K, L)) = \ln(\tau) + \beta_1 \ln(K) + \beta_2 \ln(L)$$

Hence the model to estimate is

$$y_t = \alpha + \beta_1 k_t + \beta_2 l_t + u_t$$

b. All variables are in logs, hence $\hat{\beta}_1$ and $\hat{\beta}_2$ are elasticities. If the capital stock increases by 1%, then output will increase by $\hat{\beta}_1$%. α is a constant, not an elasticity.

c.

$$\hat{\beta} = (X'X)^{-1}(X'y) = \begin{pmatrix} -6.215 \\ 0.590 \\ 0.564 \end{pmatrix}$$

where $T = (2009 - 1990 + 1) \cdot 4 = 80$. Furthermore: $\hat{u} = y - X\hat{\beta}$, $\hat{y} = X\hat{\beta}$, $S_{yy} = S_{\hat{y}\hat{y}} + S_{\hat{u}\hat{u}}$

$$y'y = \hat{y}'\hat{y} + \hat{u}'\hat{u} = \hat{\beta}'X'X\hat{\beta} + \hat{u}'\hat{u} = \hat{\beta}'X'X(X'X)^{-1}X'y + \hat{u}'\hat{u} = \hat{\beta}'X'y + \hat{u}'\hat{u}$$

Hence,

$$\hat{u}'\hat{u} = y'y - \hat{\beta}'X'y = 77$$

and

$$\hat{\sigma}^2 = \frac{\hat{u}'\hat{u}}{80 - 3} = 1$$

$$\hat{V}(\hat{\beta}) = \hat{\sigma^2}(X'X)^{-1} = (X'X)^{-1}$$

(i) $H_0 : \beta_1 = 0$ vs $H_1 : \beta_1 > 0$ (one-sided as negative elasticities are economically infeasible)

(ii) $H_0 : \beta_2 = 0$ vs $H_1 : \beta_2 > 0$ (one-sided as negative elasticicities are economically infeasible)

t-statistics:

$$t_1 = \frac{\hat{\beta}_1 - 0}{\sqrt{0.014}} = 4.986$$

$$t_2 = \frac{\hat{\beta}_2 - 0}{\sqrt{0.012}} = 5.148$$

Critical value is $t^{crit} = t_{0.95,77} = 1.684$
Decision for both: we reject H_0.

d.

$$S_{\hat{y}\hat{y}} = y'y = T\overline{y}^2 = 4012.214 - 80 \cdot (-552/80)^2 = 203.414$$

$$R^2 = 1 - \frac{S_{\hat{u}\hat{u}}}{S_{\hat{y}\hat{y}}} = 0.6215$$

62.15% of the variation in log output is explained by the variation in the exogenous variables.

e. $H_0 : R^2 = 0$ vs. $H_1 : R^2 > 0$ is equivalent to $H_0 : \beta_1 = 0$ and $\beta_2 = 0$ vs. H_1 : either $\beta_1 \neq 0$ or $\beta_2 \neq 0$ or both. We can test this with an F-test:

$$F = \frac{R^2/((K+1) - L)}{(1 - R^2)/(T - (K+1))} = \frac{0.6215/(3-1)}{(1 - 0.6215)/(80-3)} = 63.207$$

The critical value is $F^{crit} = F_{0.95,77} = [3.11, 3.12]$. Because $F > F^{crit}$ we reject the null hypothesis, the model is useful.

f. $H_0 : \beta_1 + \beta_2 = 1$ vs. $H_1 : \beta_1 + \beta_2 \neq 1$ F-test with $R\beta = q$:

$$\begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta_1 \\ \beta_2 \end{pmatrix} = 1$$

Test statistic:

$$F = \frac{1}{L}(R\hat{\beta} - q)' \left[R\hat{\sigma}^2 (X'X)^{-1} R' \right]^{-1} (R\hat{\beta} - q)$$

$$R\hat{\beta} - q = \hat{\beta}_1 + \hat{\beta}_2 - 1 = 0.154$$

With $L = 1$ and $\hat{\sigma}^2 = 1$ we get

$$F = 1.318$$

The critical value is $F^{crit} = F_{0.95,1,77} = [3.96, 3.97]$. Since $F < F^{crit}$ we cannot reject the null hypthesis, i.e. constant elasticities to scale might be true.

g.

$$Pr\left[\hat{\beta}_1 - t_{1-a/2,T-K-1} \cdot \sqrt{\hat{V}(\hat{\beta}_1)} \leq \beta_1 \leq \hat{\beta}_1 + t_{1-a/2,T-K-1} \cdot \sqrt{\hat{V}(\hat{\beta}_1)} \right] = 1 - a$$

For $a = 0.05$ we have $t_{0.975,77} = 1.96$ and

$$[0.358; 0.822]$$

For $a = 0.02$ we have $t_{1-\frac{0.02}{2},77} = 2.326$ and

$$[0.315; 0.865]$$

When a gets smaller, the type I error decreases, i.e. H_0 is rejected even if it is true. Type II errors, ie we do not reject H_0 even though it is false, increases. For the type I error to decrease, the intervall needs to get wider.

Exercise 3 (Labor Demand)

In order to estimate the labor demand, the following model is estimated:

$$\ln(n_t) = \alpha + \beta_1 \ln(p_t) + \beta_2 \ln(w_t) + u_t$$

where n_t denotes the labor demand (number of employees), p_t production (real GDP) and w_t the nominal wage. Consider a sample with yearly data covering 1970-2006.

a. Compute the value β_2 such that, when the nominal wage decreases by 5%, the number of employees will increase from 25 to 26 Millions.

b. An estimation with nominal wages in DM yields:

$$\hat{\alpha} = 0.8, \qquad \hat{\beta}_1 = 1.3, \qquad \hat{\beta}_2 = -1.5$$

Due to the change of currency to Euro the nominal wage is multiplied by the factor $1/1.956 = 0.5113$. How do the estimates change if one considers the nominal wage in Euros instead of in DM?

Answer of exercise 3

a.

$$\beta_2 = \frac{\partial \ln(n_t)}{\partial \ln(w_t)} = \frac{\frac{\partial n_t}{n_t}}{\frac{\partial w_t}{w_t}}$$

$$\frac{\partial n_t}{n_t} = \frac{26-25}{25} = 0.04$$

$$\frac{\partial w_t}{w_t} = -0.05$$

Hence $\beta_2 = \frac{0.04}{-0.05} = -0.8$.

b. New model:

$$\begin{aligned}\ln(n_t) &= \alpha + \beta_1 \ln(p_t) + \beta_2 \ln(w_t \cdot 0.5113) + u_t \\ &= \underbrace{\alpha + \beta_2 \ln(0.5113)}_{\alpha^*} + \beta_1 \ln(p_t) + \beta_2 \ln(w_t) + u_t\end{aligned}$$

only the estimate for the constant will change to $\hat{\alpha}^* = \hat{\alpha} + \hat{\beta}_2 \ln(0.5113) = 1.8062$, the other estimates will remain the same.

Exercise 4 (Estimating Functions)

Consider the estimation of a simple regression model without a constant term

$$y_t = \beta x_t + u_t, \qquad t = 1, ..., T$$

where it is assumed that the last observation may be erroneous. Therefore, the observation at time point T gets a lower (deterministic) weight, $0 \leq w < 1$, in the following estimating function:

$$\tilde{\beta}_w = \frac{\left(\sum_{t=1}^{T-1} x_t y_t\right) + w(x_T y_T)}{\left(\sum_{t=1}^{T-1} x_t^2\right) + w x_T^2}$$

a. Show that the estimating function $\tilde{\beta}_w$ is unbiased for β.
   Hint: Show that

$$\tilde{\beta}_w = \beta + \frac{(\sum_{t=1}^{T-1} x_t u_t) + w x_T u_T}{k}$$

   where $k = (\sum_{t=1}^{T-1} x_t^2) + w x_T^2$.

b. Compute the variance of $\tilde{\beta}_w$ for $w = 0.5$ provided that

$$\sum_{t=1}^{T} x_t^2 = 100, \qquad x_T = 2, \qquad \sigma^2 = 1$$

   Hint: Use the fact that $E(u_t u_s) = 0$ *for* $t \neq s$.

c. Do you expect the variance of the least squares estimator $\hat{\beta}$ to be smaller, equal or larger to the variance of $\tilde{\beta}_w$?

Answer of exercise 4

a. We need to show that $E[\tilde{\beta}_w] = \beta$. Proof:

$$\begin{aligned}
\tilde{\beta}_w &= \frac{(\sum_{t=1}^{T-1} x_t y_t) + w(x_T y_T)}{(\sum_{t=1}^{T-1} x_t^2) + w x_T^2} \\
&= \frac{(\sum_{t=1}^{T-1} x_t(\beta x_t + u_t)) + w(x_T(\beta x_T + u_T))}{(\sum_{t=1}^{T-1} x_t^2) + w x_T^2} \\
&= \frac{\beta \sum_{t=1}^{T-1} x_t^2 + \sum_{t=1}^{T-1} x_t u_t + \beta w x_T^2 + w x_T u_T}{(\sum_{t=1}^{T-1} x_t^2) + w x_T^2} \\
&= \frac{\beta(\sum_{t=1}^{T-1} x_t^2 + w x_T^2) + (\sum_{t=1}^{T-1} x_t u_t + w x_T u_T)}{(\sum_{t=1}^{T-1} x_t^2) + w x_T^2} \\
&= \beta + \frac{\sum_{t=1}^{T-1} x_t u_t + w x_T u_T}{(\sum_{t=1}^{T-1} x_t^2) + w x_T^2}
\end{aligned}$$

   As $E(u_t) = 0$ we now can show that $\tilde{\beta}_w = \beta$

b.

$$var(\tilde{\beta}_w) = E[(\tilde{\beta}_w - E(\tilde{\beta}_w))^2] = E[(\tilde{\beta}_w - \beta)^2] = E\left[\underbrace{\frac{1}{(\sum_{t=1}^{T-1} x_t^2 + w x_T^2)^2}}_{const}\left(\sum_{t=1}^{T-1} x_t u_t + w x_T u_T\right)^2\right]$$

$$= const \cdot E\left[\sum_{t=1}^{T-1} x_t u_t \sum_{t=1}^{T-1} x_t u_t + 2\sum_{t=1}^{T-1} x_t u_t w x_T u_T + w^2 x_T^2 u_T^2\right]$$

$$= const \cdot \sum_{s=1}^{T-1}\sum_{t=1}^{T-1} x_s x_t E[u_s u_t] + const \cdot 2\sum_{t=1}^{T-1} x_t w x_T E[u_t u_T] + const \cdot w^2 x_T^2 E[u_T^2]$$

Note that $E[u_t u_s] =$ for $t \neq s$, hence

$$var(\tilde{\beta}_w) = const \cdot \sum_{t=1}^{T-1} x_t^2 E[u_t u_t] + const \cdot w^2 x_T^2 \sigma^2$$
$$= const \cdot \left(\sum_{t=1}^{T-1} x_t^2 \sigma^2 + w^2 x_T^2 \sigma^2 \right)$$

Now we have that $\sum_{t=1}^{T-1} x_t^2 = \sum_{t=1}^{T} x_t^2 - x_T^2 = 100 - 4 = 96$. So for $w = 0.5$:

$$var(\tilde{\beta}_w) = \frac{96}{96 + 0.5 \cdot 4} \cdot 1 + \frac{0.25 \cdot 4}{96 + 0.5 \cdot 4} \cdot 1 = 0.9897$$

c. The least-square estimator is BLUE, hence we expect it to be smaller as the proposed estimator is not the OLS estimator unless $w = 1$.

Table of the $(1-a)$ quantiles of the t_ν-distribution

ν	a = 0.05	0.025	0.01
1	6.31380	12.70620	31.82050
2	2.92000	4.30270	6.96460
3	2.35340	3.18240	4.54070
4	2.13180	2.77640	3.74690
5	2.01500	2.57060	3.36490
6	1.94320	2.44690	3.14270
7	1.89460	2.36460	2.99800
8	1.85950	2.30600	2.89650
9	1.83310	2.26220	2.82140
10	1.81250	2.22810	2.76380
11	1.79590	2.20100	2.71810
12	1.78230	2.17880	2.68100
13	1.77090	2.16040	2.65030
14	1.76130	2.14480	2.62450
15	1.75310	2.13140	2.60250
16	1.74590	2.11990	2.58350
17	1.73960	2.10980	2.56690
18	1.73410	2.10090	2.55240
19	1.72910	2.09300	2.53950
20	1.72470	2.08600	2.52800
21	1.72070	2.07960	2.51760
22	1.71710	2.07390	2.50830
23	1.71390	2.06870	2.49990
24	1.71090	2.06390	2.49220
25	1.70810	2.05950	2.48510
26	1.70560	2.05550	2.47860
27	1.70330	2.05180	2.47270
28	1.70110	2.04840	2.46710
29	1.69910	2.04520	2.46200
30	1.69730	2.04230	2.45730
31	1.69550	2.03950	2.45280
32	1.69390	2.03690	2.44870
33	1.69240	2.03450	2.44480
34	1.69090	2.03220	2.44110
35	1.68960	2.03010	2.43770
36	1.68830	2.02810	2.43450
37	1.68710	2.02620	2.43140
38	1.68600	2.02440	2.42860
39	1.68490	2.02270	2.42580
40	1.68390	2.02110	2.42330
>40	1.645	1.960	2.326

Table of the $(1-a)$ quantiles of the χ^2_ν-distribution

ν	a = 0.05	0.025	0.01
1	3.84	5.02	6.63
2	5.99	7.38	9.21
3	7.82	9.35	11.35
4	9.49	11.14	13.28
5	11.07	12.83	15.09
6	12.59	14.45	16.81
7	14.07	16.01	18.48
8	15.51	17.54	20.09
9	16.92	19.02	21.67
10	18.31	20.48	23.21
11	19.68	21.92	24.73
12	21.03	23.34	26.22
13	22.36	24.74	27.69
14	23.68	26.12	29.14
15	25.00	27.49	30.58
16	26.30	28.84	32.00
17	27.59	30.19	33.41
18	28.87	31.53	34.81
19	30.14	32.85	36.19
20	31.41	34.17	37.57
21	32.67	35.48	38.93
22	33.92	36.78	40.29
23	35.17	38.08	41.64
24	36.41	39.36	42.98
25	37.65	40.65	44.31
26	38.88	41.92	45.64
27	40.11	43.20	46.96
28	41.34	44.46	48.28
29	42.56	45.72	49.59
30	43.77	46.98	50.89
35	49.80	53.20	57.34
40	55.76	59.34	63.69
45	61.66	65.41	69.96
50	67.50	71.42	76.15
55	73.31	77.38	82.29
60	79.08	83.30	88.38
65	84.82	89.18	94.42
70	90.53	95.02	100.43
75	96.22	100.84	106.39
80	101.88	106.63	112.33
85	107.52	112.39	118.24
90	113.15	118.14	124.12
95	118.75	123.86	129.97
100	124.34	129.56	135.81

Table of the 0.95 quantiles of the F_{ν_1,ν_2}-distribution

	ν_1									
ν_2	1	2	3	4	5	10	15	20	25	50
1	161.45	199.50	215.71	224.58	230.16	241.88	245.95	248.01	249.26	251.77
2	18.51	19.00	19.16	19.25	19.30	19.40	19.43	19.45	19.46	19.48
3	10.13	9.55	9.28	9.12	9.01	8.79	8.70	8.66	8.63	8.58
4	7.71	6.94	6.59	6.39	6.26	5.96	5.86	5.80	5.77	5.70
5	6.61	5.79	5.41	5.19	5.05	4.74	4.62	4.56	4.52	4.44
6	5.99	5.14	4.76	4.53	4.39	4.06	3.94	3.87	3.83	3.75
7	5.59	4.74	4.35	4.12	3.97	3.64	3.51	3.44	3.40	3.32
8	5.32	4.46	4.07	3.84	3.69	3.35	3.22	3.15	3.11	3.02
9	5.12	4.26	3.86	3.63	3.48	3.14	3.01	2.94	2.89	2.80
10	4.96	4.10	3.71	3.48	3.33	2.98	2.85	2.77	2.73	2.64
15	4.54	3.68	3.29	3.06	2.90	2.54	2.40	2.33	2.28	2.18
20	4.35	3.49	3.10	2.87	2.71	2.35	2.20	2.12	2.07	1.97
25	4.24	3.39	2.99	2.76	2.60	2.24	2.09	2.01	1.96	1.84
30	4.17	3.32	2.92	2.69	2.53	2.16	2.01	1.93	1.88	1.76
35	4.12	3.27	2.87	2.64	2.49	2.11	1.96	1.88	1.82	1.70
40	4.08	3.23	2.84	2.61	2.45	2.08	1.92	1.84	1.78	1.66
45	4.06	3.20	2.81	2.58	2.42	2.05	1.89	1.81	1.75	1.63
50	4.03	3.18	2.79	2.56	2.40	2.03	1.87	1.78	1.73	1.60
55	4.02	3.16	2.77	2.54	2.38	2.01	1.85	1.76	1.71	1.58
60	4.00	3.15	2.76	2.53	2.37	1.99	1.84	1.75	1.69	1.56
65	3.99	3.14	2.75	2.51	2.36	1.98	1.82	1.73	1.68	1.54
70	3.98	3.13	2.74	2.50	2.35	1.97	1.81	1.72	1.66	1.53
75	3.97	3.12	2.73	2.49	2.34	1.96	1.80	1.71	1.65	1.52
80	3.96	3.11	2.72	2.49	2.33	1.95	1.79	1.70	1.64	1.51
85	3.95	3.10	2.71	2.48	2.32	1.94	1.79	1.70	1.64	1.50
90	3.95	3.10	2.71	2.47	2.32	1.94	1.78	1.69	1.63	1.49
95	3.94	3.09	2.70	2.47	2.31	1.93	1.77	1.68	1.62	1.48
100	3.94	3.09	2.70	2.46	2.31	1.93	1.77	1.68	1.62	1.48