Econometrics 1

Exam

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- Answer **all** of the following exercises in either German or English.
- Explain your answers and derivations. All your computations and intermediate steps need to be verifiable and understandable.
- Formulas which we covered in the lecture and class need not to be derived again.
- If you prefer a notation different from the one used in the course, define it.
- Always use significance level $a = 5\%$.
- Please report 3 decimal places in numerical answers.
- If not otherwise stated, assume the validity of the assumption A, B and C given in the lecture.
- Permissible aids:
  - non-programmable pocket calculator
  - cheat sheet: one-sided A4 white sheet of paper with annotations, formulas, texts, sketches, etc.

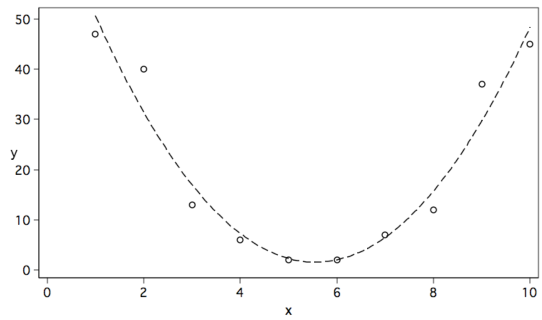
Exercise 1 (Understanding)

a. State one of the A, B and C assumptions introduced in the lecture in your own words. Provide a counter example by drawing a picture of points (scatterplot), if possible. Otherwise describe the counterexample in your own words.

b. Explain the idea of the Likelihood-Ratio test in two sentences.

Answer of exercise 1

a. The relationship between the endogenous variable y_t and x_t is linear. A counter example:



b. If the maximal likelihood under the restrictions $L(\hat{\beta}_R, \hat{\sigma}^2_R)$ is significantly lower than the maximal likelihood without restrictions $L(\hat{\beta}_{ML}, \hat{\sigma}^2_{ML})$ then we reject the null hypothesis.

Exercise 2 (Money Demand)

The demand for money is determined using the regression model

$$y_t = \alpha + \beta_1 x_{1t} + \beta_2 x_{2t} + u_t$$

where

$$\begin{aligned} y_t &: \text{real money stock in logs} \\ x_{1t} &: \text{real income in logs} \\ x_{2t} &: \text{interest rate in \%} \end{aligned}$$

Given quarterly data for the period 1970-1996 ($T = 108$), a least squares estimation shows that

$$\hat{y}_t = -8.2 + 1.5x_{1t} - 0.01x_{2t}$$

and

$$R^2 = 0.95$$

The estimated covariance matrix of $\hat{\beta}$ is given by

$$\widehat{\mathrm{Var}}(\hat{\beta}) = \hat{\sigma}^2(X'X)^{-1} = \begin{pmatrix} 0.012 & -0.002 & 0 \\ & 0.002 & 0 \\ & & 0.001 \end{pmatrix}$$

Furthermore, we have that

$$\frac{1}{T}(\mathbf{y}'\mathbf{y} - T\bar{y}^2) = 20$$

where $\mathbf{y}$ denotes the $T \times 1$ vector with y_t in the t-th row.

a. What do the estimated values $\hat{\beta}_1$ and $\hat{\beta}_2$ mean for the effect of the corresponding variables on the demand for money?

b. Test the following hypotheses:
   i. $H_0 : \beta_1 = 1$ vs. $H_1 : \beta_1 \neq 1$
   ii. $H_0 : \beta_2 = 0$ vs. $H_1 : \beta_2 < 0$
   iii. $H_0 : \beta_1 = 1$ and $\beta_2 = 0$ vs. H_1 : the null hypothesis is not true.

c. Compute the maximum likelihood estimators for α, β_1, β_2 and σ^2.
   Hint: $R^2 = 1 - \frac{S_{\hat{u}\hat{u}}}{S_{yy}}$.

Answer of exercise 2

a. Both variables, real money stock and real income, are in logs, hence $\hat{\beta}_1$ is an elasticity which measure the percentage change of real money demand if the real income changes by 1%. $\hat{\beta}_2$ measures the percentage increase of real money demand, if the interest rate increases by one point, it is a semi-elasticity.

b. i. $H_0 : \beta_1 = 1$ vs. $H_1 : \beta_1 \neq 1$ Two-sided test:

$$t = \frac{\hat{\beta}_1 - 1}{\hat{se}(\hat{\beta}_1)} = \frac{1.5 - 1}{\sqrt{0.002}} = 11.1$$

   $t^{crit} = t_{1-\alpha/2,T-3} = 1.96$. Since $|t| > t^{crit}$, we reject the null hypothesis.

   ii. $H_0 : \beta_2 = 0$ vs. $H_1 : \beta_2 < 0$ One-sided test:

$$t = \frac{-0.01 - 0}{\sqrt{0.001}} = -0.3165$$

   $t^{crit} = t_{1-\alpha,T-3} = 1.645$. Since $t < t^{crit}$, we cannot reject the null hypothesis.

iii. $H_0: \beta_1 = 1$ and $\beta_2 = 0$ vs. H_1: not H_0. F-test: $H_0: R\beta = q$ with $R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $q = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $L = 2$, then

$$F = \frac{(R\hat{\beta} - q)'[R(X'X)^{-1}R']^{-1}(R\hat{\beta} - q)/L}{\hat{\sigma}^2}$$

Note that we have $\hat{\sigma}^2(X'X)^{-1}$, so

$$F = (R\hat{\beta} - q)'[R\hat{\sigma}^2(X'X)^{-1}R']^{-1}(R\hat{\beta} - q)/L = 62.55$$

$F^{crit} = F^(1-a)_{2,101} = 3.09$. As $F > F^{crit}$ we reject the null hypothesis.

c. $R^2 = 1 - \frac{S_{\hat{u}\hat{u}}}{S_{yy}} = 0.95$, $S_{\hat{u}\hat{u}} = (1 - 0.95)S_{yy}$, $S_{yy} = \sum(y_t - \bar{y})^2 = \sum y_t^2 - T\bar{y}^2 = 108 \cdot 20 = 2160$. Hence, $S_{\hat{u}\hat{u}} = 108$, then $\hat{\sigma}^2_{ML} = \frac{1}{108}108 = 1$.
$\hat{\beta}_{1,ML} = \hat{\beta}_1 = 1.5$, $\hat{\beta}_{2,ML} = \hat{\beta}_2 = -0.01$ and $\hat{\alpha}_{ML} = \hat{\alpha} = -8.2$

Exercise 3 (Capital-Asset-Pricing-Model)

Consider the following regression model for an extended Capital-Asset-Pricing-Model (CAPM):

$$y_t = \alpha + \beta_1 x_{1t} + \beta_2 x_{2t} + u_t$$

where

$$\begin{aligned} y_t &: \text{rate of return of BMW in \%} \\ x_{1t} &: \text{rate of return of the DAX index in \%} \\ x_{2t} &: \text{price-earnings (P/E) ratio of BMW} \end{aligned}$$

Note that the P/E ratio is defined as the ratio of a company's stock price to the company's earnings per share. For daily observations we have the following intermediate results:

$$\mathbf{X}'\mathbf{X} = \begin{pmatrix} 100 & 30 & 20 \\ 30 & 110 & 7 \\ 20 & 7 & 200 \end{pmatrix},$$

$$(\mathbf{X}'\mathbf{X})^{-1} = \begin{pmatrix} 0.011 & -0.003 & -0.001 \\ -0.003 & 0.010 & 0 \\ -0.001 & 0 & 0.005 \end{pmatrix},$$

$$\mathbf{X}'\mathbf{y} = \begin{pmatrix} 29.29 \\ 58.79 \\ 45.86 \end{pmatrix},$$

$$\mathbf{y}'\mathbf{y} = 138.496$$

a. Estimate a 95% confidence interval for β_2 given the least squares estimator $\hat{\beta}_2$.

b. Your professor argues that x_{2t} is not a relevant variable and wants you to reestimate the model without it. Do you agree with your professor given the current dataset? Briefly outline the possible consequences of omitting x_{2t}.

c. Assume that the true value of β_2 is 0.2. Compute the bias of the least squares estimators $\hat{\alpha}$ and $\hat{\beta}_1$ if one estimates

$$y_t = \alpha + \beta_1 x_{1t} + v_t$$

where v_t are the error terms of the simplified regression model.

Hint: $\begin{pmatrix} 100 & 30 \\ 30 & 110 \end{pmatrix}^{-1} = \begin{pmatrix} 0.011 & -0.003 \\ -0.003 & 0.010 \end{pmatrix}$

Answer of exercise 3

a.

$$\hat{\beta} = (X'X)^{-1}(X'y) = \begin{pmatrix} 0.1 \\ 0.5 \\ 0.2 \end{pmatrix}$$

$$\hat{V}(\hat{\beta}) = \hat{\sigma}^2 (X'X)^{-1}$$

Due to the variance decomposition, we have

$$\hat{u}'\hat{u} = y'y - \hat{\beta}'X'y = 138.495 - \begin{pmatrix} 0.1 & 0.5 & 0.2 \end{pmatrix} \begin{pmatrix} 29.29 \\ 58.79 \\ 45.86 \end{pmatrix} = 97$$

$$\hat{\sigma}^2 = \frac{1}{100-3} \cdot 97 = 1$$

$$\hat{V}(\hat{\beta}_3) = 0.005\hat{\sigma}^2 = 0.005$$

$$t_{1-\alpha/2} = 1.9847$$

The interval is

$$[\hat{\beta}_3 \pm t_{1-\alpha/2}\hat{V}(\hat{\beta}_3)]$$

b. Omitted variable bias! Since $X'X$ is not diagonal, there is a bias.

c. We have

$$E(\hat{\beta}_a) - \beta_a = (X_a'X_a)^{-1}X_a'X_2\beta_2$$

Re-estimation:

$$X'X = \begin{pmatrix} X_a'X_a & X_a'X_2 \\ X_2'X_a & X_2'X_2 \end{pmatrix}$$

$$(X_a'X_a)^{-1} = \begin{pmatrix} 100 & 30 \\ 30 & 100 \end{pmatrix}^{-1} = \begin{pmatrix} 0.011 & -0.003 \\ -0.003 & 0.010 \end{pmatrix}$$

Hence the bias is equal to

$$\begin{pmatrix} 0.011 & -0.003 \\ -0.003 & 0.010 \end{pmatrix} \begin{pmatrix} 20 \\ 7 \end{pmatrix} \cdot 0.2 = \begin{pmatrix} 0.0398 \\ 0.002 \end{pmatrix}$$

Exercise 4 (Estimating Functions)

Consider the simple linear regression model

$$y_t = \alpha + \beta x_t + u_t, \qquad t = 1, ..., 20$$

and the following estimating function for the slope parameter β:

$$\tilde{\beta} = \frac{\bar{y}_2 - \bar{y}_1}{\bar{x}_2 - \bar{x}_1}$$

where

$$\bar{y}_1 = \frac{1}{10}\sum_{t=1}^{10} y_t, \quad \bar{y}_2 = \frac{1}{10}\sum_{t=11}^{20} y_t, \quad \bar{x}_1 = \frac{1}{10}\sum_{t=1}^{10} x_t, \quad \bar{x}_2 = \frac{1}{10}\sum_{t=11}^{20} x_t$$

a. Show that $\tilde{\beta}$ is an unbiased estimating function for β.
Hint: Show that

$$\tilde{\beta} = \beta + \frac{\sum_{t=11}^{20} u_t - \sum_{t=1}^{10} u_t}{k}$$

where $k = \sum_{t=11}^{20} x_t - \sum_{t=1}^{10} x_t$.

b. For the next two exercises assume that

$$\sum_{t=1}^{20}(x_t - \bar{x})^2 = 40, \quad \sigma^2 = 1, \quad \bar{x}_1 = 3, \quad \bar{x}_2 = 5.$$

   i. Compute the variance of $\tilde{\beta}$.
   Hint: Use the fact that $E(u_t u_s) = 0$ *for* $t \neq s$.

   ii. Compute the variance of the least-squares estimator $\hat{\beta}$ and compare it to the one of $\tilde{\beta}$.

Answer of exercise 4

a. We need to show that $E(\tilde{\beta}) = \beta$. Proof:

$$\begin{aligned}
\tilde{\beta} &= \frac{\bar{y}_2 - \bar{y}_1}{\bar{x}_2 - \bar{x}_1} \\
&= \frac{1/10\sum_{t=11}^{20} y_t - 1/10\sum_{t=1}^{10} y_t}{1/10\sum_{t=11}^{20} x_t - 1/10\sum_{t=1}^{10} x_t} \\
&= \frac{\sum_{t=11}^{20} y_t - \sum_{t=1}^{10} y_t}{\sum_{t=11}^{20} x_t - \sum_{t=1}^{10} x_t} \\
&= \frac{\sum_{t=11}^{20}(\alpha + \beta x_t + u_t) - \sum_{t=1}^{10}(\alpha + \beta x_t + u_t)}{\sum_{t=11}^{20} x_t - \sum_{t=1}^{10} x_t} \\
&= \frac{10\alpha + \beta\sum_{t=11}^{20} x_t + \sum_{t=11}^{20} u_t - 10\alpha - \beta\sum_{t=1}^{10} x_t - \sum_{t=1}^{10} u_t}{\sum_{t=11}^{20} x_t - \sum_{t=1}^{10} x_t} \\
&= \frac{\beta(\sum_{t=11}^{20} x_t - \sum_{t=1}^{10} x_t) + \sum_{t=11}^{20} u_t - \sum_{t=1}^{10} u_t}{\sum_{t=11}^{20} x_t - \sum_{t=1}^{10} x_t} \\
&= \beta + \frac{\sum_{t=11}^{20} u_t - \sum_{t=1}^{10} u_t}{\sum_{t=11}^{20} x_t - \sum_{t=1}^{10} x_t}
\end{aligned}$$

Since $E(u_t) = 0$, we have that

$$E(\tilde{\beta}) = \beta$$

b. i.

$$var(\tilde{\beta}) = E[(\tilde{\beta} - E(\tilde{\beta}))^2] = E[(\tilde{\beta} - \beta)^2] = E\left[\left(\frac{\sum_{t=11}^{20} u_t - \sum_{t=1}^{10} u_t}{\sum_{t=11}^{20} x_t - \sum_{t=1}^{10} x_t}\right)\right]$$

$$= \frac{1}{(\sum_{t=11}^{20} x_t - \sum_{t=1}^{10} x_t)^2} \cdot E\left[\sum_{t=11}^{20} u_t \sum_{t=11}^{20} u_t - 2\sum_{t=11}^{20} u_t \sum_{t=1}^{10} u_t + \sum_{t=1}^{10} u_t \sum_{t=1}^{10} u_t\right]$$

Note that $E[u_t u_s] =$ for $t \neq s$, hence

$$\tilde{\beta} = \frac{1}{(\sum_{t=11}^{20} x_t - \sum_{t=1}^{10} x_t)^2} \cdot \left[\sum_{t=11}^{20} E[u_t^2] + \sum_{t=1}^{10} E[u_t^2]\right]$$

$$= \frac{1}{(\sum_{t=11}^{20} x_t - \sum_{t=1}^{10} x_t)^2} \cdot \left[10\sigma^2 + 10\sigma^2\right] = \frac{20\sigma^2}{(\sum_{t=11}^{20} x_t - \sum_{t=1}^{10} x_t)^2}$$

We have that $\bar{x}_1 \cdot 10 = \sum_{t=1}^{10} x_t = 3 \cdot 10 = 30$ and $\sum_{t=11}^{20} x_t = 5 \cdot 10 = 50$. Hence

$$var(\tilde{\beta}) = \frac{20 \cdot 1}{(50 - 30)^2} = \frac{1}{20}$$

ii. Variance of OLS estimator:

$$var(\hat{\beta}) = \frac{\sigma^2}{\sum_{t=1}^{20} (x_t - \bar{x})^2} = \frac{1}{40}$$

OLS is BLUE, hence it has a smaller variance that $\tilde{\beta}$.

Table of quantiles of the t_ν-distribution, given are the $(1-a)$-quantiles

ν	a 0.05	0.025	0.01
1	6.31380	12.70620	31.82050
2	2.92000	4.30270	6.96460
3	2.35340	3.18240	4.54070
4	2.13180	2.77640	3.74690
5	2.01500	2.57060	3.36490
6	1.94320	2.44690	3.14270
7	1.89460	2.36460	2.99800
8	1.85950	2.30600	2.89650
9	1.83310	2.26220	2.82140
10	1.81250	2.22810	2.76380
11	1.79590	2.20100	2.71810
12	1.78230	2.17880	2.68100
13	1.77090	2.16040	2.65030
14	1.76130	2.14480	2.62450
15	1.75310	2.13140	2.60250
16	1.74590	2.11990	2.58350
17	1.73960	2.10980	2.56690
18	1.73410	2.10090	2.55240
19	1.72910	2.09300	2.53950
20	1.72470	2.08600	2.52800
21	1.72070	2.07960	2.51760
22	1.71710	2.07390	2.50830
23	1.71390	2.06870	2.49990
24	1.71090	2.06390	2.49220
25	1.70810	2.05950	2.48510
26	1.70560	2.05550	2.47860
27	1.70330	2.05180	2.47270
28	1.70110	2.04840	2.46710
29	1.69910	2.04520	2.46200
30	1.69730	2.04230	2.45730
31	1.69550	2.03950	2.45280
32	1.69390	2.03690	2.44870
33	1.69240	2.03450	2.44480
34	1.69090	2.03220	2.44110
35	1.68960	2.03010	2.43770
36	1.68830	2.02810	2.43450
37	1.68710	2.02620	2.43140
38	1.68600	2.02440	2.42860
39	1.68490	2.02270	2.42580
40	1.68390	2.02110	2.42330
> 40	1.645	1.960	2.326

Table of quantiles of the χ^2_ν-distribution, given are $(1-a)$-quantiles

ν	a 0.05	0.025	0.01
1	3.84	5.02	6.63
2	5.99	7.38	9.21
3	7.82	9.35	11.35
4	9.49	11.14	13.28
5	11.07	12.83	15.09
6	12.59	14.45	16.81
7	14.07	16.01	18.48
8	15.51	17.54	20.09
9	16.92	19.02	21.67
10	18.31	20.48	23.21
11	19.68	21.92	24.73
12	21.03	23.34	26.22
13	22.36	24.74	27.69
14	23.68	26.12	29.14
15	25.00	27.49	30.58
16	26.30	28.84	32.00
17	27.59	30.19	33.41
18	28.87	31.53	34.81
19	30.14	32.85	36.19
20	31.41	34.17	37.57
21	32.67	35.48	38.93
22	33.92	36.78	40.29
23	35.17	38.08	41.64
24	36.41	39.36	42.98
25	37.65	40.65	44.31
26	38.88	41.92	45.64
27	40.11	43.20	46.96
28	41.34	44.46	48.28
29	42.56	45.72	49.59
30	43.77	46.98	50.89
35	49.80	53.20	57.34
40	55.76	59.34	63.69
45	61.66	65.41	69.96
50	67.50	71.42	76.15
55	73.31	77.38	82.29
60	79.08	83.30	88.38
65	84.82	89.18	94.42
70	90.53	95.02	100.43
75	96.22	100.84	106.39
80	101.88	106.63	112.33
85	107.52	112.39	118.24
90	113.15	118.14	124.12
95	118.75	123.86	129.97
100	124.34	129.56	135.81

Table of the quantiles of the F_{ν_1,ν_2}-distribution, given are the 0.95 -quantiles (i.e. $a = 0.05$)

	ν_1									
ν_2	1	2	3	4	5	10	15	20	25	50
1	161.45	199.50	215.71	224.58	230.16	241.88	245.95	248.01	249.26	251.77
2	18.51	19.00	19.16	19.25	19.30	19.40	19.43	19.45	19.46	19.48
3	10.13	9.55	9.28	9.12	9.01	8.79	8.70	8.66	8.63	8.58
4	7.71	6.94	6.59	6.39	6.26	5.96	5.86	5.80	5.77	5.70
5	6.61	5.79	5.41	5.19	5.05	4.74	4.62	4.56	4.52	4.44
6	5.99	5.14	4.76	4.53	4.39	4.06	3.94	3.87	3.83	3.75
7	5.59	4.74	4.35	4.12	3.97	3.64	3.51	3.44	3.40	3.32
8	5.32	4.46	4.07	3.84	3.69	3.35	3.22	3.15	3.11	3.02
9	5.12	4.26	3.86	3.63	3.48	3.14	3.01	2.94	2.89	2.80
10	4.96	4.10	3.71	3.48	3.33	2.98	2.85	2.77	2.73	2.64
15	4.54	3.68	3.29	3.06	2.90	2.54	2.40	2.33	2.28	2.18
20	4.35	3.49	3.10	2.87	2.71	2.35	2.20	2.12	2.07	1.97
25	4.24	3.39	2.99	2.76	2.60	2.24	2.09	2.01	1.96	1.84
30	4.17	3.32	2.92	2.69	2.53	2.16	2.01	1.93	1.88	1.76
35	4.12	3.27	2.87	2.64	2.49	2.11	1.96	1.88	1.82	1.70
40	4.08	3.23	2.84	2.61	2.45	2.08	1.92	1.84	1.78	1.66
45	4.06	3.20	2.81	2.58	2.42	2.05	1.89	1.81	1.75	1.63
50	4.03	3.18	2.79	2.56	2.40	2.03	1.87	1.78	1.73	1.60
55	4.02	3.16	2.77	2.54	2.38	2.01	1.85	1.76	1.71	1.58
60	4.00	3.15	2.76	2.53	2.37	1.99	1.84	1.75	1.69	1.56
65	3.99	3.14	2.75	2.51	2.36	1.98	1.82	1.73	1.68	1.54
70	3.98	3.13	2.74	2.50	2.35	1.97	1.81	1.72	1.66	1.53
75	3.97	3.12	2.73	2.49	2.34	1.96	1.80	1.71	1.65	1.52
80	3.96	3.11	2.72	2.49	2.33	1.95	1.79	1.70	1.64	1.51
85	3.95	3.10	2.71	2.48	2.32	1.94	1.79	1.70	1.64	1.50
90	3.95	3.10	2.71	2.47	2.32	1.94	1.78	1.69	1.63	1.49
95	3.94	3.09	2.70	2.47	2.31	1.93	1.77	1.68	1.62	1.48
100	3.94	3.09	2.70	2.46	2.31	1.93	1.77	1.68	1.62	1.48